



ε -regularity for systems involving non-local, antisymmetric operators

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Received: 31 December 2013 / Accepted: 19 July 2015 / Published online: 20 August 2015
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Abstract We prove an epsilon-regularity theorem for critical and super-critical systems with a non-local antisymmetric operator on the right-hand side. These systems contain as special cases, both, Euler–Lagrange equations of conformally invariant variational functionals as Rivière treated them, and also Euler–Lagrange equations of fractional harmonic maps introduced by Da Lio–Rivière. In particular, the arguments give new and uniform proofs of the regularity results by Rivière, Rivière–Struwe, Da-Lio–Rivière, and also the integrability results by Sharp–Topping and Sharp, not discriminating between the classical local, and the non-local situations.

Mathematics Subject Classification 58E20 · 35B65 · 35J60 · 35S05

1 Introduction

In recent years there has been quite some research on the effect of antisymmetric potentials in the regularity theory of critical and super-critical elliptic partial differential equations. This was initiated by Rivière who in his celebrated [19] proved that solutions $u \in W^{1,2}(D, \mathbb{R}^N)$ to the equation

$$\Delta u = \Omega \cdot \nabla u \quad \text{in } D \subset \mathbb{R}^2, \quad (1.1)$$

which is a contracted notation of

$$\Delta u^i = \sum_{k=1}^N \Omega_{ik} \cdot \nabla u^k \quad 1 \leq i \leq N, \quad \text{in } D \subset \mathbb{R}^2,$$

Communicated by F. Helein.

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are Hölder continuous, under the condition that $\Omega_{ij} \in L^2(D, \mathbb{R}^2)$ and the at first sight seemingly non-descript condition

$$\Omega_{ik} = -\Omega_{ki}, \quad 1 \leq i, k \leq N. \tag{1.2}$$

As Rivière showed, (1.1) with (1.2) is essentially the general form of Euler–Lagrange equations of conformally invariant variational functionals which allow the characterization of Grüter [13], take for example a manifold $\mathcal{N} \subset \mathbb{R}^N$ and the Dirichlet energy

$$\int_{\mathbb{R}^2} |\nabla u|^2, \quad u : D \subset \mathbb{R}^2 \rightarrow \mathcal{N} \subset \mathbb{R}^N.$$

We refer the interested reader to the introduction of [19] for more details. In [20] this was generalized to an epsilon-regularity theorem for $D \subset \mathbb{R}^m, m \geq 3$.

If the antisymmetry-condition (1.2) is violated, solutions to (1.1) might exhibit singularities such as Frehse’s [10] counter-example $\log \log \frac{1}{|x|}$. In fact, the antisymmetry is shown to be closely related to the appearance of Hardy spaces, and also to Hélein’s [14] moving frame technique, cf. [22].

Motivated by this, Da Lio and Rivière [6] (for $m = 1$) showed that this regularizing effect of antisymmetry exists and appears also in the setting of $m/2$ -harmonic maps, critical points of the energy

$$\int_{\mathbb{R}^m} \left| |\nabla|^{\frac{m}{2}} u \right|^2, \quad u : \mathbb{R}^m \rightarrow \mathcal{N} \subset \mathbb{R}^N.$$

which satisfy (roughly) an Euler–Lagrange equation of the form

$$\Delta^{\frac{m}{2}} u^i = \sum_{k=1}^N \Omega_{ik} |\nabla|^{\frac{m}{2}} u^k \quad 1 \leq i \leq N, \quad \text{in } D \subset \mathbb{R}^m. \tag{1.3}$$

Here, $\Omega_{ij} \in L^2(\mathbb{R}^m)$ satisfies again (1.2), and $|\nabla|^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ is the elliptic differential operator of differential order α with the symbol $|\xi|^\alpha$, for the precise definition we refer to Sect. 1.

As well in the classical situation [14, 19], as also in the case of fractional harmonic maps, the argument relies on transforming the equation with an orthogonal matrix P . That is, one computes the respective equation $P \nabla u$ instead of ∇u , or $P \Delta^{\frac{m}{4}} u$ instead of $\Delta^{\frac{m}{4}} u$ and obtains a transformed Ω_P , which for the right choice of P exhibits better properties than the original Ω : In the classical case, $\text{div}(\Omega_P) = 0$, while in the fractional case, $\Omega_P \in L^{2,1}$ (where $L^{2,1} \subsetneq L^2$ is the Lorentz space dual to the weak L^2 , denoted by $L^{2,\infty}$). Note that while a condition like $\text{div}(f) = 0$ is destroyed under a distortion like $\tilde{f} := fg$, even for $g \in L^\infty$, the condition $f \in L^{2,1}$ is also valid for $\tilde{f} = fg$, if $g \in L^\infty$.

Thus, the techniques developed in the fractional setting [5–7, 24, 25], seem somewhat more dynamic and stable under certain distortions. For example, in [8] Da Lio and the author were able to extend some of the results to the degenerate situation of the energy

$$\int_{\mathbb{R}^m} \left| |\nabla|^\alpha u \right|^{\frac{m}{\alpha}}, \quad u : \mathbb{R}^m \rightarrow \mathcal{N} \subset \mathbb{R}^N,$$

the Euler–Lagrange equation of which have the form

$$|\nabla|^\alpha \left(\left| |\nabla|^\alpha u \right|^{\frac{m}{\alpha}-2} |\nabla|^\alpha u \right) = \left| |\nabla|^\alpha u \right|^{\frac{m}{\alpha}-2} \sum_{k=1}^N \Omega_{ik} |\nabla|^\alpha u^k \quad 1 \leq i \leq N, \quad \text{in } D \subset \mathbb{R}^m.$$

The aim of the present work is to shed more light on the connection between the two systems (1.3) and (1.1) in the critical and supercritical case, and we are going to extend the techniques developed in [6,7,24,25] to give a uniform argument for ε -regularity for quite general systems which in particular include as special cases both (1.3) and (1.1). Setting $w := (-\Delta)^{\frac{1}{2}}u \equiv |\nabla|^1 u \in L^2(\mathbb{R}^m)$, (1.1) reads as

$$\Delta^{\frac{1}{2}} w^i = \sum_{\gamma=1}^m \sum_{k=1}^N \Omega_{ik}^\gamma \mathcal{R}_\gamma[w^k], \tag{1.4}$$

where $\mathcal{R}_\gamma \equiv \partial_\gamma \Delta^{-\frac{1}{2}}$ denotes the Riesz transform. Thus, (1.1) is of the form (1.3), but Ω is not a pointwise matrix anymore, but a non-local, linear operator mapping $L^2(\mathbb{R}^m)$ into $L^1(\mathbb{R}^m)$. This was our main motivation, to study the regularity, and, in the super-critical regime, ε -regularity of solutions $w \in L^2(\mathbb{R}^m)$ of

$$\int w_i |\nabla|^\mu \varphi = - \int \Omega_{ik}[w_k] \varphi \quad \text{for all } \varphi \in C_0^\infty(D), \tag{1.5}$$

where Ω_{ik} is a linear mapping which maps $L^2(\mathbb{R}^m)$ into $L^1(\mathbb{R}^m)$. For the largest part of this article, we will restrict ourselves to Ω of the form

$$\Omega_{ij}[\] = \sum_{l=0}^m A_{ij}^l \mathcal{R}_l[\], \quad \text{where } A_{ij}^l = -A_{ji}^l \in L^2(\mathbb{R}^m), \quad i, j \in 1, \dots, m, \tag{1.6}$$

and $\mathcal{R}_l[\]$ is the l th Riesz transform for $l = 1, \dots, m$ and $\mathcal{R}_0[\]$ is the identity on \mathbb{R}^m . The arguments presented here hold also for more general potentials $\Omega : L^2 \rightarrow L^1$, under suitable conditions on quasi-locality and its commutators. But as (1.6) contains already the most interesting examples (see below), we shall restrict our attention to this setting for the sake of overview.

Our main result is then the following ε -regularity:

Theorem 1.1 *Let $\mu \leq \min\{1, \frac{m}{2}\}$ or $\mu = \frac{m}{2}$. Let $D \subset\subset \mathbb{R}^m$, $p \in (1, \infty)$, then there exists $\theta > 0$ such that the following holds: Let $w \in L^2(\mathbb{R}^m) \cap L^{(2)2\mu}(D)$, that is,*

$$\|w\|_{2, \mathbb{R}^m} + \sup_{B_\rho \subset D} \rho^{\frac{2\mu-m}{2}} \|w\|_{2, B_\rho} < \infty, \tag{1.7}$$

be a solution to (1.5), where Ω is of the form (1.6). If Ω satisfies moreover

$$\sup_{B_\rho(x), x \in D} \rho^{\frac{2\mu-m}{2}} \|A_{ij}^l\|_{2, B_\rho} \leq \theta \quad l = 0, \dots, m; \quad i, j = 1, \dots, m \tag{1.8}$$

then $w \in L_{loc}^p(D)$.

Let us remark the following corollaries from Theorem 1.1.

As mentioned above, by the representation (1.4) and the stability of the arguments as $\mu \rightarrow 1$, this gives a new proof of Rivière’s theorem [19], and also the ε -regularity theorem of [20].

Moreover, from Theorem 1.1 a new proof of Sharp and Topping’s integrability theorem [29] for (1.1) follows, and also an extension to the super-critical setting. The latter has been done independently, and by different methods by Sharp [28].

The extension of [29] to the case of non-local elliptic operators was one of the motivations for the research that led to this article. In fact, we are able to extend these integrability results

to the non-local case for $\mu \leq 1$. For $\mu > 1$ it seems already in the classical setting of the biharmonic maps, cf. [31], that for ε -regularity we need more information on the growth of Ω in terms of the solution, a fact which appeared also in our setting and forced us to restrict $\mu = \frac{m}{2}$ if $\mu > 1$.

Another corollary worth mentioning is that the arguments presented here also enable us to treat (ε) -regularity for critical points of more general non-local energies, e.g.,

$$E(u) = \int |\nabla^\alpha u|^2 \quad u : \mathbb{R}^m \rightarrow \mathcal{N} \subset \mathbb{R}^N, \tag{1.9}$$

where for $\mathcal{R} = [\mathcal{R}_1, \dots, \mathcal{R}_m]^T$, and \mathcal{R}_i being the i th Riesz transform,

$$\nabla^\alpha u := \mathcal{R}[|\nabla|^\alpha u].$$

Another remark regards the smallness condition of (1.8). In the critical setting $2\mu = m$, it is easy to verify, that this condition holds, if D is chosen appropriately small. In the super-critical regime $2\mu < m$, this condition would follow from some kind of monotonicity formula for stationary points of energies of the form (1.9), which for the non-classical settings are unknown so far, though there are some results into this direction [18].

Let us now sketch the arguments we are going to need. Firstly, motivated by the arguments in [20], we are going estimate the growth of the norm possibly far below the natural exponent 2. More precisely we estimate the growth in R of

$$\sup_{B_r \subset B_R} r^{\frac{\lambda_\kappa - m}{p_\kappa}} \|w\|_{p_\kappa, B_r}, \tag{1.10}$$

starting with $\kappa = \mu$, where

$$\lambda_\kappa := \frac{m(2\mu - \kappa)}{m - \kappa},$$

$$p_\kappa := \frac{m}{m - \kappa}.$$

The main work is to show that for any $\kappa \in [\mu, 2\mu)$ there is a good growth of these quantities, then starting for $\kappa_0 = \mu$, we can find a sequence of κ_i which converges to 2μ , such that each growth of the κ_i -norm (that is (1.10) with κ_i) is controlled by the κ_{i-1} -norm. Finally, for κ sufficiently close to 2μ , we show that we can actually have an estimate for $p > 2$. From this we have

Theorem 1.2 *There is $\theta_2 > 0$ such that if $\theta < \theta_2$, there exists $p > 2, \lambda < 2\mu$, such that*

$$w \in L_{loc}^{(p)\lambda}(D).$$

For Theorem 1.2, the antisymmetry of Ω will be crucial. Once Theorem 1.2 is established, the system (1.5) becomes subcritical, and we can drop the antisymmetry condition and just by the growth of the PDE, we have

Theorem 1.3 *Assume w as in Theorem 1.1, where we do not require the antisymmetry of Ω . Assume moreover, that $w \in L_{loc}^{p_1}(D)$ for $p_1 > 2$. Then for any $p > 2$, there is $\theta_p > 0$ such that if $\theta < \theta_p$ in (1.8), also*

$$w \in L_{loc}^p(D).$$

The main difficulty is thus Theorem 1.2 and the estimates of the Morrey norm. For the proof of this theorem we need the following two main technical ingredients: Firstly, we need an extension of earlier commutator estimates from [6, 7], and also the pointwise estimates as in [24, 25]. We consider two types of commutators: For $\varphi \in C_0^\infty(\mathbb{R}^m)$, $T : L^p(\mathbb{R}^m) \rightarrow L^q(\mathbb{R}^m)$, $1 \leq p, q \leq \infty$. We then set for $f \in L^p(\mathbb{R}^m)$ the commutator $\mathcal{C}(\varphi, T)[f]$

$$\mathcal{C}(\varphi, T)[f] := \varphi T[f] - T[\varphi f]. \tag{1.11}$$

This commutator was estimated in terms of Hardy spaces for $T = \mathcal{R}$ the Riesz transform or $T = I_s$ the Riesz potential in [3, 4], nevertheless we need more precise estimates and generalizations. The next bilinear commutator was introduced in [7], in [24] pointwise estimates were given.

$$H_s(a, b) := |\nabla|^s(ab) - a|\nabla|^s b - b|\nabla|^s a. \tag{1.12}$$

For these commutators the following holds

Theorem 1.4 *For any $\mu \in (0, 1]$, we have the following Hardy-space \mathcal{H} estimate (for $\mathcal{R}[\]$ any zero-multiplier operator; we need it for the Riesz-transform, only)*

$$\| |\nabla|^\mu (\mathcal{R}[h] I_\mu b - \mathcal{R}[h I_\mu b]) \|_{\mathcal{H}} \lesssim \|h\|_2 \|b\|_2.$$

Moreover, we have

$$\| \mathcal{C}(f, \mathcal{R})[|\nabla|^\mu \varphi] \|_2 \lesssim \| |\nabla|^\mu f \|_2 [\varphi]_{BMO},$$

and its pointwise counter-part: For any $\delta_i \in (0, 1)$ and any $\gamma_i \in (0, \delta_i)$, $i = 1, 2$,

$$| \mathcal{C}(a, \mathcal{R})[b] | \leq C_{\mathcal{R}, \delta_1, \gamma_1} I_{\delta_1 - \gamma_1} | |\nabla|^{\delta_1} a | I_{\gamma_1} |b| + C_{\mathcal{R}, \delta_2, \gamma_2} I_{\gamma_2} (I_{\delta_2 - \gamma_2} |b| | |\nabla|^{\delta_2} a |).$$

Finally we have

$$\| H_\mu(\varphi, g) \|_2 \lesssim \| |\nabla|^\mu g \|_2 [\varphi]_{BMO}.$$

and

$$\| |\nabla|^\mu H_\mu(a, b) \|_{\mathcal{H}} \lesssim \| |\nabla|^\mu a \|_2 \| |\nabla|^\mu b \|_2 \text{ for } \mu \in (0, 1], \tag{1.13}$$

as well as its pointwise counterpart: for any $\mu \in [0, m]$ there is $L \in \mathbb{N}$ such that for any $\beta \in [0, \min(\mu, 1))$, $\mu \in [0, m)$, $\tau \in (\max\{\beta, \mu + \beta - 1\}, \mu]$ there are, $s_k \in [0, \mu)$, $t_k \in [0, \tau)$, where $\tau - \beta - s_k - t_k \geq 0$, such that the following holds

$$| |\nabla|^\beta H_\mu(a, b) | \lesssim \sum_{k=1}^L I_{\tau - \beta - s_k - t_k} (I_{s_k} | |\nabla|^\mu a | I_{t_k} | |\nabla|^\tau b |).$$

Remark 1.5 For $\mu < 1$ the Hardy-space estimates above follow essentially from an obvious adaption of Da Lio and Riviere’s argument [7], and (1.13) has been proved by them. For $\mu > 1$, already from the pointwise arguments in [24] there is no hope for similar results. The interesting and new case $\mu = 1$, for which even (1.13) was unclear up to now, needs a more careful adaption of the arguments in [7].

Since the arguments leading to Theorem 1.4 are rather technical and are not otherwise needed in this paper, we will present the details of the proof in a forthcoming paper [26]. We gather a few more estimates of this sort in Sect. 1.

The second main ingredient is the choice of the “gauge” or “frame” P for our antisymmetric operator Ω .

Theorem 1.6 *Let Ω be as in (1.6), and assume that $\Omega_{ij}[\] = -\Omega_{ji}[\]$. For any $B_r \subset \mathbb{R}^m$, we can then choose $P : \mathbb{R}^m \rightarrow SO(N)$, $\text{supp}(P - I) \subset B_r$. Then for any $\varphi \in C_0^\infty(B_r)$,*

$$\begin{aligned}
 & - \int \Omega^P [|\nabla|^\mu \varphi] \leq C r^{\frac{m}{2}-\mu} \|A\|_2 [\varphi]_{BMO} \\
 & + \|A\|_2^2 \begin{cases} [\varphi]_{BMO} & \text{if } \mu \in (0, 1], \\ \| |\nabla|^\mu \varphi \|_{(2,\infty)} & \text{if } \mu > 1, \end{cases}
 \end{aligned}$$

where

$$\Omega_{ij}^P[f] := (|\nabla|^\mu P_{ik}) P_{kj}^T f + P_{ik} \Omega_{kl} [P_{lj}^T f].$$

In [22] the construction of such a P is done via minimization of $E(P) = \|P \nabla P^T + P \Omega P\|_{L^2}^2$ under the condition that P maps into $SO(N)$, a.e.. This is essentially the argument that H elein [14] used for his moving-frame technique, and it provides an alternative to Riviere’s adaption of Uhlenbecks [35] gauge-theoretic construction of P in [19]. Both techniques can be extended to the fractional case, where Ω is still a pointwise multiplication [6,24]. We adapt the arguments [22,24] to this case of a non-local operator $\Omega[\cdot]$, by minimizing in Sect. 1.6 the energy

$$E(P) := \sup_{\psi \in L^2(\mathbb{R}^m)} \int \Omega^P [\psi],$$

and showing that several terms of the Euler–Lagrange equations fall under the realm of Theorem 1.4.

Notation Let $L^{p,q}$ be the Lorentz spaces, cf., e.g. [12,15,34], whose norm we denote with $\|\cdot\|_{(p,q)}$. We set

$$\|f\|_{(p,q)_\lambda} \equiv \|f\|_{\mathcal{M}((p,q),\lambda)} := \sup_{B_r \subset \mathbb{R}^m} r^{\frac{\lambda-m}{p}} \|f\|_{(p,q),B_r}, \tag{1.14}$$

and for $A \subset \mathbb{R}^m$,

$$[f]_{(p,q)_\lambda,A} := |A|^{\frac{\lambda-m}{mp}} \|f\|_{(p,q),A}, \tag{1.15}$$

$$\|f\|_{(p,q)_\lambda,A} := \sup_{B_\rho \subset A} [f]_{(p,q),B_\rho}. \tag{1.16}$$

We say that f belongs to the Morrey space $L^{(p,q)_\lambda}(A)$, if the respective norm $\|f\|_{(p,q)_\lambda,A}$ is finite.

We will also use frequently the following annuli

$$A_{\Lambda,r}^k := B_{2^k \Lambda r} \setminus B_{2^{k-1} \Lambda r}, \quad A_r^k \equiv A_{1,r}^k. \tag{1.17}$$

In Sect. 1 we recall several facts on the fractional laplacian, which we are going to use throughout this work.

2 $L^{2+\varepsilon}$ -integrability: Proof of Theorem 1.2

It is helpful, to check once and for all,

$$m - 2\mu = \frac{m - \lambda_\kappa}{p_\kappa}, \quad \kappa \in [\mu, 2\mu), \tag{2.1}$$

where

$$\lambda_\kappa := \frac{m(2\mu - \kappa)}{m - \kappa}, \tag{2.2}$$

$$p_\kappa := \frac{m}{m - \kappa}. \tag{2.3}$$

Assume $w : \mathbb{R}^m \rightarrow \mathbb{R}^N$, $\mu \leq \frac{m}{2}$, $w \in L^2(\mathbb{R}^m)$, $|\nabla|^\mu w \in L^2(\mathbb{R}^m)$ is for $D \subset\subset \mathbb{R}^m$ a solution to (1.5). We are going to establish that for any $\kappa \in [\mu, 2\mu)$, if $\theta \equiv \theta_\kappa$ in (1.8) is suitably small, for any $\tilde{D} \subset\subset D$, we have

$$\sup_{r>0, x_0 \in \tilde{D}} r^{\frac{\lambda_\kappa - m}{p_\kappa}} \|w\|_{p_\kappa, B_r(x_0)} \leq C_{\tilde{D}, w, \kappa}. \tag{2.4}$$

Note that possibly $p_\kappa < 2$ for all $\kappa \in [\mu, 2\mu)$. In order to show (2.4), we first note that its satisfied by assumption (1.7) for $\kappa = \mu$. In fact, if $x_0 \in \tilde{D} \subset\subset D$, then for any $r > 0$, or $B_r(x_0) \subset D$ or $r > c \text{ dist}(\tilde{D}, \partial D)$.

Now, we show that for arbitrary $\kappa \in [\mu, 2\mu)$, there is $\kappa_1 > \kappa$, so that (2.4) holds. Moreover, we will show a lower bound on $\kappa_1 - \kappa$, in order to ensure that we come arbitrarily close to 2μ if we repeat this construction finitely many times.

Then we can show that if we choose $\kappa \in [\mu, 2\mu)$ close enough to 2μ , (2.4) suffices to conclude the better integrability of Theorem 1.2.

Establishing (2.4)

For mappings $P : \mathbb{R}^m \rightarrow SO(N)$, $P \equiv I$ on $\mathbb{R}^m \setminus D$ (denoting with $I = (\delta_{ij})_{ij} \in \mathbb{R}^{N \times N}$ the identity matrix) from (1.5) we have

$$\begin{aligned} \int P_{ik} w_k |\nabla|^\mu \varphi &= \int w_k |\nabla|^\mu (P_{ik} \varphi) - \int w_k (|\nabla|^\mu P_{ik}) \varphi - \int w_k H_\mu(P_{ik}, \varphi) \\ &= - \int \Omega_{kl} [w_l] P_{ik} \varphi - \int w_k (|\nabla|^\mu P_{ik}) \varphi - \int w_k H_\mu((P - I)_{ik}, \varphi), \end{aligned}$$

where H_μ is the bilinear operator defined in (1.12).

Setting $v_i := P_{ik} w_k$, this is

$$\int v_i |\nabla|^\mu \varphi = - \int (P_{ik} \Omega_{kl} [P_{jl} v_j] + (|\nabla|^\mu P_{ik}) P_{jk} v_j) \varphi - \int w_k H_\mu((P - I)_{ik}, \varphi). \tag{2.5}$$

The Growth Estimates The main difficulty is the following estimate of the right-hand side of (2.5).

Theorem 2.1 (Right-hand side estimates) *If $\mu \in (0, \min\{1, \frac{m}{2}\})$ or $2\mu = m$, there is a uniform $\Lambda \equiv \Lambda_\mu > 0$, depending only on μ , such that the following holds: Let $B_r \subset \mathbb{R}^m$, and assume (2.5) holds for all $\varphi \in C_0^\infty(B_r)$. Then there exists a choice of P such that (2.5) implies for any $\varphi \in C_0^\infty(B_{\Lambda^{-1}r})$, and for any $\tau \in (\max\{\mu - 1, 0\}, \mu]$ sufficiently close to, or greater than $2\mu - \kappa$,*

$$\begin{aligned} (\Lambda^{-1}r)^{2\mu - m} \int v |\nabla|^\mu \varphi &\leq C_\kappa \theta \| |\nabla|^\tau \varphi \|_{\left(\frac{m}{\tau + \kappa - \mu}, 1\right)} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\ &\quad + C_\kappa \theta \| |\nabla|^\tau \varphi \|_{\left(\frac{m}{\tau + \kappa - \mu}, 2\right)} \Lambda^{\kappa - 3\mu} \sum_{k=1}^\infty 2^{k(\kappa - 3\mu)} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k}. \end{aligned}$$

where we recall that the right-hand side norms were defined in (1.15), (1.16), A_r^k is as in (1.17), and λ_κ as in (2.2), p_κ as in (2.3).

Theorem 2.1 is a direct consequence of the Eq. (2.5), the choice of P and estimates on the term involving the antisymmetric potential Ω transformed by this P , see Lemma 3.2, and the estimates on the remaining term involving H_μ , see Lemma 3.1.

Once Theorem 2.1 is obtained, we proceed as follows: From Lemma 5.18 (applied to $\Lambda^{-1}r$ instead of r) we infer for any $\tau \in (0, \mu]$ sufficiently close to μ and any $\Lambda \gg \Lambda_\mu$ sufficiently large [for the right-hand side norms recall (1.15) and (1.16)], also in view of Proposition 5.14,

$$\begin{aligned}
 & (\Lambda^{-2}r)^{2\mu-m} \|\ |\nabla|^{\mu-\tau} v \|_{\left(\frac{m}{m+\mu-\tau-\kappa}, \infty\right), B_{\Lambda^{-2}r}} \\
 & \leq \Lambda^{m-2\mu} C_\kappa \theta \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\
 & \quad + \Lambda^{m-2\mu} \Lambda^{\kappa-3\mu} C_\kappa \theta \sum_{k=1}^\infty 2^{k(\kappa-3\mu)} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k} \\
 & \quad + C (\Lambda^{-2}r)^{2\mu-m} \Lambda^{\kappa-m+\tau-\mu} \|w\|_{(p_\kappa, \infty), B_{\Lambda^{-1}r}} \\
 & \quad + C (\Lambda^{-2}r)^{2\mu-m} \Lambda^{\kappa-m+\tau-\mu} \sum_{k=1}^\infty 2^{k(\kappa-m+\tau-\mu)} \|w\|_{(p_\kappa, \infty), A_{\Lambda^{-1}r}^k} \\
 & \stackrel{(2.1)}{\leq} C_\kappa \theta \Lambda^{m-2\mu} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\
 & \quad + C_\kappa \theta \Lambda^{m-2\mu} \sum_{k=1}^\infty 2^{k(\kappa-3\mu)} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k} \\
 & \quad + C \Lambda^{\kappa+\tau-3\mu} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{\Lambda^{-1}r}} \\
 & \quad + C \Lambda^{\kappa+\tau-3\mu} \sum_{k=1}^\infty 2^{k(\kappa+\tau-3\mu)} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_{\Lambda^{-1}r}^k} \\
 & \stackrel{p.A.7}{\lesssim} (C_\kappa \theta \Lambda^{m-2\mu} + C \Lambda^{\kappa+\tau-3\mu}) \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\
 & \quad + (C_\kappa \theta \Lambda^{m-2\mu} + C \Lambda^{\kappa+\tau-3\mu}) \sum_{k=1}^\infty 2^{k(\kappa+\tau-3\mu)} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k}.
 \end{aligned}$$

For later reference, we write this as

$$\begin{aligned}
 & (\Lambda^{-2}r)^{2\mu-m} \|\ |\nabla|^{\mu-\tau} v \|_{\left(\frac{m}{m+\mu-\tau-\kappa}, \infty\right), B_{\Lambda^{-2}r}} \\
 & \leq (C_{\kappa, \mu} \theta \Lambda^{m-2\mu} + C_\mu \Lambda^{\kappa+\tau-3\mu}) \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\
 & \quad + (C_{\kappa, \mu} \theta \Lambda^{m-2\mu} + C_\mu \Lambda^{\kappa+\tau-3\mu}) \sum_{k=1}^\infty 2^{k(\kappa+\tau-3\mu)} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k}. \tag{2.6}
 \end{aligned}$$

For $\tau = \mu$,

$$\begin{aligned}
 & (\Lambda^{-2}r)^{2\mu-m} \|v\|_{(p_\kappa, \infty), B_{\Lambda^{-2}r}} \\
 & \leq (C_{\kappa, \mu} \theta \Lambda^{m-2\mu} + C_\mu \Lambda^{\kappa-2\mu}) \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\
 & \quad + (C_{\kappa, \mu} \theta \Lambda^{m-2\mu} + C_\mu \Lambda^{\kappa-2\mu}) \sum_{k=1}^\infty 2^{k(\kappa-2\mu)} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k}. \tag{2.7}
 \end{aligned}$$

The Iteration Procedure Note that $|w| = |v|$, so we can use them equivalently. Equation (2.7) holds for any $B_r(x_0)$, where $x_0 \in D$ and $r < \tilde{d}(x_0) := C \text{dist}(x_0, \partial D)$ (the constant

essentially only depending on the construction of P and the set where Ω is small). For $x_0 \in D$ and $R > 0$ set

$$\Phi_{x_0}(R) := \sup_{B_\rho \subset B_R(x_0)} \rho^{2\mu-m} \|w\|_{(p_\kappa, \infty), B_\rho},$$

and its centered counter-part

$$\Psi_{x_0}(K, R) := \sup_{\rho \in (0, KR), x \in B_R(x_0)} \rho^{2\mu-m} \|w\|_{(p_\kappa, \infty), B_\rho(x)} \leq \Phi_{x_0}(2KR)$$

then from (2.7) for any $R, x_0 \in D$ with $R < d(x_0)$, we have

$$\begin{aligned} \Phi_{x_0}(\Lambda^{-2}R) &\leq (C_\kappa \theta \Lambda^{m-2\mu} + C \Lambda^{\kappa-2\mu}) \Phi_{x_0}(R) \\ &\quad + (C_\kappa \theta \Lambda^{m-2\mu} + C \Lambda^{\kappa-2\mu}) \sum_{k=1}^\infty 2^{k(\kappa-2\mu)} \Psi_{x_0}(2^k, R). \end{aligned}$$

Note that from (2.4), we know that $\Phi_{x_0}(KR) < C_{D, x_0, w}$ for all $K > 0$, whenever $x_0 \in D, R < d(x_0)$.

In order to iterate, set $\Lambda^2 := 2^L$, some $L \in \mathbb{N}$, and apply Lemma 5.19, we set (fixing R)

$$a_l := \Phi_{x_0}(2^l R), \quad b_{l,k} := \Psi_{x_0}(2^k, 2^l R),$$

then we have for any $l \leq -1$,

$$a_{l-L} \leq (C_\kappa \theta \Lambda^{m-2\mu} + C \Lambda^{\kappa-2\mu}) a_l + (C_\kappa \theta \Lambda^{m-2\mu} + C \Lambda^{\kappa-2\mu}) \sum_{k=1}^\infty 2^{k(\kappa-2\mu)} b_{l,k}.$$

Now we can iterate, Lemma 5.19, satisfying the assumption (5.22) by choosing $\tilde{\theta} := (\frac{2\mu-\kappa}{C_\mu})^2 < \theta$, and $\Lambda \equiv \Lambda_\kappa$ large enough, and then θ small enough. Then, for any $r < R$,

$$\begin{aligned} \sup_{B_\rho \subset B_r(x_0)} \rho^{2\mu-m} \|w\|_{(p_\kappa, \infty), B_\rho} &= \sup_{B_\rho \subset B_r(x_0)} \rho^{2\mu-m} \|v\|_{(p_\kappa, \infty), B_\rho} \\ &\lesssim C_{\kappa, w, \Lambda, R} r^{\sigma_\kappa}, \quad \text{where } \sigma_\kappa = \frac{1}{4} \left(\frac{2\mu - \kappa}{C_\mu} \right)^2. \end{aligned}$$

We can assume, that $\sigma_\kappa < 2\mu - \kappa$. Since

$$\sup_{\rho > R} \rho^{2\mu-\sigma_\kappa-m} \|w\|_{(p_\kappa, \infty), B_\rho(x_0)} \lesssim R^{-\sigma_\kappa} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{4r}},$$

we arrive at

$$r^{2\mu-\sigma_\kappa-m} \|w\|_{(p_\kappa, \infty), B_r(x_0)} \leq C_{\kappa, w, x_0}$$

so we get for any $B_r(x_0) \subset B_R(x_0)$

$$r^{-\sigma_\kappa} \|\chi_{B_r} w\|_{(p_\kappa, \infty)_{\lambda_\kappa}} + \sup_{\rho > 0} \rho^{2\mu-\sigma_\kappa-m} \|w\|_{(p_\kappa, \infty), B_\rho(x_0)} \leq C_{\kappa, w}.$$

Plugging this into (2.6), we have for all $\tau \in (\max\{0, \mu - 1\}, \mu]$ sufficiently close to, or greater than $2\mu - \kappa$,

$$r^{2\mu-m} \|\|\nabla\|^{\mu-\tau} v\|_{\left(\frac{m}{m+\mu-\tau-\kappa}, \infty\right), B_{\Lambda^{-1}r}(x_0)} \lesssim C_{\kappa, w} r^{\sigma_\kappa} + C_{\kappa, w} r^{\sigma_\kappa} \sum_{k=1}^\infty 2^{k(\kappa-2\mu+\sigma_\kappa)}, \quad (2.8)$$

so that we have for all small r ,

$$r^{2\mu - m - \sigma_\kappa} \|\ |\nabla|^{\mu - \tau} v \|_{\left(\frac{m}{m + \mu - \tau - \kappa}, \infty\right), B_r(x_0)} \lesssim C_{w, \kappa, R}.$$

Moving the $B_R(x_0)$, for any $D_1 \subset\subset D$, we have that

$$\|\ |\nabla|^{\mu - \tau} v \|_{\left(\frac{m}{m + \mu - \tau - \kappa}, \infty\right)_\lambda, D_1} \lesssim C_{w, \kappa, |\nabla|_1, D} \tag{2.9}$$

for λ such that (choosing τ possibly even closer to μ , ensuring that $|\mu - \tau| \leq \frac{\sigma_\kappa}{2}$)

$$\frac{\lambda}{m} = \frac{3\mu - \tau - \kappa - \sigma_\kappa}{m + \mu - \tau - \kappa} \leq \frac{3\mu - \tau - \kappa - \sigma_\kappa}{m - \kappa} \stackrel{(2.2)}{=} \frac{\lambda_\kappa}{m} + \frac{\mu - \tau - \sigma_\kappa}{m - \kappa}$$

Choosing the next κ Assume for a moment that $2\mu < m$. we can guarantee

$$0 < \lambda < \lambda_\kappa - c_m \sigma_\kappa, \tag{2.10}$$

and we choose $\kappa_{1,1} \in (\kappa, 2\mu)$ via

$$\lambda =: m \frac{2\mu - \kappa_{1,1}}{m - \kappa_{1,1}}.$$

By (2.10),

$$m \frac{2\mu - \kappa_{1,1}}{m - \kappa_{1,1}} < m \frac{2\mu - \kappa}{m - \kappa} - c_{m-2\mu} \sigma_\kappa$$

and thus we have

$$\kappa_{1,1} > \kappa + \sigma c_{m-2\mu} \frac{(m - \kappa_{1,1})(m - \kappa)}{m}.$$

On the other hand, by a localized version of Adams' [1]-argument on Riesz potentials, we infer from (2.9) that for any $D_2 \subset\subset D_1$,

$$\|v\|_{(p, \infty)_\lambda, D_2} = \|w\|_{(p, \infty)_\lambda, D_2} < \infty,$$

where

$$\frac{1}{p} = \frac{m + \mu - \tau - \kappa}{m} - \frac{\mu - \tau}{\lambda} \in (0, 1).$$

Letting

$$\frac{m}{m - \kappa_{1,2}} := p,$$

we can estimate

$$\frac{\kappa_{1,2} - \kappa}{m} = (\mu - \tau) \left(\frac{1}{\lambda} - \frac{1}{m} \right) \geq \sigma_\kappa c_\mu.$$

Thus for a certain $\alpha > 0$,

$$\kappa_1 := \min \kappa_{1,1}, \kappa_{1,2} \geq \kappa_0 + c_0 (2\mu - \kappa)^\alpha,$$

and since

$$p \geq \frac{m}{m - \kappa_1}, \quad \lambda < m \frac{2\mu - \kappa_1}{m - \kappa_1},$$

for any $D_3 \subset\subset D$, we arrive at

$$\|w\|_{(p_{\kappa_1}, \infty)_{\frac{2\mu - \kappa_1}{m - \kappa_1}}, D_3} < \infty.$$

Varying this in $D_3 \subset\subset D$, we have (2.4) for κ_1 . If $2\mu = m$, we use this same argument, to conclude that $w \in L^p(D_3)$ for some $p > 2$, which is already the claim of Theorem 1.2.

Estimating the growth of κ Iterating this procedure [for smaller and smaller θ in (1.8)], we obtain $\kappa_k \in [\mu, 2\mu)$, and

$$\kappa_{k+1} \geq \kappa_k + c_0(2\mu - \kappa)^\alpha.$$

Since the sequence $(\kappa_k)_k$ is monotone and bounded, and the only fixed point is $\kappa_\infty = 2\mu$, for any $\varepsilon > 0$ there is a step-count L such that $|\kappa_L - 2\mu| < \varepsilon$. This shows (2.4) \square

Integrability above 2

So far, it is possible, that $p_\kappa < 2$ for all $\kappa < 2\mu$. But since $\lambda_\kappa \xrightarrow{\kappa \rightarrow 2\mu} 0$, as $\kappa \rightarrow 2\mu$, we will now show that the conditions for Theorem 1.3 for w will be satisfied eventually.

By the arguments above, fixing $\tilde{D} \subset\subset D$, going back to (2.8), if $2\mu - \kappa < \varepsilon$ small enough, for $\tau \in (\max\{\varepsilon, \mu - 1\}, \mu]$, ignoring $\sigma_\kappa > 0$,

$$\sup_{B_r \subset \tilde{D}} r^{2\mu - m} \| |\nabla|^{|\mu - \tau} v \|_{\left(\frac{m}{m + \mu - \tau - \kappa}, \infty\right), B_{\Lambda^{-1}r}(x_0)} \lesssim C_{\kappa, w, \tilde{D}}.$$

If $2\mu = m$, choosing $\tau = \mu$, we have

$$\frac{m}{m + \mu - \mu - \kappa} \xrightarrow{\kappa \rightarrow 2\mu = m} \infty,$$

which proves Theorem 1.2, and in fact even Theorem 1.1. So let from now on $2\mu < m$, $\mu \leq 1$. Then for $\lambda_{s, \varepsilon} \in (0, m)$, $s := \mu - \tau$,

$$\begin{aligned} \frac{\lambda_{s, \varepsilon} - m}{m + \mu - \tau - \kappa} &= 2\mu - m \\ \Leftrightarrow \lambda_{s, \varepsilon} &= \frac{m}{m + \mu - \tau - \kappa} (3\mu - \tau - \kappa) \xrightarrow{\tau \rightarrow \mu, \kappa \rightarrow 2\mu} 0 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\tilde{p}} &:= \frac{m + \mu - \tau - \kappa}{m} - \frac{\mu - \tau}{\frac{m}{m + \mu - \tau - \kappa} (3\mu - \tau - \kappa)} \\ &= \frac{m + \mu - \tau - \kappa}{m} - \frac{(\mu - \tau)(m + \mu - \tau - \kappa)}{m(3\mu - \tau - \kappa)} \\ &= 1 + \frac{\mu - \tau - \kappa}{m(3\mu - \tau - \kappa)} (2\mu + \tau - \kappa) - \frac{(\mu - \tau)}{3\mu - \tau - \kappa} \end{aligned}$$

we have by Adams' [1],

$$v \in L_{loc}^{(\tilde{p}, \infty)_{\lambda_{s, \varepsilon}}}(D).$$

One checks that one can choose $\kappa \approx 2\mu$, and then τ suitably close to μ such that $\tilde{p} > 2$, $\lambda_{s, \varepsilon} < 2\mu$. (In fact, also in this case one can see that \tilde{p} will be arbitrarily close to ∞). Thus Theorem 1.2 is established. \square

3 Ingredients for the Proof of Theorem 2.1

3.1 Estimates of the H -term

This is to estimate for $\varphi \in C_0^\infty(B_r)$ the following term

$$\int w H_\mu(P - I, \varphi) = \int I_\beta w |\nabla|^\beta H_\mu(P, \varphi) \tag{3.1}$$

Lemma 3.1 *Let $\mu \in (0, \frac{m}{2}]$, $\mu \leq 1$ or $\mu = \frac{m}{2}$. For any $\kappa \in [\mu, 2\mu)$, there are $C_{\kappa, \mu} > 0$, $\tau \in (0, \mu)$ such for any $\varphi \in C_0^\infty(B_{\Lambda^{-1}r})$ the following holds: If $\text{supp}(P - I) \subset B_{\Lambda^{-1}r}$,*

$$\begin{aligned} & (\Lambda^{-1}r)^{2\mu-m} \int w H_\mu(P - I, \varphi) \\ & \leq C_{\kappa, \mu} \| |\nabla|^\tau \varphi \|_{\left(\frac{m}{\kappa+\tau-\mu}, 2\right)} (\Lambda^{-1}r)^{\mu-\frac{m}{2}} \| |\nabla|^\mu P \|_2 \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\ & \quad + C_{\kappa, \mu} \| |\nabla|^\tau \varphi \|_{\left(\frac{m}{\kappa+\tau-\mu}, 2\right)} (\Lambda^{-1}r)^{\mu-\frac{m}{2}} \| |\nabla|^\mu P \|_2 \sum_{k=1}^\infty (2^k \Lambda)^{\kappa-3\mu} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k} \end{aligned}$$

where we recall the definition A_r^k from (1.17), λ_κ from (2.2), and p_κ from (2.3). As for the asymptotic behavior as $\kappa \rightarrow 2\mu$, one can choose τ approaching $\max\{\mu - 1, 0\}$, and $C_{\kappa, \mu}$ blows up.

Proof of Lemma 3.1 For a somewhat clearer presentation, we are going to show the following claim for $\varphi \in C_0^\infty(B_r)$ and $\text{supp}(P - I) \subset B_r$

$$\begin{aligned} & r^{2\mu-m} \int w H_\mu(P - I, \varphi) \\ & \leq C_{\kappa, \mu} \| |\nabla|^\tau \varphi \|_{\left(\frac{m}{\kappa+\tau-\mu}, 2\right)} r^{\mu-\frac{m}{2}} \| |\nabla|^\mu P \|_2 \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{\Lambda r}} \\ & \quad + C_{\kappa, \mu} \| |\nabla|^\tau \varphi \|_{\left(\frac{m}{\kappa+\tau-\mu}, 2\right)} r^{\mu-\frac{m}{2}} \| |\nabla|^\mu P \|_2 \sum_{k=1}^\infty (2^k \Lambda)^{\kappa-3\mu} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_{\Lambda, r}^k} \end{aligned}$$

Applied to $\tilde{r} := \Lambda^{-1}r$ gives the original claim.

As usual, we decompose

$$\int w H_\mu(P - I, \varphi) = I + \sum_{k=1}^\infty II_k,$$

where

$$I := \int \chi_{B_{\Lambda r}} w H_\mu(P - I, \varphi),$$

and, denoting $A_k := A_{\Lambda, r}^k$,

$$II_k := \int w H_\mu(P - I, \varphi) \chi_{A_k}.$$

As for II_k , since $\text{supp } \varphi \cup \text{supp}(P - I) \subset \overline{B_r}$

$$H_\mu(P - I, \varphi) \chi_{A_k} = \chi_{A_k} |\nabla|^\mu ((P - I)\varphi).$$

By Lemma 5.15 we then have for any $\tau \in (0, \mu]$, using also Lemma 5.12,

$$\begin{aligned} \|H_\mu(P - I, \varphi)\|_{(\frac{m}{\kappa}, 1).A_\kappa} &\lesssim (2^k \Lambda r)^{-m-\mu} (2^k \Lambda r)^\kappa r^{\frac{m}{2}-\kappa+\mu} \|\varphi\|_{(\frac{m}{\kappa-\mu}, \infty)} \|P - I\|_2 \\ &\lesssim (2^k \Lambda r)^{-m-\mu} (2^k \Lambda r)^\kappa r^{\frac{m}{2}-\kappa+2\mu} \|\nabla|\tau\varphi\|_{(\frac{m}{\kappa+\tau-\mu}, \infty)} \|\nabla|^\mu P\|_2 \\ &= (2^k \Lambda)^{-m+\kappa-\mu} \|\nabla|\tau\varphi\|_{(\frac{m}{\kappa+\tau-\mu}, \infty)} r^{\mu-\frac{m}{2}} \|\nabla|^\mu P\|_2. \end{aligned}$$

Consequently,

$$\begin{aligned} |II_k| &\lesssim \|w\chi_{A_k}\|_{(p_\kappa, \infty)} (2^k \Lambda)^{-m+\kappa-\mu} \|\nabla|\tau\varphi\|_{(\frac{m}{\kappa+\tau-\mu}, \infty)} r^{\mu-\frac{m}{2}} \|\nabla|^\mu P\|_2 \\ &\stackrel{(2.1)}{\lesssim} (2^k \Lambda r)^{m-2\mu} [w\chi_{A_k}]_{(p_\kappa, \infty)_{\lambda_\kappa}} (2^k \Lambda)^{-m+\kappa-\mu} \|\nabla|\tau\varphi\|_{(\frac{m}{\kappa+\tau-\mu}, \infty)} r^{\mu-\frac{m}{2}} \|\nabla|^\mu P\|_2 \\ &\lesssim r^{m-2\mu} (2^k \Lambda)^{\kappa-3\mu} \|\nabla|\tau\varphi\|_{(\frac{m}{\kappa+\tau-\mu}, \infty)} r^{\mu-\frac{m}{2}} \|\nabla|^\mu P\|_2 [w\chi_{A_k}]_{(p_\kappa, \infty)_{\lambda_\kappa}}. \end{aligned}$$

As for I , set $\tilde{w} := \chi_{B_{\Lambda^k r}} w$ and write

$$\int \tilde{w} H_\mu(P, \varphi) = \int I_\beta \tilde{w} |\nabla|^\beta H_\mu(P, \varphi)$$

Actually, the claim follows quite straight forward from (5.30) for $\mu \leq 1$, $\beta := \mu$, but the pointwise estimates on H , Lemma 5.20, are strong enough to deal with our situation, and they do not make use of para-products which were necessary for the proof of (5.30): By Lemma 5.13

$$\|I_\beta \tilde{w}\|_{(p_1, \infty)_{\lambda_\kappa}} \lesssim \|\tilde{w}\|_{(p_\kappa, \infty)_{\lambda_\kappa}}$$

where for $\beta < \min(2\mu - \kappa, 1)$,

$$\frac{1}{p_1} = \frac{m - \kappa}{m} \frac{2\mu - \kappa - \beta}{2\mu - \kappa} \in (0, 1).$$

If $\mu = \frac{m}{2}$, we set $\beta = 0$, if $\mu < \frac{m}{2}$, let $\epsilon > 0$ such that $\mu + \epsilon < \frac{m}{2}$. Now we estimate $\|\nabla|^\beta H_\mu(P, \varphi)\|$, applying Lemma 5.20 for any $\tau \in (\max\{\beta, \mu + \beta - 1\}, \mu]$, we have to control terms of the form (for $s \in (0, \mu)$, $t \in (0, \tau)$, $\tau - \beta - s - t \in [0, \epsilon)$)

$$I_{\tau-\beta-s-t} (I_s |\nabla|^\mu P | I_t |\nabla|^\tau \varphi).$$

We have

$$\|I_s |\nabla|^\mu P\|_{(p_2, 2)} \lesssim \|\nabla|^\mu P\|_2, \quad \frac{1}{p_2} = \frac{1}{2} - \frac{s}{m} \in (0, 1),$$

$$\|I_t |\nabla|^\tau \varphi\|_{(p_3, 2)} \lesssim \|\nabla|\tau\varphi\|_{(\frac{m}{\kappa+\tau-\mu}, 2)}, \quad \frac{1}{p_3} = \frac{\kappa + \tau - \mu}{m} - \frac{t}{m} \in (0, 1).$$

Note that

$$0 < \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{2} + \frac{\kappa + \tau - \mu - s - t}{m} < \frac{1}{2} + \frac{\kappa + \tau - \mu + \epsilon - \tau + \beta}{m} < \frac{1}{2} + \frac{\epsilon + \mu}{m} < 1,$$

consequently,

$$\|I_{\tau-\beta-s-t}(I_s|\nabla|^\mu P|I_t|\nabla|^\tau\varphi)\|_{(p_4,1)} \lesssim \|\nabla|^\mu P\|_2 \|\nabla|^\tau\varphi\|_{\left(\frac{m}{\kappa+\tau-\mu},2\right)},$$

where

$$\frac{1}{p_4} = \frac{1}{p_2} + \frac{1}{p_3} - \frac{\tau - \beta - s - t}{m} = \frac{1}{2} + \frac{\kappa + \beta - \mu}{m} \in (0, 1).$$

Now we have to ensure that the $f(\beta) \leq 1$ for admissible β (and admissible τ):

$$f(\beta) := \frac{1}{p_1} + \frac{1}{p_4} = \frac{3}{2} - \frac{\mu}{m} - \beta \frac{m - 2\mu}{m(2\mu - \kappa)} > 0.$$

Obviously, $f(0) = 1$ holds, if $\mu = \frac{m}{2}$ (so $\beta = 0$, and τ arbitrarily between $(\mu - 1, \mu]$). As for the case $\mu < \frac{m}{2}$, $\mu \leq 1$, We have $2\mu - \kappa \leq 1$ for $\kappa \in [\mu, 2\mu)$, then

$$f(2\mu - \kappa) = \frac{1}{2} + \frac{\mu}{m} < 1.$$

so we can take $\beta < 1$ sufficiently close to $2\mu - \kappa$, so that $f(\beta) < 1$, and take $\tau \in (\beta, \mu)$ sufficiently close to or greater than $2\mu - \kappa$. Consequently,

$$\begin{aligned} |I| &\lesssim \int_{B_{4r}} I_\beta \tilde{w} |\nabla|^\beta H_\mu(P, \varphi) + \sum_{k=1}^\infty \int_{A_{4r}^k} I_\beta \tilde{w} |\nabla|^\beta H_\mu(P, \varphi) \\ &\lesssim \|I_\beta \tilde{w}\|_{(p_1, \infty), B_{4r}} \|\nabla|^\beta H_\mu(P, \varphi)\|_{(p_4, 1)} r^{m - \frac{m}{p_1} - \frac{m}{p_4}} \\ &\quad + \sum_{k=1}^\infty \|I_\beta \tilde{w}\|_{(p_1, \infty), A_{4r}^k} \|\nabla|^\beta H_\mu(P, \varphi)\|_{(p_4, 1), A_{4r}^k} (2^k r)^{m - \frac{m}{p_1} - \frac{m}{p_4}} \\ &\lesssim r^{\frac{m-\lambda_\kappa}{p_1}} \|I_\beta \tilde{w}\|_{(p_1, \infty)_{\lambda_\kappa}} \|\nabla|^\beta H_\mu(P, \varphi)\|_{(p_4, 1)} r^{m - \frac{m}{p_1} - \frac{m}{p_4}} \\ &\quad + \sum_{k=1}^\infty (2^k r)^{\frac{m-\lambda_\kappa}{p_1}} \|I_\beta \tilde{w}\|_{(p_1, \infty)_{\lambda_\kappa}} \|\nabla|^\beta H_\mu(P, \varphi)\|_{(p_4, 1), A_{4r}^k} (2^k r)^{m - \frac{m}{p_1} - \frac{m}{p_4}} \\ &\lesssim r^{\frac{m-\lambda_\kappa}{p_1}} \|\tilde{w}\|_{(p_\kappa, \infty)_{\lambda_\kappa}} \|\nabla|^\beta H_\mu(P, \varphi)\|_{(p_4, 1)} r^{m - \frac{m}{p_1} - \frac{m}{p_4}} \\ &\quad + \sum_{k=1}^\infty (2^k r)^{\frac{m-\lambda_\kappa}{p_1}} \|\tilde{w}\|_{(p_\kappa, \infty)_{\lambda_\kappa}, A_{4r}^k} \|\nabla|^\beta H_\mu(P, \varphi)\|_{(p_4, 1), A_{4r}^k} (2^k r)^{m - \frac{m}{p_1} - \frac{m}{p_4}}. \end{aligned}$$

By Proposition 5.22, for the same τ as above,

$$\|\nabla|^\beta H_\mu(P, \varphi)\|_{(p_4, 1)} \lesssim \|\nabla|^\mu P\|_2 \|\nabla|^\tau\varphi\|_{\frac{m}{\kappa-\tau-\mu}, 2}.$$

Now we apply Proposition 5.23 (using that φ and $P - I$ have support in B_r), and using

$$\begin{aligned} &\frac{m - \frac{m(2\mu-\kappa)}{m-\kappa}}{p_1} + m - \frac{m}{p_1} - \frac{m}{p_4} - m - \beta + \frac{m}{p_4} \\ &= -2\mu + \kappa, \end{aligned}$$

and

$$\frac{m - \lambda_\kappa}{p_1} + m - \frac{m}{p_1} - \frac{m}{p_4} + \frac{m}{2} - \mu = m - 2\mu$$

we conclude

$$\begin{aligned}
 |I| &\lesssim r^{m-2\mu} \|\tilde{w}\|_{(p_\kappa, \infty)_{\lambda_\kappa}} r^{\mu-\frac{m}{2}} \|\nabla|\mu P\|_2 \|\nabla|\tau\varphi\|_{\frac{m}{\kappa-\tau-\mu}, 2} \\
 &\quad + r^{m-2\mu} \sum_{k=1}^\infty 2^{k(-2\mu+\kappa)} \|\tilde{w}\|_{(p_\kappa, \infty)_{\lambda_\kappa}, A_{4r}^k} \|\nabla|\tau\varphi\|_{\left(\frac{m}{\kappa+\tau-\mu}, \infty\right)} r^{\mu-\frac{m}{2}} \|\nabla|\mu P\|_{(2, \infty)} \\
 &\lesssim C_\kappa r^{m-2\mu} \|w\chi_{B_{\Lambda r}}\|_{(p_\kappa, \infty)_{\lambda_\kappa}} r^{\mu-\frac{m}{2}} \|\nabla|\mu P\|_2 \|\nabla|\tau\varphi\|_{\frac{m}{\kappa-\tau-\mu}, 2}.
 \end{aligned}$$

□

3.2 Better integrability for transformed potential

This section is devoted to the proof of the following Lemma:

Lemma 3.2 *Let $B_r \subset \mathbb{R}^m$, Ω as in (1.6), $\Lambda > 2$. There exists $P : \mathbb{R}^m \rightarrow SO(N)$, $P \equiv I$ on $\mathbb{R}^m \setminus B_{\Lambda^{-1}r}$, with the estimate*

$$(\Lambda^{-1}r)^{\frac{2\mu-m}{2}} \|\nabla|\mu P\|_{2, \mathbb{R}^m} \lesssim \theta, \tag{3.2}$$

such that for any $\tau \in (0, \mu]$ sufficiently close or greater than $2\mu - \kappa$, $\kappa \in [\mu, 2\mu)$, $\theta > 0$ from (1.8) in $D = B_r$, and for any $\varphi \in C_0^\infty(B_{\Lambda^{-1}r})$, if $\mu \in (0, 1]$, or $\mu = \frac{m}{2}$,

$$\begin{aligned}
 &(\Lambda^{-1}r)^{2\mu-m} \int \left((\nabla|\mu P)P^T w + P\Omega[P^T w] \right) \varphi \\
 &\leq C_{\kappa, \mu} \theta \|\nabla|\tau\varphi\|_{\left(\frac{m}{\kappa+\tau-\mu}, 1\right)} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_r} \\
 &\quad + C_{\kappa, \mu} \theta \|\nabla|\tau\varphi\|_{\left(\frac{m}{\kappa+\tau-\mu}, 2\right)} \sum_{k=1}^\infty (2^k \Lambda)^{\kappa-3\mu} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_r^k}.
 \end{aligned}$$

where we recall the definition A_r^k from (1.17), λ_κ from (2.2), and p_κ from (2.3).

As in the proof of Lemma 3.1, we prove the scaled claim for replacing r by Λr which makes the presentation of the proof somewhat lighter: We are going to show the existence of P such that for $\varphi \in C_0^\infty(B_r)$

$$\begin{aligned}
 &r^{2\mu-m} \int \left((\nabla|\mu P)P^T w + P\Omega[P^T w] \right) \varphi \\
 &\leq C_{\kappa, \mu} \theta \|\nabla|\tau\varphi\|_{\left(\frac{m}{\kappa+\tau-\mu}, 1\right)} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{\Lambda r}} \\
 &\quad + C_{\kappa, \mu} \theta \|\nabla|\tau\varphi\|_{\left(\frac{m}{\kappa+\tau-\mu}, 2\right)} \sum_{k=1}^\infty (2^k \Lambda)^{\kappa-3\mu} [w]_{(p_\kappa, \infty)_{\lambda_\kappa}, A_{\Lambda r}^k}, \tag{3.3}
 \end{aligned}$$

Fix $B_r \subset \mathbb{R}^m$. In order to prove this claim, note that

$$\int \left((\nabla|\mu P)P^T w + P\Omega[P^T w] \right) \varphi = \int \left((\nabla|\mu P)P^T w + P\chi_{B_r}\Omega[P^T w] \right) \varphi,$$

so we are going to assume that the A_l in (1.6)

$$\text{supp } A_l \subset B_r, \quad \Omega[\] = \chi_{B_r}\Omega[\] \tag{3.4}$$

and consequently assuming (from (1.8)) that

$$r^{\frac{2\mu-m}{2}} \|A_l\|_{2, \mathbb{R}^m} \|f\|_2 + \sup_{\rho \in (0, \Lambda r)} \rho^{\frac{2\mu-m}{2}} \|\Omega[f]\|_{1, B_\rho} \lesssim \theta \|f\|_2 \tag{3.5}$$

Let $P : \mathbb{R}^m \rightarrow SO(N)$ be the minimizer, $P \equiv I$ on $\mathbb{R}^m \setminus B_r$, of $E(\cdot) \equiv E_{r,x,\Lambda_\mu,1,2}(\cdot)$, where Λ_μ is from Lemma 5.5. Using (5.6), (3.4), we have the estimates (for from now on fixed $\Lambda > 2$),

$$r^{\frac{2\mu-m}{2}} \|\nabla|^\mu P\|_{2,\mathbb{R}^m} \lesssim \theta, \tag{3.6}$$

which after rescaling amounts to (3.2), and with the help of (3.5),

$$r^{\frac{2\mu-m}{2}} \left\| (|\nabla|^\mu P)P^T f + P\Omega[P^T f] \right\|_{1,B_{\Lambda r}} \lesssim \theta \|f\|_{2,\mathbb{R}^m}. \tag{3.7}$$

Let

$$w = w\chi_{B_{\Lambda r}} + \sum_{k=1}^\infty w\chi_{A_{\Lambda r}^k} =: w_0 + \sum_{k=1}^\infty w_k.$$

Then,

$$\begin{aligned} & \int \left((|\nabla|^\mu P)P^T w + P\Omega[P^T w] \right) \varphi \\ &= \int (|\nabla|^\mu P)P^T w_0 \varphi + P\Omega[P^T w_0 \varphi] - \int PC(\varphi, \Omega)[P^T w_0] + \sum_{k=1}^\infty \int P\Omega[P^T w_k] \varphi \\ &=: I - II + III. \end{aligned}$$

The disjoint support part (III)

Since $\mu \leq \kappa < 2\mu$,

$$\begin{aligned} \int P\Omega[P^T w_k] \varphi &\stackrel{(1.6)}{\lesssim} \|A\|_{2,B_r} \|\varphi\|_2 \|\mathcal{R}[P^T w_k]\|_{\infty,B_r} \\ &\stackrel{p.B.1}{\lesssim} \|A\|_{2,B_r} r^{\frac{m}{2}-\kappa+\mu} \|\nabla|^\tau \varphi\|_{\frac{m}{\kappa+\tau-\mu}} (2^k \Lambda r)^{-m+\kappa} \|w_k\|_{(p_\kappa,\infty)} \\ &\stackrel{(2.1)}{\lesssim} r^{\mu-\frac{m}{2}} \|A\|_{2,B_r} \|\nabla|^\tau \varphi\|_{\frac{m}{\kappa+\tau-\mu}} (2^k \Lambda)^{-m+\kappa} (2^k \Lambda r)^{m-2\mu} [w]_{(p_\kappa,\infty)_{\lambda_\kappa}, A_{\Lambda,r}^k} \\ &\stackrel{(3.5)}{\lesssim} \theta r^{m-2\mu} (2^k \Lambda)^{\kappa-2\mu} \|\nabla|^\tau \varphi\|_{\frac{m}{\kappa+\tau-\mu}} [w]_{(p_\kappa,\infty)_{\lambda_\kappa}, A_{\Lambda,r}^k}. \end{aligned}$$

The same-support/commutator part (II)

We have

$$|II| \lesssim \|A\|_2 \|\mathcal{C}(\varphi, \mathcal{R})[P^T w_0]\|_{2,B_r} \stackrel{(3.5)}{\lesssim} r^{\frac{m-2\mu}{2}} \theta \|\mathcal{C}(\varphi, \mathcal{R})[P^T w_0]\|_{2,B_r}.$$

Now we apply Lemma 5.26, and have for arbitrary $\delta \in (0, 1)$, $\gamma_{1,2} \in (0, \delta)$,

$$\left| \mathcal{C}(\varphi, \mathcal{R})[P^T w_0] \right| \lesssim I_{\delta-\gamma_1} |\nabla|^\delta \varphi |I_{\gamma_1}| w_0| + C_{\mathcal{R},\delta,\gamma_2} I_{\gamma_2} (|\nabla|^\delta \varphi |I_{\delta-\gamma_2}| w_0|)$$

Now, if we choose $\delta < \tau$

$$\|I_{\delta-\gamma_1} |\nabla|^{\delta_1} \varphi\|_{(\frac{m}{\gamma_1+\kappa-\mu}, q)} \lesssim \|\nabla|^\tau \varphi\|_{(\frac{m}{\tau+\kappa-\mu}, q)},$$

and for $\beta < 2\mu - \kappa$, using [1], see Lemma 5.13,

$$r^{\frac{\lambda_\kappa-m}{p\gamma_1}} \|I_\beta w_0\|_{(p_\beta,\infty),B_r} \lesssim \|I_\beta w_0\|_{(p_\beta,\infty)_{\lambda_\kappa}} \lesssim \|w_0\|_{(p_\kappa,\infty)_{\lambda_\kappa}}$$

where

$$\frac{1}{p\beta} = \frac{m - \kappa}{m} \frac{2\mu - \kappa - \beta}{2\mu - \kappa} \in (0, 1).$$

Now,

$$\begin{aligned} & \frac{1}{p\gamma_1} + \frac{\gamma_1 + \kappa - \mu}{m} \\ &= \frac{\mu}{m} + (m - 2\mu) \frac{(2\mu - \kappa) - \gamma_1}{m(2\mu - \kappa)} \\ &\leq \frac{1}{2}, \end{aligned} \tag{3.8}$$

if we choose $\gamma_1 \in (0, 2\mu - \kappa)$ as follows: If $\mu = \frac{m}{2}$ we can choose γ arbitrarily. If $\mu < \frac{m}{2}$ and $\mu \leq 1$, then we pick γ_1 sufficiently close to $2\mu - \kappa \leq 1$. That is, for any $\tau < \mu$ sufficiently close or greater than $2\mu - \kappa$ such that there is a $\gamma_1 < \delta < \tau$, $\delta < 2\mu - \kappa$, satisfying the above equation, we have

$$\|I_{\delta-\gamma_1} \|\nabla\|^{\delta_1} \varphi\| I_{\gamma_1} \|w_0\|_{2, B_r} \lesssim r^{\frac{m}{2} - \frac{m}{p\gamma_1} - (\gamma_1 + \kappa - \mu)} r^{\frac{m-\lambda_\kappa}{p\gamma_1}} \|\nabla\|^\tau \varphi\| \left(\frac{m}{\tau + \kappa - \mu}, 2\right) \|w_0\|_{(p_\kappa, \infty)_{\lambda_\kappa}},$$

and

$$\frac{m}{2} - \frac{m}{p\gamma_1} - (\gamma_1 + \kappa - \mu) + \frac{m - \lambda_\kappa}{p\gamma_1} = \frac{m}{2} - (\gamma_1 + \kappa - \mu) - (2\mu - \kappa - \gamma_1) = \frac{m}{2} - \mu.$$

As for the second term, for $\delta - \gamma_2 < 2\mu - \kappa$, using the formula (3.8) with δ instead of γ_1 ,

$$\begin{aligned} \frac{1}{p_2} &:= \frac{\delta + \kappa - \mu}{m} + \frac{1}{p\delta-\gamma_2} \\ &= \frac{\delta + \kappa - \mu}{m} + \frac{1}{p_\delta} + \gamma_2 \frac{m - \kappa}{m(2\mu - \kappa)} \leq \frac{1}{2} + \gamma_2 \frac{m - \kappa}{m(2\mu - \kappa)} < 1, \end{aligned}$$

if we choose $\gamma_1 < \delta$ (as above γ_1) close enough $2\mu - \kappa$, and γ_2 very small. Consequently, if we set

$$\lambda := \lambda_\kappa,$$

and $\tilde{\lambda} \in (0, m)$ such that $\frac{\tilde{\lambda}-m}{p_2} = \frac{\lambda-m}{p\delta-\gamma_2}$, that is

$$\begin{aligned} \frac{\tilde{\lambda}}{p_2} &= \frac{\lambda - m}{p\delta-\gamma_2} + \frac{m}{p_2} = \frac{\lambda_\kappa - m}{p\delta-\gamma_2} + \delta + \kappa - \mu + \frac{m}{p\delta-\gamma} \\ &= (\lambda_\kappa) \frac{m - \kappa}{m} \frac{2\mu - \kappa - (\delta - \gamma_2)}{2\mu - \kappa} + \delta + \kappa - \mu \\ &= \mu + \gamma_2 \end{aligned} \tag{3.9}$$

then

$$\begin{aligned} \|\lvert\nabla\rvert^{\delta_2}\varphi I_{\delta_2-\gamma_2}\lvert w_0\rvert\|_{(p_2,2)_{\tilde{\lambda}}} &\approx \sup_{B_\rho} \rho^{\frac{\tilde{\lambda}-m}{p_2}} \|\lvert\nabla\rvert^{\delta_2}\varphi I_{\delta_2-\gamma_2}\lvert w_0\rvert\|_{(p_2,2),B_\rho} \\ &\lesssim \|\lvert\nabla\rvert^\tau\varphi\|_{\left(\frac{m}{\tau+\kappa-\mu},2\right)} \sup_{B_\rho} \rho^{\frac{\tilde{\lambda}-m}{p_2}} \|I_{\delta_2-\gamma_2}\lvert w_0\rvert\|_{(p_{\delta_2-\gamma_2},\infty),B_\rho} \\ &\approx \|\lvert\nabla\rvert^\tau\varphi\|_{\left(\frac{m}{\tau+\kappa-\mu},2\right)} \|I_{\delta_2-\gamma_2}\lvert w_0\rvert\|_{(p_{\delta_2-\gamma_2},\infty)_\lambda,B_\rho} \\ &\lesssim \|\lvert\nabla\rvert^\tau\varphi\|_{\left(\frac{m}{\tau+\kappa-\mu},2\right)} \|w_0\|_{(p_\kappa,\infty)_{\lambda_\kappa}}. \end{aligned}$$

Now observe

$$\begin{aligned} \frac{1}{2} - \left(\frac{1}{p_2} - \frac{\gamma_2}{\tilde{\lambda}}\right) &\stackrel{(3,9)}{=} \frac{1}{2} - \frac{\mu}{p_2(\mu + \gamma_2)} \\ &= \frac{1}{2} - \frac{\mu}{\mu + \gamma_2} \left(\frac{\delta_2 + \kappa - \mu}{m} + \frac{m - \kappa}{m} \frac{2\mu - \kappa - (\delta_2 - \gamma_2)}{2\mu - \kappa}\right) \\ &= \frac{1}{2} - \frac{\mu}{\mu + \gamma_2} \left(\frac{((2\mu - \kappa) - \delta_2)(m - 2\mu)}{m(2\mu - \kappa)} + \frac{\mu}{m} + \frac{m - \kappa}{m} \frac{\gamma_2}{2\mu - \kappa}\right) \geq 0, \end{aligned}$$

for sufficiently small γ_2 and δ_2 sufficiently close to $2\mu - \kappa$. In fact, this holds obviously, if $\frac{\mu}{m} < \frac{1}{2}$. If $\frac{\mu}{m} = \frac{1}{2}$, we have

$$\frac{\mu}{\mu + \gamma_2} \left(\frac{((2\mu - \kappa) - \delta_2)(m - 2\mu)}{m(2\mu - \kappa)} + \frac{\mu}{m} + \frac{m - \kappa}{m} \frac{\gamma_2}{2\mu - \kappa}\right) = \frac{\mu}{\mu + \gamma_2} \left(\frac{1}{2} + \frac{\gamma_2}{2\mu}\right) = \frac{1}{2}$$

Moreover, one checks

$$\frac{m}{2} - \frac{\mu}{\mu + \gamma_2} \frac{m}{p_2} + \frac{\mu}{\mu + \gamma_2} \frac{m - \tilde{\lambda}}{p_2} = \frac{m}{2} - \frac{\mu}{\mu + \gamma_2} \frac{\tilde{\lambda}}{p_2} \stackrel{(3,9)}{=} \frac{m}{2} - \mu.$$

Thus,

$$\begin{aligned} \|I_{\gamma_2}(\lvert\nabla\rvert^{\delta_2}\varphi I_{\delta_2-\gamma_2}\lvert w_0\rvert)\|_{2,B_r} &\lesssim r^{\frac{m}{2}-\mu} \|I_{\gamma_2}(\lvert\nabla\rvert^{\delta_2}\varphi I_{\delta_2-\gamma_2}\lvert w_0\rvert)\|_{\left(\frac{p_2(\mu+\gamma_2)}{\mu},2\right)_{\tilde{\lambda}}} \\ &\lesssim r^{\frac{m}{2}-\mu} \|\lvert\nabla\rvert^{\delta_2}\varphi I_{\delta_2-\gamma_2}\lvert w_0\rvert\|_{(p_2,2)_{\tilde{\lambda}}} \\ &\lesssim r^{\frac{m}{2}-\mu} \|\lvert\nabla\rvert^\tau\varphi\|_{\left(\frac{m}{\tau+\kappa-\mu},2\right)} \|w_0\|_{(p_\kappa,\infty)_{\lambda_\kappa}}. \end{aligned}$$

The same-support/commutator part (I)

Here, we decompose

$$w_0\varphi = \lvert\nabla\rvert^\mu(\eta_{\Lambda r}(I_\mu(w_0\varphi))) + \lvert\nabla\rvert^\mu((1 - \eta_{\Lambda r})(I_\mu(w_0\varphi))) =: \lvert\nabla\rvert^\mu g_1 + \lvert\nabla\rvert^\mu g_2$$

and

$$\begin{aligned} I &= \int (\lvert\nabla\rvert^\mu P)P^T \lvert\nabla\rvert^\mu g_1 + P\Omega[P^T \lvert\nabla\rvert^\mu g_1] + \int (\lvert\nabla\rvert^\mu P)P^T \lvert\nabla\rvert^\mu g_2 + P\Omega[P^T \lvert\nabla\rvert^\mu g_2] \\ &=: I_1 + I_2. \end{aligned}$$

For I_1 we use Theorem 1.6 in the form of Lemma 5.7,

$$I_1 = \int \Omega_P[\lvert\nabla\rvert^\mu g_1] \lesssim \theta r^{m-2\mu} \begin{cases} [g_1]_{\text{BMO}} & \text{if } \mu \leq 1, \\ r^{\mu-\frac{m}{2}} \|\lvert\nabla\rvert^\mu g_1\|_{(2,\infty)} & \text{if } \mu > 1. \end{cases}$$

Note that

$$\text{supp}(\varphi w_0) \subset B_r,$$

and moreover for $q_\mu = \infty$, for $\kappa > \mu$, and $q_\mu = 1$ for $\kappa = \mu$, (for arbitrary $\tau > 0$)

$$\|\varphi w_0\|_{(\frac{m}{m-\mu}, \infty)_{\frac{\mu m}{m-\mu}}} \lesssim \|\varphi\|_{(\frac{m}{\kappa-\mu}, \infty)} \|w\chi_{B_r}\|_{(p_\kappa, \infty)_{\lambda_\kappa}} \lesssim \||\nabla|^\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, q_\mu)} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{2r}}. \tag{3.10}$$

Then, the claim for I_1 follows from

Proposition 3.3 *Let $\mu \leq 1$, $g := \eta_{\Lambda r} I_\mu(f)$, $\text{supp } f \subset \overline{B_r}$, then for any $\kappa \in [\mu, 2\mu)$,*

$$[g]_{\text{BMO}} \lesssim (1 + \Lambda^{\mu-m}) \|f\|_{(\frac{m}{m-\mu}, \infty)_{\frac{\mu m}{m-\mu}}}.$$

Proof From [1, Proposition 3.3.]

$$[g]_{\text{BMO}} \lesssim \||\nabla|^\mu g\|_{(1)_\mu}.$$

Since,

$$|\nabla|^\mu g = f + |\nabla|^\mu((1 - \eta_{\Lambda r})I_\mu f),$$

we have,

$$\begin{aligned} [g]_{\text{BMO}} &\lesssim \|f\|_{(1)_\mu} + \||\nabla|^\mu((1 - \eta_{\Lambda r})I_\mu f)\|_{(1)_\mu} \\ &\lesssim \|f\|_{(\frac{m}{m-\mu}, \infty)_{\frac{\mu m}{m-\mu}}} + \||\nabla|^\mu((1 - \eta_{\Lambda r})I_\mu f)\|_{(\frac{m}{\mu}, \infty)}, \end{aligned}$$

and by Proposition 5.17

$$\begin{aligned} \||\nabla|^\mu((1 - \eta_{\Lambda r})I_\mu f)\|_{\frac{m}{\mu}} &\lesssim \sup_{\alpha \in [0, \mu]} (\Lambda r)^{-m+\mu-\alpha} \|f\|_1 \||\nabla|^{\mu-\alpha}((1 - \eta_{\Lambda r}))\|_{\frac{m}{\mu}} \\ &\lesssim (\Lambda r)^{\mu-m} \|f\|_1. \end{aligned} \tag{3.11}$$

Since $\text{supp } f \subset B_r$,

$$r^{\mu-m} \|f\|_1 \lesssim \|f\|_{(1)_\mu} \lesssim \|f\|_{(\frac{m}{m-\mu}, \infty)_{\frac{\mu m}{m-\mu}}}. \tag{3.12}$$

□

Moreover, as in (3.11), from Proposition 5.17 and (3.10),

$$\||\nabla|^\mu g_2\|_2 \lesssim (\Lambda r)^{-\frac{m}{2}} \|\varphi w_0\|_1 \stackrel{(3.12)}{\lesssim} r^{\frac{m}{2}-\mu} \||\nabla|^\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, q_\mu)} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{2r}},$$

implying

$$\begin{aligned} |I_2| &\lesssim \|\Omega_P\|_{2 \rightarrow 1} (\Lambda r)^{\frac{m}{2}-\mu} \||\nabla|^\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, q_\mu)} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{2r}} \\ &\stackrel{(3.7)}{\lesssim} \theta r^{m-2\mu} \||\nabla|^\tau \varphi\|_{(\frac{m}{\kappa+\tau-\mu}, q_\mu)} \|w\|_{(p_\kappa, \infty)_{\lambda_\kappa}, B_{2r}}. \end{aligned}$$

This proves the claim (3.3) and thus Lemma 3.2.

4 Higher integrability: Proof of Theorem 1.3

This section treats the regularity arguments, which can be used once the equation becomes sub-critical, that is once we have obtained a sufficient initial integrability of the solution. In that case, the antisymmetry of the right-hand side operator is irrelevant, and the regularity follows from a bootstrapping argument, which nevertheless might be of independent interest.

Let $w \in L^{(p)\lambda}_{loc}(D) \cap L^2(\mathbb{R}^m)$ be a solution to

$$|\nabla|^\mu w = \Omega[w] \text{ in } D \subset\subset \mathbb{R}^m.$$

Choosing for any domain $\tilde{D} \subset\subset D$, we can choose a domain $D_2, \tilde{D} \subset\subset D_2 \subset\subset D$ and a cutoff function $\eta_{\tilde{D}} \in C_0^\infty(D_2), \eta_{\tilde{D}} \equiv 1$ in \tilde{D} . Then $w_{\tilde{D}} := \eta_{\tilde{D}} w \in L^{(p)\lambda}(\mathbb{R}^m)$ is a solution to

$$|\nabla|^\mu w_{\tilde{D}} = \Omega[w_{\tilde{D}}] + \Omega[w - w_{\tilde{D}}] + |\nabla|^\mu(w_{\tilde{D}} - w) \text{ in } \tilde{D},$$

and in \tilde{D} ,

$$\|\Omega[w - w_{\tilde{D}}] + |\nabla|^\mu(w_{\tilde{D}} - w)\|_{\infty, \tilde{D}} \leq C_{\tilde{D}, D, D_2, \eta, \|w\|_2}.$$

So Theorem 1.3 follows from the following argument.

Lemma 4.1 *Let $p > 2$, and $0 < \mu \leq \frac{m}{2}, \lambda \leq 2\mu$, and let $w \in L^{(p)\lambda}$ be a solution to*

$$|\nabla|^\mu w = \Omega[w] + f \text{ in } D \subset\subset \mathbb{R}^m, \tag{4.1}$$

where $f \in L^\infty(D)$. Then, for any $\tilde{p} \in [p, \infty)$ there exists $\varepsilon \in (0, 1)$ such that if θ from (1.8) satisfies $\theta < \varepsilon$, then $w \in L^{\tilde{p}}_{loc}(D)$.

Proof In order to keep the presentation short, we are going to assume that $\Omega[\cdot] = A\mathcal{R}[\cdot]$. Also note that if $w \in L^{(p)\lambda}$ for some $p > 2$, than for some $\tilde{p} \in (2, p), w \in L^{\tilde{p}, \tilde{\lambda}}$, for some $\tilde{\lambda} < \lambda$, so we can assume w.l.o.g. that $\lambda < 2\mu$. From (4.1) we have for any $B_r \subset B_R \subset \tilde{D}$,

$$\begin{aligned} \||\nabla|^\mu w\|_{\frac{2p}{p+2}, B_r} &\lesssim \|A\|_{2, B_r} \|\mathcal{R}[w]\|_{p, B_r} + \|f\|_\infty r^m \frac{p+2}{2p} \\ &\stackrel{(1.8)}{\lesssim} r^{\frac{m-2\mu}{2}} \theta \|w\|_{p, B_{2r}} + r^{\frac{m-2\mu}{2}} \theta \sum_{k=2}^\infty 2^{-k\frac{m}{p}} \|w\|_{p, A_k^r} + \|f\|_\infty r^m \frac{p+2}{2p} \\ &\lesssim r^{\frac{m-2\mu}{2}} r^{\frac{m-\lambda}{p}} \theta \|w\|_{(p)\lambda, B_R} \\ &\quad + r^{\frac{m-2\mu}{2}} r^{\frac{m-\lambda}{p}} \theta \sum_{k=2}^\infty 2^{-k\frac{\lambda}{p}} [w]_{(p)\lambda, B_{2^{k+1}R}} + \|f\|_\infty r^m \frac{p+2}{2p} \end{aligned}$$

That is, for

$$\lambda_N := \lambda \frac{2}{2+p} + 2\mu \frac{p}{2+p} \in (\lambda, 2\mu), \tag{4.2}$$

$$\||\nabla|^\mu w\|_{(\frac{2p}{p+2})\lambda_N, B_{\frac{R}{2}}} \lesssim \theta \|w\|_{(p)\lambda, B_R} + \theta \sum_{k=2}^\infty 2^{-k\frac{\lambda}{p}} [w]_{(p)\lambda, B_{2^{k+1}R}} + \|f\|_\infty R^{\lambda_N} \frac{p+2}{2p}$$

Consequently, by Proposition 4.2 (note that $\frac{2p}{p+2} > 1$), for $p_2 = 2p/(p + 2)$ and $p_1 > p$ (since $\lambda_N < 2\mu$) defined by

$$\frac{1}{p_1} = \frac{1}{p} + \frac{1}{2} - \frac{\mu}{\lambda_N} \tag{4.3}$$

$$\begin{aligned} \|w\|_{p_1, B_{\Lambda^{-1}r}} &\lesssim \Lambda^{-\frac{m}{p_1}} r^{\frac{m}{p_1}-m} \|w\|_{1, B_r} + (\Lambda^{-1}r)^{-\frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu + \frac{m}{p_1}} \|\nabla|\mu w\|_{\left(\frac{2p}{p+2}\right)_{\lambda_N}, B_r} \\ &\quad + \Lambda^{-\frac{m}{p_1}} \sum_{k=1}^{\infty} 2^{-km} r^{\frac{m}{p_1}-m} \|w\|_{1, A_r^k} \\ &\lesssim \Lambda^{-\frac{\lambda}{p}} (\Lambda^{-1}r)^{\frac{m}{p_1} - \frac{\lambda}{p}} \|w\|_{(p)\lambda, B_r} \\ &\quad + (\Lambda^{-1}r)^{-\frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu + \frac{m}{p_1}} \theta \|w\|_{(p)\lambda, B_{2r}} \\ &\quad + (\Lambda^{-1}r)^{-\frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu + \frac{m}{p_1}} \theta \sum_{k=2}^{\infty} 2^{-k\frac{\lambda}{p}} [w]_{(p)\lambda, B_{2^{k+2}r}} \\ &\quad + (\Lambda^{-1}r)^{-\frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu + \frac{m}{p_1}} \|f\|_{\infty} r^{\lambda_N \frac{p+2}{2p}} \\ &\quad + (\Lambda^{-1}r)^{\frac{m}{p_1} - \frac{\lambda}{p}} \Lambda^{-\frac{\lambda}{p}} \sum_{k=1}^{\infty} 2^{-k\frac{\lambda}{p}} [w]_{(p)\lambda, B_{2^{k+1}r}} \\ &\lesssim (\Lambda^{-1}r)^{-\frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu + \frac{m}{p_1}} (\theta + \Lambda^{-\frac{\lambda}{p}}) \|w\|_{(p)\lambda, B_{2r}} \\ &\quad + (\Lambda^{-1}r)^{-\frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu + \frac{m}{p_1}} (\theta + \Lambda^{-\frac{\lambda}{p}}) \sum_{k=1}^{\infty} 2^{-k\frac{\lambda}{p}} [w]_{(p)\lambda, B_{2^{k+1}r}} \\ &\quad + (\Lambda^{-1}r)^{-\frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu + \frac{m}{p_1}} \|f\|_{\infty} r^{\lambda_N \frac{p+2}{2p}}. \end{aligned}$$

Consequently,

$$\begin{aligned} \|w\|_{p, B_{\Lambda^{-1}r}} &\lesssim (\Lambda^{-1}r)^{\frac{m}{p} - \frac{m}{p_1}} \|w\|_{p_1, B_{\Lambda^{-1}r}} \\ &\lesssim (\Lambda^{-1}r)^{\frac{m}{p} - \frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu} (\theta + \Lambda^{-\frac{\lambda}{p}}) \|w\|_{(p)\lambda, B_{2r}} \\ &\quad + (\Lambda^{-1}r)^{\frac{m}{p} - \frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu} (\theta + \Lambda^{-\frac{\lambda}{p}}) \sum_{k=1}^{\infty} 2^{-k\frac{\lambda}{p}} [w]_{(p)\lambda, B_{2^{k+1}r}} \\ &\quad + (\Lambda^{-1}r)^{\frac{m}{p} - \frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu} \|f\|_{\infty} r^{\lambda_N \frac{p+2}{2p}}. \end{aligned}$$

Now

$$\frac{m}{p} - \frac{\lambda_N}{p} - \frac{\lambda_N}{2} + \mu = \frac{m - \lambda}{p},$$

which implies finally, for any $B_{2r} \subset D$,

$$\begin{aligned} \|w\|_{(p)\lambda, B_{\Lambda^{-1}r}} &\lesssim (\theta + \Lambda^{-\frac{\lambda}{p}}) \|w\|_{(p)\lambda, B_{2r}} \\ &\quad + (\theta + \Lambda^{-\frac{\lambda}{p}}) \sum_{k=1}^{\infty} 2^{-k\frac{\lambda}{p}} [w]_{(p)\lambda, B_{2^{k+1}r}} + \|f\|_{\infty} r^{\lambda_N \frac{p+2}{2p}}. \end{aligned}$$

Now we argue similar to the iteration in Sect. 2: Choose $\Lambda_\lambda := 2^{C_{p,\mu}\lambda^{-4}}$, assume that $\theta < \Lambda_\lambda^{-\frac{\lambda}{p}}$, and choose $C_{p,\mu}$ so that (5.22) is satisfied. Then we can choose a new $\lambda_1 = \lambda - c\lambda^4$ for which the above estimate holds and the right-hand side is finite. Repeating this argument (for smaller and smaller θ), we obtain a monotone decreasing sequence of $\lambda_{i+1} = \lambda_i - c\lambda_i^4 \geq 0$, which has as only fixed point 0. Thus, for any $\lambda > 0$ there exists $\theta > 0$ such that for any $\tilde{D} \subset\subset D$,

$$\|w\|_{(p)_\lambda, \tilde{D}} \leq C_{\tilde{D}, D, \lambda, w}.$$

Note that for $\lambda \rightarrow 0$, $\lambda_N \rightarrow \mu \frac{2p}{p+2}$ and thus p_1 in (4.3) tends to infinity. Thus, we have obtain for any $\tilde{p} > 1$ a $\lambda_{\tilde{p}} > 0$ such that $p_1 \equiv p_1(\lambda_{\tilde{p}}) > \tilde{p}$, and if θ is small enough, we have to iterate the above argument finitely many steps to obtain that $w \in L_{loc}^{p_1}(\tilde{D})$. \square

Proposition 4.2 *For any $f, \mu \in (0, m)$ we have for $p_1 \in (1, \infty), p_2 \in (1, \infty), \lambda \in (0, m)$ such that*

$$\frac{1}{p_1} = \frac{1}{p_2} - \frac{\mu}{\lambda},$$

the following estimate for any $\Lambda > 2$

$$\begin{aligned} \|f\|_{p_1, B_{\Lambda^{-1}r}} &\lesssim \Lambda^{-\frac{m}{p_1}} r^{\frac{m}{p_1}-m} \|f\|_{1, B_r} + (\Lambda^{-1}r)^{-\frac{\lambda}{p_2} + \mu + \frac{m}{p_1}} \|\nabla|^\mu f\|_{(p_2)_\lambda, B_r} \\ &\quad + \sum_{i=1}^\infty 2^{-im} \Lambda^{-\frac{m}{p_1}} r^{\frac{m}{p_1}-m} \|f\|_{1, A_r^i}. \end{aligned}$$

Proof Let $1 < p_4 \leq p'_1$,

$$\frac{1}{p_3} + \frac{1}{p_4} = 1.$$

$$\frac{1}{p_3} = \frac{1}{p_2} - \frac{\mu}{\lambda} \in (0, 1).$$

There exists $\varphi \in C_0^\infty(B_{\Lambda^{-1}r}), \|\varphi\|_{p'_1} \leq 1$, such that

$$\begin{aligned} \|f\|_{p_1, B_{\Lambda^{-1}r}} &\lesssim \int_{\mathbb{R}^m} f \varphi = \int_{B_{\Lambda^{-1}r}} I_\mu(\eta_{B_r} |\nabla|^\mu f) \varphi + \sum_{k=1}^\infty \int_{\mathbb{R}^m} f |\nabla|^\mu (\eta_{A_r^k} I_\mu \varphi) \\ &= \int_{B_{\Lambda^{-1}r}} I_\mu(\eta_{B_r} |\nabla|^\mu f) \varphi + \sum_{k=1}^\infty \int_{B_r} f |\nabla|^\mu (\eta_{A_r^k} I_\mu \varphi) + \sum_{k=1}^\infty \sum_{i=1}^\infty \int_{A_r^i} f |\nabla|^\mu (\eta_{A_r^k} I_\mu \varphi) \\ &\lesssim \|I_\mu(\eta_r |\nabla|^\mu f)\|_{p_3, B_{\Lambda^{-1}r}} \|\varphi\|_{p_4} + \sum_{k=1}^\infty \|f\|_{1, B_r} \|\nabla|^\mu (\eta_{A_r^k} I_\mu \varphi)\|_{\infty, B_r} \\ &\quad + \sum_{k=1}^\infty \sum_{i=1}^\infty \|f\|_{1, A_r^i} \|\nabla|^\mu (\eta_{A_r^k} I_\mu \varphi)\|_{\infty, A_r^i}. \end{aligned}$$

The claim follows then from the following estimates: Firstly, (this argument holds, if $k \geq 2$ by Lemma 5.15, if $k = 1$ one has to apply Lemma 5.17 to get the same estimate)

$$\begin{aligned} \|\ |\nabla|^\mu (\eta_{A_r^k} I_\mu \varphi)\|_{\infty, B_r} \|f\|_{1, B_r} &\lesssim (2^k r)^{-m-\mu} \|I_\mu \varphi\|_{1, A_r^k} \|f\|_{1, B_r} \\ &\lesssim (2^k r)^{-m-\mu} (2^k r)^m \|I_\mu \varphi\|_{\infty, A_r^k} \|f\|_{1, B_r} \lesssim (2^k r)^{-m-\mu} (2^k r)^m (2^k r)^{-m-\mu} \|\varphi\|_1 \|f\|_{1, B_r} \\ &\lesssim (2^k r)^{-m-\mu} (2^k r)^m (2^k r)^{-m+\mu} (\Lambda^{-1} r)^{\frac{m}{p_1}} \|f\|_{1, B_r} = r^{\frac{m}{p_1}-m} 2^{-km} \Lambda^{-\frac{m}{p_1}} \|f\|_{1, B_r}. \end{aligned}$$

By Lemma 5.17,

$$\|\ |\nabla|^\mu (\eta_{A_r^k} I_\mu \varphi)\|_{\infty} \lesssim 2^{-km} \Lambda^{-\frac{m}{p_1}} r^{\frac{m}{p_1}-m}.$$

And for $|i - k| \geq 2$, twice using Lemma 5.15

$$\begin{aligned} \|\ |\nabla|^\mu (\eta_{A_r^k} I_\mu \varphi)\|_{\infty, A_r^i} &\lesssim (2^{\max\{i, k\}} r)^{-m-\mu} (2^k r)^\mu \|\varphi\|_1 \\ &\lesssim 2^{\max\{i, k\}(-\mu-m)+k\mu} \Lambda^{-\frac{m}{p_1}} r^{-m-\frac{m}{p_1}}. \end{aligned}$$

Since $p_4 \leq p'_1$,

$$\|\varphi\|_{p_4, B_r} \lesssim (\Lambda^{-1} r)^{\frac{m}{p_4}-\frac{m}{p'_1}}.$$

And using Lemma 5.13

$$\|I_\mu(\eta_r |\nabla|^\mu f)\|_{p_3, B_{\Lambda^{-1}r}} \lesssim (\Lambda^{-1} r)^{\frac{m-\lambda}{p_3}} \|I_\mu(\eta_r |\nabla|^\mu f)\|_{(p_3)_\lambda} \lesssim (\Lambda^{-1} r)^{\frac{m-\lambda}{p_3}} \|\ |\nabla|^\mu f\|_{(p_2)_\lambda, B_r}.$$

Consequently, we have shown the claim. □

5 Energy approach for optimal frame: Proof of Theorem 1.6

In this section we construct a suitable frame P for our equation, transforming the *antisymmetric* (essentially) L^2 -potential $\Omega[\]$ into an $L^{2,1}$ - or even better in an $I_\mu \mathcal{H}$ -potential $\Omega^P[\]$. Here, \mathcal{H} is the Hardy space, and with the previous statement we essentially mean that

$$\int \Omega^P[f] \leq C_{\Omega^P} \|f\|_{(2, \infty)}, \quad \text{or} \quad \int \Omega^P[|\nabla|^\mu \varphi] \leq C_{\Omega^P} \|\varphi\|_{BMO}, \quad \text{respectively,} \quad (5.1)$$

where BMO is the space dual to \mathcal{H} . This is an improvement, since for the non-transformed Ω , we only had the estimate

$$\int \Omega[f] \leq C_\Omega \|f\|_2. \quad (5.2)$$

For motivation of the arguments presented here, let us recall the classical setting [19], where we have the equation (usually for $w^i := \nabla u^i \in L^2(\mathbb{R}^m, \mathbb{R}^2)$)

$$-\operatorname{div}(w^i) = \tilde{\Omega}_{ik} \cdot w^k,$$

for $\tilde{\Omega}_{ik} = -\tilde{\Omega}_{ki} \in L^2(\mathbb{R}^m, \mathbb{R}^2)$, and we look for an orthogonal transformation $P \in W^{1,2}(\mathbb{R}^m, SO(N))$, $SO(N) \subset \mathbb{R}^{N \times N}$ being the special orthogonal group, such that

$$\int \tilde{\Omega}_{ik}^P \cdot \nabla \varphi = 0, \quad (5.3)$$

where

$$\tilde{\Omega}_{ij}^P = P_{ik} \nabla P_{kj}^T + P_{ik} \tilde{\Omega}_{kl} P_{lj}^T, \quad \text{or equivalently,} \quad -\operatorname{div}(P_{il} w^l) = \tilde{\Omega}_{ik}^P \cdot P_{kl} w^l.$$

Also in this case, the estimate (5.3) is an improvement from the estimate for the non-transformed $\tilde{\Omega}$

$$\int \tilde{\Omega} \cdot \nabla \varphi \leq C_{\tilde{\Omega}} \|\nabla \varphi\|_2,$$

philosophically similar to the improvement (5.1) from the starting point (5.2).

For the construction of P such that (5.3) holds, there are two different arguments known: Rivière [19] adapted a result by Uhlenbeck [35] which is based on the continuity method (for the set $t\Omega$, $t \in [0, 1]$) and relies on non-elementary a-priori estimates for $\tilde{\Omega}^P$, which also needs L^2 -smallness of $\tilde{\Omega}$. In [22] the author proposed to use arguments from Hélein’s moving frame method [14]: Then the construction of P relies on the fact that (5.3) is the Euler–Lagrange equation of the energy

$$\tilde{E}(Q) := \|\tilde{\Omega}^Q\|_2^2, \quad Q \in SO(N), \text{ a.e.}, \tag{5.4}$$

the minimizer of which exists by the elementary direct method.

Both construction arguments have been generalized to the fractional setting for $\Omega[\cdot] \equiv \Omega \cdot$ a pointwise multiplication-operator [6,24]. In our situation, where $\Omega[\cdot]$ is allowed to be a linear bounded operator from L^2 to L^1 , we adapt the argument in [14,22,24], and minimize essentially the energy

$$E(Q) := \sup_{\psi \in L^2} \int \Omega^Q[\psi], \quad Q \in SO(N), \text{ a.e.}$$

While the construction of a minimizer of E , see Lemma 5.5, is not much more difficult as in the earlier situations [14,22,24], when computing the Euler–Lagrange equations, see Lemma 5.6, we have several error terms, which stem from commutators of the form $f\Omega[g] - \Omega[fg]$, which are trivial if $\Omega[\cdot]$ is a pointwise-multiplication operator $\Omega[\cdot] = \Omega \cdot$. In Lemma 5.7 we then show that these error terms all behave well enough, if we take the for us relevant case of $\Omega[\cdot]$ being of the form $A\mathcal{R}[\cdot]$.

5.1 Preliminary propositions

Here we recall some elementary statements, which enter into the proof of Theorem 1.6. Proposition 5.1 and Proposition 5.2 are simple duality arguments for linear, bounded mappings between Banach spaces. Proposition 5.4 is a quantified embedding from BMO into L^1 .

Proposition 5.1 *For any $s > 0$ there exists $\Lambda_0, C_s > 1$ such that the following holds: Let $f \in L^2(\mathbb{R}^m)$, $|\nabla|^s f \in L^2(\mathbb{R}^m)$ and assume $f \equiv 0$ on $\mathbb{R}^m \setminus B_r$ for some $B_r \subset \mathbb{R}^m$. Then for any $\Lambda \geq \Lambda_0$,*

$$\| |\nabla|^s f \|_{2, \mathbb{R}^m \setminus B_{\Lambda r}} \leq C_s \Lambda^{-\frac{m}{2}-s} \| |\nabla|^s f \|_{2, B_{\Lambda r}}.$$

Proof Using Corollary 5.16,

$$\| |\nabla|^s f \|_{2, \mathbb{R}^m \setminus B_{\Lambda r}} \lesssim \Lambda^{-\frac{m}{2}-s} \| |\nabla|^s f \|_{2, \mathbb{R}^m \setminus B_{\Lambda r}} + \Lambda^{-\frac{m}{2}-s} \| |\nabla|^s f \|_{2, B_{\Lambda r}}.$$

Thus, if $\Lambda > \Lambda_0$ for a Λ_0 depending only on s , we can absorb and conclude. □

Let us also recall the following observations which can be proven via duality and Riesz representation theorem

Proposition 5.2 *Let $A : L^2(\mathbb{R}^m) \rightarrow L^1(\mathbb{R}^m)$ be a linear, bounded operator. Then there exists $\bar{g} \in L^2(\mathbb{R}^m)$, $\|\bar{g}\|_{2,\mathbb{R}^m} = 1$ such that*

$$\sup_{\|\psi\|_{2,\mathbb{R}^m} \leq 1} \int A[\psi] = \int A[\bar{g}].$$

In particular (taking instead of A the operator $\tilde{A} := A[\chi_D \cdot]$, for any $D \subset \mathbb{R}^m$ there exists $\bar{g}_D \in L^2(D)$, $\|\bar{g}_D\|_{2,D} \leq 1$, $\text{supp } \bar{g} \subset \bar{D}$, such that

$$\sup_{\|\psi\|_{2,\mathbb{R}^m} \leq 1, \text{supp } \psi \subset \bar{D}} \int A[\psi] = \int A[\bar{g}_D].$$

Proposition 5.3 *Let $A : L^2(\mathbb{R}^m) \rightarrow L^1(\mathbb{R}^m)$ be a linear, bounded operator. Then there exists a linear, bounded operator $A^* : L^\infty(\mathbb{R}^m) \rightarrow L^2(\mathbb{R}^m)$ such that*

$$\int g A[f] = \int f A^*[g] \text{ for any } f \in L^2(\mathbb{R}^m), g \in L^\infty(\mathbb{R}^m).$$

Moreover, $\bar{g} = \|A(1)\|_2^{-1} A^(1)$ for the \bar{g} from Proposition 5.2.*

Finally, we have the following well-known fact:

Proposition 5.4 *Let $\varphi \in C_0^\infty(B_r)$, then*

$$\|\varphi\|_1 \leq C_m r^m [\varphi]_{BMO}.$$

5.2 Energy with potentials

Let $\Omega^{i,j} : L^2(\mathbb{R}^m) \rightarrow L^1(\mathbb{R}^m)$, $1 \leq i, j \leq N$ be a linear bounded Operator. And set

$$\Omega_{ij}^Q[f] := (|\nabla|^\mu(Q - I)_{ik}) Q_{kj}^T f + Q_{ik} \Omega_{kl}[Q_{ij}^T f],$$

for $\text{supp}(Q - I) \subset B_r$, $|\nabla|^\mu Q \in L^2(\mathbb{R}^{N \times N})$ and $Q \in SO(N)$ almost everywhere. For $\psi : \mathbb{R}^m \rightarrow \mathbb{R}^{N \times N}$, we write

$$\Omega^Q[\psi] := (|\nabla|^\mu(Q - I)_{ik}) Q_{kj}^T \psi_{ij} + Q_{ik} \Omega_{kl}[Q_{ij}^T \psi_{ij}],$$

Having in mind (5.4), we then define the energy

$$E(Q) \equiv E_{r,x,\Lambda,s,2}(Q) := \begin{cases} \sup_{\substack{\psi \in C_0^\infty(B_{\Lambda r}(x), \mathbb{R}^{N \times N}) \\ \|\psi\|_{2 \leq 1}} \int_{\mathbb{R}^m} (\Omega^Q)[\psi] & \text{if } \text{supp}(Q - I) \subset \overline{B_r(x)}, \\ \infty & \text{else.} \end{cases} \tag{5.5}$$

Obviously, $Q \equiv I$ is admissible and $E(I) < \infty$. Since $E() \geq 0$, there exists a minimizing sequence, and one can hope for a minimizer:

Lemma 5.5 *For any $\mu > 0$ there exists $\Lambda_0 > 1$ such that for any $\Lambda \geq \Lambda_0$, the following holds: There exists an admissible function P for E such that $E(P) \leq E(Q)$ for any other admissible function Q . Moreover,*

$$\| |\nabla|^\mu P \|_{2, B_{\Lambda r}(x)} + \Lambda^{\frac{m}{2} + \mu} \| |\nabla|^\mu P \|_{2, \mathbb{R}^m \setminus B_{\Lambda r}(x)} \leq C_\mu \|\Omega\|_{2 \rightarrow 1, B_{\Lambda r}(x)}. \tag{5.6}$$

Here,

$$\|\Omega\|_{2 \rightarrow 1, D} := \sup_{\psi \in C_0^\infty(D, \mathbb{R}^{N \times N}), \|\psi\|_2 \leq 1} \|\Omega[\psi]\|_1.$$

Proof Take Λ_0 from Proposition 5.1 and assume $\Lambda \geq \Lambda_0$. We have for any $\psi \in C_0^\infty(B_{\Lambda r}, \mathbb{R}^{N \times N}), \|\psi\|_2 \leq 1$

$$\begin{aligned} E(Q) &\geq \int (|\nabla|^\mu(Q - I) Q^T)_{ij} \psi_{ij} + \int Q \Omega[Q^T \psi] \\ &\geq \int (|\nabla|^\mu(Q - I) Q^T)_{ij} \psi_{ij} - \|\Omega\|_{2 \rightarrow 1, B_{\Lambda r}}, \end{aligned}$$

which (taking the supremum over such ψ) implies

$$\|\nabla|^\mu(Q - I)\|_{2, B_{\Lambda r}} \leq E(Q) + \|\Omega\|_{2 \rightarrow 1, B_{\Lambda r}}.$$

According to Proposition 5.1, this implies (as $Q \equiv I$ on $\mathbb{R}^n \setminus B_r$),

$$\|\nabla|^\mu(Q - I)\|_{2, \mathbb{R}^m} \leq C_\mu (E(Q) + \|\Omega\|_{2 \rightarrow 1, B_{\Lambda r}}).$$

Consequently, for a minimizing sequence P_k ,

$$\|\nabla|^\mu(P_k - I)\|_{2, \mathbb{R}^m} \leq C_\mu \|\Omega\|_{2 \rightarrow 1, B_{\Lambda r}},$$

and up to taking a subsequence, we may assume that there is an admissible function P such that $|\nabla|^\mu P_k$ converges L^2 -weakly to $|\nabla|^\mu P$ and P_k converges pointwise and strongly to P .

Then, for any fixed $\psi \in C_0^\infty(B_{\Lambda r}), \|\psi\|_{2, \mathbb{R}^{N \times N}} \leq 1$

$$E(P_k) \geq \int \Omega^P[\psi] + \int \Omega^{P_k}[\psi] - \Omega^P[\psi].$$

We claim that

$$\int \Omega^{P_k}[\psi] - \Omega^P[\psi] \xrightarrow{k \rightarrow \infty} 0, \tag{5.7}$$

which, once proven, implies that

$$\inf E(\cdot) \geq \int \Omega^P \psi,$$

which by the arbitrary choice of ψ implies that P is a minimizer. In order to show (5.7), note that

$$\begin{aligned} \Omega^{P_k}[\psi] - \Omega^P[\psi] &= |\nabla|^\mu P_k (P_k^T - P^T) \psi + |\nabla|^\mu (P_k - P) P^T \psi \\ &\quad + (P_k - P) \Omega [P_k^T \psi] + P \Omega [(P_k^T - P^T) \psi] \\ &=: I_k + II_k + III_k + IV_k. \end{aligned}$$

Since $|P_k|, |P| \leq 1$, all terms of the form $(P_k^T - P^T) \psi \xrightarrow{k \rightarrow \infty} 0$ in L^2 , by Lebesgue’s dominated convergence. Thus, $\int I_k + \int IV_k \xrightarrow{k \rightarrow \infty} 0$. By the weak L^2 -convergence of $|\nabla|^\mu P_k$, also $\int III_k \xrightarrow{k \rightarrow \infty} 0$. Since $P_k^T \psi \rightarrow P^T \psi$ in $L^2(\mathbb{R}^m)$, also $\Omega [P_k^T \psi] \xrightarrow{k \rightarrow \infty} \Omega [P^T \psi]$ in L^1 and in particular pointwise almost everywhere. Then also $\int III_k \xrightarrow{k \rightarrow \infty} 0$. \square

Lemma 5.6 *Let P be a minimizer of $E(\cdot)$ as in (5.5), and assume that*

$$\Omega_{ij}[\cdot] = -\Omega_{ji}[\cdot] \quad 1 \leq i, j \leq N. \tag{5.8}$$

Then for any $\varphi \in C_0^\infty(B_r(x))$,

$$\begin{aligned} - \int \Omega^P [|\nabla|^\mu \varphi] &= \frac{1}{2} \int H_\mu(P - I, P^T - I) |\nabla|^\mu \varphi \\ &\quad - \int so \left(PC(\varphi, \Omega) \left[P^T \overline{\Omega}^{P^T} \chi_{D_\Lambda} \right] \right) \\ &\quad + \int so(\overline{\Omega}^P \chi_{D_\Lambda} P H_\mu(\varphi, P^T - I)) \\ &\quad - \int so(\mathcal{C}(P, \Omega)[|\nabla|^\mu \varphi] P^T) \\ &\quad + \int \Omega^P [(1 - \chi_{D_\Lambda})|\nabla|^\mu \varphi]. \end{aligned}$$

Here, we denote for a matrix $A \in R^{N \times N}$, the antisymmetric part with $so(A) = 2^{-1}(A - A^T)$, and for a mapping $g : L^2 \rightarrow L^1$, we denote \bar{g} as in Proposition 5.2.

Proof We set $D = B_r(x)$ and $D_\Lambda = B_{\Lambda r}(x)$. Let $\varphi \in C_0^\infty(D)$, $\omega \in so(N)$. We distort the minimizer P of $E(\cdot)$ by

$$Q_\varepsilon = e^{\varepsilon\varphi\omega} P = P + \varepsilon\varphi \omega P + o(\varepsilon) \in H_1^{\frac{n}{2}}(D, SO(N)),$$

that is we know that

$$E(Q_\varepsilon) - E(P) \geq 0 \tag{5.9}$$

We compute

$$\begin{aligned} &|\nabla|^\mu(Q_\varepsilon - I) Q^T \\ &= |\nabla|^\mu(P - I) P^T + \varepsilon\varphi \left(\omega |\nabla|^\mu(P - I) P^T - |\nabla|^\mu(P - I) P^T \omega \right) \\ &\quad + \varepsilon |\nabla|^\mu \varphi \omega + \varepsilon \omega H_\mu(\varphi, P - I) P^T + o(\varepsilon), \end{aligned} \tag{5.10}$$

and

$$Q_\varepsilon \Omega \left[Q_\varepsilon^T \cdot \right] = P \Omega \left[P^T \cdot \right] + \varepsilon \left(\varphi \omega P \Omega \left[P^T \cdot \right] - P \Omega \left[P^T \omega \varphi \cdot \right] \right) + o(\varepsilon). \tag{5.11}$$

Together, we infer from (5.10) and (5.11) (denoting the Hilbert-Schmidt matrix product $A : B = A_{ij} B_{ij}$)

$$\begin{aligned} \Omega^{Q_\varepsilon}[\psi] &= \Omega^P[\psi] + \varepsilon \left(\varphi \omega \Omega^P[\psi] - \Omega^P[\omega \psi \varphi] \right) + \varepsilon |\nabla|^\mu \varphi \omega : \psi \\ &\quad + \varepsilon \omega H_\mu(\varphi, P - I) P^T : \psi + o(\varepsilon)[\psi]. \end{aligned}$$

Thus, for any $\varepsilon > 0$, $\psi \in C_0^\infty(D_\Lambda, \mathbb{R}^{N \times N})$, $\|\psi\|_2 \leq 1$,

$$\begin{aligned} \frac{1}{\varepsilon}(E(Q_\varepsilon) - E(P)) &\geq \frac{1}{\varepsilon} \left(\int \Omega^P[\psi] - E(P) \right) \\ &\quad + \int (\varphi \omega \Omega^P[\psi] - \Omega^P[\omega\psi\varphi]) \\ &\quad + \int |\nabla|^\mu \varphi \omega : \psi \\ &\quad + \int \omega H_\mu(\varphi, P - I) P^T : \psi \\ &\quad + o(1). \end{aligned}$$

Let $\bar{\psi} \in L^2(D_\Lambda)$ such that $E(P) = \int \Omega^P[\bar{\psi}]$ (cf. Proposition 5.2), this implies for the choice $\psi = \bar{\psi}$

$$\begin{aligned} 0 &\stackrel{(5.9)}{\geq} \frac{1}{\varepsilon}(E(Q_\varepsilon) - E(P)) \geq \int (\varphi \omega \Omega^P[\bar{\psi}] - \Omega^P[\omega\bar{\psi}\varphi]) \\ &\quad + \int |\nabla|^\mu \varphi \omega : \bar{\psi} \\ &\quad + \int \omega H_\mu(\varphi, P - I) P^T : \bar{\psi} \\ &\quad + o(1). \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we then have

$$\begin{aligned} - \int |\nabla|^\mu \varphi \omega : \bar{\psi} &\geq \int \varphi \omega \Omega^P[\bar{\psi}] - \Omega^P[\omega\bar{\psi}\varphi] \\ &\quad + \int \omega H_\mu(\varphi, P - I) P^T : \bar{\psi} \end{aligned}$$

which holds for any $\varphi \in C_0^\infty(B_r)$. Replacing φ by $-\varphi$, we arrive at

$$- \int |\nabla|^\mu \varphi \omega : \bar{\psi} = \int \varphi \omega \Omega^P[\bar{\psi}] - \Omega^P[\omega\bar{\psi}\varphi] + \int \omega H_\mu(\varphi, P - I) P^T : \bar{\psi}. \tag{5.12}$$

Now we need to be more specific about the characteristics of $\bar{\psi}$. We have

$$E(P) = \sup_\psi \int \Omega^P[\psi] = \sup_\psi \int |\nabla|^\mu P_{ik} P_{kj}^T \psi_{ij} + P_{ik} \Omega_{kl} [P_{lj}^T \psi_{ij}].$$

Let $\Omega_{kl}^* : L^\infty(\mathbb{R}^m) \rightarrow L^2(\mathbb{R}^m)$ be the linear bounded operator such that (cf. Proposition 5.3)

$$\int_{\mathbb{R}^m} g \Omega_{kl} [f] = \int_{\mathbb{R}^m} \Omega_{kl}^* [g] f, \quad \text{for any } f \in L^2(\mathbb{R}^m), g \in L^\infty(\mathbb{R}^m).$$

Set then,

$$\left((\Omega^P)^* \right)_{ij} [f] = |\nabla|^\mu P_{ik} P_{kj}^T f + \Omega_{kl}^* [f P_{ik}] P_{lj}^T \in L^2(\mathbb{R}^m),$$

and

$$\left(\overline{\Omega^P} \right)_{ij} = \left((\Omega^P)^* \right)_{ij} [1] \in L^2(\mathbb{R}^m).$$

Since

$$\int g \left(\Omega^P \right)_{ij} [f] = \int \left(\left(\Omega^P \right)^* \right)_{ij} [g] f \quad \text{for all } f \in L^2(\mathbb{R}^m), g \in L^\infty(\mathbb{R}^m),$$

we have

$$E(P) = \sup_{\psi} \int \overline{\Omega^P} : \psi \chi_{D_\Lambda} = c \int \overline{\Omega^P} : \overline{\Omega^P} \chi_{D_\Lambda} = c \int \Omega^P [\overline{\Omega^P} \chi_{D_\Lambda}],$$

for some normalizing constant c . That is,

$$(E(P))^2 = \int \left(\Omega^P \right)_{ij} [\chi_{D_\Lambda} \overline{\Omega^P}_{ij}],$$

and we can assume $\overline{\psi} = c \chi_{D_\Lambda} \overline{\Omega^P} = c \chi_{D_\Lambda} \overline{\Omega^P}$ for some normalizing constant c . Now,

$$\begin{aligned} - \int |\nabla|^\mu \varphi \omega : \overline{\psi} &= -c \int |\nabla|^\mu \varphi \omega_{ij} \chi_{D_\Lambda} \overline{\Omega^P}_{ij} \\ &= -\omega_{ij} \int \Omega^P_{ij} [|\nabla|^\mu \varphi] + \int \omega : \Omega^P [(1 - \chi_{D_\Lambda}) |\nabla|^\mu \varphi]. \end{aligned}$$

Consequently, (5.12) reads as

$$\begin{aligned} - \int \omega : \Omega^P [|\nabla|^\mu \varphi] &= \int \varphi \omega_{ik} \left(\Omega^P \right)_{kj} [(\overline{\Omega^P})_{ij} \chi_{D_\Lambda}] - \left(\Omega^P \right)_{ij} [\omega_{ik} (\overline{\Omega^P})_{kj}] \varphi \\ &\quad + \int_{D_\Lambda} \omega H_\mu(\varphi, P - I) P^T : \overline{\Omega^P} \\ &\quad + \int \omega : \Omega^P [(1 - \chi_{D_\Lambda}) |\nabla|^\mu \varphi]. \end{aligned}$$

Note that, since $\varphi \in C_0^\infty(\mathbb{R}^m) \subset L^\infty$,

$$\omega_{ik} \int \left(\Omega^P \right)_{ij} [(\overline{\Omega^P})_{kj} \varphi] = \omega_{ik} \int \left(\overline{\Omega^P} \right)_{ij} \left(\overline{\Omega^P} \right)_{kj} \varphi \stackrel{\omega \in \text{so}}{=} 0. \tag{5.13}$$

By the same argument,

$$\begin{aligned} &\omega_{ik} \int \varphi \left(\Omega^P \right)_{kj} [\chi_{D_\Lambda} (\overline{\Omega^P})_{ij}] \\ &= \omega_{ik} \int \varphi \left(\Omega^P \right)_{kj} [\chi_{D_\Lambda} (\overline{\Omega^P})_{ij}] - \omega_{ik} \int \left(\Omega^P \right)_{kj} [(\overline{\Omega^P})_{ij} \varphi] \end{aligned}$$

and

$$\begin{aligned} &\omega_{ik} \int \varphi \left(\Omega^P \right)_{kj} [(\overline{\Omega^P})_{ij} \chi_{D_\Lambda}] \\ &= \omega_{ik} \int \left(\left(\Omega^P \right)^* \right)_{kj} [\varphi] (\overline{\Omega^P})_{ij} \chi_{D_\Lambda} = \omega_{ik} \int \varphi \left(\left(\Omega^P \right)^* \right)_{kj} [1] (\overline{\Omega^P})_{ij} \chi_{D_\Lambda} \\ &\quad - \omega_{ik} \int \left(\varphi \left(\left(\Omega^P \right)^* \right)_{kj} [1] - \left(\left(\Omega^P \right)^* \right)_{kj} [\varphi] \right) (\overline{\Omega^P})_{ij} \chi_{D_\Lambda} \\ &\stackrel{\text{supp } \varphi}{=} \omega_{ik} \int \varphi \left(\left(\Omega^P \right)^* \right)_{kj} [1] (\overline{\Omega^P})_{ij} \end{aligned}$$

$$\begin{aligned}
 & -\omega_{ik} \int \left(\varphi \left((\Omega^P)^* \right)_{kj} [1] - \left((\Omega^P)^* \right)_{kj} [\varphi] \right) (\overline{\Omega^P})_{ij} \chi_{D_\Lambda} \\
 \stackrel{(5.13)}{=} & 0 - \omega_{ik} \int \left(\varphi \left((\Omega^P)^* \right)_{kj} [1] - \left((\Omega^P)^* \right)_{kj} [\varphi] \right) (\overline{\Omega^P})_{ij} \chi_{D_\Lambda} \\
 = & \omega_{ik} \int \mathcal{C}(\varphi, \Omega_{kj}^P) [(\overline{\Omega^P})_{ij} \chi_{D_\Lambda}],
 \end{aligned}$$

where we denote the commutator \mathcal{C}

$$\mathcal{C}(b, T)[f] = b T f - T (b f).$$

Thus, we arrive at

$$\begin{aligned}
 - \int \omega : so(\Omega^P [|\nabla|^\mu \varphi])_{ij} & = \omega_{ik} \int \mathcal{C} \left(\varphi, (\Omega^P)_{kj} \right) [(\overline{\Omega^P})_{ij} \chi_{D_\Lambda}] \\
 & + \int \omega H_\mu(\varphi, P - I) P^T : \overline{\Omega^P} \chi_{D_\Lambda} \\
 & + \int \omega : \Omega^P [(1 - \chi_{D_\Lambda}) |\nabla|^\mu \varphi].
 \end{aligned}$$

One checks, that

$$\mathcal{C} \left(\varphi, (\Omega^P)_{kj} \right) [(\overline{\Omega^P})_{ij} \chi_{D_\Lambda}] = P_{kl} \mathcal{C}(\varphi, \Omega^{ls}) [P_{sj}^T (\overline{\Omega^P})_{ij} \chi_{D_\Lambda}]$$

Next, [and here the antisymmetry of Ω , (5.8), plays its role]

$$\begin{aligned}
 so(\Omega^P [|\nabla|^\mu \varphi])_{ij} & = so(|\nabla|^\mu (P - I) P^T)_{ij} |\nabla|^\mu \varphi + \frac{1}{2} P_{ik} \Omega_{kl} [P_{jl} |\nabla|^\mu \varphi] \\
 & - \frac{1}{2} P_{jk} \Omega_{kl} [P_{il} |\nabla|^\mu \varphi] \\
 \stackrel{(5.8)}{=} & so(|\nabla|^\mu (P - I) P^T)_{ij} |\nabla|^\mu \varphi + \frac{1}{2} P_{ik} \Omega_{kl} [P_{jl} |\nabla|^\mu \varphi] \\
 & + \frac{1}{2} P_{jl} \Omega_{kl} [P_{ik} |\nabla|^\mu \varphi] \\
 = & so(|\nabla|^\mu (P - I) P^T)_{ij} |\nabla|^\mu \varphi + \frac{1}{2} P_{ik} \Omega_{kl} [P_{jl} |\nabla|^\mu \varphi] \\
 & + \frac{1}{2} P_{jl} P_{ik} \Omega_{kl} [|\nabla|^\mu \varphi] - \frac{1}{2} P_{jl} \mathcal{C}(P_{ik}, \Omega_{kl}) [|\nabla|^\mu \varphi] \\
 = & so(|\nabla|^\mu (P - I) P^T)_{ij} |\nabla|^\mu \varphi + P_{ik} \Omega_{kl} [P_{jl} |\nabla|^\mu \varphi] \\
 & + \frac{1}{2} P_{ik} \mathcal{C}(P_{jl}, \Omega_{kl}) [|\nabla|^\mu \varphi] - \frac{1}{2} P_{jl} \mathcal{C}(P_{ik}, \Omega_{kl}) [|\nabla|^\mu \varphi], \\
 & + \int \Omega^P [(1 - \chi_{D_\Lambda}) |\nabla|^\mu \varphi].
 \end{aligned}$$

and

$$\begin{aligned}
 so(|\nabla|^\mu (P - I) P^T) & = \frac{1}{2} |\nabla|^\mu (P - I) P^T - \frac{1}{2} P |\nabla|^\mu (P^T - I) \\
 & = |\nabla|^\mu (P - I) P^T + \frac{1}{2} \left(-|\nabla|^\mu (P - I) P^T - P |\nabla|^\mu (P^T - I) \right)
 \end{aligned}$$

$$\begin{aligned}
 &= |\nabla|^\mu(P - I) P^T + \frac{1}{2} \left(|\nabla|^\mu(P P^T) - |\nabla|^\mu(P - I) P^T - P |\nabla|^\mu(P^T - I) \right) \\
 &= |\nabla|^\mu(P - I) P^T + \frac{1}{2} H_\mu(P - I, P^T - I)
 \end{aligned}$$

This implies finally (going with $\omega_{ij} \in \{-1, 0, 1\}$ through all the possible matrices with two non-zero entries)

$$\begin{aligned}
 - \int \Omega^P [|\nabla|^\mu \varphi] &= \frac{1}{2} \int H_\mu(P - I, P^T - I) |\nabla|^\mu \varphi \\
 &\quad + \int so \left(PC(\varphi, \Omega) \left[P^T \overline{\Omega}^{P^T} \chi_{D_\Lambda} \right] \right) \\
 &\quad + \int so \left(\overline{\Omega}^P \chi_{D_\Lambda} P H_\mu(\varphi, P^T - I) \right) \\
 &\quad - \int so(\mathcal{C}(P, \Omega) [|\nabla|^\mu \varphi] P^T) \\
 &\quad + \int \Omega^P [(1 - \chi_{D_\Lambda}) |\nabla|^\mu \varphi].
 \end{aligned}$$

□

Then, using the commutator estimates in [4], (5.28), (5.29), and (5.30), we have shown the following Lemma, which implies Theorem 1.6

Lemma 5.7 *Let P be a minimizer of $E(\cdot)$ as in (5.5), Lemma 5.6. Assume moreover, that Ω satisfies (1.6). Then for any $\varphi \in C_0^\infty(B_r)$*

$$- \int \Omega^P [|\nabla|^\mu \varphi] \lesssim \Lambda^{-\frac{m}{2}-\mu} r^{\frac{m}{2}-\mu} \|A\|_2 [\varphi]_{BMO} + \|A\|_2^2 \begin{cases} [\varphi]_{BMO} & \text{if } \mu \in (0, 1], \\ \||\nabla|^\mu \varphi\|_{(2,\infty)} & \text{if } \mu > 1. \end{cases}$$

Proof By Lemma 5.5 and Lemma 5.6,

$$\|\Omega^P\|_{2 \rightarrow 1} + \|\overline{\Omega}^P\|_2 + \||\nabla|^\mu P\|_2 \lesssim \|\Omega[\]\|_{2 \rightarrow 1} \lesssim \|A\|_2,$$

and by Lemma 5.6 we need to estimate

$$\int H_\mu(P - I, P^T - I) |\nabla|^\mu \varphi \tag{5.14}$$

$$\left| \int so \left(PC(\varphi, \Omega) \left[P^T \overline{\Omega}^{P^T} \chi_{D_\Lambda} \right] \right) \right| \lesssim \|A\|_2 \| \mathcal{C}(\varphi, \mathcal{R}) \left[P^T \overline{\Omega}^{P^T} \chi_{D_\Lambda} \right] \|_2, \tag{5.15}$$

$$\left| \int so \left(\overline{\Omega}^P \chi_{D_\Lambda} P H_\mu(\varphi, P^T - I) \right) \right| \lesssim \|\overline{\Omega}^P\|_2 \|H_\mu(\varphi, P^T - I)\|_2, \tag{5.16}$$

$$\left| \int so(\mathcal{C}(P, \Omega) [|\nabla|^\mu \varphi] P^T) \right| \lesssim \|A\|_2 \| \mathcal{C}(P, \mathcal{R}) [|\nabla|^\mu \varphi] \|_2, \tag{5.17}$$

$$\left| \int \Omega^P [(1 - \chi_{D_\Lambda}) |\nabla|^\mu \varphi] \right| \lesssim \|\Omega^P\|_{2 \rightarrow 1} \|(1 - \chi_{D_\Lambda}) |\nabla|^\mu \varphi\|_2. \tag{5.18}$$

The estimate of (5.14) is immediate from (5.30), for the estimate of (5.15) we apply [4]. For the estimate of (5.16) we use (5.29), for (5.17) we have (5.28). It remains to estimate (5.18), which follows from

$$\begin{aligned} \|(1 - \chi_{D_\Lambda})|\nabla|^\mu\varphi\|_2 &\lesssim \sum_{k=1}^\infty \|\nabla|^\mu\varphi\|_{2, A_{\Lambda r}^k} \stackrel{l.B.1}{\lesssim} \sum_{k=1}^\infty (2^k \Lambda r)^{-\frac{m}{2}-\mu} \|\varphi\|_1 \\ &\stackrel{p.5.4}{\lesssim} \sum_{k=1}^\infty (2^k \Lambda r)^{-\frac{m}{2}-\mu} r^m [\varphi]_{BMO}. \end{aligned}$$

□

Acknowledgments The author has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no 267087, DAAD PostDoc Program (D/10/50763) and the Forschungsinstitut für Mathematik, ETH Zürich. He would like to thank Tristan Rivière and the FIM for their hospitality. Also we would like to thank the anonymous referee for their suggestions.

Appendix 1: Some facts on our fractional operators

The fractional laplacian $\Delta^{\frac{s}{2}}$ is usually defined via its Fourier-symbol $-|\xi|^s$. Here, we will mostly use the negative fractional laplacian $(-\Delta)^{\frac{s}{2}} \equiv |\nabla|^s$ (which here plays the role of the gradient, or the divergence and rotation in the classical settings), defined via its symbol $|\xi|^s$. These operators are defined for $s \in (-m, m)$, if $s < 0$, we write $\Delta^{\frac{s}{2}} \equiv I_{|s|}$. Most of the time, we will use the potential definition: For Schwartz functions f ,

$$\begin{aligned} |\nabla|^s f(x) &= c_s \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{f(x) - f(y)}{|x-y|^{m+s}} dy \\ &= c_s \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{|x-y|>\varepsilon} \frac{f(x+z) + f(x-z) - 2f(x)}{|z|^{m+s}} dy \quad \text{for } s \in (0, 2). \end{aligned}$$

For $s \in (2, m)$ one can easily extend this formula. For example, $|\nabla|^3 f = |\nabla|^1(-\Delta)f$. The inverse is the Riesz potential,

$$I_s f(x) = \tilde{c}_s \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{|x-y|^{m-s}} dy \quad \text{for } s \in (0, m).$$

We refer, e.g., to [17,21] on hyper-singular operators, generalizations, and different representation formulas. For interpolation (in particular fractional Sobolev spaces), Tartar’s monograph [34] might be very useful.

Next, we state some useful facts about the fractional laplacian, which we are going to use throughout our paper, as standard repertoire.

We have the standard Poincaré inequality, for a proof, we refer, e.g., to [23].

Lemma 5.8 [Poincaré inequality with compact support] *Let $s \in [0, m)$, $p \in (1, \infty)$, $q \in [1, \infty]$, then for any $B_r \subset \mathbb{R}^m$, and any $f \in C_0^\infty(B_r)$*

$$\|f\|_{(p_1, q_1)} \leq C_s r^s \|\nabla|^s f\|_{(p_1, q_1)}.$$

The (scaling invariant) Sobolev inequality takes the form

Lemma 5.9 [Sobolev inequality] *Let $s \in [0, m)$, $p_1, p_2 \in [1, \infty)$, $q \in [1, \infty]$, for any $f \in \mathcal{S}(\mathbb{R}^m)$,*

$$\|f\|_{(p_1, q)} \leq \|\nabla|^s f\|_{(p_2, q)},$$

where

$$\frac{1}{p_2} = \frac{1}{p_1} + \frac{s}{m}.$$

For $p_1 = \infty$, we have the following limiting version of Sobolev’s inequality:

Lemma 5.10 [Limiting Sobolev inequality] *Let $s \in (0, m)$. For any $f \in \mathcal{S}(\mathbb{R}^m)$,*

$$\|f\|_\infty \leq \| |\nabla|^s f \|_{(\frac{m}{s}, 1)},$$

Also, we have the following Hölder-like inequality

Lemma 5.11 [Hölder inequality] *Let $s \in [0, m)$, then for any $p_1 < p_2$, for any $B_r \subset \mathbb{R}^m$, and any $f \in C_0^\infty(B_r)$*

$$\| |\nabla|^s f \|_{(p_1, q_1)} \leq C_{s, p_1, p_2} r^{\frac{m}{p_1} - \frac{m}{p_2}} \| |\nabla|^s f \|_{(p_2, \infty)}$$

Proof Let $\Lambda > 2$, then

$$\| |\nabla|^s f \|_{(p_1, q_1), B_{\Lambda r}} \lesssim C_{s, p_1, p_2, \Lambda} r^{\frac{m}{p_1} - \frac{m}{p_2}} \| |\nabla|^s f \|_{(p_2, \infty)}.$$

On the other hand, for some $\theta > 0$, by Lemma 5.15, Lemma 5.8,

$$\| |\nabla|^s f \|_{(p_1, q_1), \mathbb{R}^n \setminus B_{\Lambda r}} \lesssim \Lambda^{-\theta} r^{-s} \|f\|_{(p_1, q_1)} \lesssim \Lambda^{-\theta} \| |\nabla|^s f \|_{(p_1, q_1)}.$$

For sufficiently large Λ we can absorb the latter term into the left-hand side, and obtain the claim. □

From the Lemmata before, we also have

Lemma 5.12 [Poincaré-Sobolev inequality with compact support] *Let $s \in (0, m)$, $p_1, q_1 \in (1, \infty)$, then we have $s \leq t$, for any $B_r \subset \mathbb{R}^m$, and any $f \in C_0^\infty(B_r)$*

$$\| |\nabla|^s f \|_{(p_1, q_1)} \leq C_{p_1, q_1, p_2, q_2, s, t} r^{\frac{m}{p_1} - \frac{m}{p_2} + s - t} \| |\nabla|^t f \|_{(p_2, q_2)},$$

where $p_2 \in (1, \infty)$ such that

$$\frac{1}{p_2} \leq \frac{1}{p_1} + \frac{s - t}{m}$$

and $q_2 = \infty$ if the above inequality is strict, else $q_1 = q_2$.

A very important ingredient in our arguments is the boundedness of the Riesz potential on Morrey spaces.

Lemma 5.13 [1] *Let $s \in [0, m)$, $p_1, p_2 \in (1, \infty)$, $q \in [1, \infty]$, $\lambda \in (0, m]$, such that*

$$\frac{1}{p_1} = \frac{1}{p_2} - \frac{s}{\lambda}.$$

Then for any $f \in \mathcal{S}(\mathbb{R}^m)$,

$$\|I_s f\|_{(p_1, q)_\lambda} \lesssim \|f\|_{(p_2, q)_\lambda}.$$

The following is an easy equivalence result, recall (1.17).

Proposition 5.14 *Let $\Lambda > 2, \sigma > 0$. Then,*

$$\sum_{k=K_0}^{\infty} 2^{-k\sigma} \|f\|_{(p,q),A_{2^k}^k} \leq C_{\sigma} \|f\|_{(p,q),B_{\Lambda r}} + \sum_{k=0}^{\infty} 2^{-k\sigma} \|f\|_{(p,q),A_{2^k}^k}$$

Proof Let $k_0 = \lfloor \log_2 \Lambda \rfloor \geq 1$, then

$$2^{k_0} \leq \Lambda < 2^{k_0+1}$$

We have

$$2^{-\sigma(l+k_0)} \|f\|_{(p,q),A_{2^{l+k_0}}^k} \lesssim 2^{-\sigma k_0} 2^{-\sigma(l-1)} \|f\|_{(p,q),A_{2^{l-1}\Lambda r}^k} + 2^{-\sigma k_0} 2^{\sigma l} \|f\|_{(p,q),A_{2^l\Lambda r}^k}.$$

□

Appendix 2: Quasi-locality

In this section we gather some facts which quantify the quasi-local behaviour of operators like fractional laplacians $|\nabla|^\alpha$, Riesz transforms \mathcal{R} , and Riesz potentials I_s . With ‘‘quasi-local’’ we mean the following: Let $A \subset \mathbb{R}^m$ be some domain and assume that $\text{supp } f \subset A$. If we take T to be any of the above mentioned operators, then there is no reason why $\text{supp } Tf \subset A$, nor $\text{supp } Tf \subset B_\delta A$ for some finite $\delta > 0$. Nevertheless, if we take a domain $B \subset \mathbb{R}^m$, $\text{dist}(A, B) > \epsilon > 0$, then $Tf \in C^\infty(B)$. In fact, in this case

$$Tf(x) = k * f(x) \quad \text{for } x \in B,$$

where k is a kernel of the form $k(y) = h(y/|y|) |y|^{-m-s}$ for some $s \in (-m, m)$, h some smooth function on \mathbb{S}^{m-1} . Since $\text{supp } f \subset A$ and $x \in B$, we can replace

$$Tf(x) = \tilde{k} * f(x),$$

where $\tilde{k}(y) = (1 - \eta(y))k(y)$, and $\eta \in C_0^\infty(B_\epsilon(0))$, $\eta \equiv 1$ on $B_{\epsilon/2}(0)$. Obviously, $\tilde{k} \in C^\infty(\mathbb{R}^m)$, and consequently so is Tf . In fact, by the usual Young-inequality, we have

$$\|Tf\|_{\infty, B} \leq \|\tilde{k}\|_{\infty} \|f\|_1 \leq \|k\|_{\infty, \mathbb{R}^m \setminus B_{\epsilon/2}(0)} \|f\|_1 \leq C_{\|h\|_{\infty}} \epsilon^{-m-s} \|f\|_1.$$

That is, although we cannot ensure that $Tf \equiv 0$ in B (as it would be, e.g., the case for local operators like ∇), we can at least quantify that the farther away B is from A , the smaller becomes the norm of Tf on B . In particular, we have

Lemma 5.15 [Quasi-locality (I)] *Let $p_1, p_2, q_1, q_2 \in [1, \infty]$, $s \in (-m, m)$ and $\Omega_1, \Omega_2 \subset \mathbb{R}^m$ be disjoint domains with $d = \text{dist}(\Omega_1, \Omega_2) > 0$ and with positive and finite Lebesgue measure. Then, for any $f \in \mathcal{S}(\mathbb{R}^m)$,*

$$\|\Delta^{\frac{s}{2}}(f \chi_{\Omega_2})\|_{(p_1, q_1), \Omega_1} \leq d^{-m-s} |\Omega_1|^{1/p_1} |\Omega_2|^{1-1/p_2} \|f\|_{(p_2, q_2), \Omega_2},$$

where we set

$$\Delta^{\frac{s}{2}} = \begin{cases} |\nabla|^s & \text{if } s > 0, \\ Id \text{ or } \mathcal{R} & \text{if } s = 0, \\ I_{|s|} & \text{if } s < 0. \end{cases}$$

Often we will use the above also for Ω_1 or Ω_2 to be a complement of some ball B_r . This is valid, since $\mathbb{R}^m \setminus B_r = \bigcup_{k=1}^\infty A_r^k$, recall (1.17). Then

$$\chi_{\mathbb{R}^m \setminus B_r} = \sum_{k=1}^\infty \chi_{A_r^k},$$

and for each A_r^k we have the correct estimate, so that for $s \in (-m, m)$ the sum on k is convergent. Consequently, as a special case, using also Poincaré inequality (cf. Sect. 1), we have

Corollary 5.16 [Quasi-locality (II)] *Let $p_1, p_2 \in (1, \infty)$, $q_1, q_2 \in [1, \infty]$, $s, t \in [0, m)$. Then, for any $B_r \subset \mathbb{R}^m$, $f \in \mathcal{S}(\mathbb{R})$, $\Lambda > 1$,*

$$\| |\nabla|^s (f \chi_{B_r}) \|_{(p_1, q_1), \mathbb{R}^m \setminus B_{\Lambda r}} \leq C_{s, p_1, p_2, q_1} \Lambda^{-m-s+\frac{m}{p_1} - \frac{m}{p_2} - s+t} \| |\nabla|^t (\chi_{B_r} f) \|_{(p_2, q_2), B_r}.$$

Lemma 5.17 [Quasilocality (III)] *Let $f, g \in \mathcal{S}(\mathbb{R}^m)$, $\Omega_1, \Omega_2 \subset \mathbb{R}^m$ be disjoint domains with $d = \text{dist}(\Omega_1, \Omega_2) > 0$ and with positive and finite Lebesgue measure.*

$$\begin{aligned} & \| |\nabla|^s ((\Delta^{\frac{t}{2}} f \chi_{\Omega_1}) g \chi_{\Omega_2}) \|_{(p_1, q_1)} \\ & \lesssim \sup_{\alpha \in [0, s]} d^{-m-t-\alpha} \| f \chi_{\Omega_1} \|_1 \| |\nabla|^{s-\alpha} (g \chi_{\Omega_2}) \|_{(p_1, q_1)} \end{aligned}$$

for any $t \in (-m, m)$, $s \in (0, m)$.

Appendix 3: Left-hand side estimates

Lemma 5.18 [Left-hand side estimates] *For a uniform constant C , and any $\kappa \in [\mu, 2\mu)$, $\mu \leq \frac{m}{2}$, $\Lambda \geq 4$,*

$$\begin{aligned} \| v \|_{(\frac{m}{m-\kappa}, \infty), B_{\Lambda^{-1}r}} & \leq C \sup_{\varphi \in C_0^\infty(B_r, \mathbb{R}^N)} \frac{1}{\| |\nabla|^\mu \varphi \|_{(\frac{m}{\kappa}, 1)}} \int v \cdot |\nabla|^\mu \varphi \\ & + C \Lambda^{\kappa-m} \| v \|_{(\frac{m}{m-\kappa}, \infty), B_r} \\ & + C \Lambda^{\kappa-m} \sum_{k=0}^\infty 2^{k(\kappa-m)} \| v \|_{(\frac{m}{m-\kappa}, \infty), B_{A_r^k}}. \end{aligned}$$

More generally, for $\tau \in (0, \mu]$,

$$\begin{aligned} \| |\nabla|^{\mu-\tau} v \|_{(\frac{m}{m+\mu-\tau-\kappa}, \infty), B_{\Lambda^{-1}r}} & \leq C \sup_{\varphi \in C_0^\infty(B_r, \mathbb{R}^N)} \frac{1}{\| |\nabla|^\tau \varphi \|_{(\frac{m}{\tau+\kappa-\mu}, 1)}} \int v \cdot |\nabla|^\mu \varphi \\ & + C \Lambda^{\kappa-m+\tau-\mu} \| v \|_{(\frac{m}{m-\kappa}, \infty), B_r} \\ & + C \Lambda^{\kappa-m+\tau-\mu} \sum_{k=1}^\infty 2^{k(\kappa-m+\tau-\mu)} \| v \|_{(\frac{m}{m-\kappa}, \infty), A_r^k}. \end{aligned}$$

Similar versions of this estimate have been appearing throughout the literature regarding fractional harmonic maps, we give a sketched argument for the convenience of the reader:

Proof Let $f \in C_0^\infty(B_{\Lambda^{-1}r}, \mathbb{R}^N)$, $\|f\|_{\left(\frac{m}{\tau+\kappa-\mu}, 1\right)} \leq 1$ such that

$$\| |\nabla|^{\mu-\tau} v \|_{\left(\frac{m}{m+\mu-\tau-\kappa}, \infty\right), B_{\Lambda^{-1}r}} \leq 2 \int |\nabla|^{\mu-\tau} v \cdot f.$$

Decompose for the usual cutoff $\eta_{r/2} \in C_0^\infty(B_{\frac{r}{2}})$, $\eta \equiv 1$ on $B_{\frac{r}{4}}$,

$$f = |\nabla|^\tau (\eta_{\frac{r}{2}} I_\tau f) + |\nabla|^\tau ((1 - \eta_{\frac{r}{2}}) I_\tau f) =: |\nabla|^\tau g_1 + |\nabla|^\tau g_2.$$

As usual, using Lemma 5.20 (for $\beta = 0$) as an approximate product rule, for finitely many $s_k \in [0, \tau]$, say $k = 1, \dots, L$ for some $L \in \mathbb{N}$,

$$\| |\nabla|^\tau g_1 \|_{\left(\frac{m}{\tau+\kappa-\mu}, 1\right)} \lesssim \sum_k \| I_{s_k} \left| |\nabla|^\tau \eta_{\frac{r}{2}} \right| \|_{\left(\frac{m}{\tau-s_k}, \infty\right)} \| I_{\tau-s_k} |f| \|_{\left(\frac{m}{s_k+\kappa-\mu}, 1\right)} \lesssim \|f\|_p.$$

As for g_2 , for a usual decomposition unity $\eta_l \in C_0^\infty(B_{2^l r} \setminus B_{2^{l-2} r})$, that is pointwise $\sum_{l=-2}^\infty \eta_l + \eta_{\frac{r}{2}} \equiv 1$,

$$|\nabla|^\tau g_2 = \sum_{l=-2}^\infty |\nabla|^\tau (\eta_l I_\tau f) =: \sum_{l=-2}^\infty |\nabla|^\tau \tilde{g}_l$$

and with the help of Lemma 5.17,

$$\| |\nabla|^\mu \tilde{g}_l \|_{\left(\frac{m}{\kappa}, 1\right)} \lesssim (2^l \Lambda)^{\kappa-m-\mu+\tau} \|f\|_{\frac{m}{\tau+\kappa-\mu}} \leq (2^l \Lambda)^{\kappa-m+\tau-\mu} \|f\|_{\frac{m}{\tau+\kappa-\mu}},$$

and for $k \geq 1$,

$$\| |\nabla|^\mu \tilde{g}_l \|_{\left(\frac{m}{\kappa}, 1\right), B_{2^k r} \setminus B_{2^{k-1} r}} \lesssim \Lambda^{\kappa-m+\tau-\mu} 2^{k\kappa+\max\{k,l\}(-m-\mu)+l\tau} \|f\|_{\frac{m}{\tau+\kappa-\mu}}.$$

Consequently, for any $k \in \mathbb{N}_0$,

$$\| |\nabla|^\mu g_2 \|_{\left(\frac{m}{\kappa}, 1\right), A_r^k} \lesssim (2^k \Lambda)^{\kappa-m+\tau-\mu} \|f\|_{\frac{m}{\tau+\kappa-\mu}}.$$

So we conclude using

$$\int v \cdot |\nabla|^\mu g_2 \lesssim \|v\|_{\left(\frac{m}{m-\kappa}, \infty\right), B_r} \| |\nabla|^\mu g_2 \|_{\left(\frac{m}{\kappa}, 1\right), B_r} + \sum_{k=1}^\infty \|v\|_{\left(\frac{m}{m-\kappa}, \infty\right), A_r^k} \| |\nabla|^\mu g_2 \|_{\left(\frac{m}{\kappa}, 1\right), A_r^k}.$$

□

Appendix 4: Iteration

The following is a version of the usual iteration lemma used to establish Dirichlet growth (cf., e.g., [11]). The proof is based on the arguments in [32, p. 11]. Similar arguments also appear in [7]. We leave the details of the proof to the reader, and refer to the presentation in [2].

Lemma 5.19 *Let $(a_l)_{l=-\infty}^\infty, (b_{l,k})_{l,k=-\infty}^\infty$ be positive sequences, such that*

$$b_{l,k} \leq C_1 a_{l+k} \tag{5.19}$$

$$\sup_{k,l \in \mathbb{Z}} b_{k,l} + \sup_{k \leq 0} a_k \leq C_2. \tag{5.20}$$

Assume that

$$a_{l-L} \leq \varepsilon a_l + \varepsilon \sum_{k=1}^{\infty} 2^{-\theta k} b_{l,k}. \tag{5.21}$$

If moreover for some $\tilde{\theta} \in (0, \theta)$,

$$\varepsilon 2^{L\tilde{\theta}} + C_1 \Sigma_{\theta-\tilde{\theta}} 2^{(L+1)\tilde{\theta}-\theta} \varepsilon \leq \frac{1}{4}, \tag{5.22}$$

where

$$\Sigma_{\theta} = \sum_{l=0}^{\infty} 2^{-\theta l}.$$

Then, there exists a constant $C > 0$ such that

$$a_l \leq C 2^{-\frac{\theta}{4}l} \text{ for all } l \leq 0.$$

Appendix 5: Commutators and fractional product rules: Theorem 1.4

In this section we state some commutator estimates and non-local expansion rules which were introduced in [24], motivated by the results in [7,25]. The proofs can be found in [26]. The for us most important commutators are

$$H_{\alpha}(a, b) = |\nabla|^{\alpha}(ab) - a|\nabla|^{\alpha}b - b|\nabla|^{\alpha}a,$$

and for a linear operator T

$$\mathcal{C}(a, T)[b] = aT[b] - T[ab].$$

The commutator $H_{\alpha}(a, b)$ was introduced by Da Lio and Rivière in [7], where Hardy-space \mathcal{H} and BMO -estimates where shown, making use of the Hardy–Littlewood decomposition and paraproducts. This is also somewhat related to the techniques of the T1-Theorem cf. [16]. If one is interested in L^2 -estimates only (e.g., in the sphere case) then there is an extremely elementary argument [25] somewhat inspired by Tartar’s proof of Wente’s inequality [33]. For general Lorentz space estimates there is also an argument using potential arguments, which even gives pointwise estimates, and was introduced in [24]. As it is a direct, pointwise argument not involving the Fourier transform, it is easier to apply in non-linear situations, cf. [2].

The commutator $\mathcal{C}(a, T)[b]$ and its Hardy-space/BMO estimates were introduced in [4] for the Riesz transform \mathcal{R} , and later generalized to the Riesz potential I_{α} in [3]. Again for pointwise estimates the arguments in [25] can be adapted.

Here, we are going to state in “Pointwise fractional product rules via potentials” section of Appendix 5 pointwise estimates on $H_{\alpha}(a, b)$, and in “Pointwise commutator estimates via potentials section” of Appendix 5 pointwise estimates on $\mathcal{C}(a, T)[b]$ which can be proved using and extending the techniques from [25]. For Hardy-space/BMO estimates, in “Fractional product rules in the Hardy-space via para-products: including the limit case” section of Appendix 5, the techniques in [7] have to be adapted.

Let us shortly recall the notion for Hardy space \mathcal{H} and BMO . The latter space BMO is defined as

$$g \in BMO \quad :\Leftrightarrow [g]_{BMO} = \sup_{B_r \subset \mathbb{R}^m} |B_r|^{-1} \int_{B_r} \left| g - |B_r|^{-1} \int_{B_r} g \right| < \infty.$$

Our interest in BMO stems from the fact, that it is a bigger space than L^∞ , and we have the nice embedding

$$[g]_{BMO} \lesssim \sup_{r>0} r^{\frac{p\tau-m}{p}} \|\lvert\nabla\rvert^\tau g\|_{(p,\infty),B_r} \quad \text{for } \tau > 0, p > 1, \tag{5.23}$$

whereas for L^∞ we only have the following embedding, which is more difficult to control,

$$\|g\|_\infty \lesssim \|\lvert\nabla\rvert^\tau g\|_{(\frac{m}{\tau},1)} \quad \text{for } \tau \in (0, m). \tag{5.24}$$

The Hardy space \mathcal{H} , on the other hand, is a slightly smaller space than L^1 , with the (for us) most important property that

$$\int f g \lesssim \|f\|_{\mathcal{H}} [g]_{BMO}. \tag{5.25}$$

That is, if we know that a quantity belongs to the Hardy space, it allows us to control the integral of (5.25) in terms of the right-hand side of (5.23), instead of having to deal with the terms on the right-hand side of (5.24).

The norm of the Hardy space \mathcal{H} is usually defined via

$$\|f\|_{\mathcal{H}} = \|\sup_{t>0} \phi_t * f\|_1,$$

where $\phi \in C_0^\infty(B_1)$, $\int \phi = 1$, and $\phi_t(x) = t^{-m}\phi(x/t)$, cf. [9,30], another very readable overview in the context with Partial Differential Equations is given in [27].

Pointwise fractional product rules via potentials

Lemma 5.20 *For any $\alpha \in (0, m)$ there is $L \in \mathbb{N}$ such that the following holds: For any $\beta \in [0, \min(\alpha, 1))$, $\beta \leq m - \alpha$, $\tau \in (\max\{\beta, \alpha + \beta - 1\}, \alpha]$, $\epsilon > 0$, there are, $s_k \in (0, \alpha)$, $t_k \in (0, \tau)$, where $\tau - \beta - s_k - t_k \in [0, \epsilon)$, such that the following holds*

$$\|\lvert\nabla\rvert^\beta H_\alpha(a, b)\| \lesssim \sum_{k=1}^L I_{\tau-\beta-s_k-t_k} (I_{s_k} \lvert\nabla\rvert^\alpha a \lvert I_{t_k} \lvert\nabla\rvert^\tau b).$$

Lemma 5.21 *Let $\alpha \in (0, m)$, $\epsilon > 0$ and assume that $\tau_1, \tau_2 \in (\max\{\alpha - 1, 0\}, \alpha]$, $\tau_1 + \tau_2 > \alpha$. Then for some $L \in \mathbb{N}$, there are $s_k \in (0, \tau_1)$, $t_k \in (0, \tau_2)$, $\tau_1 + \tau_2 - s_k - t_k - \alpha \in [0, \epsilon)$ such that*

$$\|H_\alpha(a, b)\| \lesssim \sum_{k=1}^L I_{\tau_1+\tau_2-s_k-t_k-\alpha} (I_{s_k} \lvert\nabla\rvert^{\tau_1} a \lvert I_{t_k} \lvert\nabla\rvert^{\tau_2} b). \tag{5.26}$$

Proposition 5.22 *Let $f, g \in \mathcal{S}(\mathbb{R}^m)$, Then*

$$\|\lvert\nabla\rvert^\beta H_\mu(f, g)\|_{(p_0,1),\mathbb{R}^m} \lesssim \|\lvert\nabla\rvert^\tau f\|_{\frac{m}{\kappa+\tau-\mu},2} \|\lvert\nabla\rvert^\mu g\|_2,$$

where τ is chosen as in Lemma 5.20

$$\frac{1}{p_0} = \frac{1}{2} + \frac{\kappa + \beta - \mu}{m}.$$

Proposition 5.23 *Let $f, g \in \mathcal{S}(\mathbb{R}^m)$, $\text{supp } f \subset \overline{B_r}$. Then for any $k \geq 2$,*

$$\|\lvert\nabla\rvert^\beta H_\mu(f, g)\|_{(p_0,1),A_r^k} \lesssim 2^{k(-\frac{m}{2}+\kappa-\mu)} \|\lvert\nabla\rvert^\tau f\|_{\frac{m}{\kappa+\tau-\mu}} \|\lvert\nabla\rvert^\mu g\|_2,$$

where

$$\frac{1}{p_0} = \frac{1}{2} + \frac{\kappa + \beta - \mu}{m}.$$

Pointwise commutator estimates via potentials

In this section, we discuss commutators of which special cases have been appearing in [3, 4]. There, usually estimates in the Hardy-space and BMO were proven. In contrast, in [26], we prove the following pointwise estimates adapting our arguments from [24].

Lemma 5.24 *Let $\beta + \delta < \min(\tau, 1)$, $\delta > 0$, $\epsilon > 0$. There exists a finite number L , and $s_k, \tilde{s}_k > 0$, $t_k, \tilde{t}_k \in (0, \tau)$, $\tilde{s}_k + \tilde{t}_k = s_k + t_k = \tau - \beta - \delta$, $\tilde{s}_k < \epsilon$,*

$$C(I_\tau A, |\nabla|^\beta)[|\nabla|^\delta B] \lesssim \sum_{k=1}^L I_{s_k} |A| I_{t_k} |B| + \sum_{k=1}^L I_{\tilde{s}_k} (I_{\tilde{t}_k} |A| |B|). \tag{5.27}$$

The following estimate should be compared to the estimates in [3], who extended arguments in [4] from Riesz transforms to Riesz Potentials. Their estimates treat cases in which one of the involved functions b belongs to BMO , which one usually uses in applications for estimates of that expression in terms of $|\nabla|^s b$. But if one knows that $|\nabla|^s b$ exists, then the following estimates are more precise than their BMO -counterparts in terms of Lorentz space estimates.

Lemma 5.25 *For any $\delta > 0$ such that $s + \delta < 1$ and any $\gamma \in (s, s + \delta)$, we have*

$$\begin{aligned} \left| |\nabla|^s C(a, I_s)[b] \right| \leq & C_{s, \delta, \gamma} I_{s+\delta-\gamma} |\nabla|^{s+\delta} a \big| I_{\gamma-s} I_s b \big| \\ & + C_{s, \delta, \gamma} \min \left\{ I_{\gamma-s} (I_s b | I_{s+\delta-\gamma} |\nabla|^{s+\delta} a |), I_{\gamma-s} (I_{s+\delta-\gamma} | I_s b | |\nabla|^{s+\delta} a |) \right\}. \end{aligned}$$

For $s = 0$, a (non-trivial) version of Lemma 5.25, is the following result, for any Riesz transform \mathcal{R} . Like Lemma 5.25 was related to Chanillo’s [3], this estimate is related to [4].

Lemma 5.26 *Then, for any $\delta \in (0, 1)$ and any $\gamma_i \in (0, \delta)$, $i = 1, 2$, we have*

$$|C(a, \mathcal{R})[b]| \leq C_{\mathcal{R}, \delta, \gamma_1} I_{\delta-\gamma_1} |\nabla|^\delta a \big| I_{\gamma_1} |b| + C_{\mathcal{R}, \delta, \gamma_2} I_{\gamma_2} (I_{\delta-\gamma_2} |b| |\nabla|^\delta a |).$$

Fractional product rules in the Hardy-space via para-products: including the limit case

In this section we introduce and state Hardy-space estimates on several commutators. In order to prove these, one has to extend techniques developed by Da Lio and Rivière in [7] in order to estimate their behavior involving the Hardy spaces \mathcal{H} . The details are given in [26]. Technically, for the case $\mu < 1$ one uses a straight-forward generalization of the arguments by Da Lio and Rivière. In the case $\mu = 1$, these arguments have to be extended.

$$\|C(\mathcal{F}, \mathcal{R})[|\nabla|^\mu \varphi]\|_2 \lesssim \| |\nabla|^\mu f \|_2 [\varphi]_{BMO}, \tag{5.28}$$

$$\|H_\mu(\varphi, g)\|_2 \lesssim \| |\nabla|^\mu g \|_2 [\varphi]_{BMO}, \tag{5.29}$$

$$\| |\nabla|^\mu H_\mu(a, b) \|_{\mathcal{H}} \lesssim \| |\nabla|^\mu a \|_2 \| |\nabla|^\mu b \|_2 \quad \text{for } \mu \in (0, 1]. \tag{5.30}$$

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