Calc. Var. (2016) 55:32
DOI 10.1007/s00526-016-0970-2

Calculus of Variations

Uniqueness and nondegeneracy of positive radial solutions of div $(\rho\nabla\mathbf{u}) + \rho(-\mathbf{gu} + \mathbf{hu^p}) = 0$

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Received: 13 November 2014 / Accepted: 24 January 2016 / Published online: 7 March 2016

Abstract We study the uniqueness and nondegeneracy of positive solutions of div $(\rho\nabla u) + \rho(-gu + hu^p) = 0$ in a ball, the entire space, an annulus, or an exterior domain under the Dirichlet boundary condition.

Mathematics Subject Classification 35A02 · 34B18 · 35B09 · 35J15

1 Introduction

We study the uniqueness and nondegeneracy of positive radial solutions of the problem

$$\operatorname{div}(\rho(|x|)\nabla u(x)) + \rho(|x|)(-g(|x|)u(x) + h(|x|)u(x)^p) = 0 \quad \text{in } B_R \tag{1.1}$$

under the boundary condition

$$\begin{cases} u(x) = 0 \quad \text{for} \quad |x| = R \quad \text{in the case of} \quad 0 < R < \infty, \\ u(x) \to 0 \quad \text{as} \quad |x| \to \infty \quad \text{in the case of} \quad R = \infty. \end{cases}$$

Communicated by C. S. Lin.

Naoki Shioji: This work is partially supported by the Grant-in-Aid for Scientific Research (C) (No. 26400160) from Japan Society for the Promotion of Science. K. Watanabe: This work is partially supported by the Grant-in-Aid for Scientific Research (C) (No. 24540199) from Japan Society for the Promotion of Science.

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Here, $n \geq 2$, $p > 1$, $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ with $R \in (0, \infty]$, and $\rho, g, h : (0, R) \to \mathbb{R}$ are appropriate functions. If $\rho(r) \equiv 1$, then (1.1) is

$$\Delta u(x) - g(|x|)u(x) + h(|x|)u(x)^p = 0 \quad \text{in } B_R.$$

These problems include many important equations, like the scalar field equation, and they were studied by many researchers; see [8,9,11,14,22–26,32,33,36–38,42,44,45,47–49,53–56] and the references therein. Recently, in [47], we introduced a new generalized Pohožaev function and we studied the uniqueness of positive solutions of problem (1.1). We showed the result is applicable to various examples. However, for some examples, in the case when $n = 2$, we could not show the uniqueness of positive solutions. For instance, we consider the problem

$$\Delta u - (\lambda + |x|^2)u + u^p = 0 \quad \text{in } \mathbb{R}^n \quad \text{and} \quad u(x) \to 0 \quad \text{as } |x| \to \infty, \tag{1.2}$$

where $n \in \mathbb{N}$ with $n \geq 2$, $\lambda > -n$, $1 < p < \infty$ in the case $n = 2$ and $1 < p \leq (n+2)/(n-2)$ in the case $n \geq 3$. In [47], if $n > 2$, we could show the uniqueness of positive solutions of (1.2), but we could not show its uniqueness in the case $n = 2$.

In this paper, first, we show that if our generalized Pohožaev function is nontrivial and nonnegative for each positive radial solution of (1.1), then the problem has at most one positive radial solution. We can see that (B5) (i) (with other assumptions) in Theorem 1, given in [47], is one of sufficient conditions. We give another sufficient condition (B5) (ii) in Theorem 1. The new condition seems to be useful in the case of $n = 2$. Next, we study the existence of a unique positive radial solution of (1.1). We note that Theorem 1 says nothing on the existence of a solution. Adding some assumptions to those of Theorem 1 and applying the variational method, we show the existence of a unique positive radial solution of problem (1.1). One of the difficulties is to show that the solution obtained by the variational method does not diverge at the origin. Using a subsolution estimate, we show that the obtained solution is continuous at the origin. Next, we study the nondegeneracy of the unique positive solution in a radial function space. Our assumptions for the nondegeneracy result are essentially same as those for the unique existence result, and our uniqueness theorem plays an important role to show the nondegeneracy in a radial function space. Moreover, we study the nondegeneracy of the positive radial solution of (1.1) in a full space (including nonradial functions). Further, we study the uniqueness of positive radial solutions of

$$\operatorname{div}(\rho(|x|)\nabla u(x)) + \rho(|x|)(-g(|x|)u(x) + h(|x|)u(x)^p) = 0 \quad \text{in } A_{R',R} \tag{1.3}$$

under the boundary condition

$$u(x) = 0 \quad \text{for} \quad |x| = R' \quad \text{and} \quad \begin{cases} u(x) = 0 & \text{for} \quad |x| = R \quad \text{if} \quad R' < R < \infty, \\ u(x) \to 0 & \text{as} \quad |x| \to \infty \quad \text{if} \quad R = \infty, \end{cases}$$

where $0 < R' < R \leq \infty$, $p > 1$, $n \geq 1$ and $A_{R',R} = \{x \in \mathbb{R}^n : R' < |x| < R\}$. We also study the nondegeneracy of the unique positive solution of (1.3) in a radial function space. We note that our uniqueness and nondegeneracy results cover the results in [8,14,22], in which the cases $\rho(r) \equiv 1$ and $h(r) \equiv 1$ were studied.

This paper is organized as follows. In the next section, we recall the generalized Pohožaev identity introduced in [47]. In Sect. 3, we show our uniqueness and nondegeneracy results for (1.1), and in Sect. 4, we give the proofs of them. In Sect. 5, we study the nondegeneracy in the case when our Pohožaev function is identically zero. In Sect. 6, we show our uniqueness and nondegeneracy results for (1.3). In the final section, we show some examples to which our results are applicable and we give some new uniqueness results.

2 Generalized Pohožaev identity

We recall the generalized Pohožaev identity introduced in [47]. Setting $f(r) = r^{n-1}\rho(r)$, we can consider

$$u_{rr}(r) + \frac{f_r(r)}{f(r)}u_r - g(r)u(r) + h(r)u(r)^p = 0$$

for radial solutions of (1.1). We note that $g(r)$ was used in [47] instead of $-g(r)$.

Proposition 1 *Let $-\infty < R' < R \le \infty$, $g \in C^1((R', R))$ and $f, h \in C^3((R', R))$ such that f, h are positive in (R', R). If $p > 1$ and $u \in C^2((R', R))$ is a positive solution of*

$$u_{rr}(r) + \frac{f_r(r)}{f(r)}u_r(r) - g(r)u(r) + h(r)u(r)^p = 0 \quad in \ (R', R),$$

then

$$\frac{d}{dr}J(r; u) = G(r)u(r)^2 \quad in\,(R', R),$$

where

$$\begin{aligned} J(r; u) &= \frac{1}{2}a(r)u_r(r)^2 + b(r)u_r(r)u(r) + \frac{1}{2}c(r)u(r)^2 \\ &\quad - \frac{1}{2}a(r)g(r)u(r)^2 + \frac{1}{p+1}a(r)h(r)u(r)^{p+1}, \\ a(r) &= f(r)^{\frac{2(p+1)}{p+3}}h(r)^{-\frac{2}{p+3}}, \\ b(r) &= -\frac{1}{2}a_r(r) + \frac{f_r(r)}{f(r)}a(r), \\ c(r) &= -b_r(r) + \frac{f_r(r)}{f(r)}b(r), \\ G(r) &= b(r)g(r) + \frac{1}{2}c_r(r) - \frac{1}{2}\big(a(r)g(r)\big)_r(r). \end{aligned}$$

Remark 1 The following are detailed expressions of $b(r)$, $c(r)$ and $G(r)$:

$$\begin{aligned} b(r) &= (p+3)^{-1}f(r)^{\frac{p-1}{p+3}}h(r)^{-\frac{p+5}{p+3}}\left(2h(r)f_r(r) + f(r)h_r(r)\right), \\ c(r) &= (p+3)^{-2}f(r)^{-\frac{4}{p+3}}h(r)^{-\frac{2(p+4)}{p+3}}\Big(h(r)^2[8f_r(r)^2 - 2(p+3)f(r)f_{rr}(r)] \\ &\qquad + (p+5)f(r)^2h_r(r)^2 - f(r)h(r)[(p-5)f_r(r)h_r(r) + (p+3)f(r)h_{rr}(r)]\Big), \\ G(r) &= \frac{1}{2}(p+3)^{-3}f(r)^{-\frac{p+7}{p+3}}h(r)^{-\frac{2}{p+3}-3} \\ &\qquad \times \Big[-\Big[32f_r(r)^3 + 2(p-9)(p+3)f(r)f_r(r)f_{rr}(r) + 2(p+3)^2f(r)^2f_{rrr}(r)\Big]h(r)^3 \\ &\qquad - \Big[2(p-1)(p+3)^2f(r)^2f_r(r)h(r)^3 - 4(p+3)^2f(r)^3h(r)^2h_r(r)\Big]g(r) \\ &\qquad - (p+3)^3f(r)^3g_r(r)h(r)^3 \\ &\qquad - \Big[(p(p-6)+21)f(r)f_r(r)^2 + ((p(p-6)-27))f(r)^2f_{rr}(r)\Big]h(r)^2h_r(r) \end{aligned}$$

$$+ 3(p-1)(p+5)f(r)^2 f_r(r)h(r)h_r(r)^2 - 2(p+4)(p+5)f(r)^3 h_r(r)^3$$
$$- 3(p-1)(p+3)f(r)^2 f_r(r)h(r)^2 h_{rr}(r)$$
$$+ 3(p+3)(p+5)f(r)^3 h(r)h_r(r)h_{rr}(r) - (p+3)^2 f(r)^3 h(r)^2 h_{rrr}(r)\Big].$$

For the reader's convenience, we show the expressions $a(r), b(r), c(r)$ and $G(r)$ for specified $f(r), g(r), h(r)$ in Appendix 1.

3 Ball or entire space case

In this section, we study the problem

$$\begin{cases} u_{rr}(r) + \frac{f_r(r)}{f(r)}u_r - g(r)u(r) + h(r)u(r)^p = 0, & R' < r < R, \\ u(R') \in (0, \infty), \quad u(R) = 0, \end{cases} \tag{3.1}$$

where $-\infty < R' < R \leq \infty$, $p > 1$ and f, g, h are some functions. In the case of $R = \infty$, $u(R) = 0$ means that $u(r) \to 0$ as $r \to \infty$. We note that $u_r(R') = 0$ is not included in (3.1). However, we impose conditions that each positive solution in the following sense satisfies it; see Lemma 1. We say u is a positive solution of (3.1) if

$$\begin{cases} u \in C([R', \infty)) \cap C^2((R', \infty)) & \text{in the case of } R = \infty, \\ u \in C([R', R]) \cap C^2((R', R)) & \text{in the case of } R < \infty, \end{cases}$$

$u(r) > 0$ for each $r \in [R', R)$, and u satisfies (3.1). For the sake of completeness, we note that if u is a positive radial solution of (1.1) then u is a positive solution of (3.1) with $R' = 0$ and $f(r) = r^{n-1}\rho(r)$. We impose the following conditions on f, g and h.

(B1) (i) $-\infty < R' < R \leq \infty$, $g \in C^1((R', R))$, $f, h \in C^3((R', R))$, and f, h are positive in (R', R).

(ii) $\overline{\lim}_{r \to R'} f(r) < \infty$.

(iii) $\lim_{r \to R'} \dfrac{1}{f(r)} \displaystyle\int_{R'}^{r} f(\tau)(|g(\tau)| + h(\tau))\, d\tau = 0.$

(iv) There exists $\bar{R} \in (R', R)$ such that

(a) $fg, fh \in L^1((R', \bar{R}))$,

(b) $\tau \mapsto f(\tau)(|g(\tau)| + h(\tau)) \displaystyle\int_{\tau}^{\bar{R}} \frac{d\sigma}{f(\sigma)} \in L^1((R', \bar{R}))$,

(c) $1/f \notin L^1((R', \bar{R}))$.

(v) In the case of $R < \infty$, $g \in C((R', R])$, $f, h \in C^2((R', R])$, $f(R) > 0$ and $h(R) > 0$ are also satisfied.

Remark 2 Let $g \in C^1((0, \infty)) \cap C([0, \infty))$, $\rho, h \in C^3((0, \infty)) \cap C^2([0, \infty))$, ρ, h are positive on $[0, \infty)$ and $n \in \mathbb{R}$ with $n \geq 2$. Set $f(r) = r^{n-1}\rho(r)$. Then it is easy to see that (B1) is satisfied with $R' = 0 < R \leq \infty$.

Now, we state our uniqueness theorem. In the following, $a(r), b(r), c(r), G(r)$ and $J(r; u)$ are the ones given in Proposition 1. We note that the case (B5) (i) is essentially same as [47, Theorem 1] and that a similar condition to (B5) (ii) was studied by Byeon–Oshita [8].

Theorem 1 *Let* $p > 1$. *Assume* (B1) *and the following.*

(B2) $\overline{\lim}_{r\to R'} a(r) < \infty$, $\overline{\lim}_{r\to R'} |b(r)| < \infty$, $\lim_{r\to R'} a(r)g(r) = 0$ *and* $\lim_{r\to R'} a(r)h(r) = 0$.
(B3) $\underline{\lim}_{r\to R'} c(r) \in [0, \infty]$.
(B4) *In the case of* $R = \infty$, $G^- \not\equiv 0$ *is satisfied, where* $G^-(r) = \min\{G(r), 0\}$ *for* $r \in (R', R)$.
(B5) *One of the following conditions is satisfied.*

(i) *There exists* $\kappa \in [R', R]$ *such that*

$$G(r) \geq 0 \quad \text{on } (R', \kappa) \quad \text{and} \quad G(r) \leq 0 \quad \text{on } (\kappa, R).$$

(ii) $\{R' < r < R : G(r) = 0,\ D(r) > 0\} = \emptyset$, *where*

$$D(r) = b(r)^2 - a(r)\big(c(r) - a(r)g(r)\big).$$

Then in the case of $R < \infty$, *problem* (3.1) *has at most one positive solution, and in the case of* $R = \infty$, *problem* (3.1) *has at most one positive solution* u *which satisfies* $J(r; u) \to 0$ *as* $r \to \infty$.

Remark 3 If $p > 1$ and the assumptions in Remark 2 are satisfied, then (B2) and (B3) hold. See Proposition 2 in the next section.

Remark 4 The following is a detailed expression of $D(r)$:

$$\begin{aligned} D(r) &= (p+3)^{-2} f(r)^{\frac{-2(p+1)}{p+3}} h(r)^{-\frac{2(p+5)}{p+3}} \\ &\quad \times \Big[(p+3)^2 f(r)^2 h(r)^2 g(r) + \big(-4 f_r(r)^2 + 2(p+3) f(r) f_{rr}(r)\big) h(r)^2 \\ &\quad - (p+4) f(r)^2 h_r(r)^2 + (p-1) f(r) f_r(r) h(r) h_r(r) + (p+3) f(r)^2 h(r) h_{rr}(r)\Big]. \end{aligned}$$

For the reader's convenience, we show the expressions $D(r)$ for specified $f(r)$, $g(r)$, $h(r)$ in Appendix 1.

Next, we study the existence of a unique positive solution of (3.1) by the variational method. We introduce function spaces $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{L}, \|\cdot\|_{\mathcal{L}})$ such that $\mathcal{X}$ is continuously imbedded into $\mathcal{L}$, and we define a functional I on $\mathcal{X}$ by

$$\begin{aligned} I(u) &= \int_{R'}^{R} \left(\frac{1}{2}(|u_r(r)|^2 + g(r)|u(r)|^2) - \frac{1}{p+1} h(r)|u(r)|^{p+1}\right) f(r)\, dr \\ &= \frac{1}{2}\|u\|_{\mathcal{X}}^2 - \frac{1}{p+1}\|u\|_{\mathcal{L}}^{p+1} \end{aligned} \tag{3.2}$$

whose positive critical point corresponds to a positive solution of (3.1). Now, we give them in detail. We set

$$\mathcal{D} = \left\{\varphi \in C^\infty([R', R)) : \operatorname{supp} \varphi \subset [R', R),\ \frac{d^{2k-1}\varphi}{dr^{2k-1}}(R') = 0 \text{ for each } k \in \mathbb{N}\right\}.$$

We define

$$\|\varphi\|_{\mathcal{X}} = \left(\int_{R'}^{R} \left(\varphi_r(r)^2 + g(r)\varphi(r)^2\right) f(r)\, dr\right)^{\frac{1}{2}} \quad \text{for each } \varphi \in \mathcal{D},$$

$$\|\varphi\|_{\mathcal{L}} = \left(\int_{R'}^{R} h(r)|\varphi(r)|^{p+1} f(r)\, dr\right)^{\frac{1}{p+1}} \quad \text{for each } \varphi \in \mathcal{D},$$

and we impose the following conditions.

(B6) $$\inf_{\varphi\in\mathcal{D}\setminus\{0\}} \frac{\|\varphi\|_{\mathcal{X}}}{\|\varphi\|_{\mathcal{L}}} > 0, \quad \inf_{\varphi\in\mathcal{D}\setminus\{0\}} \frac{\|\varphi\|_{\mathcal{X}}^2}{\int_{R'}^{R}\left(\varphi_r(r)^2 + |g(r)|\varphi(r)^2\right) f(r)\, dr} > 0.$$

We denote by $\mathcal{X}$ and $\mathcal{L}$ the completion of $\mathcal{D}$ with respect to $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{L}}$, respectively. We can see that both inequalities in (B6) hold even if the infimums are taken on $\mathcal{X}\setminus\{0\}$. So, under these assumptions, the embedding from $\mathcal{X}$ into $\mathcal{L}$ is continuous, and the norm defined by

$$\left(\int_{R'}^{R} \left(\varphi_r(r)^2 + |g(r)|\varphi(r)^2\right) f(r)\, dr\right)^{\frac{1}{2}} \quad \text{for each } \varphi \in \mathcal{X}$$

is equivalent to $\|\cdot\|_{\mathcal{X}}$ on $\mathcal{X}$.

Remark 5 Under these assumptions, we can see that each equivalence class in $\mathcal{X}$ is the standard almost everywhere equivalence class. We can also see that for each $u \in \mathcal{X}$, there is $v \in L^{p+1}_{\mathrm{loc}}(R', R)$ such that

$$\int_{R'}^{R} u\varphi_r\, dr = -\int_{R'}^{R} v\varphi\, dr \quad \text{for each} \quad \varphi \in C_0^\infty(R', R),$$

and we denote this v by u_r. Consequently, we have $\mathcal{X} \subset H^1_{\mathrm{loc}}(R', R)$.

Remark 6 Even if we assume (B6) with $C_0^\infty((R', R))$ instead of $\mathcal{D}$, we can show that the completion of $C_0^\infty((R', R))$ with respect to the norm $\|\cdot\|_{\mathcal{X}}$ is exactly $\mathcal{X}$ by assumption (B1) (iv) (c). See Lemma 9 in Appendix 2.

Remark 7 Since we consider the problems not only like the scalar field equation but also like Matukuma's equation, see Sect. 7, we do not assume the condition such as

$$\inf_{\varphi\in\mathcal{D}\setminus\{0\}} \frac{\int_{R'}^{R}(\varphi_r(r)^2 + g(r)\varphi(r)^2) f(r)\, dr}{\int_{R'}^{R} h(r)u(r)^2 f(r)\, dr} > 0.$$

Now, we show our existence result. In many applications, the following assumption (B9) holds.

Theorem 2 *Let* $p > 1$. *Assume* (B1)–(B6) *and the following.*

(B7) *One of the following conditions is satisfied.*

(i) *The embedding* $\mathcal{X} \hookrightarrow \mathcal{L}$ *is compact.*
(ii) *There exists* $\hat{g} \in C((R', R))$ *such that*

$$S_g \equiv \inf_{u\in\mathcal{X}\setminus\{0\}} \frac{\|u\|_{\mathcal{X}}^2}{\|u\|_{\mathcal{L}}^2} < S_{\hat{g}} \equiv \inf_{u\in\mathcal{X}\setminus\{0\}} \frac{\int_{R'}^{R}(u_r(r)^2 + \hat{g}(r)u(r)^2) f(r)\, dr}{\|u\|_{\mathcal{L}}^2},$$

and for each $\{u_m\} \subset \mathcal{X}$ *converging weakly to some* $u \in \mathcal{X}$*, there holds*

$$\int_{R'}^{R} (\hat{g}(r) - g(r))|u_m(r) - u(r)|^2 f(r)\, dr \to 0.$$

(B8) *There exist* $\bar{p} \in [p, \infty)$, $q \in (1, \bar{p})$ *and* $\bar{R} \in (R', R)$ *such that*

$$\inf_{u \in \mathcal{X} \setminus \{0\}} \frac{\left(\int_{R'}^{\bar{R}} \left(u_r(r)^2 + g(r)u(r)^2\right) f(r)\, dr\right)^{\frac{1}{2}}}{\left(\int_{R'}^{\bar{R}} |u|^{\bar{p}+1} hf\, dr\right)^{\frac{1}{\bar{p}+1}}} > 0, \tag{3.3}$$

$$\int_{R'}^{\bar{R}} (|g^-|/h)^{\frac{q+1}{q-1}} hf\, dr < \infty. \tag{3.4}$$

(B9) *In the case of* $R = \infty$*, each positive solution* $u \in \mathcal{X} \cap C^2((R', \infty)) \cap C([R', \infty))$ *of* (3.1) *satisfies* $J(r; u) \to 0$ *as* $r \to \infty$.

Then problem (3.1) *has a unique positive solution* $\bar{u}$ *in* $\mathcal{X}$.

Remark 8 Under assumption (B6), (3.3) holds with $\bar{p} = p$. In applications, if p is a so called subcritical exponent, letting $\bar{p} > p$ be the critical one, we usually have (3.3).

Remark 9 We need assumption (3.4) to show that the solution given by the variational method does not diverge at $r = R'$. We note that each of the following conditions is a sufficient condition for (3.4).

(i) $g^- \equiv 0$.
(ii) f, g, h are continuous at R' and $h(R') > 0$.

Next, we show a nondegeneracy result for the unique positive solution $\bar{u}$ of (3.1) in the space $\mathcal{X}$.

Theorem 3 *Let* $p > 1$. *Assume* (B1)–(B8) *and the following.*
(B9′) *In the case of* $R = \infty$*, for each* $u \in \mathcal{X} \cap C^2((R', \infty)) \cap C([R', \infty))$ *which is positive on* $[R', \infty)$ *and satisfies*

$$u_{rr}(r) + \frac{f_r(r)}{f(r)} u_r(r) - g(r)u(r) + h(r)u(r)^p = 0 \quad \text{for each } r \in (R_u, \infty)$$

with some $R_u \in (R', \infty)$*, there holds*

$$\lim_{r \to \infty} J(r; u) = 0.$$

(B10) $G \not\equiv 0$ *in* (R', R).
Then the unique positive solution $\bar{u}$ *of problem* (3.1) *is a nondegenerate critical point of the* C^2*-functional* I *defined by* (3.2) *for each* $u \in \mathcal{X}$.

Remark 10 In the case of $R = \infty$, (B10) is already assumed in (B4).

Remark 11 Even if $G \equiv 0$ with $R < \infty$, we have a nondegeneracy result. Since it is a little bit complicated, we postpone it to Sect. 5.

Next, we study the nondegeneracy of the unique positive solution of (3.1) in a general function space. Let $n \in \mathbb{N}$ with $n \geq 2$. We set $R' = 0$ and $\rho(r) = f(r)/r^{n-1}$. We define

$$\|\varphi\|_{\mathcal{X}_\rho} = \left(\int_{B_R} \big(|\nabla\varphi(x)|^2 + g(|x|)|\varphi(x)|^2 \big) \rho(|x|)\, dx \right)^{\frac{1}{2}} \quad \text{for each } \varphi \in C_0^\infty(B_R),$$
$$\|\varphi\|_{\mathcal{L}_\rho} = \left(\int_{B_R} h(|x|)|\varphi(x)|^{p+1} \rho(|x|)\, dx \right)^{\frac{1}{p+1}} \quad \text{for each } \varphi \in C_0^\infty(B_R),$$

where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$, and we impose the following conditions.

$$(B6') \qquad \inf_{\varphi \in C_0^\infty(B_R)\setminus\{0\}} \frac{\|\varphi\|_{\mathcal{X}_\rho}}{\|\varphi\|_{\mathcal{L}_\rho}} > 0, \quad \inf_{\varphi \in C_0^\infty(B_R)\setminus\{0\}} \frac{\|\varphi\|^2_{\mathcal{X}_\rho}}{\int_{B_R} \big(|\nabla\varphi|^2 + |g|\,|\varphi|^2\big)\rho\, dx} > 0.$$

As before, we denote by $\mathcal{X}_\rho$ and $\mathcal{L}_\rho$ the completion of $C_0^\infty(B_R)$ with respect to $\|\cdot\|_{\mathcal{X}_\rho}$ and $\|\cdot\|_{\mathcal{L}_\rho}$, respectively. We can see that both inequalities in (B6') hold even if the infimums are taken on $\mathcal{X}_\rho\setminus\{0\}$.

Now, we show our nondegeneracy result in $\mathcal{X}_\rho$. In many applications, (B12) and (B13) below hold by the elliptic regularity.

Theorem 4 *Let $p > 1$ and $n \in \mathbb{N}$ with $n \geq 2$. Assume* (B1)–(B5) *with $R' = 0$,* (B6') *with $\rho(r) = f(r)/r^{n-1}$,* (B8), (B9$'$) *and* (B10). *Let $\bar{u}$ be the unique radially symmetric, positive solution of* (3.1). *Assume also the following.*

(B11) $f_r \geq 0$ *in* $(0, R)$,

$$(\log \rho(r))_{rr} \geq 0, \quad g_r(r) \geq 0 \quad \text{and} \quad h_r(r) \leq 0 \quad \text{in } (0, R), \tag{3.5}$$

and in the case $R = \infty$, at least one inequality in (3.5) *is not identically equal.*

(B12) (i) $\overline{\lim}_{r\to 0} \max\{|f_r(r)\bar{u}_r(r)|,\ f(r)|g(r)|,\ f(r)h(r)\} < \infty$.

(ii) *In the case of $R = \infty$,*

$$\overline{\lim_{r\to\infty}} \max\big\{|f_r(r)\bar{u}_r(r)|,\ f(r)|g(r)|\bar{u}(r),\ f(r)h(r)\bar{u}(r)^p,\ f(r)|\bar{u}_r(r)|\big\} < \infty.$$

(B13) *For each weak solution $w \in \mathcal{X}_\rho$ of*

$$\Delta w + \frac{\nabla\rho\nabla w}{\rho} - gw + ph\bar{u}^{p-1}w = 0 \quad \text{in } B_R,$$

i.e.,

$$\int_{B_R} \big(\nabla w \nabla v + gwv - ph\bar{u}^{p-1}wv\big)\rho\, dx = 0 \quad \text{for each } v \in \mathcal{X}_\rho,$$

there hold w is in $C^1(B_R)$, and in the case of $R = \infty$,

$$\lim_{|x|\to\infty} w(x) = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \frac{\partial w}{\partial x_i}(x) = 0 \quad \text{for each } i = 1, \ldots, n.$$

Then $\bar{u}$ is a nondegenerate critical point of the C^2-functional $\mathcal{I}$ defined by

$$\mathcal{I}(u) = \int_{B_R} \left(\frac{1}{2}(|\nabla u(x)|^2 + g(|x|)|u(x)|^2) - \frac{1}{p+1} h(|x|)|u(x)|^{p+1} \right) \rho(|x|)\, dx$$

for $u \in \mathcal{X}_\rho$.

Remark 12 For each radially symmetric $u \in \mathcal{X}_\rho$, there holds $\mathcal{I}(u) = |S^{n-1}|I(u)$, where $|S^{n-1}|$ is the surface measure of S^{n-1} and I is the functional defined by (3.2).

4 Proof of Theorems 1–4

First, we give the following.

Proposition 2 *Let $g \in C^1((0,\infty)) \cap C([0,\infty))$, $\rho, h \in C^2([0,\infty)) \cap C^3((0,\infty))$, ρ, h are positive on $[0,\infty)$, $n \in \mathbb{R}$ with $n \geq 2$, and $p > 1$. Set $f(r) = r^{n-1}\rho(r)$. Then* (B1)–(B3) *are satisfied with $R' = 0 < R \leq \infty$.*

Proof It is easy to see that (B1) is satisfied. Under the assumptions, we have

$$a(r) = \left(\rho(r)r^{n-1}\right)^{\frac{2(p+1)}{p+3}} h(r)^{-\frac{2}{p+3}},$$

$$b(r) = \frac{\rho(r)^{\frac{2(p+1)}{p+3}-1} r^{\frac{2(n-1)(p+1)}{p+3}-1}}{(p+3)h(r)^{\frac{p+5}{p+3}}} \Big(2h(r)[(n-1)\rho(r) + r\rho_r(r)] + r\rho(r)h_r(r)\Big),$$

$$\begin{aligned} c(r) = \frac{\rho(r)^{\frac{2(p+1)}{p+3}-2} r^{\frac{2(n-1)(p+1)}{p+3}-2}}{(p+3)^2 h(r)^{\frac{2(p+4)}{p+3}}} \cdot \Big[2h(r)^2\Big((n-1)[n+2-(n-2)p]\rho(r)^2 + 4r^2\rho_r(r)^2 \\ - r\rho(r)\big((p+3)r\rho_{rr}(r) + 2(n-1)(p-1)\rho_r(r)\big)\Big) + (p+5)r^2\rho(r)^2h_r(r)^2 \\ - r\rho(r)h(r)\big[(p+3)r\rho(r)h_{rr}(r) + (p-5)h_r(r)\big(r\rho_r(r) + (n-1)\rho(r)\big)\big]\Big]. \end{aligned}$$

We set

$$\nu = \frac{2(n-1)(p+1)}{p+3}.$$

From $\nu > 1$, we can easily see $a(r) \to 0$ and $b(r) \to 0$ as $r \to 0$, which yields (B2). Since we have $\nu > 2$ if $n \geq 3$ and $\nu < 2$ if $n = 2$, we can find

$$\lim_{r\to 0} c(r) = \begin{cases} 0 & \text{if } \ n \geq 3, \\ \infty & \text{if } \ n = 2. \end{cases}$$

We consider the case $2 < n < 3$. Since $\nu > 2$ is equivalent to $p > (4-n)/(n-2)$, we have

$$\lim_{r\to 0} c(r) = \begin{cases} 0 & \text{if } 2 < n < 3 \text{ and } p > (4-n)/(n-2), \\ \frac{(n-2)^2\rho(0)^{\frac{2}{n-2}}}{h(0)^{\frac{n-2}{n-1}}} & \text{if } 2 < n < 3 \text{ and } p = (4-n)/(n-2), \\ \infty & \text{if } 2 < n < 3 \text{ and } p < (4-n)/(n-2). \end{cases}$$

Hence we have shown (B3). □

We give the following three lemmas. Although Lemmas 1 and 2 are slightly different from [47, Lemmas 1 and 2], the proofs of [47, Lemmas 1 and 2] work well. Lemma 3 is same as [47, Lemma 3].

Lemma 1 *Let $p > 1$. Assume* (B1) (i) *and a nonnegative function $u \in C^2((R', R))$ satisfies*

$$u_{rr}(r) + \frac{f_r(r)}{f(r)}u_r(r) - g(r)u(r) + h(r)u(r)^p = 0 \quad \textit{for each } r \in (R', R). \tag{4.1}$$

If (B1) (iii) *and* (B1) (iv) *are satisfied and u is bounded in a neighborhood of R', then $u_r(r) \to 0$ as $r \to R'$. If $R < \infty$,* (B1) (iii) *is satisfied, and u is a positive solution of* (3.1), *then u is continuously differentiable at R and $u_r(R) \in (-\infty, 0)$.*

Lemma 2 *Let $p > 1$. Assume* (B1) (i), (B1) (iii) *and* (B1) (iv). *If nonnegative functions $u, v \in C([R', R)) \cap C^2((R', R))$ satisfies* (4.1) *and $u(R') = v(R')$, then they coincide.*

Lemma 3 *Let $p > 1$. Assume* (B1) (i)–(iv). *Let u and v be positive solutions of* (3.1). *Then*

$$\frac{d}{dr}\left(\frac{v(r)}{u(r)}\right) = \frac{1}{u(r)^2}\int_{R'}^{r}\frac{f(\tau)}{f(r)}h(\tau)\left(u(\tau)^{p-1} - v(\tau)^{p-1}\right)u(\tau)v(\tau)\,d\tau$$

for each $r \in (R', R)$.

Proposition 3 *Let $p > 1$. Assume* (B1) *and* (B2). *Let u and v be positive solutions of* (3.1) *such that $u(R') < v(R')$ and $J(r; u) \geq 0$ on (R', R). Then*

$$\frac{d}{dr}\left(\frac{v(r)}{u(r)}\right) < 0 \quad \textit{for each } r \in (R', R).$$

Proof Assume that the conclusion does not hold. We set $w(r) = v(r)/u(r)$ for $r \in (R', R)$. Then by Lemma 3, there exists $r_* \in (R', R)$ such that $w_r(r_*) = 0$ and $w_r(r) < 0$ on (R', r_*). We note $w(r_*) < 1$. We define

$$X(r) = w(r)^2 J(r; u) - J(r; v) \quad \text{for each } r \in (R', R). \tag{4.2}$$

Then we have

$$\begin{aligned} X(r) = &\frac{1}{2}a(r)\left(\frac{v(r)^2 u_r(r)^2}{u(r)^2} - v_r(r)^2\right) + b(r)\left(\frac{v(r)^2 u_r(r)}{u(r)} - v_r(r)v(r)\right) \\ &+ \frac{1}{p+1}a(r)h(r)v(r)^2\left(u(r)^{p-1} - v(r)^{p-1}\right) \end{aligned} \tag{4.3}$$

for each $r \in (R', R)$. From (B2) and Lemma 1, we have

$$\lim_{r \to R'} X(r) = 0. \tag{4.4}$$

From $w_r(r_*) = 0$ and $w(r_*) < 1$, we also have

$$X(r_*) = \frac{1}{p+1}a(r_*)h(r_*)v(r_*)^2\left(u(r_*)^{p-1} - v(r_*)^{p-1}\right) > 0.$$

On the other hand, from $w_r(r) < 0$ on (R', r_*) and $J(r; u) \geq 0$ on (R', R), we have

$$X_r(r) = 2w(r)w_r(r)J(r; u) \leq 0 \quad \text{on} (R', r_*),$$

which contradicts (4.4) and $X(r_*) > 0$. Hence we have shown our assertion. □

Now, we assume all assumptions in Theorem 1.

Proposition 4 *Let u be a positive solution of* (3.1). *In the case of $R = \infty$, assume $J(r; u) \to 0$ as $r \to \infty$. Then $J(\cdot; u) \not\equiv 0$, and $J(r; u) \geq 0$ for each $r \in (R', R)$.*

Proof From (B2) and (B3), we have $\varliminf_{r\to R'} J(r;u) \geq 0$. In the case $R < \infty$, we have $\lim_{r\to R} J(r;u) = (1/2)a(R)u_r(R)^2 > 0$, and hence $J(\cdot;u) \not\equiv 0$. In the case of $R = \infty$, we have $\lim_{r\to\infty} J(r;u) = 0$ by the assumption, and we can find $J(\cdot;u) \not\equiv 0$ by $G^- \not\equiv 0$.

We will show $J(r;u) \geq 0$ for each $r \in (R', R)$. Since it is trivial in the case (B5) (i), we consider the case (B5) (ii). If $J(r;u) \geq 0$ does not hold, then there exists $r_0 \in (R', R)$ such that

$$J(r_0;u) < 0 \quad \text{and} \quad 0 = \frac{d}{dr} J(r_0;u) = G(r_0)u(r_0)^2.$$

Then we can find $G(r_0) = 0$, and

$$0 > \frac{2J(r_0;u)}{u(r_0)^2} > a(r_0)\frac{u_r(r_0)^2}{u(r_0)^2} + 2b(r_0)\frac{u_r(r_0)}{u(r_0)} + c(r_0) - a(r_0)g(r_0).$$

So we have $D(r_0) > 0$, which contradicts (B5) (ii). Hence we have shown $J(r;u) \geq 0$ for each $r \in (R', R)$. □

Proof of Theorem 1 Suppose that the conclusion does not hold. Then there exist distinct positive solutions u, v of (3.1), and in the case of $R = \infty$, they satisfy $J(r;u) \to 0$ and $J(r;v) \to 0$ as $r \to \infty$. We may assume $u(R') < v(R')$. By Proposition 4, we have $J(r;u) \geq 0$, $J(r;v) \geq 0$ for each $r \in (R', R)$ and $J(\cdot;u) \not\equiv 0$, $J(\cdot;v) \not\equiv 0$. We define w and X as in the proof of Proposition 3. From its proof, we have (4.4). We also have $\lim_{r\to R} X(r) = 0$, which is obtained by (4.3) in the case of $R < \infty$. In the case of $R = \infty$, it is obtained by (4.2), $w_r(r) < 0$, $\lim_{r\to\infty} J(r;u) = 0$ and $\lim_{r\to\infty} J(r;v) = 0$. However, we have

$$X_r(r) = (w(r)^2)_r J(r;u) \leq 0 \quad \text{for each} \quad \in (R', R) \quad \text{and} \quad X_r(\cdot) \not\equiv 0,$$

which is a contradiction. Therefore, we have shown our assertion. □

Remark 13 Assumptions (B3) and (B5) were only used to show Proposition 4. So if there is another condition which yields the consequence of Proposition 4, we can obtain another uniqueness theorem. This fact will be used in the proof of Theorem 5.

Remark 14 In the proof of Theorem 1 with $R = \infty$, we used $J(r;u) \to 0$ as $r \to \infty$ but we did not use $u(r) \to 0$ as $r \to \infty$.

Next, we give a proof of the existence of a unique positive solution of (3.1).

Proposition 5 *Under the assumptions of Theorem 2, there exists $u \in \mathcal{X}$ such that $\|u\|_{\mathcal{L}} = 1$, $u \geq 0$ in (R', R) and*

$$\mathscr{R}(u) = \inf\{\mathscr{R}(v) : v \in \mathcal{X},\ \|v\|_{\mathcal{L}} = 1\}, \tag{4.5}$$

where $\mathscr{R}$ is a C^2-functional defined by

$$\mathscr{R}(u) = \frac{\|u\|_{\mathcal{X}}^2}{\|u\|_{\mathcal{L}}^2} \quad \text{for each } u \in \mathcal{X}\backslash\{0\}.$$

Proof Since we have $\mathscr{R}(|u|) = \mathscr{R}(u)$ for each $u \in \mathcal{X}\backslash\{0\}$, it is enough to show that there is $u \in \mathcal{X}$ satisfying $\|u\|_{\mathcal{L}} = 1$ and (4.5), which is easily shown in the case (B7) (i). So we consider the case (B7) (ii). Let $\{u_m\} \subset \mathcal{X}$ such that $\|u_m\|_{\mathcal{L}} = 1$ for each $m \in \mathbb{N}$ and $\|u_m\|_{\mathcal{X}}^2 \to S_g$. We may assume that $\{u_m\}$ converges weakly to $u \in \mathcal{X}$. Noting

$$\lim_{m\to\infty} \left(\|u_m\|_{\mathcal{L}}^{p+1} - \|u_m - u\|_{\mathcal{L}}^{p+1}\right) = \|u\|_{\mathcal{L}}^{p+1},$$

see [5], we have

$$\begin{aligned} S_g &= \|u_m\|_{\mathcal{X}}^2 + o(1) = \left(\frac{S_{\hat{g}} - S_g}{S_{\hat{g}}} + \frac{S_g}{S_{\hat{g}}} \right) \|u_m - u\|_{\mathcal{X}}^2 + \|u\|_{\mathcal{X}}^2 + o(1) \\ &= \frac{S_{\hat{g}} - S_g}{S_{\hat{g}}} \|u_m - u\|_{\mathcal{X}}^2 + \frac{S_g}{S_{\hat{g}}} \int_{R'}^{R} \left(|u_{m,r} - u_r|^2 + \hat{g}(r)|u_m - u|^2 \right) f(r)\, dr + \|u\|_{\mathcal{X}}^2 + o(1) \\ &\geq \frac{S_{\hat{g}} - S_g}{S_{\hat{g}}} \|u_m - u\|_{\mathcal{X}}^2 + S_g (\|u_m - u\|_{\mathcal{L}}^2 + \|u\|_{\mathcal{L}}^2) + o(1) \\ &\geq \frac{S_{\hat{g}} - S_g}{S_{\hat{g}}} \|u_m - u\|_{\mathcal{X}}^2 + S_g \left(\|u_m - u\|_{\mathcal{L}}^{p+1} + \|u\|_{\mathcal{L}}^{p+1} \right)^{\frac{2}{p+1}} + o(1) \\ &= \frac{S_{\hat{g}} - S_g}{S_{\hat{g}}} \|u_m - u\|_{\mathcal{X}}^2 + S_g + o(1). \end{aligned}$$

So we obtain $\|u_m - u\|_{\mathcal{X}} \to 0$, and we can find that u satisfies $\|u\|_{\mathcal{L}} = 1$ and (4.5). □

Proof of Theorem 2 Let $u \in \mathcal{X}$ be the function obtained in the previous proposition. Setting $\bar{u} = \|u\|_{\mathcal{X}}^{-2/(p-1)} u$, we can find $\bar{u}$ is a nontrivial, nonnegative critical point of I. By the standard regularity arguments, we can see that $\bar{u} \in C^2((R', R))$ and

$$\bar{u}_{rr}(r) + \frac{f_r(r)}{f(r)} \bar{u}_r(r) - g(r)\bar{u}(r) + h(r)\bar{u}(r)^p = 0 \quad \text{for } r \in (R', R),$$

and that in the case of $R < \infty$, $\bar{u}$ also belongs to $C^1((R', R])$. By (B8) and a subsolution estimate, we can see that $\bar{u}$ is bounded in a neighborhood of R'; see Proposition 6. So, from Lemma 1, we can consider that $\bar{u}$ and $\bar{u}_r$ are continuous at R' and $\bar{u}_r(R') = 0$. We have $\bar{u}(r) > 0$ in (R', R). If not, there is $r_0 \in (R', R)$ with $\bar{u}(r_0) = 0$. Then we can find $\bar{u}_r(r_0) = 0$ and hence we have $\bar{u} \equiv 0$, which is a contradiction. From Lemma 2, we also have $\bar{u}(R') > 0$. Hence $\bar{u}$ is a positive solution of (3.1). By (B9) and Theorem 1, we can see that $\bar{u}$ is a unique positive solution of (3.1) in $\mathcal{X}$. □

Next, we give a proof of Theorem 3 which shows if $G \not\equiv 0$ then the unique positive solution of (3.1) is a nondegenerate critical point of I. For a critical point u of I, we define the Morse index of $I''(u)$ by

$$\max \left\{ \dim H : H \text{ is a subspace of } \mathcal{X} \text{ such that } I''(u)[v, v] < 0 \text{ for each } v \in H \backslash \{0\} \right\}.$$

Proof of Theorem 3 From $G \not\equiv 0$ in (R', R), we can find a closed interval $[r_1, r_2] \subset (R', R)$ such that

$$\min_{r \in [r_1, r_2]} |G(r)| > 0.$$

We choose $\gamma \in C_0^\infty((R', R)) \backslash \{0\}$ such that $\gamma \geq 0$ and $\operatorname{supp} \gamma = [r_1, r_2]$. Let $\delta > 0$, which will be fixed later. We define

$$g_\delta(r) = g(r) + \delta\gamma(r)h(r)\bar{u}(r)^{p-1}, \quad h_\delta(r) = (1 + \delta\gamma(r))h(r) \tag{4.6}$$

in (R', R). Using these functions instead of g and h, we define a_δ, b_δ, c_δ, G_δ and D_δ in (R', R) as follows:

$$a_\delta = f^{\frac{2(p+1)}{p+3}} h_\delta^{-\frac{2}{p+3}}, \quad b_\delta = -\frac{1}{2}a_{\delta,r} + \frac{f_r}{f}a_\delta, \quad c_\delta = -b_{\delta,r} + \frac{f_r}{f}b_\delta,$$
$$G_\delta = b_\delta g_\delta + \frac{1}{2}c_{\delta,r} - \frac{1}{2}\big(a_\delta g_\delta\big)_r, \quad D_\delta = b_\delta^2 - a_\delta\big(c_\delta - a_\delta g_\delta\big). \tag{4.7}$$

Since $g_\delta = g$ and $h_\delta = h$ in $(R', R)\backslash[r_1, r_2]$, we can easily see

$$a_\delta = a, \quad b_\delta = b, \quad c_\delta = c, \quad G_\delta = G, \quad D_\delta = D \quad \text{in } (R', R)\backslash[r_1, r_2].$$

Now we fix $\delta > 0$ small enough such that

$$\min_{r\in[r_1,r_2]} |G_\delta(r)| > 0.$$

In the case (B5) (i), we can easily see that

$$G_\delta(r) \geq 0 \quad \text{in } (R', \kappa) \quad \text{and} \quad G_\delta(r) \leq 0 \quad \text{in } (\kappa, R). \tag{4.8}$$

In the case (B5) (ii), we have

$$\begin{aligned}&\{r \in (R', R)\backslash[r_1, r_2] : G_\delta(r) = 0,\ D_\delta(r) > 0\}\\ &\quad = \{r \in (R', R)\backslash[r_1, r_2] : G(r) = 0,\ D(r) > 0\} = \emptyset\end{aligned}$$

and

$$\{r \in [r_1, r_2] : G_\delta(r) = 0,\ D_\delta(r) > 0\} \subset \{r \in [r_1, r_2] : G_\delta(r) = 0\} = \emptyset,$$

which yields

$$\{r \in (R', R) : G_\delta(r) = 0,\ D_\delta(r) > 0\} = \emptyset. \tag{4.9}$$

Since we can easily see $\bar{u}$ is a positive solution of

$$\begin{cases} u_{rr}(r) + \frac{f_r(r)}{f(r)}u_r - g_\delta(r)u(r) + h_\delta(r)u(r)^p = 0, & R' < r < R,\\ u(R') \in (0, \infty), \quad u(R) = 0, & \end{cases} \tag{4.10}$$

from (4.8), (4.9), (B9') and Theorem 1, we can find that $\bar{u}$ is its unique positive solution.

Now, we will show that $\bar{u}$ is a nondegenerate critical point of I. Suppose not, i.e., there exists $\varphi \in \mathcal{X}\backslash\{0\}$ satisfying

$$I''(\bar{u})[\varphi, \psi] = 0 \quad \text{for each } \psi \in \mathcal{X}, \tag{4.11}$$

which yields

$$\varphi_{rr} + \frac{f_r(r)}{f(r)}\varphi_r(r) - g(r)\varphi(r) + ph(r)\bar{u}(r)^{p-1}\varphi(r) = 0 \quad \text{for } R' < r < R. \tag{4.12}$$

We define a C^2-functional I_δ by

$$I_\delta(u) = \int_{R'}^R \left(\frac{1}{2}(|u_r|^2 + g_\delta(r)|u|^2) - \frac{1}{p+1}h_\delta(r)|u|^{p+1}\right) f(r)\,dr \quad \text{for } u \in \mathcal{X}.$$

We can easily see

$$I_\delta''(\bar{u})[\psi, \psi] = I''(\bar{u})[\psi, \psi] - \delta(p-1)\int_{R'}^R \gamma(r)h(r)\bar{u}(r)^{p-1}\psi(r)^2 f(r)\,dr$$

for each $\psi \in \mathcal{X}$, and

$$I''(\bar{u})[\bar{u}, \bar{u}] = -(p-1) \int_{R'}^{R} h(r)\bar{u}(r)^{p+1} f(r)\, dr < 0.$$

Moreover, we have

$$\int_{R'}^{R} \gamma(r)h(r)\bar{u}(r)^{p-1}|\varphi(r)|^2 f(r)\, dr > 0.$$

Indeed, if not, we have $\varphi \equiv 0$ on supp γ. Then from (4.12), we have $\varphi \equiv 0$ in (R', R), which contradicts $\varphi \in \mathcal{X}\backslash\{0\}$. Using (4.11) and the two inequalities above, we can see

$$I''_\delta(\bar{u})[\alpha\bar{u} + \beta\varphi, \alpha\bar{u} + \beta\varphi] < 0 \quad \text{for each } (\alpha, \beta) \in \mathbb{R}^2\backslash\{(0, 0)\},$$

which yields that the Morse index of $I''_\delta(\bar{u})$ is at least two. However, since the Morse index of $I''_\delta(\bar{u})$ is one, we obtain a contradiction. Hence we have shown that $\bar{u}$ is a nondegenerate critical point of I.

Although it is well known, for the reader's convenience, we briefly show that the Morse index of $I''_\delta(\bar{u})$ is one. Since $\bar{u}$ is the unique positive solution of (4.10), for each $v \in \mathcal{X}$, it satisfies $\alpha(t; v) \geq \alpha(0; v)$ for each $t \in \mathbb{R}$ with $|t| \ll 1$, where

$$\alpha(t; v) = \frac{\int_{R'}^{R} (|\bar{u}_r + tv_r|^2 + g_\delta(r)|\bar{u} + tv|^2) f(r)\, dr}{\left(\int_{R'}^{R} h_\delta(r)|\bar{u} + tv|^{p+1} f(r)\, dr\right)^{\frac{2}{p+1}}}.$$

Since we have $\alpha_t(0; v) = 0$ and $\alpha_{tt}(0; v) \geq 0$, we obtain

$$I''_\delta(\bar{u})[v, v] \geq -\frac{(p-1)\left(\int_{R'}^{R} (\bar{u}_r v_r + g_\delta(r)\bar{u}v) f(r)\, dr\right)^2}{\int_{R'}^{R} (|\bar{u}_r|^2 + g_\delta(r)|\bar{u}|^2) f(r)\, dr}.$$

We can consider that

$$(w, z) \mapsto \int_{R'}^{R} (w_r z_r + g_\delta(r)wz) f(r)\, dr : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$$

is an inner product on $\mathcal{X}$, which induces an equivalent norm on $\mathcal{X}$. For each $v \in \mathcal{X}$ which is orthogonal to $\bar{u}$ by this inner product, we have $I''_\delta(\bar{u})[v, v] \geq 0$. Since $I''_\delta(\bar{u})[\bar{u}, \bar{u}] < 0$, we find that the Morse index of $I''_\delta(\bar{u})$ is one. □

Next, we give a proof of the nondegeneracy result for the unique positive solution of (3.1) in the space $\mathcal{X}_\rho$.

Lemma 4 *Suppose the assumptions in Theorem 4 and let $\bar{u}$ be the unique positive solution which is obtained in Theorem 2. Then there holds $\bar{u}_r(r) < 0$ for each $r \in (0, R)$.*

Proof We set

$$E(r) = \frac{1}{2}\bar{u}_r(r)^2 - \frac{1}{2}g(r)\bar{u}(r)^2 + \frac{1}{p+1}h(r)\bar{u}(r)^{p+1} \quad \text{for } 0 < r < R.$$

From $f_r \geq 0$, $g_r \geq 0$ and $h_r \leq 0$, we have

$$E_r(r) = \bar{u}_r\bar{u}_{rr} - \frac{1}{2}g_r\bar{u}^2 - g\bar{u}\bar{u}_r + \frac{1}{p+1}h_r\bar{u}^{p+1} + h\bar{u}^p\bar{u}_r$$
$$= -\frac{f_r}{f}\bar{u}_r^2 - \frac{1}{2}g_r\bar{u}^2 + \frac{1}{p+1}h_r\bar{u}^{p+1} \leq 0$$

for each $r \in (0, R)$. We will show $E(r) \geq 0$ for each $r \in (0, R)$. In the case $R < \infty$, from $E(r) \to \bar{u}_r(R)^2/2$ as $r \to R$, we have $E(r) \geq 0$ for $r \in (0, R)$. In the case $R = \infty$, from $f_r(r) \geq 0$ and $f(r) > 0$ for each $r \in (0, R)$, we have $\lim_{r\to\infty} f(r) \in (0, \infty]$. So, from

$$\int_0^\infty |E(r)|f(r)\,dr \leq \int_0^\infty \left(\frac{1}{2}(\bar{u}_r(r)^2 + |g(r)|\bar{u}(r)^2) + \frac{1}{p+1}h(r)\bar{u}(r)^{p+1}\right) f(r)\,dr < \infty,$$

we can infer $E(r) \to 0$ as $r \to \infty$, and hence we have $E(r) \geq 0$ for $r \in (0, \infty)$. Next, we will show $\bar{u}_r(r) < 0$ for $0 < r \ll 1$. From $h_r(r) \leq 0$ and $h(r) > 0$ for $r \in (0, R)$, we have $\lim_{r\to 0} h(r) \in (0, \infty]$. Since $\bar{u}(r) \to \bar{u}(0) \in (0, \infty)$ and $\bar{u}_r(r) \to 0$ as $r \to 0$, $E(r) \geq 0$ for each $r \in (0, R)$, and $p > 1$, we can infer

$$-g(r)\bar{u}(r) + h(r)\bar{u}(r)^p > 0 \quad \text{for } 0 < r \ll 1,$$

which yields

$$\bar{u}_r(r) = -\frac{1}{f(r)}\int_0^r (-g(s)\bar{u}(s) + h(s)\bar{u}(s)^p) f(s)\,ds < 0 \quad \text{for } 0 < r \ll 1.$$

Now, we will show $\bar{u}_r(r) < 0$ for each $r \in (0, R)$. Assume not. Then there is $r_0 \in (0, R)$ such that $\bar{u}_r(r_0) = 0$ and $\bar{u}_r(r) < 0$ for each $r \in (0, r_0)$. So we have $\bar{u}_{rr}(r_0) \geq 0$. On the other hand, from $E(r_0) \geq 0$, we have

$$\bar{u}_{rr}(r_0) = -\big(-g(r_0)\bar{u}(r_0) + h(r_0)\bar{u}(r_0)^p\big) < 0,$$

which is a contradiction. Hence, we have shown our assertion. □

Proof of Theorem 4 Suppose that the conclusion does not hold. Then there exists $w \in \mathcal{X}_\rho \backslash \{0\}$ such that

$$\mathcal{I}''(\bar{u})[w, v] = \int_{B_R} \big(\nabla w \nabla v + gwv - ph\bar{u}^{p-1}wv\big)\rho\,dx = 0 \quad \text{for each } v \in \mathcal{X}_\rho. \tag{4.13}$$

That is, $w \in \mathcal{X}_\rho \backslash \{0\}$ is a weak solution of

$$\Delta w + \frac{\nabla w \nabla \rho}{\rho} - gw + ph\bar{u}^{p-1}w = 0 \quad \text{in } B_R.$$

We note that such w satisfies the following in the weak sense:

$$0 = \Delta w + \frac{\nabla w \nabla \rho}{\rho} - gw + ph\bar{u}^{p-1}w$$
$$= w_{rr} + \frac{n-1}{r}w_r + \frac{1}{r^2}\Delta_{S^{n-1}}w + \frac{r^{n-1}}{f}w_r(r^{1-n}f)_r - gw + ph\bar{u}^{p-1}w$$
$$= w_{rr} + \frac{f_r}{f}w_r + \frac{1}{r^2}\Delta_{S^{n-1}}w - gw + ph\bar{u}^{p-1}w.$$

Let $\{\mu_k\}$ be the eigenvalues of the Laplace-Beltrami operator on S^{n-1} and let $\{e_k\}$ be their corresponding eigenfunctions whose $L^2(S^{n-1})$ norm is one. Then it is well known that

$$\mu_0 = 0 < \mu_1 = \cdots = \mu_n = n-1 < \mu_{n+1} \leq \cdots$$

and $\{e_k\}$ is a complete orthogonal basis of $L^2(S^{n-1})$. We put

$$w_k(r) = \int_{S^{n-1}} w(r,\omega)e_k(\omega)\,d\omega \quad \text{for each } k \in \mathbb{N}\cup\{0\}. \tag{4.14}$$

From $w \in \mathcal{X}_\rho$, we can infer $w_k \in \mathcal{X}$. Since w_k satisfies

$$w_{k,rr} + \frac{f_r}{f}w_{k,r} + \left(-g(r) + ph(r)\bar{u}^{p-1} - \frac{\mu_k}{r^2}\right)w_k = 0 \quad \text{in } (0,R)$$

in the weak sense, it belongs to $C^2((0,R))$ and it satisfies the differential equation above in the classical sense in $(0,R)$. In the case $R<\infty$, we can see that w_k is continuously differentiable at R. By assumption (B13), we can see that w_k is continuously differentiable at 0, and in the case of $R=\infty$, $w_k(r)\to 0$ and $w_{k,r}(r)\to 0$ as $r\to\infty$. From $w \not\equiv 0$, there is $k\in\mathbb{N}\cup\{0\}$ such that $w_k\not\equiv 0$. Since $\mu_0=0$ and $w_0\in\mathcal{X}$, we have $w_0\equiv 0$ by Theorem 3. So we have $k\in\mathbb{N}$. From (4.14), we have $w_k(0)=0$ and $w_{k,r}(0)\in\mathbb{R}$. Let α and β be consecutive zeros of w_k such that $0\le\alpha<\beta\le R$, and in the case $R=\infty$, at least one inequality in (3.5) is not identically equal in (α,β). Without loss of generality, we may assume $w_k>0$ in (α,β). From

$$\bar{u}_{rr} + \frac{f_r}{f}\bar{u}_r - g\bar{u} + h\bar{u}^p = 0 \quad \text{in } (0,R),$$

we have

$$\bar{u}_{rrr} + \frac{f_r}{f}\bar{u}_{rr} + \left(\left(\frac{f_r}{f}\right)_r - g + ph\bar{u}^{p-1}\right)\bar{u}_r + (-g_r\bar{u} + h_r\bar{u}^{p-1}) = 0 \quad \text{in } (0,R).$$

Since we have $g_r\ge 0$, $h_r\le 0$ and

$$0 \le (\log\rho(r))_{rr} = \left(\frac{f_r}{f}\right)_r + \frac{n-1}{r^2} \le \left(\frac{f_r}{f}\right)_r + \frac{\mu_k}{r^2},$$

we obtain

$$\begin{aligned} 0 &= \Big[f(\bar{u}_{rr}w_k - \bar{u}_r w_{k,r})\Big]_\alpha^\beta + \int_\alpha^\beta \left(\left(\frac{f_r}{f}\right)_r + \frac{\mu_k}{r^2}\right)\bar{u}_r w_k f\,dr + \int_\alpha^\beta (-g_r\bar{u} + h_r\bar{u}^p)w_k f\,dr \\ &< \Big[f(\bar{u}_{rr}w_k - \bar{u}_r w_{k,r})\Big]_\alpha^\beta = \xi(\beta) - \xi(\alpha) \end{aligned}$$

in the case $R=\infty$, and $0\le\xi(\beta)-\xi(\alpha)$ in the case $R<\infty$, where

$$\begin{aligned} \xi(r) &= f(r)(\bar{u}_{rr}(r)w_k(r) - \bar{u}_r(r)w_{k,r}(r)) \\ &= -(f_r(r)\bar{u}_r(r) + f(r)g(r)\bar{u}(r) - f(r)h(r)\bar{u}(r)^p)w_k(r) - f(r)\bar{u}_r(r)w_{k,r}(r). \end{aligned}$$

From assumption (B12), we can see

$$\begin{cases} \xi(\alpha) > 0 \text{ in the case of } \alpha > 0, \\ \xi(\alpha) = 0 \text{ in the case of } \alpha = 0, \end{cases}$$

and

$$\begin{cases} \xi(\beta) < 0 \text{ in each case of } \beta \le R < \infty \text{ and } \beta < R = \infty, \\ \xi(\beta) = 0 \text{ in the case of } \beta = R = \infty. \end{cases}$$

So we have $\xi(\beta)-\xi(\alpha)\le 0$ in the case $R=\infty$ and $\xi(\beta)-\xi(\alpha)<0$ in the case $R<\infty$, which is a contradiction. Hence we have shown our assertion. □

5 Nondegeneracy in the case $R < \infty$ and $G \equiv 0$

We continue to study the nondegeneracy of the unique positive solution $\bar{u}$ of (3.1) in the case $G \equiv 0$ with $R < \infty$. Although assumption (B10′) seems to be complicated, it works for some examples; see Remarks 16 and 17.

Theorem 5 *Let $p > 1$. Assume* (B1)–(B3) *with $R < \infty$,* (B6)–(B8) *and the following.*

(B10′) (i) *g, h are continuous at R', f is monotone increasing in a neighborhood of R'.*
(ii) *$G \equiv 0$ in (R', R).*
(iii) *For each $\{\theta_m\}(\subset (0,\infty))$ with $\theta_m \to \infty$ and $\{r_m\}(\subset [R', R))$ with $r_m \to R'$, there exist a subsequence $\{m_i\}$ of $\{m\}$ and $\tilde{f} \in C^1((0,\infty)) \cap C([0,\infty))$ such that $\tilde{f}$ is positive in $(0,\infty)$,*

$$\lim_{i\to\infty} \frac{(f(\theta_{m_i}^{-\frac{p-1}{2}} t + r_{m_i}))_t}{f(\theta_{m_i}^{-\frac{p-1}{2}} t + r_{m_i})} = \frac{\tilde{f}_t(t)}{\tilde{f}(t)} \quad \text{in } C_{\text{loc}}((0,\infty))$$

and the problem

$$\begin{cases} w_{tt}(t) + \frac{\tilde{f}_t(t)}{\tilde{f}(t)} w_t(t) + h(R')|w(t)|^{p-1}w(t) = 0, & t \in (0,\infty), \\ w(0) = 1, \\ 0 \le w(t) \le 1, & t \in (0,\infty) \end{cases} \tag{5.1}$$

does not admit a solution in $C^2((0,\infty)) \cap C([0,\infty))$.

Then the unique positive solution $\bar{u}$ of problem (3.1) *is a nondegenerate critical point of the C^2-functional I defined by* (3.2).

Theorem 6 *Assume the assumptions of Theorem 4 with $R < \infty$ and* (B10′) *instead of* (B10). *Then the conclusion of Theorem 4 holds.*

Remark 15 In (B10′) (iii), $\tilde{f}$ may depend on $\{\theta_m\}$, $\{r_m\}$ and $\{m_i\}$.

Remark 16 Let $R' = 0$ and let $f(r) = r^{n-1}\rho(r)$ such that $n \ge 1$, $\rho \in C^2([0,\infty)) \cap C^3((0,\infty))$ and $\rho > 0$ in $[0,\infty)$. In this case, the function $\tilde{f}$ defined by

$$\tilde{f}(t) = \begin{cases} (t + C)^{n-1} & \text{in the case when } \theta_{m_i}^{(p-1)/2} r_{m_i} \to C \in [0,\infty), \\ 1 & \text{in the case when } \theta_{m_i}^{(p-1)/2} r_{m_i} \to \infty \end{cases}$$

satisfies the properties in (B10′) (iii).

Remark 17 The following are examples which satisfy $G(r) \equiv 0$ and (B1)–(B3).

(i) $n > 5/2$, $f(r) = r^{n-1}$, $h(r) = r^{(n-2)p+n-4}$ and $g(r) = C_1 r^{2(n-3)}$ with $C_1 \in \mathbb{R}$.
(ii) $n > 5/2$, $f(r) = r^{n-1}\exp(r^2/4)$, $h(r) = r^{(n-2)p+n-4}$ and

$$g(r) = C_1 \exp\left(-\frac{p-1}{2(p+3)} r^2\right) r^{2(n-3)} - \frac{(n-2)p+n+2}{2(p+3)} - \frac{p+1}{2(p+3)^2} r^2$$

with $C_1 \in \mathbb{R}$.

These examples can be found through the next remark. See also Appendix 1.

Remark 18 By direct calculations, it holds that

$$\frac{d}{dr}\left(f(r)^{-2}D(r)\right) = -2f(r)^{-2}a(r)G(r)$$

and that $f(r)^{-2}D(r) \equiv C_1 \in \mathbb{R}$ is equivalent to

$$\begin{aligned} g(r) = \frac{1}{(p+3)^2 f(r)^2 h(r)^2}\Big(&C_1(p+3)^2 f(r)^{\frac{8}{p+3}} h(r)^{\frac{4}{p+3}+2} + 4f_r(r)^2 h(r)^2 \\ &- f(r)h(r)\left(2(p+3)f_{rr}(r)h(r) + (p-1)f_r(r)h_r(r)\right) \\ &+ f(r)^2\left((p+4)h_r(r)^2 - (p+3)h(r)h_{rr}(r)\right)\Big). \end{aligned}$$

Once Theorem 5 is given, we can obtain Theorem 6 by the same proof of Theorem 4. So we give a proof of Theorem 5 only. We assume its assumptions. For each $\delta > 0$, we define g_δ, h_δ, a_δ, b_δ, c_δ by (4.6) and (4.7) with $\gamma \equiv 1$, and we define

$$\begin{aligned} J_\delta(r;u) = &\frac{1}{2}a_\delta(r)u_r(r)^2 + b_\delta(r)u_r(r)u(r) + \frac{1}{2}c_\delta(r)u(r)^2 \\ &- \frac{1}{2}a_\delta(r)g_\delta(r)u(r)^2 + \frac{1}{p+1}a_\delta(r)h_\delta(r)u(r)^{p+1}. \end{aligned} \tag{5.2}$$

We also define S_δ as the set of all positive solutions of

$$\begin{cases} u_{rr}(r) + \frac{f_r(r)}{f(r)}u_r - g_\delta(r)u + h_\delta(r)u^p = 0, \quad R' < r < R, \\ u(R') \in (0,\infty), \quad u(R) = 0. \end{cases} \tag{5.3}$$

We can see that $\bar{u}$ is a positive solution of (5.3) for each $\delta > 0$.

Lemma 5 *It holds that*

$$\inf_{0<\delta<1} \inf_{u \in S_\delta} \|u\|_{\mathcal{X}} > 0.$$

Proof Let $\sqrt{C_1}$ be the infimum value of the left hand side inequality in (B6). For each $\delta \in (0,1)$ and $u \in S_\delta$, we have

$$\begin{aligned} (1+\delta)\int_{R'}^{R} u(r)^{p+1}h(r)f(r)dr &= \int_{R'}^{R} u(r)^{p+1}h_\delta(r)f(r)dr \\ = \int_{R'}^{R}\left(u_r(r)^2 + g_\delta(r)u(r)^2\right)f(r)dr &\geq \int_{R'}^{R}\left(u_r(r)^2 + g(r)u(r)^2\right)f(r)dr \\ \geq C_1\left(\int_{R'}^{R} u(r)^{p+1}h(r)f(r)dr\right)^{\frac{2}{p+1}}, \end{aligned}$$

which yields

$$\left(\int_{R'}^{R} u(r)^{p+1}h(r)f(r)dr\right)^{\frac{p-1}{p+1}} \geq \frac{C_1}{1+\delta} \geq \frac{C_1}{2}.$$

Thus we have shown our assertion. □

Lemma 6 *There exist* $\delta_0 \in (0,1)$ *such that*

$$\sup_{0<\delta<\delta_0} \sup_{u \in S_\delta} \max_{R' \leq r \leq R} u(r) < \infty. \tag{5.4}$$

Proof Suppose that the conclusion does not hold. Then there exist $\{\delta_m\} \subset (0,1)$ with $\delta_m \to 0$ and $\{u_m\} \subset C^1([R', R]) \cap C^2((R', R))$ such that $u_m \in S_{\delta_m}$ for each $m \in \mathbb{N}$ and $\theta_m \equiv \max_{R' \le r \le R} u_m(r) \to \infty$ as $m \to \infty$. For each $m \in \mathbb{N}$, we choose $r_m \in (R', R)$ with $\theta_m = u_m(r_m)$ and we define

$$v_m(t) = \frac{1}{\theta_m} u_m(\theta_m^{-\frac{p-1}{2}} t + r_m) \quad \text{for each } t \in \overline{L_m},$$

where $L_m = \left(\theta_m^{\frac{p-1}{2}}(R' - r_m), \theta_m^{\frac{p-1}{2}}(R - r_m)\right)$. Without loss of generality, we may assume $r_m \to r_* \in [R', R]$. We set

$$\beta_m(t) = \theta_m^{-\frac{p-1}{2}} t + r_m \quad \text{for } m \in \mathbb{N} \text{ and } t \in \overline{L_m}.$$

Without loss of generality, we may assume that $\lim_{m\to\infty} \theta_m^{(p-1)/2}(R' - r_m)$ exists in $[-\infty, 0]$ and $\lim_{m\to\infty} \theta_m^{(p-1)/2}(R - r_m)$ exists in $[0, \infty]$. Let $L (\subset \mathbb{R})$ be the limit closed interval of $\{L_m\}$. Then we have

$$\begin{cases} L \supset (-\infty, 0] & \text{in the case } r_* > R', \\ L \supset [0, \infty) & \text{in the case } r_* = R'. \end{cases}$$

For each $m \in \mathbb{N}$, we have $v_m(0) = 1$, $v_{m,t}(0) = 0$ and

$$\begin{aligned} v_{m,tt}(t) &+ \frac{(f(\beta_m(t)))_t}{f(\beta_m(t))} v_{m,t}(t) + (1 + \delta_m) h(\beta_m(t)) v_m(t)^p \\ &- \theta_m^{-p+1}\left[g(\beta_m(t)) + \delta_m h(\beta_m(t)) \bar{u}(\beta_m(t))^{p-1}\right] v_m(t) = 0 \end{aligned} \tag{5.5}$$

for each $t \in L_m$, and hence we have

$$\begin{aligned} v_{m,t}(t) f(\beta_m(t)) = \int_0^t & f(\beta_m(s))\Big[-(1+\delta_m) h(\beta_m(s)) v_m(s)^p \\ &+ \left[g(\beta_m(s)) + \delta_m h(\beta_m(s)) \bar{u}(\beta_m(s))^{p-1}\right] \theta_m^{1-p} v_m(s)\Big] ds \end{aligned} \tag{5.6}$$

for each $t \in L_m$. In the case of $r_* > R'$, from (5.5) and (5.6), we can see that for each $\alpha > 0$,

$$\overline{\lim_{m\to\infty}} \sup_{t \in [-\alpha, 0]} |v_{m,t}(t)| < \infty \quad \text{and} \quad \overline{\lim_{m\to\infty}} \sup_{t \in [-\alpha, 0]} |v_{m,tt}(t)| < \infty.$$

Taking a subsequence $\{v_{m_i}\}$ of $\{v_m\}$, we can infer that there exists $v \in C^2((-\infty, 0])$ such that $\|v_{m_i} - v\|_{C^1_{\mathrm{loc}}((-\alpha,0])} \to 0$, v is nonnegative on $(-\infty, 0]$, and

$$\begin{cases} v_{tt}(t) + h(r_*)|v(t)|^{p-1} v(t) = 0 & \text{for each } t \in (-\infty, 0], \\ v(0) = 1, \quad v_t(0) = 0. \end{cases}$$

However, such v never exists. So, we can find that the case $r_* > R'$ does not occur. Next, we consider the case $r_* = R'$. From (B10′) (i), (5.5) and (5.6), for each $\alpha > 0$ and $\varepsilon \in (0, \alpha)$, we have

$$\overline{\lim_{m\to\infty}} \sup_{t \in [0, \alpha]} |v_{m,t}(t)| < \infty \quad \text{and} \quad \overline{\lim_{m\to\infty}} \sup_{t \in [\varepsilon, \alpha]} |v_{m,tt}(t)| < \infty.$$

Using assumption (B10′) (iii) and taking a subsequence $\{v_{m_j}\}$ of $\{v_m\}$, we can infer that there exist $\tilde{f} \in C([0,\infty)) \cap C^1((0,\infty))$ and $w \in C^2((0,\infty)) \cap C([0,\infty))$ such that $\tilde{f}$ is positive in $(0, \infty)$, $\|v_{m_j} - w\|_{C_{\mathrm{loc}}([0,\infty))} \to 0$, $\|v_{m_j} - w\|_{C^1_{\mathrm{loc}}((0,\infty))} \to 0$, and w satisfies (5.1). By (B10′) (iii), such w does not exist. So we have shown our assertion. □

Lemma 7 *It holds that*

$$\sup_{\delta\in(0,\delta_0)} \sup_{u\in S_\delta} \|u\|_{C^1([R',R])} < \infty \tag{5.7}$$

and

$$\lim_{\delta\to 0} \sup_{u\in S_\delta} \|u-\bar{u}\|_{C^1([R',R])} = 0.$$

Proof For each $\delta \in (0,\delta_0)$ and $u \in S_\delta$, we have

$$f(r)u_r(r) = \int_{R'}^{r} f(s)\big(-g_\delta(s)u(s) + h_\delta(s)u(s)^p\big)\,ds \quad \text{for each } r \in (R',R).$$

Since $f, g, h, \bar{u} \in C([R',R])$, f is monotone increasing in a neighborhood of R' and f is positive on $(R',R]$, we can infer (5.7). From (5.3) and (5.7), we have

$$\sup_{\delta\in(0,\delta_0)} \sup_{u\in S_\delta} \|u\|_{C^2([R'+\varepsilon,R])} < \infty \quad \text{for each } \varepsilon > 0. \tag{5.8}$$

Let $\{\delta_m\} \subset (0,\delta_0)$ and $\{u_m\} \subset C^1([R',R]) \cap C^2((R',R))$ such that $\delta_m \to 0$ as $m \to \infty$ and $u_m \in S_{\delta_m}$ for each $m \in \mathbb{N}$. Since we have (5.7) and (5.8), taking a subsequence if necessary, we may assume that there exists $\psi \in C([R',R]) \cap C^2((R',R))$ such that ψ is nonnegative on $[R',R]$, $\{u_m\}$ converges to ψ in $C([R',R])$ and $C^1_{\mathrm{loc}}((R',R])$, and ψ satisfies $\psi(R) = 0$ and

$$\psi_{rr}(r) + \frac{f_r(r)}{f(r)}\psi_r(r) - g(r)\psi(r) + h(r)\psi(r)^p = 0, \quad R' < r < R.$$

Since $\psi \not\equiv 0$ by Lemma 5, ψ must be positive on $[R',R)$. Since (3.1) has the unique positive solution $\bar{u}$, we have $\psi = \bar{u}$. From (B1) (iii) and

$$\begin{aligned}
&|u_{m,r}(r) - \bar{u}_r(r)| \\
&= \left|\tfrac{1}{f(r)} \textstyle\int_{R'}^{r} f(s)(-g_{\delta_m}(s)u_m(s) + h_{\delta_m}(s)u_m(s)^p + g(s)\bar{u}(s) - h(s)\bar{u}(s)^p)\,ds\right| \\
&\le \tfrac{1}{f(r)} \textstyle\int_{R'}^{r} f(s)(|g(s)| + h(s))\,ds \cdot \Big(\|u_m - \bar{u}\|_{C([R',R])} + \|u_m^p - \bar{u}^p\|_{C([R',R])} \\
&\quad + \delta_m \|\bar{u}^{p-1}u_m\|_{C([R',R])} + \delta_m \|u_m^p\|_{C([R',R])}\Big)
\end{aligned}$$

for each $r \in [R',R]$, we can infer that our assertion holds. □

Proof of Theorem 5 From $(d/dr)J(r;\bar{u}) = G(r)\bar{u}(r)^2 = 0$ for each $r \in (R',R)$, we have

$$J(r;\bar{u}) = \frac{1}{2}a(R)\bar{u}_r(R)^2 > 0 \quad \text{for each } r \in [R',R].$$

So we have $c(R') \equiv \lim_{r\to R'} c(r) \in (0,\infty)$ and

$$J(R';\bar{u}) \equiv \lim_{r\to R'} J(r;\bar{u}) = \frac{1}{2}c(R')u(R')^2.$$

Noting

$$\begin{cases} a_\delta(r) = \frac{a(r)}{(1+\delta)^{\frac{2}{p+3}}}, \quad b_\delta(r) = \frac{b(r)}{(1+\delta)^{\frac{2}{p+3}}}, \quad c_\delta(r) = \frac{c(r)}{(1+\delta)^{\frac{2}{p+3}}}, \\ J_\delta(r;u) = \frac{J(r;u)}{(1+\delta)^{\frac{2}{p+3}}} + \frac{\delta}{(1+\delta)^{\frac{2}{p+3}}}\Big(-\frac{1}{2}a(r)h(r)\bar{u}(r)^{p-1}u(r)^2 + \frac{1}{p+1}a(r)h(r)u(r)^{p+1}\Big), \end{cases} \tag{5.9}$$

and using the previous lemma, we have

$$\lim_{\delta \to 0} \sup_{u \in S_\delta} \sup_{r \in [R', R]} |J_\delta(r; u) - J(r; \bar{u})| = 0.$$

Then we can choose $0 < \delta \ll 1$ satisfying

$$\inf_{u \in S_\delta} \inf_{r \in [R', R]} J_\delta(r; u) > 0.$$

By Remark 13, we can see that $\bar{u}$ is the unique positive solution of (5.3). As in the proof of Theorem 3, we can show that if $\bar{u}$ is a degenerate critical point of I, then the Morse index $I''_\delta(\bar{u})$ is at least two, which is a contradiction. Hence, $\bar{u}$ is a nondegenerate critical point of I. □

6 Annulus or exterior domain case

In this section, we study the problem

$$\begin{cases} u_{rr}(r) + \frac{f_r(r)}{f(r)} u_r - g(r)u + h(r)u^p = 0, & R' < r < R, \\ u(R') = 0, \quad u(R) = 0, \end{cases} \tag{6.1}$$

where $-\infty < R' < R \leq \infty$, $p > 1$ and f, g, h are some functions. In the case of $R = \infty$, $u(R) = 0$ means that $u(r) \to 0$ as $r \to \infty$. We say u is a positive solution of (6.1) if

$$\begin{cases} u \in C([R', \infty)) \cap C^2((R', \infty)) & \text{in the case of } R = \infty, \\ u \in C([R', R]) \cap C^2((R', R)) & \text{in the case of } R < \infty, \end{cases}$$

$u(r) > 0$ for each $r \in (R', R)$, and u satisfies (6.1). We impose the following conditions on f, g and h.

(A1) (i) $-\infty < R' < R \leq \infty$, $g \in C([R', R)) \cap C^1((R', R))$, $f, h \in C^2([R', R)) \cap C^3((R', R))$, and f, h are positive on $[R', R)$.
(ii) In the case of $R < \infty$, $g \in C([R', R])$, $f, h \in C^2([R', R])$, $f(R) > 0$ and $h(R) > 0$.

In the following, $a(r)$, $b(r)$, $c(r)$, $G(r)$ and $J(r; u)$ are the ones given in Proposition 1. By similar arguments as in the proof of Theorem 1, we can prove the next theorem. So we omit its proof.

Theorem 7 *Let* $p > 1$. *Assume* (A1) *and the following.*

(A2) *One of the following conditions is satisfied.*

(i) *There exists* $\kappa \in [R', R]$ *such that*

$$G(r) \geq 0 \text{ in } (R', \kappa) \quad \textit{and} \quad G(r) \leq 0 \text{ in } (\kappa, R).$$

(ii) $\{R' < r < R : G(r) = 0,\ D(r) > 0\} = \emptyset$.

Then in the case of $R < \infty$, *problem* (6.1) *has at most one positive solution, and in the case of* $R = \infty$, *problem* (6.1) *has at most one positive solution* u *which satisfies* $J(r; u) \to 0$ *as* $r \to \infty$.

Remark 19 In the case of $R = \infty$, $G^- \not\equiv 0$ is not assumed as in Theorem 1. However, if (6.1) has a positive solution u such that $J(r; u) \to 0$ as $r \to \infty$, it must be $G^- \not\equiv 0$. Indeed, we have $J(R'; u) = (1/2)a(R')\bar{u}_r(R')^2 > 0$ and $J(r; u) \to 0$ as $r \to \infty$. From $(d/dr)J(r; u) = G(r)u(r)^2$, we have $G^- \not\equiv 0$.

As in Sect. 3, we define

$$\|\varphi\|_{\mathcal{X}} = \left(\int_{R'}^{R} \left(\varphi_r(r)^2 + g(r)\varphi(r)^2\right) f(r)\, dr \right)^{\frac{1}{2}} \quad \text{for each } \varphi \in C_0^\infty(R', R),$$

$$\|\varphi\|_{\mathcal{L}} = \left(\int_{R'}^{R} h(r)|\varphi(r)|^{p+1} f(r)\, dr \right)^{\frac{1}{p+1}} \quad \text{for each } \varphi \in C_0^\infty(R', R),$$

and we impose the following conditions.

(A3) $$\inf_{\varphi \in C_0^\infty((R',R))\setminus\{0\}} \frac{\|\varphi\|_{\mathcal{X}}}{\|\varphi\|_{\mathcal{L}}} > 0, \qquad \inf_{\varphi \in C_0^\infty((R',R))\setminus\{0\}} \frac{\|\varphi\|_{\mathcal{X}}^2}{\int_{R'}^{R} \left(\varphi_r^2 + |g|\varphi^2\right) f\, dr} > 0.$$

We denote by $\mathcal{X}$ and $\mathcal{L}$ the completion of $C_0^\infty((R', R))$ with respect to $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{L}}$, respectively. We can see that both inequalities in (A3) hold even if the infimums are taken on $\mathcal{X}\setminus\{0\}$.

Theorem 8 *Let $p > 1$ and assume* (A1)–(A3). *Assume also the following.*

(A5) *The embedding $\mathcal{X} \hookrightarrow \mathcal{L}$ is compact.*
(A6) *In the case of $R = \infty$, for each $u \in \mathcal{X} \cap C^2((R', \infty)) \cap C([R', \infty))$ which is positive in (R', ∞) and satisfies*

$$u_{rr}(r) + \frac{f_r(r)}{f(r)} u_r(r) - g(r)u(r) + h(r)u(r)^p = 0 \quad \text{for each } r \in (R_u, \infty)$$

with some $R_u \in (R', \infty)$, there holds

$$\lim_{r\to\infty} J(r; u) = 0.$$

Then problem (6.1) *has a unique positive solution in $\mathcal{X}$ and it is a nondegenerate critical point of C^2-functional I defined by*

$$I(u) = \int_{R'}^{R} \left(\frac{1}{2}(|u_r(r)|^2 + g(r)|u(r)|^2) - \frac{1}{p+1} h(r)|u(r)|^{p+1} \right) f(r)\, dr \quad \text{for } u \in \mathcal{X}.$$

Remark 20 In the case $R = \infty$ and $G^- \equiv 0$, if (A1)–(A3) are satisfied, then (A5) or (A6) must not hold; see Remark 19.

By similar arguments as in the proof of Theorem 2, we can show that there exists a unique positive solution $\bar{u}$ of (6.1). If $G \not\equiv 0$ in (R', R), as in the proof of Theorem 3, we can show that $\bar{u}$ is a nondegenerate critical point of I. Even if $G \equiv 0$ in (R', R), by similar arguments as in the proof of Theorem 5, we can show that $\bar{u}$ is a nondegenerate critical point of I. However, for the reader's convenience, in Appendix 3, we show the nondegeneracy in the case $G \equiv 0$ in (R', R).

7 Applications

For a given function space, by adding subscript "rad", we denote its restriction to radial functions. If $q \geq 1$ and a function α is given, we denote by L^q_α, the space consists of functions such that the integral of $|u(\cdot)|^q \alpha(\cdot)$ is finite.

7.1 The scalar field equation

Although the results in this subsection are well known, for the reader's convenience, we consider the problem

$$\Delta u(x) - u(x) + u(x)^p = 0 \quad \text{in } \mathbb{R}^n, \qquad u(x) \to 0 \quad \text{as } |x| \to \infty, \tag{7.1}$$

where $n \in \mathbb{N}$ with $n \geq 2$ and $1 < p < (n+2)/(n-2)$. The uniqueness of a positive solution (7.1) up to translation was established by Kwong [25]. Setting $R' = 0$, $R = \infty$, $\rho(r) = 1$, $f(r) = r^{n-1}$, $g(r) = 1$ and $h(r) = 1$, we apply our results. Since we have

$$\begin{aligned} G(r) = {}& \frac{n-1}{(p+3)^3} r^{\frac{2(n-1)(p+1)}{p+3}-3} \\ & \times \Big(2((n-2)p + n - 4)(n + 2 - (n-2)p) - (p-1)(p+3)^2 r^2 \Big), \end{aligned}$$

we can see that in the case of $n = 2$, $G(r) < 0$ in $(0, \infty)$, and in the case of $n \geq 3$, there is $\kappa \in (0, \infty)$ such that $G(r) > 0$ in $(0, \kappa)$ and $G(r) < 0$ in (κ, ∞). Each radially symmetric, positive solution u of (7.1) decays exponentially and so does u_r, which implies $J(r; u) \to 0$ as $r \to \infty$ and u belongs to $H^1_{\text{rad}}(\mathbb{R}^n)$. Moreover, it is well known that the embedding $H^1_{\text{rad}}(\mathbb{R}^n) \hookrightarrow L^{p+1}(\mathbb{R}^n)$ is compact. Hence by our theorems, there is a unique radially symmetric, positive solution $\bar{u}$ of (7.1) and it is a nondegenerate critical point of $\mathcal{I}|_{H^1_{\text{rad}}(\mathbb{R}^n)}$, where $\mathcal{I}$ is defined by

$$\mathcal{I}(u) = \int_{\mathbb{R}^n} \left(\frac{1}{2}(|\nabla u(x)|^2 + |u(x)|^2) - \frac{1}{p+1}|u(x)|^{p+1} \right) dx \quad \text{for } u \in H^1(\mathbb{R}^n).$$

Since $(\log \rho(r))_{rr} = 0$, $g_r(r) = 0$ and $h_r(r) = 0$, we can not apply Theorem 4. Actually, we know that the kernel of $\mathcal{I}''(\bar{u})$ is spanned by $\partial \bar{u}/\partial x_1, \ldots, \partial \bar{u}/\partial x_n$, see [39, Lemma 4.2], and $\bar{u}$ is a degenerate critical point of $\mathcal{I}$.

7.2 Matukuma's equation

Let $n \in \mathbb{N}$ with $n \geq 3$ and $1 < p < (n+2)/(n-2)$. We study

$$u \in \dot{H}^1(\mathbb{R}^n) \quad \text{and} \quad \Delta u(x) + \frac{u(x)^p}{1 + |x|^2} = 0 \quad \text{in } \mathbb{R}^n. \tag{7.2}$$

Here, $\dot{H}^1(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm defined by

$$\|u\| = \left(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.$$

For the problem, we refer to [27–31,40,41,47,52,53]. Since $\dot{H}^1(\mathbb{R}^n)$ is continuously embedded into $L^{2n/(n-2)}(\mathbb{R}^n)$, we can easily see that there is $C_1 > 0$ such that

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{p+1}}{1 + |x|^2} \, dx \right)^{\frac{1}{p+1}} \leq C_1 \|u\| \quad \text{for each } u \in \dot{H}^1(\mathbb{R}^n).$$

So we can define a C^2-functional $\mathcal{I}$ on $\dot{H}^1(\mathbb{R}^n)$ by

$$\mathcal{I}(u) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla u(x)|^2 - \frac{1}{p+1} \frac{|u(x)|^{p+1}}{1 + |x|^2} \right) dx \quad \text{for } u \in \dot{H}^1(\mathbb{R}^n).$$

We apply our results with $R' = 0, R = \infty, \rho(r) = 1, f(r) = r^{n-1}, g(r) = 0$ and $h(r) = 1/(1+r^2)$. Since we have

$$|u(r)| \le \int_r^\infty |u_t(t)|\,dt \le \left(\int_r^\infty |u_t(t)|^2 t^{n-1}\,dt\right)^{\frac{1}{2}} \frac{r^{\frac{2-n}{2}}}{\sqrt{n-2}}$$

for each $r > 0$ and $u \in C_0^\infty(\mathbb{R}^n)$, we can see that there exists $C_2 > 0$ which satisfies

$$\int_r^\infty \frac{1}{1+s^2}|u(s)|^{p+1}s^{n-1}\,ds \le C_2\|u\|^{p+1}\int_r^\infty s^{\frac{2-n}{2}(p+1)+n-3}\,ds$$

and

$$\int_0^r \frac{1}{1+s^2}|u(s)|^{p+1}s^{n-1}\,ds \le C_2\|u\|^{p+1}\int_0^r s^{\frac{2-n}{2}(p+1)+n-1}\,ds$$

for each $r > 0$ and $u \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^n)$. Noting

$$\frac{2-n}{2}(p+1)+n-3 < -1 \Leftrightarrow p > 1, \quad \frac{2-n}{2}(p+1)+n-1 > -1 \Leftrightarrow p < \frac{n+2}{n-2},$$

we can find that the embedding $\dot{H}^1_{\mathrm{rad}}(\mathbb{R}^n) \hookrightarrow L_h^{p+1}(\mathbb{R}^n)$ is compact.

Next, we will show (B9'). Let u be a positive solution of

$$u \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^n) \quad \text{and} \quad (r^{n-1}u_r(r))_r + \frac{r^{n-1}}{1+r^2}u(r)^p = 0 \quad \text{for} \quad r \ge R_u, \tag{7.3}$$

where $R_u > 0$. Without loss of generality, we may assume $u \in C([R_u,\infty)) \cap C^2((R_u,\infty))$. We know $u(r) = O(r^{(2-n)/2})$ as $r \to \infty$, and we will show $u(r) = O(r^{2-n})$ and $u_r(r) = O(r^{1-n})$ as $r \to \infty$. Let $\varepsilon > 0$ be any number satisfying $n-2 < (n-2-\varepsilon)p$. Assume $u(r) = O(r^{-\alpha})$ as $r \to \infty$ with $(n-2)/2 \le \alpha < n-2-\varepsilon$. Setting $v(r) = r^{-s}$ with $0 < s < n-2$ and $s \le \alpha p$, we can see

$$(u - Cv)(R_u) \le 0 \quad \text{and} \quad (u-Cv)_{rr} + \frac{n-1}{r}(u-Cv)_r \ge 0 \quad \text{for} \quad r > R_u$$

with some $C > 0$, which yields $u(r) = O(r^{-s})$ as $r \to \infty$. Applying this procedure several times, we can infer $u(r) = O(r^{2-n+\varepsilon})$ as $r \to \infty$. Since we can take any small $\varepsilon > 0$, from (7.3), we can easily see that $r^{n-1}u_r(r) \to \beta \in \mathbb{R}$ as $r \to \infty$. By l'Hôptail's rule, we have $r^{n-2}u(r) \to -\beta/(n-2)$ as $r \to \infty$. So we have shown $u(r) = O(r^{2-n})$ and $u_r(r) = O(r^{1-n})$ as $r \to \infty$. On the other hand, by Appendix 1, we can see

$$a(r) = O(r^\nu), \quad b(r) = O(r^{\nu-1}) \quad \text{and} \quad c(r) = O(r^{\nu-2}) \quad \text{as } r \to \infty,$$

where $\nu = 2(n-1)(p+1)/(p+3)$. Hence we have shown (B9').

In [47, Section 5.2], we have shown that there is $\kappa \in (0,\infty)$ such that $G(r) > 0$ in $(0,\kappa)$ and $G(r) < 0$ in (κ,∞). Hence there is a unique positive solution $\bar{u} \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^n)$ of (7.2). From $\bar{u}(r) = O(r^{2-n})$ and $\bar{u}_r(r) = O(r^{1-n})$ as $r \to \infty$, we have (B12) (ii). By the elliptic regularity, we can infer that (B13) holds. Since it is easy to see that other assumptions are satisfied, $\bar{u}$ is a nondegenerate critical point of $\mathcal{I}$. Summing up, we have shown the following.

Theorem 9 *Let $n \in \mathbb{N}$ with $n \ge 3$ and $1 < p < (n+2)/(n-2)$. Then there exists a unique positive radial solution of (7.2) and it is a nondegenerate critical point of $\mathcal{I}$.*

Remark 21 In the argument above, we can show $\beta < 0$. Indeed, from $(ru_r(r) + (n-2)u(r))_r < 0$ for $r > R_u$, we can see $ru_r(r) + (n-2)u(r) > 0$ for $r > R_u$. Since $(r^{n-2}u(r))_r = r^{n-3}(ru_r(r) + (n-2)u(r))$, we have $\lim_{r\to\infty} r^{n-2}u(r) > 0$, which yields $\beta < 0$.

Remark 22 The existence of a unique positive solution of (7.2) was obtained by Yanagida [52, 53]. He showed the problem has a unique positive radial solution u with finite total mass, i.e., $\int_{\mathbb{R}^n} u(x)^p/(1+|x|^2)\,dx < \infty$.

7.3 Nonlinear Schrödinger equation with harmonic potential

We study the problem

$$\Delta u - (\lambda + |x|^2)u + u^p = 0 \quad \text{in } \mathbb{R}^n \quad \text{and} \quad u(x) \to 0 \quad \text{as } |x| \to \infty, \tag{7.4}$$

where $n \in \mathbb{N}$ with $n \geq 2$, $\lambda > -n$, $1 < p < \infty$ in the case $n = 2$ and $1 < p \leq (n+2)/(n-2)$ in the case $n \geq 3$. For the problem, we refer to [15,16,19–21,47]. We know that each positive solution u of (7.4) is radially symmetric, and u and u_r decay exponentially; so we have $u \in \Sigma$, where

$$\Sigma = \left\{u \in H^1(\mathbb{R}^n) : |x|u \in L^2(\mathbb{R}^n)\right\}.$$

We set

$$\|u\|_\Sigma = \left(\int_{\mathbb{R}^n} (|\nabla u|^2 + |x|^2|u|^2)\,dx\right)^{\frac{1}{2}} \quad \text{for each } u \in \Sigma.$$

It is well known that the embedding $(\Sigma, \|\cdot\|_\Sigma) \hookrightarrow L^2(\mathbb{R}^n)$ is compact,

$$n = \inf_{u \in \Sigma\setminus\{0\}} \frac{\int_{\mathbb{R}^n} (|\nabla u|^2 + |x|^2|u|^2)\,dx}{\int_{\mathbb{R}^n} |u|^2\,dx},$$

and the infimum is attained by $x \mapsto \exp(-|x|^2/2)$. Since $\lambda > -n$, the norm defined by

$$\|u\| = \left(\int_{\mathbb{R}^n} (|\nabla u|^2 + (\lambda + |x|^2)|u|^2)\,dx\right)^{\frac{1}{2}} \quad \text{for } u \in \Sigma$$

is equivalent to $\|\cdot\|_\Sigma$. We define

$$\mathcal{I}(u) = \int_{\mathbb{R}^n} \left(\frac{1}{2}(|\nabla u|^2 + (\lambda + |x|^2)|u|^2) - \frac{1}{p+1}|u|^{p+1}\right) dx \quad \text{for } u \in \Sigma.$$

We note that $\mathcal{I} \in C^2(\Sigma, \mathbb{R})$ by the Sobolev embedding theorem. Setting $R' = 0$, $R = \infty$, $f(r) = r^{n-1}$, $g(r) = \lambda + r^2$ and $h(r) = 1$, we apply our results. In the case of $n \geq 3$, we have shown in [47, Section 5.4] that there is $\kappa \in [0, \infty)$ such that $G(r) > 0$ in $(0, \kappa)$ and $G(r) < 0$ in (κ, ∞). In the case of $n = 2$, we have

$$G(r) = -\frac{r^{-\frac{p+7}{p+3}}}{(p+3)^3}\Big((p-1)(p+3)^2r^2(\lambda + r^2) + (p+3)^3r^4 + 16\Big),$$

$$D(r) = \frac{r^{2-\frac{8}{p+3}}}{(p+3)^2}\left((p+3)^2r^2\left(\lambda + r^2\right) - 4\right),$$

which yields $\{r \in (0, \infty) : G(r) = 0,\ D(r) > 0\} = \emptyset$. So we have shown (B5). By the compactness of $\Sigma \hookrightarrow L^2(\mathbb{R}^n)$, in the subcritical case, it is easy to see that (B7) (i) holds. In the critical case, assuming $\lambda < 0$ in the case $n \geq 4$ and $\lambda < -1$ in the case $n = 3$, and setting $\hat{g}(r) = r^2$, we have $S_g < S_{\hat{g}}$; see [15, Section 5]. Using the compactness of $\Sigma \hookrightarrow L^2(\mathbb{R}^n)$ again, we can see that (B7) (ii) holds. Since it is easy to see that other assumptions hold, we can obtain the following.

Theorem 10 *Let $n \in \mathbb{N}$ with $n \geq 2$, $p > 1$ and $\lambda > -n$. If $p < (n+2)/(n-2)$, or*

$$p = \frac{n+2}{n-2} \quad \text{and} \quad \lambda < \begin{cases} 0 & \text{in the case of } n \geq 4, \\ -1 & \text{in the case of } n = 3, \end{cases} \tag{7.5}$$

then problem (7.4) has a unique positive solution, it is radially symmetric, and it is a nondegenerate critical point of $\mathcal{I}$.

Remark 23 In the critical case, if λ does not satisfy the inequality in (7.5), problem (7.4) does not have a positive solution. See [16] and [47, Theorem 7].

Remark 24 In the subcritical case with $n \geq 2$, the uniqueness of a positive solution was studied by Hirose–Ohta [19–21]. In [47, Section 5.4], we studied the uniqueness of a positive solution including the critical case, but we could not treat the case $n = 2$. Here, applying the condition (B5) (ii), we show its uniqueness even in the case $n = 2$.

7.4 The Haraux–Weissler equation

We study the problem

$$u \in \dot{H}^1_\rho(\mathbb{R}^n) \quad \text{and} \quad \Delta u(x) + \frac{1}{2} x \cdot \nabla u(x) + \lambda u(x) + u(x)^p = 0 \quad \text{in } \mathbb{R}^n. \tag{7.6}$$

Here, $n \in \mathbb{N}$ with $n \geq 2$, $\lambda < n/2$, $1 < p \leq (n+2)/(n-2)$, and

$$\dot{H}^1_\rho(\mathbb{R}^n) = \left\{ u \in \dot{H}^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\nabla u|^2 \rho(x) dx < \infty \right\},$$

where $\rho(x) = \exp(|x|^2/4)$ and $\dot{H}^1_\rho(\mathbb{R}^n)$ is endowed with the norm

$$\|u\|_{\dot{H}^1_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\nabla u(x)|^2 \rho(x)\, dx \right)^{\frac{1}{2}} \quad \text{for each } u \in \dot{H}^1_\rho(\mathbb{R}^n).$$

For the problem, we refer to [1,12,13,17,18,43,47,50,51]. It is well known that the embedding $\dot{H}^1_\rho(\mathbb{R}^n) \hookrightarrow L^2_\rho(\mathbb{R}^n)$ is compact,

$$\inf_{v \in \dot{H}^1_\rho(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla v(x)|^2 \rho(x)\, dx}{\int_{\mathbb{R}^n} |v(x)|^2 \rho(x)\, dx} = \frac{n}{2},$$

and the infimum is attained by $x \mapsto \exp(-|x|^2/4) \in \dot{H}^1_\rho(\mathbb{R}^n)$. We define

$$\|u\| = \left(\int_{\mathbb{R}^n} (|\nabla u(x)|^2 - \lambda |u(x)|^2) \rho(x)\, dx \right)^{\frac{1}{2}} \quad \text{for each } u \in \dot{H}^1_\rho(\mathbb{R}^n).$$

Since $\lambda < n/2$, the norm is equivalent to $\|\cdot\|_{\dot{H}^1_\rho(\mathbb{R}^n)}$. We define

$$\mathcal{I}(u) = \int_{\mathbb{R}^n} \left(\frac{1}{2} (|\nabla u|^2 - \lambda |u|^2) - \frac{1}{p+1} |u|^{p+1} \right) \rho(x)\, dx \quad \text{for } u \in \dot{H}^1_\rho(\mathbb{R}^n).$$

Setting $R' = 0$, $R = \infty$, $\rho(r) = \exp(r^2/4)$, $f(r) = r^{n-1}\exp(r^2/4)$, $g(r) = -\lambda$ and $h(r) = 1$, we apply our results. In the case of $n \geq 3$, we have shown in [47, Section 5.5] that there is $\kappa \in [0, \infty)$ such that $G(r) > 0$ in $(0, \kappa)$ and $G(r) < 0$ in (κ, ∞). In the case of $n = 2$, we have

$$\begin{aligned} G(r) = &-\frac{\left(e^{\frac{r^2}{4}} r\right)^{\frac{2(p+1)}{p+3}}}{4(p+3)^3 r^3}\Big((p^2-1)r^6 + 2(5p^2+6p-3)r^4 + 12(p^2-1)r^2 + 64 \\ &-2\lambda(p-1)(p+3)^2(r^2+2)\Big), \\ D(r) = &\frac{\left(e^{\frac{r^2}{4}} r\right)^{\frac{4(p+1)}{p+3}}}{2(p+3)^2 r^2}\left((p+1)r^4 + 2(3p+5)r^2 - 8 - 2\lambda(p+3)^2 r^2\right). \end{aligned}$$

Since

$$\begin{aligned} &(p^2-1)r^6 + 2(5p^2+6p-3)r^4 + 12(p^2-1)r^2 + 64 - 2\lambda(p-1)(p+3)^2(r^2+2) \\ &\quad -(p-1)(r^2+2)\left((p+1)r^4 + 2(3p-5)r^2 - 8 - 2\lambda(p+3)^2 r^2\right) \\ &= 2(p+3)\left((p+1)r^4 + 8\right), \end{aligned}$$

we can see $\{r \in (0, \infty) : G(r) = 0,\ D(r) > 0\} = \emptyset$, which yields (B5) (ii). We know that if $2 \leq q < 2n/(n-2)$, the embedding $\dot{H}^1_\rho(\mathbb{R}^n) \hookrightarrow L^q_\rho(\mathbb{R}^n)$ is compact, and if $n \geq 3$, the embedding $\dot{H}^1_\rho(\mathbb{R}^n) \hookrightarrow L^{2n/(n-2)}_\rho(\mathbb{R}^n)$ is continuous. So in the subcritical case, we have (B7) (i). In the critical case, assuming $\lambda > \max\{1, n/4\}$ and setting $\hat{g}(r) = 0$, we have $S_g < S_{\hat{g}}$; see [13, Theorem 4.10]. So we can see (B7) (ii). Next, we will show (B9'). Let u be a positive solution of

$$\begin{cases} u \in \dot{H}^1_{\rho,\mathrm{rad}}(\mathbb{R}^n), \\ (r^{n-1}\exp(r^2/4)u_r(r))_r + r^{n-1}\exp(r^2/4)(\lambda u(r) + u(r)^p) = 0 \quad \text{for} \quad r > R_u, \end{cases}$$

where R_u is some positive real number. By [47, (5.26)], we can find

$$u(r) = O\left(r^{2\lambda-n}\exp(-r^2/4)\right) \quad \text{and} \quad u_r(r) = O(r^{2\lambda-n+1}\exp(-r^2/4)) \quad \text{as} \quad r \to \infty.$$

On the other hand, by Appendix 1, we have

$$a(r) = O(\alpha(r)), \quad b(r) = O(r\alpha(r)) \quad \text{and} \quad c(r) = O(r^2\alpha(r)) \quad \text{as} \quad r \to \infty,$$

where $\alpha(r) = (r^{n-1}\exp(r^2/4))^{2(p+1)/(p+3)}$. So we can see that (B9') holds. Hence there is a unique radially symmetric, positive solution of (7.6). Since we have $f_r(r) = (r^2/2 + n - 1)r^{n-2}\exp(r^2/4)$, $(\log\rho(r))_{rr} = 1/2$, $g_r(r) = 0$ and $h_r(r) = 0$, we can see that (B11) holds. By [34, Proposition A.1] and [35, Proposition A.1], we can see that (B13) holds. Hence by our theorems, we can obtain the following.

Theorem 11 *Let $n \in \mathbb{N}$ with $n \geq 2$, $p > 1$ and $\lambda < n/2$. If $p < (n+2)/(n-2)$, or $p = (n+2)/(n-2)$ and $\lambda > \max\{1, n/4\}$, then problem (7.6) has a unique positive radial solution and it is a nondegenerate critical point of $\mathcal{I}$.*

Remark 25 In the case of $n \geq 3$ and $1 < p < (n+2)/(n-2)$, with an additional assumption $\lambda > 0$, Hirose [18, Theorem 1.1] obtained the uniqueness of a positive radial solution of (7.6).

7.5 The Brezis–Nirenberg problem on a spherical cap and a spherical band

Let $n \in \mathbb{N}$ with $n \geq 2$ and let Δ_{S^n} be the Laplace–Beltrami operator on S^n, where $S^n = \{X = (X_1, \ldots, X_n, X_{n+1}) \in \mathbb{R}^{n+1} : |X| = 1\}$. Let $p > 1$ and we consider the Brezis–Nirenberg problem on a spherical cap

$$\begin{cases} \Delta_{S^n} u + \lambda u + u^p = 0 & \text{in} \ \ \Omega_{\theta_1}, \\ u = 0 & \text{on} \ \ \partial\Omega_{\theta_1}, \end{cases} \tag{7.7}$$

where $\Omega_{\theta_1} = \{X \in S^n : X_{n+1} > \cos\theta_1\}$ with $\theta_1 \in (0, \pi)$ and $\lambda < \lambda_1$. Here, λ_1 is the first eigenvalue of $-\Delta_{S^n}$ on Ω_{θ_1} with the Dirichlet boundary condition. For the problem, we refer to [2–4,7,10,47]. Let $P : S^n \backslash \{(0, \ldots, 0, -1)\} \to \mathbb{R}^n$ be the stereographic projection defined by

$$P(X_1, \ldots, X_n, X_{n+1}) = \frac{1}{X_{n+1} + 1}(X_1, \ldots, X_n), \quad X \in S^n \backslash \{(0, \ldots, 0, -1)\}. \tag{7.8}$$

We set

$$R = \tan\frac{\theta_1}{2} \quad \text{and} \quad B_R = \{x \in \mathbb{R}^n : |x| < R\}. \tag{7.9}$$

We can easily see $B_R = P(\Omega_{\theta_1})$. We consider the problem

$$\begin{cases} \Delta v + \frac{n(n-2)+4\lambda}{(1+|x|^2)^2} v + 4(1+|x|^2)^{\frac{(n-2)p-(n+2)}{2}} v^p = 0 & \text{in} \ \ B_R, \\ v = 0 & \text{on} \ \ \partial B_R. \end{cases} \tag{7.10}$$

Then we can see that u is a positive solution of (7.7) if and only if the function v defined by

$$u(P^{-1}x) = (1+|x|^2)^{\frac{n-2}{2}} v(x) \quad \text{for } x \in \overline{B_R} \tag{7.11}$$

is a positive solution of (7.10). Setting $f(r) = r^{n-1}$,

$$g(r) = -\frac{n(n-2)+4\lambda}{(1+r^2)^2} \quad \text{and} \quad h(r) = 4(1+r^2)^{\frac{(n-2)p-(n+2)}{2}}, \tag{7.12}$$

we have

$$G(r) = \frac{2^{\frac{p-1}{p+3}}(n-1)}{(p+3)^3} r^{\frac{2(n-1)(p+1)}{p+3}-3} (1+r^2)^{\frac{n+2-(n-2)p}{p+3}-3} (1-r^2)(Ar^4 + Br^2 + A),$$

where

$$\begin{aligned} A &= \big(n+2-(n-2)p\big)\big((n-2)p+n-4\big) \\ &= (p+3)[3n^2 - 6n - (n^2 - 4n + 4)p] - 8(n-1)^2, \\ B &= (p+3)[-6n^2 + 12n + (2n^2 + 4\lambda - 4)p + 2\lambda p^2 - 6\lambda - 12] + 16(n-1)^2. \end{aligned}$$

We note that $A < 0$ for $n = 2$, and that for $n \geq 3$,

$$A > 0 \Leftrightarrow p < \frac{n+2}{n-2}, \quad A = 0 \Leftrightarrow p = \frac{n+2}{n-2}, \quad A < 0 \Leftrightarrow p > \frac{n+2}{n-2}.$$

We set

$$\lambda_{n,p} = \frac{6 + (6-4n)p}{(p+3)(p-1)}.$$

Then we can easily see

$$\lambda > \lambda_{n,p} \Leftrightarrow 2A + B > 0, \quad \lambda = \lambda_{n,p} \Leftrightarrow 2A + B = 0, \quad \lambda < \lambda_{n,p} \Leftrightarrow 2A + B < 0,$$

and for $n \geq 3$,

$$\lambda_{n,p} < 0 \quad \text{and} \quad \lambda_{n,(n+2)/(n-2)} = -\frac{n(n-2)}{4}.$$

For the reader's convenience, we give the following.

Lemma 8 *There hold the following.*

(i) *In the case of $A > 0$,*

(a) *if $2A + B \geq 0$, $G(r)$ changes its sign only at $r = 1$ from plus to minus,*
(b) *if $2A + B < 0$, then there exists unique $\bar{r} \in (0, 1)$ with $G(\bar{r}) = 0$ and $G(r)$ changes its sign as follows:*

r	0		$\bar{r}$		1		$1/\bar{r}$		∞
$G(r)$		$+$		$-$		$+$		$-$	

(ii) *In the case of $A = 0$,*

(a) *if $B > 0$, then $G(r)$ changes its sign only at $r = 1$ from plus to minus,*
(b) *if $B = 0$, then $G(r) \equiv 0$,*
(c) *if $B < 0$, then $G(r)$ changes its sign only at $r = 1$ from minus to plus.*

(iii) *In the case of $A < 0$,*

(a) *if $2A + B \leq 0$, then $G(r)$ changes its sign only at $r = 1$ from minus to plus,*
(b) *if $2A + B > 0$, then there exists unique $\bar{r} \in (0, 1)$ with $G(\bar{r}) = 0$ and $G(r)$ changes its sign as follows:*

r	0		$\bar{r}$		1		$1/\bar{r}$		∞
$G(r)$		$-$		$+$		$-$		$+$	

Proof We set $z(r) = Ar^4 + Br^2 + A$ for $r \in (0, \infty)$. We note that the sign of $G(r)$ equals to the sign of $(1-r)z(r)$ and that $z(r) = A(r^2 - 1)^2 + (2A + B)r^2$ and $z(1) = 2A + B$. We consider the case $A > 0$. If $2A + B \geq 0$, we have $z(r) > 0$ in $(0, 1) \cup (1, \infty)$, which yields (i) (a). If $2A + B < 0$, there is unique $\bar{r} \in (0, 1)$ with $z(\bar{r}) = 0$. Since we have $z(r) > 0$ in $(0, \bar{r}) \cup (1/\bar{r}, \infty)$ and $z(r) < 0$ in $(\bar{r}, 1/\bar{r})$, we can see that the conclusion of (i) (b) holds. We can show other cases, similarly. □

Now, we apply our results to problem (7.7) for the subcritical and critical cases. We note that in the case of $n = 3$ and $p = 5$, Bandle and Benguria [4] studied the existence of a positive radial solution of (7.7) and its uniqueness, and that the uniqueness results except for the case $n = 2$ was also studied in [47]. In the following, recall that P, R, B_R, g and h are the ones given in (7.8), (7.9) and (7.12).

Theorem 12 *Let $n \in \mathbb{N}$ with $n \geq 2$ and $\theta_1 \in (0, \pi)$. Assume one of the following conditions.*

(i) *$n \geq 3$, $1 < p < (n+2)/(n-2)$ and one of the following holds:*

(a) *$\theta_1 \leq \pi/2$ and $\lambda \in (-\infty, \lambda_1)$,*
(b) *$\theta_1 > \pi/2$ and $\lambda \in [\lambda_{n,p}, \lambda_1)$.*

(ii) *$n \geq 4$, $p = (n+2)/(n-2)$ and $\lambda \in (\lambda_{n,(n+2)/(n-2)}, \lambda_1)$.*

(iii) $n = 3$, $p = 5$ *and* $\lambda \in (\mu_1, \lambda_1)$, *where* $\mu_1 = (\pi^2 - 4\theta_1^2)/(4\theta_1^2)$ *and* $\lambda_1 = (\pi^2 - \theta_1^2)/\theta_1^2$.
(iv) $n = 2$, $p > 1$ *and one of the following holds:*

(a) $\theta_1 \le \pi/2$ *and* $\lambda \in (-\infty, \lambda_1)$,
(b) $\theta_1 > \pi/2$ *and* $\lambda \in [-2/(p+3), \lambda_1)$.

Then problem (7.7) *has a unique positive radial solution* u. *Moreover, let* v *be the positive radial solution to* (7.10) *defined by* (7.11). *Then* v *is a nondegenerate critical point of* $\mathcal{I}|_{H^1_{0,\mathrm{rad}}(B_R)}$ *on* $H^1_{0,\mathrm{rad}}(B_R)$, *and if* $\lambda \ge -n(n-2)/4$ *is additionally assumed, then* v *is a nondegenerate critical point of* $\mathcal{I}$ *on* $H^1_0(B_R)$, *where* $\mathcal{I}$ *is defined by*

$$\mathcal{I}(w) = \int_{B_R} \left(\frac{1}{2}(|\nabla w|^2 + g(|x|)w^2) - \frac{1}{p+1} h(|x|)|w|^{p+1} \right) dx \quad \text{for } w \in H^1_0(B_R).$$

Proof of Theorem 12 For the case of $n \ge 3$ and $1 < p < (n+2)/(n-2)$ and for the case of $n = 2$ and $p > 1$, we know that the embedding $H^1_0(B_R) \hookrightarrow L^p(B_R)$ is compact, and hence (B7) (i) holds. Even for the case of $n \ge 4$ and $p = (n+2)/(n-2)$, it can be shown by similar arguments in [6], we can see that (B7) (ii) holds with $\hat{g} \equiv 0$. For the case $n = 3$ and $p = 5$, from [4, Proof of Lemma 1], we can find that (B7) (ii) holds with

$$\hat{g}(r) = -\frac{n(n-2) + 4\mu_1}{(1+r^2)^2}.$$

Hence, for the cases (i)–(iii), we can find that the problem has a unique positive solution by using Theorem 2 and Lemma 8. We consider the case $n = 2$ and $p > 1$. In this case, we have

$$D(r) = -\frac{1}{(p+3)^2} \left(\frac{2r}{1+r^2} \right)^{\frac{2(p-1)}{p+3}} \left((r^2-1)^2 + (p+3)(2 + (p+3)\lambda)r^2 \right).$$

If $\theta_1 \in (0, \pi)$ and $-2/(p+3) \le \lambda < \lambda_1$, we can easily see $\{r \in (0, R) : G(r) = 0,\ D(r) > 0\} = \emptyset$, and if $\theta_1 \le \pi/2$ and $\lambda \le \lambda_{2,p}$, we have $G(r) \le 0$ in $(0, R)$ from Lemma 8. Noting $\lambda_{2,p} \ge -2/(p+3)$ and using Theorem 2 and Lemma 8, we can show that the problem has a unique positive solution as written in (iv). Finally, noting $g_r(r) \ge 0$ in the case $\lambda \ge -n(n-2)/4$ and $h_r(r) \le 0$, we can obtain the nondegeneracy results from Theorems 3, 4 and 5. □

Next, we consider the problem on a spherical band

$$\begin{cases} \Delta_{S^n} u + \lambda u + u^p = 0 & \text{in } \Omega_{\theta_1', \theta_1}, \\ u = 0 & \text{on } \partial\Omega_{\theta_1', \theta_1}, \end{cases} \tag{7.13}$$

where $n \in \mathbb{N}$ with $n \ge 2$, $p > 1$, $\Omega_{\theta_1', \theta_1} = \{X \in S^n : \cos\theta_1' > X_{n+1} > \cos\theta_1\}$ with $0 < \theta_1' < \theta_1 < \pi$, and $\lambda < \lambda_1$. Here, λ_1 is the first eigenvalue of $-\Delta_{S^n}$ on $\Omega_{\theta_1', \theta_1}$ with the Dirichlet boundary condition. As before, u is a positive radial solution to (7.13) if and only if the function v defined by (7.11) is a positive solution of

$$\begin{cases} \Delta v + \frac{n(n-2)+4\lambda}{(1+|x|^2)^2} v + 4(1+|x|^2)^{\frac{(n-2)p-(n+2)}{2}} v^p = 0 & \text{in } A_{R',R}, \\ v = 0 & \text{on } \partial A_{R',R}, \end{cases} \tag{7.14}$$

where $R' = \tan\theta_1'/2$, $R = \tan\theta_1/2$ and $A_{R',R} = P(\Omega_{\theta_1', \theta_1})$.

First, we study the subcritical and critical cases for (7.13).

Theorem 13 *Let $n \in \mathbb{N}$ with $n \geq 2$ and $0 < \theta_1' < \theta_1 < \pi$. Assume one of the following conditions.*

(i) *$n \geq 3$, $1 < p \leq (n+2)/(n-2)$ and one of the following holds:*

- (a) *$\pi/2 \notin (\theta_1', \theta_1)$ and $\lambda \in (-\infty, \lambda_1)$,*
- (b) *$\pi/2 \in (\theta_1', \theta_1)$ and $\lambda \in [\lambda_{n,p}, \lambda_1)$.*

(ii) *$n = 2$, $p > 1$ and one of the following holds:*

- (a) *$\pi/2 \notin (\theta_1', \theta_1)$ and $\lambda \in (-\infty, \lambda_1)$.*
- (b) *$\pi/2 \in (\theta_1', \theta_1)$ and $\lambda \in [-2/(p+3), \lambda_1)$,*

Then problem (7.13) has a unique positive radial solution u, and the positive radial solution v to (7.14) defined by (7.11) is a nondegenerate critical point of I on $H^1_{0,\mathrm{rad}}(A_{R',R})$, where I is defined by

$$I(w) = \int_{A_{R',R}} \left(\frac{1}{2}(|\nabla w|^2 + g(|x|)w^2) - \frac{1}{p+1} h(|x|)|w|^{p+1} \right) dx \tag{7.15}$$

for $w \in H^1_{0,\mathrm{rad}}(A_{R',R})$.

Remark 26 For the sake of completeness, under assumption $0 < \theta_1' < \theta_1 < \pi$, $\pi/2 \notin (\theta_1', \theta_1)$ is equivalent to $0 < \theta_1' < \theta_1 \leq \pi/2$ or $\pi/2 \leq \theta_1' < \theta_1 < \pi$.

Proof of Theorem 13 We note that the embedding $H^1_{0,\mathrm{rad}}(A_{R',R}) \hookrightarrow L^p(A_{R',R})$ is compact. So we can show that the problem has a unique positive solution by Theorem 8 and Lemma 8, and we can show its nondegeneracy by Theorem 8. □

Remark 27 In the case when $n \geq 3$, $p = (n+2)/(n-2)$ and $\lambda = \lambda_{n,p} = -n(n-2)/4$, we have $G(r) \equiv 0$, and this case is not excluded in the theorem above.

Next, we study the supercritical case. Even in this case, since the embedding $H^1_{0,\mathrm{rad}}(A_{R',R}) \hookrightarrow L^p(A_{R',R})$ is compact, we can obtain the following as before.

Theorem 14 *Let $n \in \mathbb{N}$ with $n \geq 3$ and $p > (n+2)/(n-2)$. Let $0 < \theta_1' < \theta_1 < \pi$ and $\lambda \in \mathbb{R}$ which satisfy one of the following conditions*

(i) *$(\theta_1', \theta_1) \cap [\theta_\lambda, \pi - \theta_\lambda] = \emptyset$ and $\lambda \in (-\infty, \lambda_1)$,*

(ii) *$[\theta_1', \theta_1] \subset [\theta_\lambda, \pi - \theta_\lambda]$ and $\lambda \in (\lambda_{n,p}, \lambda_1)$,*

where θ_λ is defined by

$$\theta_\lambda = \begin{cases} \text{unique } \theta \in (0, \pi/2) \text{ satisfying } G(\tan(\theta/2)) = 0 & \text{for } \lambda > \lambda_{n,p}, \\ \frac{\pi}{2} & \text{for } \lambda \leq \lambda_{n,p}. \end{cases}$$

Then problem (7.13) has a unique positive radial solution u, and the positive radial solution v to (7.14) defined by (7.11) is a nondegenerate critical point of I on $H^1_{0,\mathrm{rad}}(A_{R',R})$, where I is defined by (7.15).

Remark 28 For the sake of completeness, under assumption $0 < \theta_1' < \theta_1 < \pi$, $(\theta_1', \theta_1) \cap [\theta_\lambda, \pi - \theta_\lambda] = \emptyset$ is equivalent to $0 < \theta_1' < \theta_1 \leq \theta_\lambda$ or $\pi - \theta_\lambda \leq \theta_1' < \theta_1 < \pi$.

Remark 29 In Theorem 13, we can choose any $n \geq 2$, any $p > 1$ which is subcritical or critical, and any $\theta_1', \theta_1 \in (0, \pi)$ with $\theta_1' < \theta_1$. Once they are chosen, λ_1 is determined by θ_1', θ_1, and we can obtain a subinterval of $(-\infty, \lambda_1)$ in which problem (7.13) has a unique positive solution. However, in Theorem 14, after we choose $n \geq 3$ and $p > (n+2)/(n-2)$, in order to obtain an interval in which there exists at most one positive solution of (7.13), we also need to choose $0 < \theta_1' < \theta_1 < \pi$ and $\lambda \in \mathbb{R}$ which satisfy one of the conditions in Theorem 14.

Acknowledgments The authors are grateful to the referee for his/her careful reading and invaluable comments.

Appendix 1: The functions $a(r)$, $b(r)$, $c(r)$, $G(r)$ and $D(r)$

In this appendix, we give detailed expressions of $a(r)$, $b(r)$, $c(r)$, $G(r)$ and $D(r)$ for some specified $f(r)$, $g(r)$ and $h(r)$. In the case $f(r) = r^{n-1}$ (g and h are any functions), we have

$$
\begin{aligned}
a(r) &= r^{\frac{2(n-1)(p+1)}{p+3}} h(r)^{\frac{-2}{p+3}}, \qquad b(r) = \frac{r^{\frac{2(n-1)(p+1)}{p+3}-1}}{(p+3)h(r)^{\frac{p+5}{p+3}}}\big(2(n-1)h(r) + rh_r(r)\big),\\
c(r) &= \frac{r^{\frac{2(n-1)(p+1)}{p+3}-2}}{(p+3)^2 h(r)^{\frac{2(p+4)}{p+3}}}\Big(2(n-1)\big[n+2-(n-2)p\big]h(r)^2 + (p+5)r^2 h_r(r)^2\\
&\quad - (n-1)(p-5)rh(r)h_r(r) - (p+3)r^2 h(r)h_{rr}(r)\Big),\\
G(r) &= \frac{r^{\frac{2(n-1)(p+1)}{p+3}-3}}{2(p+3)^3 h(r)^{\frac{2}{p+3}+3}}\Big(4(n-1)\big[n+2-(n-2)p\big]\big[n-4+(n-2)p\big]h(r)^3\\
&\quad - \Big[2(n-1)(p-1)(p+3)^2 r^2 h(r)^3 - 4(p+3)^2 r^3 h(r)^2 h_r(r)\Big]g(r)\\
&\quad - (p+3)^3 r^3 g_r(r)h(r)^3 + (n-1)\big[(2n-3)p(6-p) + 6n - 33\big]rh(r)^2 h_r(r)\\
&\quad + 3(n-1)(p-1)(p+5)r^2 h(r)h_r(r)^2 - 2(p+4)(p+5)r^3 h_r(r)^3\\
&\quad - 3(n-1)(p-1)(p+3)r^2 h(r)^2 h_{rr}(r)\\
&\quad + 3(p+3)(p+5)r^3 h(r)h_r(r)h_{rr}(r) - (p+3)^2 r^3 h(r)^2 h_{rrr}(r)\Big),\\
D(r) &= \frac{r^{\frac{4(n-1)(p+1)}{p+3}-2}}{(p+3)^2 h(r)^{\frac{2(p+5)}{p+3}}}\Big(2(n-1)[(n-2)p+n-4]h(r)^2 + (p+3)^2 r^2 g(r)h(r)^2\\
&\quad - (p+4)r^2 h_r(r)^2 + (p+3)r^2 h(r)h_{rr}(r) + (n-1)(p-1)rh(r)h_r(r)\Big).
\end{aligned}
$$

In the case $f(r) = r^{n-1}$ and $h(r) = 1$, we have

$$
\begin{aligned}
a(r) &= r^{\frac{2(n-1)(p+1)}{p+3}}, \qquad b(r) = \frac{2(n-1)}{p+3} r^{\frac{2(n-1)(p+1)}{p+3}-1},\\
c(r) &= \frac{2(n-1)[n+2-(n-2)p]}{(p+3)^2} r^{\frac{2(n-1)(p+1)}{p+3}-2},\\
G(r) &= \frac{r^{\frac{2(n-1)(p+1)}{p+3}-3}}{2(p+3)^3}\Big(4(n-1)[(n-2)p+n-4][n+2-(n-2)p]\\
&\quad - 2(n-1)(p-1)(p+3)^2 r^2 g(r) - (p+3)^3 r^3 g_r(r)\Big),\\
D(r) &= \frac{r^{\frac{4(n-1)(p+1)}{p+3}-2}}{(p+3)^2}\Big(2(n-1)[(n-2)p+n-4] + (p+3)^2 r^2 g(r)\Big).
\end{aligned}
$$

In the case $f(r) = r^{n-1}\exp(r^2/4)$ and $h(r) = 1$, we have

$$a(r) = \left(e^{\frac{r^2}{4}} r^{n-1}\right)^{\frac{2(p+1)}{p+3}}, \qquad b(r) = \frac{\left(2n + r^2 - 2\right)\left(e^{\frac{r^2}{4}} r^{n-1}\right)^{\frac{2(p+1)}{p+3}}}{(p+3)r},$$

$$c(r) = \frac{\left(e^{\frac{r^2}{4}} r^{n-1}\right)^{\frac{2(p+1)}{p+3}}}{2(p+3)^2 r^2}\Big(4(n-1)[n+2-(n-2)p] \\ - 2(2n(p-1) - p + 5)r^2 - (p-1)r^4\Big),$$

$$G(r) = \frac{\left(e^{\frac{r^2}{4}} r^{n-1}\right)^{\frac{2(p+1)}{p+3}}}{4(p+3)^3 r^3}\Big(8(n-1)[(n-2)p+n-4][n+2-(n-2)p] \\ - 12(n-1)^2(p^2-1)r^2 - 4(n-1)(p-1)(p+3)^2 r^2 g(r) \\ - 2(p+3)^3 r^3 g_r(r) - 2(p-1)(p+3)^2 r^4 g(r) \\ - 2r^4\left(3n\left(p^2-1\right) - (p-6)p + 3\right) - (p^2-1)r^6\Big),$$

$$D(r) = \frac{\left(e^{\frac{r^2}{4}} r^{n-1}\right)^{\frac{4(p+1)}{p+3}}}{2(p+3)^2 r^2}\Big(4(n-1)[(n-2)p+n-4] + 2(2n(p+1) - p + 1)r^2 \\ + 2(p+3)^2 r^2 g(r) + (p+1)r^4\Big).$$

Next, we study the case $G(r) \equiv 0$. Letting $f(r) = r^{n-1}$, $h(r) = r^q$ with $q \in \mathbb{R}$ and

$$g(r) = \frac{r^{-2(n+1)}}{(p+3)^2}\left(C_1(p+3)^2 r^{\frac{4(2n+p+q+1)}{p+3}} - (2n+q-2)r^{2n}[(n-2)p+n-4-q]\right)$$

with $C_1 \in \mathbb{R}$, we have

$$a(r) = r^{\frac{2((n-1)(p+1)-q)}{p+3}}, \qquad b(r) = \frac{2n+q-2}{p+3} r^{\frac{(2n-3)(p+1)-2(q+1)}{p+3}},$$

$$c(r) = \frac{(2n+q-2)[n+2-(n-2)p+2q]}{(p+3)^2} r^{\frac{2[(n-2)p+n-4-q]}{p+3}},$$

$$G(r) = 0, \qquad D(r) = C_1 r^{2n-2}.$$

Letting $f(r) = r^{n-1}\exp(r^2/4)$, $h(r) = r^q$ with $q \in \mathbb{R}$ and

$$g(r) = C_1 \exp\left(-\frac{(p-1)r^2}{2(p+3)}\right) r^{\frac{2(-(n-1)(p-1)+2q)}{p+3}} - \frac{p+1}{2(p+3)^2} r^2 \\ + \frac{(-4n(p+1) - (p-1)(q-2))}{2(p+3)^2} - \frac{2(2n+q-2)[(n-2)p+n-4-q]}{2(p+3)^2 r^2},$$

we have

$$
\begin{aligned}
a(r) &= \left(\exp(r^2/4)r^{n-1}\right)^{\frac{2(p+1)}{p+3}} r^{-\frac{2q}{p+3}}, \\
b(r) &= \frac{\left(\exp(r^2/4)r^{n-1}\right)^{\frac{2(p+1)}{p+3}} r^{-\frac{2q}{p+3}-1}}{p+3}(r^2+2n+q-2), \\
c(r) &= \frac{\left(\exp(r^2/4)r^{n-1}\right)^{\frac{2(p+1)}{p+3}} r^{-\frac{2(p+q+3)}{p+3}}}{2(p+3)^2}\Big(2(2n+q-2)[n+2-(n-2)p+2q], \\
&\quad -(4n(p-1)+(p-5)(q-2))r^2-(p-1)r^4\Big), \\
G(r) &= 0, \qquad D(r) = C_1\exp(r^2/2)r^{2n-2}.
\end{aligned}
$$

Appendix 2: Some properties of $\mathcal{X}$

In this appendix, we assume $p > 1$, (B1) and (B6), and we understand that $\mathcal{D}$, $\mathcal{X}$ and $\mathcal{L}$ are the spaces defined in Sect. 3.

Lemma 9 *The space $\mathcal{X}$ coincides with the completion of $C_0^\infty((R', R))$ with respect to the norm $\|\cdot\|_{\mathcal{X}}$.*

Proof Since $C_0^\infty((R', R)) \subset \mathcal{D}$, it is enough to show that each element in $\mathcal{D}$ is approximated by an element in $C_0^\infty((R', R))$. Let $u \in \mathcal{D}$ and let $\varepsilon > 0$. We set $C = \max_{R' \le r < R} |u(r)|$. We define $\eta \in W^{1,\infty}(R', R)$ by $\eta(r) = 0$ for $R' < r \le \hat{\delta}$, $\eta(r) = 1$ for $\delta \le r < R$, and

$$
\eta(r) = \frac{\int_{\hat{\delta}}^{r} \frac{ds}{f(s)}}{\int_{\hat{\delta}}^{\delta} \frac{ds}{f(s)}} \quad \text{for } \hat{\delta} \le r \le \delta,
$$

where $\hat{\delta}, \delta \in (R', R)$ with $\hat{\delta} < \delta$ are chosen to be

$$
2\int_{R'}^{\delta} \left(|u_r|^2 + |g||u|^2\right) f\, dr < \varepsilon \quad \text{and} \quad \int_{\hat{\delta}}^{\delta} \frac{dr}{f(r)} > \frac{C^2}{\varepsilon}.
$$

Then we obtain

$$
\begin{aligned}
&\int_{R'}^{R} \left(|(\eta u)_r - u_r|^2 + |g||\eta u - u|^2\right) f\, dr \\
&\quad \le 2\int_{R'}^{R} |\eta(r)-1|^2 (|u_r|^2 + |g||u|^2) f(r)\, dr + 2\int_{R'}^{R} |\eta_r(r)|^2 |u|^2 f(r)\, dr \\
&\quad \le \varepsilon + C^2 \left(\int_{\hat{\delta}}^{\delta} \frac{dr}{f(r)}\right)^{-2} \int_{\hat{\delta}}^{\delta} \frac{dr}{f(r)} \le 2\varepsilon.
\end{aligned}
$$

By the standard argument with mollifiers, we can infer that our assertion holds. □

Lemma 10 *Let $u \in \mathcal{X}$. Then u^+, u^-, $|u| \in \mathcal{X}$.*

Proof Since $u \in H^1_{\mathrm{loc}}(R', R)$, we have

$$
(u^+)_r = u_r \cdot 1_{\{u>0\}} \quad \text{and} \quad (u^-)_r = u_r \cdot 1_{\{u<0\}},
$$

where 1 is the characteristic function. Let $u \in \mathcal{X}$. By the previous lemma, there exists $\{u_k\} \subset C_0^\infty((R', R))$ such that $u_k \to u$ in $\mathcal{X}$. We choose $K \in C^\infty(\mathbb{R})$ such that

$$K(t) = \begin{cases} t & \text{for } t \ge 1, \\ 0 & \text{for } t \le 0, \end{cases} \quad 0 \le K(t) \le t \quad \text{and} \quad 0 \le K_t(t) \le 2 \quad \text{for } t \in \mathbb{R}.$$

Let $k, m \in \mathbb{N}$. We claim

$$\begin{aligned}
&\int_{R'}^{R} \left| \left(\frac{1}{m} K(mu_k(r)) \right)_r - (u^+)_r \right|^2 f(r)\, dr + \int_{R'}^{R} |g(r)| \left| \frac{1}{m} K(mu_k) - u^+ \right|^2 f(r)\, dr \\
&\quad \le 2 \int_{\{r : 0 < u_k(r) < 1/m\}} (u_k^+)_r^2 f(r)\, dr + 4 \int_{R'}^{R} |u_{k,r} - u_r|^2 f(r)\, dr \\
&\qquad + 4 \int_{\{u>0\}} |1_{\{u_k>0\}} - 1|^2 u_r(r)^2 f(r)\, dr + 4 \int_{\{u<0\}} 1_{\{u_k>0\}} u_r(r)^2 f(r)\, dr \\
&\qquad + 2 \int_{\{r : 0 < u_k(r) < 1/m\}} |g(r)| u_k(r)^2 f(r)\, dr + 2 \int_{R'}^{R} |g(r)|\, |u_k - u|^2 f(r)\, dr.
\end{aligned}$$

Once the claim was shown, choosing a suitable sequence $\{m_k\}$, we can obtain $\{K(m_k u_k(r)) / m_k\} \subset C_0^\infty((R', R))$ which converges to u in $\mathcal{X}$. We will show the claim. We have

$$\begin{aligned}
&\int_{R'}^{R} \left| \left(\frac{1}{m} K(mu_k) \right)_r - (u^+)_r \right|^2 f(r)\, dr \\
&\quad \le 2 \int_{R'}^{R} |K_r(mu_k(r))(u_k)_r - (u_k^+)_r|^2 f(r)\, dr + 2 \int_{R'}^{R} |(u_k^+)_r - (u^+)_r|^2 f(r)\, dr \\
&\quad \le 2 \int_{\{r : 0 < u_k(r) < 1/m\}} (u_k^+)_r^2 f(r)\, dr \\
&\qquad + 2 \int_{R'}^{R} |(u_{k,r} - u_r) 1_{\{u_k>0\}} + u_r (1_{\{u_k>0\}} - 1_{\{u>0\}})|^2 f(r)\, dr \\
&\quad \le 2 \int_{\{r : 0 < u_k(r) < 1/m\}} (u_k^+)_r^2 f(r)\, dr + 4 \int_{R'}^{R} |u_{k,r} - u_r|^2 f(r)\, dr \\
&\qquad + 4 \int_{\{u>0\}} |1_{\{u_k>0\}} - 1|^2 u_r(r)^2 f(r)\, dr + 4 \int_{\{u<0\}} 1_{\{u_k>0\}} u_r(r)^2 f(r)\, dr.
\end{aligned}$$

Here, we used $\int_{\{u=0\}} u_r(r)^2 f(r)\, dr = 0$. On the other hand, we have

$$\begin{aligned}
&\int_{R'}^{R} |g(r)| \left| \frac{1}{m} K(mu_k) - u^+ \right|^2 f(r)\, dr \\
&\quad \le 2 \int_{R'}^{R} |g(r)| \left| \frac{1}{m} K(mu_k) - u_k^+ \right|^2 f(r)\, dr + 2 \int_{R'}^{R} |g(r)|\, |u_k^+ - u^+|^2 f(r)\, dr \\
&\quad \le 2 \int_{\{r : 0 < u_k(r) < 1/m\}} |g(r)| u_k(r)^2 f(r)\, dr + 2 \int_{R'}^{R} |g(r)|\, |u_k - u|^2 f(r)\, dr.
\end{aligned}$$

Hence we have shown the claim, and we finish our proof. □

Now, we also assume (B8). We give the following subsolution estimate.

Proposition 6 *Let $u \in \mathcal{X}$ satisfy*

$$\int_{R'}^{\bar{R}} (u_r \varphi_r + g u \varphi) f \, dr \le \int_{R'}^{\bar{R}} |u|^{p-1} u \varphi h f \, dr \quad \text{foreach } \varphi \in \mathcal{X} \text{with } \varphi \ge 0.$$

Then $\sup_{r \in (R', (R'+\bar{R})/2]} u^+(r) < \infty$.

Lemma 11 *Let $u \in \mathcal{X}$ and let z be a measurable function such that*

$$\int_{R'}^{\bar{R}} |z^+|^{\frac{\beta+1}{\beta-1}} h f \, dr < \infty$$

with some $\beta \in (1, \bar{p})$. Assume

$$\int_{R'}^{\bar{R}} (u_r \varphi_r + g u \varphi) f \, dr \le \int_{R'}^{\bar{R}} z u \varphi h f \, dr \quad \text{for each } \varphi \in \mathcal{X} \quad \text{with} \quad \varphi \ge 0.$$

Then $\sup_{r \in (R', (R'+\bar{R})/2]} u^+(r) < \infty$.

Proof We follow the argument in [46, Theorem 2.26]. Let $1/\sqrt{C_1}$ be the infimum value in (3.3), and set

$$\gamma = \max\{\beta, q\}, \quad \bar{z} = z^+ + \frac{|g^-|}{h} \quad \text{and} \quad C_2 = \left(\int_{R'}^{R'+2t} |\bar{z}|^{\frac{\gamma+1}{\gamma-1}} h f \, dr \right)^{\frac{\gamma-1}{\gamma+1}}.$$

Set also

$$\sigma = \frac{(\bar{p}+1)(\gamma-1)}{2(\bar{p}-\gamma)} \quad \text{and} \quad C_3 = \max\left\{ \max_{R'+t \le r \le R'+2t} \frac{144 C_1}{h(r)}, \ (12 C_1 C_2)^{\sigma+1} \right\}.$$

We claim that for each $l > 0$, $s > 0$ and r_1, r_2 with $(R'+\bar{R})/2 \le r_1 < r_2 \le \bar{R}$, there holds

$$\left(\int_{R'}^{r_1} \left| u |u_l|^s \right|^{\bar{p}+1} h f \, dr \right)^{\frac{2}{\bar{p}+1}} \le C_3 \left(\frac{s+1}{(r_2-r_1)^2} + (s+1)^{\sigma+1} \right) \int_{R'}^{r_2} \left| u |u_l|^s \right|^2 h f \, dr, \tag{7.16}$$

where

$$u_l(r) = \max\{0, \ \min\{u(r), l\}\}.$$

We will show the claim. Let $l > 0$, $s > 0$ and $(R'+\bar{R})/2 \le r_1 < r_2 \le \bar{R}$. Let $\eta \in \mathcal{D}$. Since we can show $\eta^2 u |u_l|^{2s} \in \mathcal{X}$ by a similar proof of the previous lemma, we have

$$\begin{aligned}
&\int_{R'}^{\bar{R}} z \eta^2 u^2 |u_l|^{2s} h f \, dr \\
&\ge \int_{R'}^{\bar{R}} \left(u_r (2\eta \eta_r u |u_l|^{2s} + \eta^2 u_r |u_l|^{2s} + 2s \eta^2 u |u_l|^{2s-2} u_l u_{l,r}) + g \eta^2 u^2 |u_l|^{2s} \right) f \, dr \\
&\ge \int_{R'}^{\bar{R}} \left(\frac{1}{2} \eta^2 |u_r|^2 |u_l|^{2s} - 2\eta_r^2 u^2 |u_l|^{2s} + 2s \eta^2 |u_{l,r}|^2 |u_l|^{2s} + g \eta^2 u^2 |u_l|^{2s} \right) f \, dr,
\end{aligned}$$

which yields

$$\int_{R'}^{\bar{R}} \left(\frac{1}{2}\eta^2 |u_r|^2 |u_l|^{2s} + 2s\eta^2 |u_{l,r}|^2 |u_l|^{2s} + g^+ \eta^2 u^2 |u_l|^{2s} \right) f\, dr \\ \leq \int_{R'}^{\bar{R}} \left(2|\eta_r|^2 u^2 |u_l|^{2s} f + \bar{z}\eta^2 u^2 |u_l|^{2s} hf \right) dr.$$

Noting

$$\left| \left(\eta u |u_l|^s \right)_r \right|^2 \leq 3|\eta_r|^2 u^2 |u_l|^{2s} + 3\eta^2 |u_r|^2 |u_l|^{2s} + 3s^2 \eta^2 |u_{l,r}|^2 |u_l|^{2s}$$

and $\eta u |u_l|^s \in \mathcal{X}$, we have

$$\left(\int_{R'}^{\bar{R}} \left| \eta u |u_l|^s \right|^{\bar{p}+1} hf\, dr \right)^{\frac{2}{\bar{p}+1}} \leq C_1 \int_{R'}^{\bar{R}} \left(\left| \left(\eta u |u_l|^s \right)_r \right|^2 + g^+ \left| \eta u |u_l|^s \right|^2 \right) f\, dr \\ \leq 6(s+1)C_1 \int_{R'}^{\bar{R}} \left(3|\eta_r|^2 u^2 |u_l|^{2s} f + \bar{z} \left| \eta u |u_l|^s \right|^2 hf \right) dr.$$

For each $\varepsilon > 0$, we have

$$\int_{R'}^{\bar{R}} \left(3|\eta_r|^2 u^2 |u_l|^{2s} f + \bar{z} \left| \eta u |u_l|^s \right|^2 hf \right) dr \leq 3 \int_{R'}^{\bar{R}} |\eta_r|^2 u^2 |u_l|^{2s} f\, dr \\ + C_2 \varepsilon^2 \left(\int_{R'}^{\bar{R}} \left| \eta u |u_l|^s \right|^{\bar{p}+1} hf\, dr \right)^{\frac{2}{\bar{p}+1}} + C_2 \varepsilon^{-2\sigma} \int_{R'}^{\bar{R}} \left| \eta u |u_l|^s \right|^2 hf\, dr.$$

Choosing $\varepsilon^{-2} = 12(s+1)C_1C_2$ and using two inequalities above, we obtain

$$\left(\int_{R'}^{\bar{R}} \left| \eta u |u_l|^s \right|^{\bar{p}+1} hf\, dr \right)^{\frac{2}{\bar{p}+1}} \leq 36(s+1)C_1 \int_{R'}^{\bar{R}} |\eta_r|^2 u^2 |u_l|^{2s} f\, dr \\ + (12(s+1)C_1C_2)^{\sigma+1} \int_{R'}^{\bar{R}} \left| \eta u |u_l|^s \right|^2 hf\, dr.$$

Now, letting η satisfy

$$\eta(r) = \begin{cases} 1 & \text{for } R' \leq r \leq r_1, \\ 0 & \text{for } r \geq r_2, \end{cases} \quad \text{and} \quad |\eta_r(r)| \leq \frac{2}{r_2 - r_1},$$

we can infer that claim (7.16) holds. We set

$$\chi = \frac{\bar{p}+1}{2} \quad \text{and} \quad t = \frac{\bar{R} - R'}{2}.$$

Applying (7.16) with $m \in \mathbb{N}$, $s = \chi^m - 1$, $r_1 = R' + (1 + 2^{-m})t$ and $r_2 = R' + (1 + 2^{-m+1})t$, and using the Lebesgue convergence theorem, we can infer

$$\left(\int_{R'}^{R' + (1+2^{-m})t} |u^+|^{2\chi^{m+1}} hf\, dr \right)^{\frac{1}{2\chi^{m+1}}} \\ \leq \left[C_3 \left(\frac{(4\chi)^m}{t^2} + \chi^{m(\sigma+1)} \right) \right]^{\frac{1}{2\chi^m}} \left(\int_{R'}^{R' + (1+2^{-m+1})t} |u^+|^{2\chi^m} hf\, dr \right)^{\frac{1}{2\chi^m}} \\ \leq \left[C_3 \left(\frac{1}{t^2} + 1 \right) C_4^m \right]^{\frac{1}{2\chi^m}} \left(\int_{R'}^{R' + (1+2^{-m+1})t} |u^+|^{2\chi^m} hf\, dr \right)^{\frac{1}{2\chi^m}},$$

where $C_4 = \max\{4\chi, \chi^{\sigma+1}\}$. So we have

$$\left(\int_{R'}^{R'+(1+2^{-m})t} |u^+|^{2\chi^{m+1}} hf\, dr\right)^{\frac{1}{2\chi^{m+1}}}$$
$$\leq \left(C_3\left(\frac{1}{t^2}+1\right)\right)^{\frac{1}{2\chi}\sum_{i=1}^m \frac{1}{\chi^{i-1}}} C_4^{\frac{1}{2\chi}\sum_{i=1}^m \frac{i}{\chi^{i-1}}} \left(\int_{R'}^{R'+2t} |u^+|^{2\chi} hf\, dr\right)^{\frac{1}{2\chi}}$$
$$\leq \left(C_3\left(\frac{1}{t^2}+1\right)\right)^{\frac{1}{2(\chi-1)}} C_4^{\frac{1}{2(\chi-1)^2}} \left(\int_{R'}^{\bar{R}} |u^+|^{\bar{p}+1} hf\, dr\right)^{\frac{1}{\bar{p}+1}}.$$

Letting $m \to \infty$, we can find that our assertion holds. □

Proof of Proposition 6 First, we note that

$$\int_{R'}^{\bar{R}} |u|^{p-1} u\varphi hf\, dr \leq \int_{R'}^{\bar{R}} (u^+)^{p-1} u\varphi hf\, dr \quad \text{for each } \varphi \in \mathcal{X} \text{ with } \varphi \geq 0.$$

In the case of $p < \bar{p}$, applying Lemma 11 with $z = (u^+)^{p-1}$ and $\beta = p$, we can see that our assertion holds. So we consider the case $p = \bar{p}$. We set $s = (\bar{p}-1)/2$. Let $l > 0$. Using the notations is Lemma 11 and noting $\int_{R'}^{\bar{R}} |u|u_l|^s|^{\bar{p}+1} hf\, dr < \infty$, we have

$$\left(\int_{R'}^{\bar{R}} \left|\eta u|u_l|^s\right|^{\bar{p}+1}\right)^{\frac{2}{\bar{p}+1}} \leq C_1 \int_{R'}^{\bar{R}} \left(\left|\left(\eta u|u_l|^s\right)_r\right|^2 + g^+\eta^2 u^2 |u_l|^{2s}\right) f\, dr$$
$$\leq 18(s+1)C_1 \int_{R'}^{\bar{R}} |\eta_r|^2 u^2 |u_l|^{2s} f\, dr + 6(s+1)C_1 \int_{R'}^{\bar{R}} \bar{z}\eta^2 u^2 |u_l|^{2s} hf\, dr$$
$$\leq 18(s+1)C_1 \int_{R'}^{\bar{R}} |\eta_r|^2 u^2 |u_l|^{2s} f\, dr$$
$$+ 6(s+1)C_1 \left(\int_{R'}^{\bar{R}} |\bar{z}|^{\frac{\bar{p}+1}{\bar{p}-1}} hf\, dr\right)^{\frac{\bar{p}-1}{\bar{p}+1}} \left(\int_{R'}^{\bar{R}} \left|\eta u|u_l|^s\right|^{\bar{p}+1} hf\, dr\right)^{\frac{2}{\bar{p}+1}}.$$

We choose $\delta > 0$ satisfying

$$6(s+1)C_1 \left(\int_{R'}^{R'+2\delta} |\bar{z}|^{\frac{\bar{p}+1}{\bar{p}-1}} hf\, dr\right)^{\frac{\bar{p}-1}{\bar{p}+1}} < \frac{1}{2}.$$

Then we have

$$\left(\int_{R'}^{R'+2\delta} \left|\eta u|u_l|^s\right|^{\bar{p}+1}\right)^{\frac{2}{\bar{p}+1}} \leq 36(s+1)C_1 \int_{R'}^{R'+2\delta} |\eta_r|^2 u^2 |u_l|^{2s} f\, dr$$
$$\leq \max_{r_1 \leq r \leq r_2} \frac{72(\bar{p}+1)C_1}{\delta^2 h(r)} \int_{R'}^{R'+2\delta} |u^+|^{\bar{p}+1} hf\, dr.$$

Letting $l \to \infty$, we obtain

$$\int_{R'}^{R'+\delta} |u^+|^{\frac{(\bar{p}+1)^2}{2}} hf\, dr < \infty.$$

Since h, f and u are continuous in (R', R), we have $\int_{R'}^{\bar{R}} |u^+|^{\frac{(\bar{p}+1)^2}{2}} hf\, dr < \infty$. Choosing $\beta \in (1, \bar{p})$ such that

$$(\bar{p}-1)\frac{\beta+1}{\beta-1} \leq \frac{(\bar{p}+1)^2}{2},$$

recalling $\bar{p} = p$, and applying Lemma 11 with $z = (u^+)^{\bar{p}-1}$, we can infer that our assertion holds. □

Appendix 3: Proof of Theorem 8

It is enough to show that the unique positive solution $\bar{u}$ is a nondegenerate critical point of I in the case $R < \infty$ and $G \equiv 0$ in (R', R); see Remark 19. For each $\delta > 0$, we define g_δ, h_δ, a_δ, b_δ, c_δ and J_δ by (4.6), (4.7) and (5.2) with $\gamma \equiv 1$; see also (5.9). We also define S_δ as the set of all positive solutions of

$$\begin{cases} u_{rr}(r) + \frac{f_r(r)}{f(r)} u_r + g_\delta(r)u + h_\delta(r)u^p = 0, & R' < r < R, \\ u(R') = 0, \quad u(R) = 0. \end{cases} \tag{7.17}$$

We can see that $\bar{u}$ is a positive solution of (7.17) for each $\delta > 0$.

Since we can prove the next lemma as in Lemma 5, we omit its proof.

Lemma 12 *It holds that*

$$\inf_{0<\delta<1} \inf_{u \in S_\delta} \|u\|_{\mathcal{X}} > 0.$$

Lemma 13 *There exist* $\delta_0 \in (0, 1)$ *such that*

$$\sup_{0<\delta<\delta_0} \sup_{u \in S_\delta} \max_{R' \le r \le R} u(r) < \infty. \tag{7.18}$$

Proof Suppose that the conclusion does not hold. Then there exist $\{\delta_m\} \subset (0, 1)$ with $\delta_m \to 0$ and $\{u_m\} \subset C([R', R]) \cap C^2((R', R))$ such that $u_m \in S_{\delta_m}$ for each $m \in \mathbb{N}$ and $\theta_m \equiv \max_{R' \le r \le R} u_m(r) \to \infty$ as $m \to \infty$. For each $m \in \mathbb{N}$, we choose $r_m \in (R', R)$ with $\theta_m = u_m(r_m)$ and we define $\{v_m\}$, $\{L_m\}$ and $\{\beta_m\}$ as in the proof of Lemma 6. Without loss of generality, we may assume that $r_m \to r_* \in [R', R]$, $\lim_{m\to\infty} \theta_m^{(p-1)/2}(R' - r_m)$ exists in $[-\infty, 0]$ and $\lim_{m\to\infty} \theta_m^{(p-1)/2}(R - r_m)$ exists in $[0, \infty]$. Let $L (\subset \mathbb{R})$ be the limit closed interval of $\{L_m\}$. We can see that L is unbounded and $0 \in L$. For each $m \in \mathbb{N}$, we have $v_m(0) = 1$, $v_{m,t}(0) = 0$ and

$$\begin{aligned} v_{m,tt}(t) + \frac{(f(\beta_m(t)))_t}{f(\beta_m(t))} v_{m,t}(t) + (1 + \delta_m) h(\beta_m(t)) v_m(t)^p \\ - \theta_m^{-p+1}\left[g(\beta_m(t)) + \delta_m h(\beta_m(t)) \bar{u}(\beta_m(t))^{p-1}\right] v_m(t) = 0 \end{aligned}$$

for each $t \in L_m$, and hence we have

$$\begin{aligned} v_{m,t}(t) f(\beta_m(t)) = \int_0^t f(\beta_m(s)) \Big[-(1 + \delta_m) h(\beta_m(s)) v_m(s)^p \\ + \left[g(\beta_m(s)) + \delta_m h(\beta_m(s)) \bar{u}(\beta_m(s))^{p-1}\right] \theta_m^{1-p} v_m(s) \Big] ds \end{aligned}$$

for each $t \in L_m$. We recall $R < \infty$, $f, h \in C^2([R', R])$, $g \in C([R', R])$ and $f > 0$ on $[R', R]$. From

$$\sup_{t \in L_m} \left| \frac{(f(\beta_m(t)))_t}{f(\beta_m(t))} \right| \le \frac{\max_{r \in [R', R]} |f_r(r)|}{\min_{r \in [R', R]} f(r)} \cdot \theta_m^{\frac{-(p-1)}{2}} \to 0 \quad \text{as } m \to \infty,$$

and the two equalities above, we can see

$$\varlimsup_{m\to\infty} \sup_{t\in K} |v_{m,t}(t)| < \infty \quad \text{and} \quad \varlimsup_{m\to\infty} \sup_{t\in K} |v_{m,tt}(t)| < \infty$$

for each bounded closed interval $K \subset L$ with $0 \in K$. Taking a subsequence $\{v_{m_i}\}$ of $\{v_m\}$, we can infer that there exists $v \in C^2(L)$ such that $\|v_{m_i} - v\|_{C^1(K)} \to 0$ for each bounded closed interval $K \subset L$ with $0 \in K$, v is nonnegative on L, and

$$\begin{cases} v_{tt}(t) + h(r_*)|v(t)|^{p-1}v(t) = 0 & \text{for each } t \in L, \\ v(0) = 1, \quad v_t(0) = 0. \end{cases}$$

However, since L is unbounded, v must be negative somewhere, which is a contradiction. Thus we have shown our assertion. □

Lemma 14 *It holds that*

$$\lim_{\delta\to 0} \sup_{u\in S_\delta} \|u - \bar{u}\|_{C^1([R',R])} = 0.$$

Proof For each $\delta \in (0, \delta_0)$, $u \in S_\delta$ and $r_u \in (R', R)$ with $u_r(r_u) = 0$, we have

$$f(r)|u_r(r)| \le \left| \int_{r_u}^{r} \left(h_\delta(s)u(s)^p + |g_\delta(s)|\bar{u}(s)^{p-1}u(s)\right) f(s)\, ds \right| \quad \text{for each } r \in (R', R),$$

So we have

$$\sup_{\delta\in(0,\delta_0)} \sup_{u\in S_\delta} \|u\|_{C^1([R',R])} < \infty,$$

and hence

$$\sup_{\delta\in(0,\delta_0)} \sup_{u\in S_\delta} \|u\|_{C^2([R',R])} < \infty.$$

Hence by similar arguments as in the proof of Lemma 7, we can infer that our assertion holds. □

Proof of Theorem 8 As already said, it is enough to show that in the case of $R < \infty$ and $G \equiv 0$, $\bar{u}$ is a nondegenerate critical point of I. From $(d/dr)J(r; \bar{u}) = G(r)\bar{u}(r)^2 = 0$ for each $r \in (R', R)$, we have

$$J(r; \bar{u}) = \frac{1}{2}a(R)\bar{u}_r(R)^2 > 0 \quad \text{for each } r \in [R', R].$$

Letting $\delta \in (0, \delta_0)$ be sufficiently small, we have

$$\inf_{u\in S_\delta} \inf_{r\in[R',R]} J_\delta(r; u) > 0$$

by the previous lemma. By a similar proof of that of Theorem 1, we can see that $\bar{u}$ is the unique positive solution of (7.17). Hence, by a similar proof of that of Theorem 3, we can find that $\bar{u}$ is a nondegenerate critical point of I. □

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