Calc. Var. (2015) 54:3251–3276
DOI 10.1007/s00526-015-0902-6

Calculus of Variations

Global smooth axisymmetric solutions of 3-D inhomogeneous incompressible Navier–Stokes system

Hammadi Abidi[1] · **Ping Zhang**[2]

Received: 19 November 2014 / Accepted: 20 July 2015 / Published online: 12 August 2015

Abstract In this paper, we investigate the global regularity to 3-D inhomogeneous incompressible Navier–Stokes system with axisymmetric initial data which does not have swirl component for the initial velocity. We first prove that the L^∞ norm to the quotient of the inhomogeneity by r, namely $a/r \stackrel{\text{def}}{=} (1/\rho - 1)/r$, controls the regularity of the solutions. Then we prove the global regularity of such solutions provided that the L^∞ norm of a_0/r is sufficiently small. Finally, with additional assumption that the initial velocity belongs to L^p for some $p \in [1, 2)$, we prove that the velocity field decays to zero with exactly the same rate as the classical Navier–Stokes system.

Mathematics Subject Classification 35Q30 · 76D03

1 Introduction

In this paper, we consider the global existence of smooth solutions to the following 3-D inhomogeneous incompressible Navier–Stokes equations with axisymmetric initial data which does not have swirl component for the initial velocity:

Communicated by F. H. Lin.

✉ Ping Zhang
zp@amss.ac.cn

Hammadi Abidi
habidi@univ-evry.fr

1 Département de Mathématiques, Faculté des Sciences de Tunis Campus Universitaire, 2092 Tunis, Tunisia

2 Academy of Mathematics & Systems Science and Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences, Beijing 100190, China

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \Delta u + \nabla \Pi = 0, \\ \mathrm{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases} \tag{1.1}$$

where $\rho, u = (u^1, u^2, u^z)$ stand for the density and velocity of the fluid respectively, and Π is a scalar pressure function. Such system describes a fluid that is incompressible but has non-constant density. Basic examples are mixture of incompressible and non reactant flows, flows with complex structure (e.g. blood flow or model of rivers), fluids containing a melted substance, etc.

A lot of recent works have been dedicated to the mathematical study of the above system. Global weak solutions with finite energy have been constructed by Simon in [22] (see also the book by Lions [18] for the variable viscosity case). In the case of smooth data with no vacuum, the existence of strong unique solutions goes back to the work of Ladyzhenskaya and Solonnikov in [16]. More precisely, they considered the system (1.1) in a bounded domain Ω with homogeneous Dirichlet boundary condition for u. Under the assumption that $u_0 \in W^{2-\frac{2}{p},p}(\Omega)$ $(p > d)$ is divergence free and vanishes on $\partial\Omega$ and that $\rho_0 \in C^1(\Omega)$ is bounded away from zero, then they [16] proved

- Global well-posedness in dimension $d = 2$;
- Local well-posedness in dimension $d = 3$. If in addition u_0 is small in $W^{2-\frac{2}{p},p}(\Omega)$, then global well-posedness holds true.

Lately, Danchin and Mucha [9] established the well-posedness of (1.1) in the whole space $\mathbb{R}^d$ in the so-called *critical functional framework* for small perturbations of some positive constant density. The basic idea are to use functional spaces (or norms) that is *scaling invariant* under the following transformation:

$$(\rho, u, \Pi)(t,x) \longmapsto (\rho, \lambda u, \lambda^2 \Pi)(\lambda^2 t, \lambda x), \qquad (\rho_0, u_0)(x) \longmapsto (\rho_0, \lambda u_0)(\lambda x). \tag{1.2}$$

One may check [5,10] and the references therein for the recent progresses along this line.

On the other hand, we recall that except the initial data have some special structure, it is still not known whether or not the System (1.1) has a unique global smooth solution with large smooth initial data, even for the classical Navier–Stokes system (NS), which corresponds to $\rho = 1$ in (1.1). For instance, Ukhovskii and Yudovich [23], and independently Ladyzhenskaya [15] proved the global existence of generalized solution along with its uniqueness and regularity for (NS) with initial data which is axisymmetric and without swirl. Leonardi et al. [17] gave a refined proof of the same result in [15,23]. The first author [1] improved the regularity of the initial data to be $u_0 \in H^{\frac{1}{2}}$. In general, the global wellposedness of (NS) with axisymmetric initial data is still open (see [7,24] for instance).

Let $x = (x_1, x_2, z) \in \mathbb{R}^3$, we denote the cylindrical coordinates of x by (r, θ, z), i. e., $r(x_1, x_2) \stackrel{\text{def}}{=} \sqrt{x_1^2 + x_2^2}$, $\theta(x_1, x_2) \stackrel{\text{def}}{=} \tan^{-1}\frac{x_2}{x_1}$ with $r \in [0, \infty)$, $\theta \in [0, 2\pi]$ and $z \in \mathbb{R}$, and

$$e_r \stackrel{\text{def}}{=} (\cos\theta, \sin\theta, 0), \quad e_\theta \stackrel{\text{def}}{=} (-\sin\theta, \cos\theta, 0), \quad e_z \stackrel{\text{def}}{=} (0,0,1).$$

We are concerned here with the global existence of axisymmetric smooth solutions to (1.1) which does not have the swirl component for the velocity field. This means solution of the form:

$$\rho(t, x_1, x_2, z) = \rho(t, r, z), \quad \Pi(t, x_1, x_2, z) = \Pi(t, r, z),$$
$$u(t, x_1, x_2, z) = u^r(t, r, z)e_r + u^z(t, r, z)e_z. \tag{1.3}$$

By virtue of (1.1) and (1.3), we find that (ρ, u, Π) verifies

$$\begin{cases} \partial_t \rho + u^r \partial_r \rho + u^z \partial_z \rho = 0, \\ \rho \partial_t u^r + \rho u^r \partial_r u^r + \rho u^z \partial_z u^r + \partial_r \Pi - \left(\frac{1}{r}\partial_r(r\partial_r u^r) + \partial_z^2 u^r - \frac{u^r}{r^2}\right) = 0, \\ \rho \partial_t u^z + \rho u^r \partial_r u^z + \rho u^z \partial_z u^z + \partial_z \Pi - \left(\frac{1}{r}\partial_r(r\partial_r u^z) + \partial_z^2 u^z\right) = 0, \\ \partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0, \\ \rho|_{t=0} = \rho_0 \quad \text{and} \quad (u^r, u^z)|_{t=0} = (u_0^r, u_0^z). \end{cases} \tag{1.4}$$

Equation of vorticity $\omega \overset{\text{def}}{=} \partial_z u^r - \partial_r u^z$: we get, by taking $\partial_z(1.4)_2 - \partial_r(1.4)_3$, that

$$\partial_t \omega + u^r \partial_r \omega + u^z \partial_z \omega - \frac{1}{r} u^r \omega + \partial_z \left(\frac{\partial_r \Pi}{\rho}\right) - \partial_r \left(\frac{\partial_z \Pi}{\rho}\right) - \partial_z \left(\frac{\partial_z \omega}{\rho}\right) - \partial_r \left(\frac{\partial_r \omega + \omega/r}{\rho}\right) = 0. \tag{1.5}$$

Equation of $\Gamma \overset{\text{def}}{=} \frac{\omega}{r}$: in view of (1.5), one has

$$\partial_t \Gamma + u^r \partial_r \Gamma + u^z \partial_z \Gamma + \frac{1}{r}\partial_z \left(\frac{\partial_r \Pi}{\rho}\right) - \frac{1}{r}\partial_r \left(\frac{\partial_z \Pi}{\rho}\right) - \partial_z \left(\frac{\partial_z \Gamma}{\rho}\right) - \frac{1}{r}\partial_r \left(\frac{r\partial_r \Gamma + 2\Gamma}{\rho}\right) = 0. \tag{1.6}$$

As for the classical Navier–Stokes system (NS) in [15,23], the quantity Γ will play a crucial role to prove the global well-posedness of (1.4). The main result of this paper states as follows:

Theorem 1.1 *Let* $a_0 \overset{def}{=} \frac{1}{\rho_0} - 1 \in L^2 \cap L^\infty$ *with* $\frac{a_0}{r} \in L^\infty$, *and there exist positive constants* m, M *so that*

$$0 < m \le \rho_0 \le M. \tag{1.7}$$

Let $u_0 = u_0^r e_r + u_0^z e_z \in H^1$ *be a solenoidal vector filed with* $\frac{u_0^r}{r}$ *and* $\Gamma_0 \overset{def}{=} \frac{\omega_0}{r}$ *belonging to* L^2. *Then*

(1) *there exists a positive time* T^* *so that* (1.4) *has a unique solution* (ρ, u) *on* $[0, T^*)$ *which satisfies for any* $T < T^*$

$$\rho \in L^\infty((0, T) \times \mathbb{R}^3), \quad u \in \mathcal{C}([0, T]; H^1(\mathbb{R}^3)) \quad \text{with} \quad \nabla u \in L^2((0, T); H^1(\mathbb{R}^3))$$
$$\sup_{t \in (0, T]} \left(t\langle t\rangle \left(\|u_t(t)\|_{L^2}^2 + \|u(t)\|_{\dot H^2}^2 + \|\nabla \Pi(t)\|_{L^2}^2\right) + \int_0^t t'\langle t'\rangle \|\nabla u_t(t')\|_{L^2}^2 \, dt' \right) < \infty. \tag{1.8}$$

If $T^* < \infty$, *there holds*

$$\lim_{t \to T^*} \left\| \frac{a(t)}{r} \right\|_{L^\infty} = \infty. \tag{1.9}$$

(2) *If we assume moreover that*

$$\left\| \frac{a_0}{r} \right\|_{L^\infty} \le \varepsilon_0 \tag{1.10}$$

for some sufficiently small positive constant ε_0, *we have* $T^* = \infty$, *and*

$$\begin{aligned}&\|u\|^2_{L^\infty(\mathbb{R}^+;H^1)} + \left\|\frac{u^r}{r}\right\|^2_{L^\infty(\mathbb{R}^+;L^2)} + \|\nabla u\|^2_{L^2(\mathbb{R}^+;H^1)} + \|\partial_t u\|^2_{L^2(\mathbb{R}^+;L^2)}\\ &\quad + \|\nabla\Pi\|^2_{L^2(\mathbb{R}^+;L^2)} \le C\mathcal{G}_0 + 1 \quad \textit{with}\\ &\mathcal{G}_0 \overset{def}{=} \exp\left(C\|u_0\|^2_{L^2}\left(1+\|u_0\|^6_{L^2}\right)\right)\left(\|u_0\|^2_{H^1} + \left\|\frac{u_0^r}{r}\right\|^2_{L^2} + 2\|\Gamma_0\|^2_{L^2}\right),\end{aligned} \tag{1.11}$$

and

$$\left\|\frac{a}{r}\right\|_{L^\infty(\mathbb{R}^+;L^\infty)} \le C\left\|\frac{a_0}{r}\right\|_{L^\infty}. \tag{1.12}$$

(3) *Besides* (1.10), *if* $u_0 \in L^p$ *for some* $p \in [1, 2)$, *let* $\beta(p) \overset{def}{=} \frac{3}{4}\left(\frac{2}{p} - 1\right)$, *one has*

$$\begin{aligned}&\|u(t)\|^2_{L^2} \le C\langle t\rangle^{-2\beta(p)}, \quad \|\nabla u(t)\|^2_{L^2} \le C\langle t\rangle^{-1-2\beta(p)},\\ &\|u_t(t)\|^2_{L^2} + \|u(t)\|^2_{\dot H^2} + \|\nabla\Pi(t)\|^2_{L^2} \le Ct^{-1}\langle t\rangle^{-1-2\beta(p)}.\end{aligned} \tag{1.13}$$

Remark 1.1 (1) Let us recall that the reason why one can prove the global well-posedness of classical 3-D Navier–Stokes system with axisymmetric data and without swirl is that $\Gamma \overset{\text{def}}{=} \frac{\omega}{r}$ satisfies

$$\partial_t\Gamma + u^r\partial_r\Gamma + u^z\partial_z\Gamma - \partial_r^2\Gamma - \partial_z^2\Gamma - \frac{3}{r}\partial_r\Gamma = 0,$$

which implies for all $p \in [1, \infty]$ that

$$\|\Gamma(t)\|_{L^p} \le \|\Gamma_0\|_{L^p}.$$

Nevertheless in the case of inhomogeneous Navier–Stokes system, Γ verifies (1.6). Then to get a global in time estimate for $\|\Gamma(t)\|_{L^2}$, we need the smallness condition (1.10). We remark that in order to prove the global regularity for the axisymmetric Navier–Stokes–Boussinesq system without swirl, the authors [3] require the support of the initial density ρ_0 does not intersect the axis (Oz) and the projection of suppρ_0 on the axis is a compact set, which seems stronger than (1.10) near the axis (Oz). Finally since we shall not use the vorticity equation (1.5), here we do not require the initial density to be close enough to some positive constant.

(2) We remark that the decay estimates (1.13) is in fact proved for general global smooth solutions of (1.1), which does not use the axisymmetric structure of the solutions, whenever $u_0 \in L^p$ for some $p \in [1, 2)$. In particular, we get rid of the technical assumption in [4] that (1.13) holds for $p \in (1, 6/5)$ and moreover the proof here is more concise than that in [4].

Let us complete this section with the notations we are going to use in this context.

Notations: $\dot H^s$ (resp. H^s) denotes the homogeneous (resp. inhomogeneous) Sobolev space with norm given by $\|f\|_{\dot H^s} \overset{\text{def}}{=} \left(\int_{\mathbb{R}^3} |\xi|^{2s}|\widehat{f}(\xi)|^2\,d\xi\right)^{\frac12}$ (resp. $\|f\|_{H^s} \overset{\text{def}}{=} \left(\int_{\mathbb{R}^3}(1+|\xi|^2)^s|\widehat{f}(\xi)|^2\,d\xi\right)^{\frac12}$). For X a Banach space and I an interval of $\mathbb{R}$, we denote by $\mathcal{C}(I;\ X)$ the set of continuous functions on I with values in X. For $q \in [1, +\infty]$, the notation $L^q(I;\ X)$ stands for the set of measurable functions on I with values in X, such that $t \longmapsto \|f(t)\|_X$ belongs to $L^q(I)$. Let $\mathbb{R}^2_+ = (0, \infty) \times \mathbb{R}$, we denote $\|f\|_{\widetilde{L}^q} \stackrel{\text{def}}{=} (\int_{\mathbb{R}^2_+} |f|^q\, dr\, dz)^{\frac{1}{q}}$. For $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$. We shall denote by $(a|b)$ (or $\int_{\mathbb{R}^3} a|b\, dx$) the $L^2(\mathbb{R}^3)$ inner product of a and b, and finally $\widetilde{\nabla} \stackrel{\text{def}}{=} (\partial_r, \partial_z)$.

2 The global H^1 estimate

In this section, we shall prove the a priori globally in time H^1 estimate for the velocity of (1.1) provided that there holds (1.10). Before proceeding, let us first rewrite the momentum equation of (1.4).

Due to $\partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0$ and $\operatorname{curl} u = \omega e_\theta$ with $\omega \stackrel{\text{def}}{=} \partial_z u^r - \partial_r u^z$, we have

$$\begin{aligned} \frac{1}{r}\partial_r(r\partial_r u^r) + \partial_z^2 u^r - \frac{u^r}{r^2} &= -\frac{1}{r}\partial_r(r\partial_z u^z + u^r) + \partial_z^2 u^r - \frac{u^r}{r^2} \\ &= -\frac{1}{r}\left(r\partial_z\partial_r u^z - \frac{u^r}{r}\right) + \partial_z^2 u^r - \frac{u^r}{r^2} \\ &= \partial_z(\partial_z u^r - \partial_r u^z) = \partial_z\omega. \end{aligned}$$

Similarly, one has

$$\begin{aligned} \frac{1}{r}\partial_r(r\partial_r u^z) + \partial_z^2 u^z &= \partial_r^2 u^z + \frac{\partial_r u^z}{r} - \partial_z\left(\partial_r u^r + \frac{u^r}{r}\right) \\ &= -\partial_r(\partial_z u^r - \partial_r u^z) - \frac{1}{r}(\partial_z u^r - \partial_r u^z) \\ &= -\partial_r\omega - \frac{1}{r}\omega. \end{aligned}$$

So that we can reformulate the momentum equation of (1.4) as

$$\begin{cases} \rho\partial_t u^r + \rho u^r\partial_r u^r + \rho u^z\partial_z u^r + \partial_r\Pi - \partial_z\omega = 0, \\ \rho\partial_t u^z + \rho u^r\partial_r u^z + \rho u^z\partial_z u^z + \partial_z\Pi + \partial_r\omega + \frac{1}{r}\omega = 0. \end{cases} \tag{2.1}$$

2.1 Local in time H^1 estimate

The purpose of this subsection is to present the estimate of $\|u\|_{L^\infty_T(H^1)}$ with T going to ∞ when ε_0 in (1.10) tending to zero.

• L^2 energy estimate

We first deduce from the transport equation of (1.4) and (1.7) that

$$m \leq \rho(t, r, z) \leq M. \tag{2.2}$$

While by first multiplying the u^r equation of (1.4) by u^r and then integrating the resulting equation over $\mathbb{R}^2_+$ with respect to the measure $r\, dr\, dz$, we write

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2_+} \rho(u^r)^2\, r dr\, dz - \int_{\mathbb{R}^2_+} \big(r\partial_t\rho + \partial_r(\rho u^r r) + \partial_z(\rho u^z r)\big)(u^r)^2\, dr\, dz \\ &\quad - \int_{\mathbb{R}^2_+} \Pi\partial_r(u^r r)\, dr\, dz + \int_{\mathbb{R}^2_+} \left((\partial_r u^r)^2 + (\partial_z u^r)^2 + \frac{(u^r)^2}{r^2}\right) r\, dr\, dz = 0. \end{aligned}$$

Whereas using the transport equation and $\partial_r(u^r r)+\partial_z(u^z r)=0$ of (1.4), we find

$$r\partial_t\rho+\partial_r(\rho u^r r)+\partial_z(\rho u^z r)=r\left(\partial_t\rho+u^r\partial_r\rho+u^z\partial_z u^r\right)+\rho\left(\partial_r(u^r r)+\partial_z(u^z r)\right)=0,$$

so that we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2_+}\rho(u^r)^2\,r dr\,dz+\int_{\mathbb{R}^2_+}\left((\partial_r u^r)^2+(\partial_z u^r)^2+\frac{(u^r)^2}{r^2}\right)r\,dr\,dz\\&=\int_{\mathbb{R}^2_+}\Pi\partial_r(u^r r)\,dr\,dz.\end{aligned}$$

Along the same line, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2_+}\rho(u^z)^2\,r dr\,dz+\int_{\mathbb{R}^2_+}\left((\partial_r u^z)^2+(\partial_z u^z)^2\right)r\,dr\,dz\\&=\int_{\mathbb{R}^2_+}\Pi\partial_z(u^z r)\,dr\,dz.\end{aligned}$$

Hence due to $\partial_r(ru^r)+\partial_z(ru^z)=0$, we achieve

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2_+}\rho\left((u^r)^2+(u^z)^2\right)r\,dr\,dz+\int_{\mathbb{R}^2_+}\left(|\widetilde{\nabla}u^r|^2+|\widetilde{\nabla}u^z|^2+\frac{(u^r)^2}{r^2}\right)r\,dr\,dz=0.$$

Integrating the above inequality over $[0,t]$ and using (2.2) gives rise to

$$\|u\|^2_{L^\infty_t(L^2)}+\|\widetilde{\nabla}u\|^2_{L^2_t(L^2)}+\left\|\frac{u^r}{r}\right\|^2_{L^2_t(L^2)}\leq C\|u_0\|^2_{L^2}. \tag{2.3}$$

- $\dot{H}^1$ energy estimate

By taking $L^2(\mathbb{R}^2_+, r\,dr\,dz)$ inner product of the u^r equation of (1.4) with $\partial_t u^r$ and using integration by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2_+}\left((\partial_r u^r)^2+(\partial_z u^r)^2+\frac{(u^r)^2}{r^2}\right)r\,dr\,dz+\int_{\mathbb{R}^2_+}\rho(\partial_t u^r)^2 r\,dr\,dz\\&=-\int_{\mathbb{R}^2_+}\rho\left(u^r\partial_r u^r+u^z\partial_z u^r\right)\partial_t u^r r\,dr\,dz+\int_{\mathbb{R}^2_+}\Pi\partial_r(\partial_t u^r r)\,dr\,dz.\end{aligned}$$

Similarly we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2_+}\left((\partial_r u^z)^2+(\partial_z u^z)^2\right)r\,dr\,dz+\int_{\mathbb{R}^2_+}\rho(\partial_t u^z)^2 r\,dr\,dz\\&=-\int_{\mathbb{R}^2_+}\rho\left(u^r\partial_r u^z+u^z\partial_z u^z\right)\partial_t u^z r\,dr\,dz+\int_{\mathbb{R}^2_+}\Pi\partial_z(\partial_t u^z r)\,dr\,dz,\end{aligned}$$

which together $\partial_r(ru^r)+\partial_z(ru^z)=0$ gives rise to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2_+}\left(|\widetilde{\nabla}u^r|^2+|\widetilde{\nabla}u^z|^2+\frac{(u^r)^2}{r^2}\right)r\,dr\,dz+\int_{\mathbb{R}^2_+}\rho\left((\partial_t u^r)^2+(\partial_t u^z)^2\right)r\,dr\,dz\\&=-\int_{\mathbb{R}^2_+}\rho\left(u^r\partial_r u^r+u^z\partial_z u^r\right)\partial_t u^r r\,dr\,dz-\int_{\mathbb{R}^2_+}\rho\left(u^r\partial_r u^z+u^z\partial_z u^z\right)\partial_t u^z r\,dr\,dz\\&\leq C\left(\|\sqrt{\rho}u^r\partial_r u^r\|^2_{L^2}+\|\sqrt{\rho}u^z\partial_z u^r\|^2_{L^2}+\|\sqrt{\rho}u^r\partial_r u^z\|^2_{L^2}+\|\sqrt{\rho}u^z\partial_z u^z)\|^2_{L^2}\right)\\&\quad+\frac{1}{2}\left(\|\sqrt{\rho}\partial_t u^r\|^2_{L^2}+\|\sqrt{\rho}\partial_t u^z\|^2_{L^2}\right),\end{aligned}$$

which along with (2.2) implies

$$\frac{d}{dt}\int_{\mathbb{R}^2_+}\left(|\widetilde{\nabla}u^r|^2+|\widetilde{\nabla}u^z|^2+\frac{(u^r)^2}{r^2}\right)r\,dr\,dz+\|\partial_t u^r\|_{L^2}^2+\|\partial_t u^z\|_{L^2}^2$$
$$\leq C\left(\|u^r\partial_r u^r\|_{L^2}^2+\|u^z\partial_z u^r\|_{L^2}^2+\|u^r\partial_r u^z\|_{L^2}^2+\|\sqrt{\rho}u^z\partial_z u^z\|_{L^2}^2\right). \qquad (2.4)$$

• <u>The second derivative estimate of the velocity</u>

By taking $L^2(\mathbb{R}^2_+; r\,dr\,dz)$ inner product of the u^r equation of (2.1) with $\partial_z\omega$ and using integration by parts, one has

$$\int_{\mathbb{R}^2_+}(\partial_z\omega)^2 r\,dr\,dz=-\int_{\mathbb{R}^2_+}\partial_z\partial_r\Pi\mid\omega r\,dr\,dz$$
$$-\int_{\mathbb{R}^2_+}\left(\rho\partial_t u^r+\rho u^r\partial_r u^r+\rho u^z\partial_z u^r\right)\mid\partial_z\omega r\,dr\,dz.$$

Similarly taking $L^2(\mathbb{R}^2_+; r\,dr\,dz)$ inner product of the u^z equation of (2.1) with $\partial_r(r\omega)r^{-1}$ leads to

$$\int_{\mathbb{R}^2_+}(\partial_r(r\omega))^2 r^{-1}\,dr\,dz=\int_{\mathbb{R}^2_+}\partial_z\partial_r\Pi\mid\omega r\,dr\,dz$$
$$-\int_{\mathbb{R}^2_+}\left(\rho\partial_t u^z+\rho u^r\partial_r u^z+\rho u^z\partial_z u^z\right)\mid\partial_r(\omega r)\,dr\,dz.$$

Yet notice that

$$\int_{\mathbb{R}^2_+}(\partial_r(r\omega))^2 r^{-1}\,dr\,dz=\int_{\mathbb{R}^2_+}\left(\frac{\omega^2}{r}+2\omega\partial_r\omega+(\partial_r\omega)^2 r\right)dr\,dz$$
$$=\int_{\mathbb{R}^2_+}\left(\frac{\omega^2}{r^2}+(\partial_r\omega)^2\right)r\,dr\,dz.$$

As a consequence, for Γ given by (1.6), we obtain

$$\int_{\mathbb{R}^2_+}\left((\partial_r\omega)^2+(\partial_z\omega)^2+\Gamma^2\right)r\,dr\,dz\leq C\left(\|u_t^r\|_{L^2}^2+\|u_t^z\|_{L^2}^2+\|u^r\partial_r u^r\|_{L^2}^2\right.$$
$$\left.+\|u^z\partial_z u^r\|_{L^2}^2+\|u^r\partial_r u^z\|_{L^2}^2+\|u^z\partial_z u^z)\|_{L^2}^2\right). \qquad (2.5)$$

Along the same line, we have

$$\|\widetilde{\nabla}\Pi\|_{L^2}^2\leq C\left(\|u_t^r\|_{L^2}^2+\|u_t^z\|_{L^2}^2+\|u^r\partial_r u^r\|_{L^2}^2\right.$$
$$\left.+\|u^z\partial_z u^r\|_{L^2}^2+\|u^r\partial_r u^z\|_{L^2}^2+\|u^z\partial_z u^z\|_{L^2}^2\right). \qquad (2.6)$$

• <u>The combined estimate</u>

Let $\delta>0$ be a small positive constant, which will be chosen hereafter. By summing up (2.4) with $\delta\times((2.5)+(2.6))$ leads to

$$\frac{d}{dt}\int_{\mathbb{R}^2_+}\left(|\widetilde{\nabla}u^r|^2+|\widetilde{\nabla}u^z|^2+\frac{(u^r)^2}{r^2}\right)r\,dr\,dz+(1-2C\delta)\left(\|\partial_t u^r\|_{L^2}^2+\|\partial_t u^z\|_{L^2}^2\right)$$
$$+\delta\left(\int_{\mathbb{R}^2_+}\left(|\widetilde{\nabla}\omega|^2+\Gamma^2\right)r\,dr\,dz+\|\widetilde{\nabla}\Pi\|_{L^2}^2\right)$$
$$\leq C\left(\|u^r\partial_r u^r\|_{L^2}^2+\|u^z\partial_z u^r\|_{L^2}^2+\|u^r\partial_r u^z\|_{L^2}^2+\|u^z\partial_z u^z\|_{L^2}^2\right).$$

Taking $\delta = \frac{1}{4C}$ in the above inequality yields

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2_+}\left(|\widetilde{\nabla} u^r|^2 + |\widetilde{\nabla} u^z|^2 + \frac{(u^r)^2}{r^2}\right) r\,dr\,dz + \|\partial_t u^r\|_{L^2}^2 + \|\partial_t u^z\|_{L^2}^2 \\
&\quad + \int_{\mathbb{R}^2_+}\left(|\widetilde{\nabla}\omega|^2 + \Gamma^2\right) r\,dr\,dz + \|\widetilde{\nabla}\Pi\|_{L^2}^2 \\
&\leq C\left(\|u^r\partial_r u^r\|_{L^2}^2 + \|u^z\partial_z u^r\|_{L^2}^2 + \|u^r\partial_r u^z\|_{L^2}^2 + \|u^z\partial_z u^z\|_{L^2}^2\right). \qquad (2.7)
\end{aligned}
$$

In order to cope with the right hand side terms in (2.7), we take cut-off functions $\varphi \in C_0^\infty[0,\infty)$ and $\psi \in C^\infty[0,\infty)$ with

$$
\varphi(r) = \begin{cases} 1 & r \in [0, 1/2], \\ 0 & r \in [1, \infty), \end{cases} \quad \text{and} \quad \psi(r) = \begin{cases} 1 & r \in [1/2, \infty), \\ 0 & r \in [0, 1/4), \end{cases} \qquad (2.8)
$$

and present the lemma as follows:

Lemma 2.1 *Let $f(r,z)$ be a smooth enough function which decays sufficiently fast at infinity. Then for $\varphi(r)$ given by (2.8), one has*

$$
\int_{\mathbb{R}^2_+} f^4\varphi(r) r^3\,dr\,dz \leq C\|f\|_{L^2}^2\left(\|f\|_{L^2} + \|\partial_r f\|_{L^2}\right)\|\partial_z f\|_{L^2}. \qquad (2.9)
$$

Proof It is easy to observe that

$$
\begin{aligned}
r^2 f^2\varphi(r) &\leq \int_0^\infty |\partial_r(r^2 f^2\varphi(r))|\,dr \\
&\leq C\int_0^\infty |f|(|f| + |\partial_r f|) r\,dr,
\end{aligned}
$$

and

$$
r f^2 \leq \int_{\mathbb{R}} |\partial_z f^2| r\,dz = 2\int_{\mathbb{R}} |f||\partial_z f| r\,dz,
$$

from which, we infer

$$
\begin{aligned}
\int_{\mathbb{R}^2_+} f^4\varphi(r) r^3\,dr\,dz &\leq C\int_{\mathbb{R}^2_+}\int_0^\infty |f|(|f| + |\partial_r f|) r\,dr \int_{\mathbb{R}} |f||\partial_z f| r\,dz\,dr\,dz \\
&\leq C\int_{\mathbb{R}^2_+} |f|(|f| + |\partial_r f|) r\,dr\,dz \int_{\mathbb{R}^2_+} |f||\partial_z f| r\,dr\,dz.
\end{aligned}
$$

Applying Hölder inequality gives rise to (2.9). □

Now let us turn to the estimate of the nonlinear terms in (2.7). We first get, by applying Hölder's inequality and the 2-D interpolation inequality,

$$
\|f\|_{L^4(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\nabla f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}, \qquad (2.10)
$$

that

$$
\begin{aligned}
\|u^r\partial_r u\|_{L^2}^2 &\leq \|\partial_r u\|_{\widetilde{L}^4}^2\|\sqrt{r}u^r\|_{\widetilde{L}^4}^2 \\
&\leq C\left(\int_{\mathbb{R}^3}\omega^4 r^{-1}\,dx\right)^{\frac{1}{2}}\|\sqrt{r}u^r\|_{\widetilde{L}^2}\|\widetilde{\nabla}(\sqrt{r}u^r)\|_{\widetilde{L}^2},
\end{aligned}
$$

where we used Biot–Sarvart's law

$$u(t,x)=\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(y-x)\wedge e_\theta\omega(t,y)}{|y-x|^3}\,dy$$

and the fact that r^{-1} is in A^p class (see [11] for instance) so that

$$\|\partial_r u\|_{\widetilde{L}^4}=\left(\int_{\mathbb{R}^3}|\partial_r u|^4 r^{-1}\,dx\right)^{\frac14}\le C\left(\int_{\mathbb{R}^3}\omega^4 r^{-1}\,dx\right)^{\frac14}.$$

Then by virtue of (2.9) and (2.10), we infer

$$\begin{aligned}\left(\int_{\mathbb{R}^3}\omega^4 r^{-1}\,dx\right)^{\frac14}&\le\left(\int_{\mathbb{R}^2_+}\Gamma^4 r^4\varphi(r)\,dr\,dz\right)^{\frac14}+\left(\int_{\mathbb{R}^2_+}\omega^4(1-\varphi(r))\,dr\,dz\right)^{\frac14}\\&\lesssim\left(\int_{\mathbb{R}^2_+}\Gamma^4 r^3\varphi(r)\,dr\,dz\right)^{\frac14}+\|\omega\psi\|_{\widetilde{L}^4}\\&\lesssim\|\Gamma\|_{L^2}^{\frac12}\left(\|\Gamma\|_{L^2}^{\frac12}+\|\widetilde{\nabla}\Gamma\|_{L^2}^{\frac12}\right)+\|\omega\psi\|_{\widetilde{L}^2}^{\frac12}\|\widetilde{\nabla}(\omega\psi)\|_{\widetilde{L}^2}^{\frac12}\\&\lesssim\|\Gamma\|_{L^2}^{\frac12}\left(\|\Gamma\|_{L^2}^{\frac12}+\|\widetilde{\nabla}\Gamma\|_{L^2}^{\frac12}\right)+\|\omega\|_{L^2}^{\frac12}\left(\|\omega\|_{L^2}^{\frac12}+\|\widetilde{\nabla}\omega\|_{L^2}^{\frac12}\right).\end{aligned}$$

Moreover, note that

$$\|\widetilde{\nabla}(\sqrt{r}u^r)\|_{\widetilde{L}^2}\le C\left(\|\widetilde{\nabla}u^r\|_{L^2}+\left\|\frac{u^r}{r}\right\|_{L^2}\right),$$

for any $\delta>0$, we write

$$\begin{aligned}\|u^r\partial_r u\|_{L^2}^2&\le C\left(\|\Gamma\|_{L^2}\left(\|\Gamma\|_{L^2}+\|\widetilde{\nabla}\Gamma\|_{L^2}\right)+\|\omega\|_{L^2}\left(\|\omega\|_{L^2}+\|\widetilde{\nabla}\omega\|_{L^2}\right)\right)\\&\quad\times\|u^r\|_{L^2}\left(\|\widetilde{\nabla}u^r\|_{L^2}+\left\|\frac{u^r}{r}\right\|_{L^2}\right)\\&\le C_\delta\|u^r\|_{L^2}^2\left(\|\widetilde{\nabla}u^r\|_{L^2}^2+\left\|\frac{u^r}{r}\right\|_{L^2}^2\right)\left(\|\omega\|_{L^2}^2+\|\Gamma\|_{L^2}^2\right)\\&\quad+\delta\left(\|\omega\|_{L^2}^2+\|\widetilde{\nabla}\omega\|_{L^2}^2+\|\Gamma\|_{L^2}^2+\|\widetilde{\nabla}\Gamma\|_{L^2}^2\right).\end{aligned}\tag{2.11}$$

To deal with $\|u^z\partial_z u\|_{L^2}$, we split $\int_{\mathbb{R}^2_+}(u^z\partial_z u)^2 r\,dr\,dz$ as

$$\int_{\mathbb{R}^2_+}(u^z\partial_z u)^2 r\,dr\,dz=\int_{\mathbb{R}^2_+}(u^z\partial_z u)^2\varphi(r)r\,dr\,dz+\int_{\mathbb{R}^2_+}(u^z\partial_z u)^2(1-\varphi(r))r\,dr\,dz.\tag{2.12}$$

By applying (2.10) and convexity inequality, we get for any $\delta>0$

$$\begin{aligned}&\int_{\mathbb{R}^2_+}(u^z\partial_z u)^2(1-\varphi(r))r\,dr\,dz\\&\quad\lesssim\int_{\mathbb{R}^2_+}(u^z\partial_z u\psi(r)r^{\frac12})^2\,dr\,dz\\&\quad\lesssim\left\|u^z\psi(r)r^{\frac14}\right\|_{\widetilde{L}^4}^2\left\|\partial_z u\psi(r)r^{\frac14}\right\|_{\widetilde{L}^4}^2\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left\|u^z\psi(r)r^{\frac14}\right\|_{\widetilde{L}^2}\left\|\widetilde{\nabla}(u^z\psi(r)r^{\frac14})\right\|_{\widetilde{L}^2}\left\|\partial_z u\psi r^{\frac14}\right\|_{\widetilde{L}^2}\left\|\widetilde{\nabla}(\partial_z u\psi r^{\frac14})\right\|_{\widetilde{L}^2}\\
&\le C_\delta\|u^z\|_{L^2}^2\left(\|u^z\|_{L^2}^2+\|\widetilde{\nabla}u^z\|_{L^2}^2\right)\|\partial_z u\|_{L^2}^2+\delta\left(\|\partial_z u\|_{L^2}^2+\|\widetilde{\nabla}\partial_z u\|_{L^2}^2\right). \qquad (2.13)
\end{aligned}
$$

Before proceeding, let us recall from (2.22) of [19] that

$$
\frac{u^r}{r}=\partial_z\Delta^{-1}\Gamma-2\frac{\partial_r}{r}\Delta^{-1}\partial_z\Delta^{-1}\Gamma, \qquad (2.14)
$$

and from (21) of [13] that

$$
\frac{\partial_r}{r}\Delta^{-1}W=\frac{x_2^2}{r^2}\mathcal{R}_{11}W+\frac{x_1^2}{r^2}\mathcal{R}_{22}W-2\frac{x_1x_2}{r^2}\mathcal{R}_{12}W \qquad (2.15)
$$

for every axisymmetric smooth function W, and where $\mathcal{R}_{ij}\overset{\text{def}}{=}\partial_i\partial_j\Delta^{-1}$.

By virtue of (2.9), we infer

$$
\begin{aligned}
\int_{\mathbb{R}^2_+}(u^z\partial_z u^r)^2\varphi(r)r\,dr\,dz&=\int_{\mathbb{R}^2_+}(u^z)^2r^{\frac32}\varphi^{\frac12}(r)\left(\partial_z\frac{u^r}{r}\right)^2r^{\frac32}\varphi^{\frac12}(r)\,dr\,dz\\
&\le\left(\int_{\mathbb{R}^2_+}(u^z)^4r^3\varphi(r)\,dr\,dz\right)^{\frac12}\left(\int_{\mathbb{R}^2_+}\left(\partial_z\frac{u^r}{r}\right)^4r^3\varphi(r)\,dr\,dz\right)^{\frac12}\\
&\lesssim\|u^z\|_{L^2}\left(\|u^z\|_{L^2}^{\frac12}+\|\partial_r u^z\|_{L^2}^{\frac12}\right)\|\partial_z u^z\|_{L^2}^{\frac12}\left\|\partial_z\frac{u^r}{r}\right\|_{L^2}\\
&\quad\times\left(\left\|\partial_z\frac{u^r}{r}\right\|_{L^2}^{\frac12}+\left\|\partial_z\partial_r\frac{u^r}{r}\right\|_{L^2}^{\frac12}\right)\left\|\partial_z^2\frac{u^r}{r}\right\|_{L^2}^{\frac12}.
\end{aligned}
$$

Yet it follows from (2.14) and (2.15) that

$$
\left\|\partial_z\frac{u^r}{r}\right\|_{L^2}\lesssim\|\Gamma\|_{L^2},\quad\left\|\partial_z^2\frac{u^r}{r}\right\|_{L^2}\lesssim\|\partial_z\Gamma\|_{L^2}\quad\text{and}\quad\left\|\partial_z\partial_r\frac{u^r}{r}\right\|_{L^2}\lesssim\|\widetilde{\nabla}\Gamma\|_{L^2}.
$$

Therefore, for any $\delta>0$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2_+}(u^z\partial_z u^r)^2\varphi(r)r\,dr\,dz&\le C\|u^z\|_{L^2}^2\left(1+\|u^z\|_{L^2}^4\right)\|\widetilde{\nabla}u^z\|_{L^2}^2\|\Gamma\|_{L^2}^2\\
&\quad+\delta\left(\|\Gamma\|_{L^2}^2+\|\widetilde{\nabla}\Gamma\|_{L^2}^2\right). \qquad (2.16)
\end{aligned}
$$

While since $\partial_r(ru^r)+\partial_z(ru^z)=0$, we have

$$
\int_{\mathbb{R}^2_+}(u^z\partial_z u^z)^2\varphi(r)r\,dr\,dz=\int_{\mathbb{R}^2_+}\left(u^z\left(\partial_r u^r+\frac{u^r}{r}\right)\right)^2\varphi(r)r\,dr\,dz. \qquad (2.17)
$$

Due to (2.14) and (2.15), we have

$$
\begin{aligned}
\int_{\mathbb{R}^2_+}\left(\frac{u^zu^r}{r}\right)^2\varphi(r)r\,dr\,dz&\le\left(\int_{\mathbb{R}^2_+}(u^z)^3\varphi^2(r)r\,dr\,dz\right)^{\frac23}\left\|\frac{u^r}{r}\right\|_{L^6}^2\\
&\le\left(\int_{\mathbb{R}^3}\frac{(u^z)^3}{r^{\frac32}}\,dx\right)^{\frac23}\left\|\partial_z\Delta^{-1}\Gamma\right\|_{L^6}^2\\
&\le C\|\nabla u^z\|_{L^2}^2\|\Gamma\|_{L^2}^2.
\end{aligned}
$$

where we used Sobolev–Hardy inequality from [6] that

$$\int_{\mathbb{R}^N} \frac{|u|^{q_*(s)}}{|x'|^s}\,dx \le C(s,q,N,k)\left(\int_{\mathbb{R}^N} |\nabla u|^q\,dx\right)^{\frac{N-s}{N-q}}, \tag{2.18}$$

where $x=(x',z)\in R^N = R^k\times\mathbb{R}^{N-k}$ with $2\le k\le N$, $1<q<N$, $0\le s\le q$ and $s<k$, $q_* \overset{\text{def}}{=} \frac{q(N-s)}{N-q}$, so that there holds

$$\left(\int_{\mathbb{R}^3} \frac{(u^z)^3}{r^{\frac32}}\,dx\right)^{\frac13} \le C\|\nabla u^z\|_{L^2}.$$

Whereas it follows from (2.14) that

$$\partial_r u^r = \partial_z\Delta^{-1}\Gamma + r\partial_z\partial_r\Delta^{-1}\Gamma - 2\partial_r^2\Delta^{-1}\partial_z\Delta^{-1}\Gamma.$$

Applying Hardy's inequality (2.18) once again yields

$$\begin{aligned}\int_{\mathbb{R}^2_+} (u^z)^2\left(\partial_z\Delta^{-1}\Gamma\right)^2\varphi(r)r\,dr\,dz &\le \left(\int_{\mathbb{R}^2_+} |u^z|^3\varphi^{\frac32}(r)r\,dr\,dz\right)^{\frac23}\|\partial_z\Delta^{-1}\Gamma\|_{L^6}^2\\ &\le \left(\int_{\mathbb{R}^3}\frac{|u^z|^3}{r^{\frac32}}\,dx\right)^{\frac23}\|\Gamma\|_{L^2}^2\\ &\lesssim \|\nabla u^z\|_{L^2}^2\|\Gamma\|_{L^2}^2.\end{aligned}$$

Similarly, by applying Lemma 2.2, one has

$$\begin{aligned}&\int_{\mathbb{R}^2_+} (u^z)^2\left(r\partial_z\partial_r\Delta^{-1}\Gamma\right)^2\varphi(r)r\,dr\,dz\\ &\le \left(\int_{\mathbb{R}^2_+}(u^z)^4\varphi(r)r^3\,dr\,dz\right)^{\frac12}\left(\int_{\mathbb{R}^2_+}|\partial_z\partial_r\Delta^{-1}\Gamma|^4\varphi(r)r^3\,dr\,dz\right)^{\frac12}\\ &\le C\|u^z\|_{L^2}\left(\|u^z\|_{L^2}^{\frac12}+\|\partial_r u^z\|_{L^2}^{\frac12}\right)\|\partial_z u^z\|_{L^2}^{\frac12}\|\Gamma\|_{L^2}\left(\|\Gamma\|_{L^2}^{\frac12}+\|\widetilde{\nabla}\Gamma\|_{L^2}^{\frac12}\right)\|\partial_z\Gamma\|_{L^2}^{\frac12}\\ &\le C_\delta\|u^z\|_{L^2}^2\left(1+\|u^z\|_{L^2}^4\right)\|\widetilde{\nabla}u^z\|_{L^2}^2\|\Gamma\|_{L^2}^2+\delta\left(\|\Gamma\|_{L^2}^2+\|\widetilde{\nabla}\Gamma\|_{L^2}^2\right).\end{aligned} \tag{2.19}$$

Let $W \overset{\text{def}}{=} \partial_z\Delta^{-1}\Gamma$. Then by virtue of (2.14), we find

$$\begin{aligned}\partial_r^2\Delta^{-1}W &= \partial_r\left(\frac{x_2^2}{r}\mathcal{R}_{11}W+\frac{x_1^2}{r}\mathcal{R}_{22}W-2\frac{x_1x_2}{r}\mathcal{R}_{12}W\right)\\ &= \sin^2\theta\mathcal{R}_{11}W+\cos^2\theta\mathcal{R}_{22}W-2\sin\theta\cos\theta\mathcal{R}_{12}W\\ &\quad + r\left(\sin^2\theta\partial_r\mathcal{R}_{11}W+\cos^2\theta\partial_r\mathcal{R}_{22}W-2\sin\theta\cos\theta\partial_r\mathcal{R}_{12}W\right).\end{aligned}$$

It is easy to observe that

$$\begin{aligned}&\int_{\mathbb{R}^2_+}(u^z)^2\left(\sin^2\theta\mathcal{R}_{11}W+\cos^2\theta\mathcal{R}_{22}W-2\sin\theta\cos\theta\mathcal{R}_{12}W\right)^2\varphi(r)r\,dr\,dz\\ &\lesssim \left(\int_{\mathbb{R}^3}|u^z|^3r^{-\frac32}\,dx\right)^{\frac23}\|\partial_z\Delta^{-1}\Gamma\|_{L^6}^2 \lesssim \|\nabla u^z\|_{L^2}^2\|\Gamma\|_{L^2}^2,\end{aligned}$$

and it follows from a similar derivation of (2.19) that

$$\begin{aligned}
&\int_{\mathbb{R}^2_+} (u^z)^2 \left(\sin^2\theta\partial_r\mathcal{R}_{11}W + \cos^2\theta\partial_r\mathcal{R}_{22}W - 2\sin\theta\cos\theta\partial_r\mathcal{R}_{12}W\right)^2 \varphi(r)r^3\,dr\,dz\\
&\leq \left(\int_{\mathbb{R}^2_+} |u^z|^4\varphi(r)r^3\,dr\,dz\right)^{\frac12}\\
&\quad\times\left(\int_{R^2_+} \left(\sin^2\theta\partial_r\mathcal{R}_{11}W + \cos^2\theta\partial_r\mathcal{R}_{22}W - 2\sin\theta\cos\theta\partial_r\mathcal{R}_{12}W\right)^4 \varphi(r)r^3\,dr\,dz\right)^{\frac12}\\
&\leq C_\delta\|u^z\|^2_{L^2}\left(1+\|u^z\|^4_{L^2}\right)\|\widetilde{\nabla}u^z\|^2_{L^2}\|\Gamma\|^2_{L^2} + \delta\left(\|\Gamma\|^2_{L^2} + \|\widetilde{\nabla}\Gamma\|^2_{L^2}\right).
\end{aligned}$$

By resuming the above estimates into (2.17), we obtain

$$\int_{\mathbb{R}^2_+} (u^z\partial_z u^z)^2\varphi(r)r\,dr\,dz \leq C_\delta\left(1+\|u^z\|^6_{L^2}\right)\|\widetilde{\nabla}u^z\|^2_{L^2}\|\Gamma\|^2_{L^2} + \delta\left(\|\Gamma\|^2_{L^2} + \|\widetilde{\nabla}\Gamma\|^2_{L^2}\right). \tag{2.20}$$

Therefore, by substituting the Estimates (2.13), (2.16) and (2.20) into (2.12), we obtain

$$\begin{aligned}
\|u^z\partial_z u\|^2_{L^2} &\leq C_\delta\Big(\left(1+\|u^z\|^6_{L^2}\right)\|\widetilde{\nabla}u^z\|^2_{L^2}\left(\|\widetilde{\nabla}u\|^2_{L^2} + \|\Gamma\|^2_{L^2}\right)\\
&\qquad + (1+\|u^z\|^4_{L^2})\|\partial_z u\|^2_{L^2}\Big) + \delta\left(\|\Gamma\|^2_{L^2} + \|\widetilde{\nabla}\partial_z u\|^2_{L^2} + \|\widetilde{\nabla}\Gamma\|^2_{L^2}\right).
\end{aligned}\tag{2.21}$$

Note that for the axisymmetric flow, we have for $1 < q < \infty$

$$\begin{aligned}
&\text{(i)} \quad \|\omega\|_{L^q} \approx \|\nabla u\|_{L^q} \quad \text{and}\\
&\text{(ii)} \quad \|\nabla\omega\|_{L^q} + \left\|\frac{\omega}{r}\right\|_{L^q} \approx \|\nabla^2 u\|_{L^q}.
\end{aligned}\tag{2.22}$$

Thanks to (2.22), by resuming the Estimates (2.11) and (2.21) into (2.7) and taking δ to be sufficiently small, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|\widetilde{\nabla}u(t)\|^2_{L^2} + \left\|\frac{u^r(t)}{r}\right\|_{L^2}\right) + \|\partial_t u\|^2_{L^2} + \|u\|^2_{\dot{H}^2} + \|\Gamma\|^2_{L^2} + \|\widetilde{\nabla}\Pi\|^2_{L^2}\\
&\leq C_\delta\Bigg((1+\|u\|^6_{L^2})\left(\|\widetilde{\nabla}u\|^2_{L^2} + \left\|\frac{u^r}{r}\right\|^2_{L^2}\right)\left(\|\widetilde{\nabla}u\|^2_{L^2} + \|\Gamma\|^2_{L^2}\right)\\
&\quad + (1+\|u^z\|^4_{L^2})\|\widetilde{\nabla}u\|^2_{L^2}\Bigg) + \delta\|\widetilde{\nabla}\Gamma\|^2_{L^2}.
\end{aligned}\tag{2.23}$$

By applying Gronwall's inequality to (2.23), we write

$$\begin{aligned}
&\|\nabla u\|^2_{L^\infty_t(L^2)} + \left\|\frac{u^r}{r}\right\|^2_{L^\infty_t(L^2)} + \|\partial_t u\|^2_{L^2_t(L^2)} + \|u\|^2_{L^2_t(\dot{H}^2)} + \|\Gamma\|^2_{L^2_t(L^2)} + \|\nabla\Pi\|^2_{L^2_t(L^2)}\\
&\leq C\exp\left(C\left(1+\|u\|^6_{L^\infty_t(L^2)}\right)\left(\|\nabla u\|^2_{L^2_t(L^2)} + \left\|\frac{u^r}{r}\right\|^2_{L^2_t(L^2)}\right)\right)\\
&\quad\times\left(\|\nabla u_0\|^2_{L^2} + \left\|\frac{u^r_0}{r}\right\|^2_{L^2} + \left(1+\|u^z\|^4_{L^\infty_t(L^2)}\right)\|\widetilde{\nabla}u\|^2_{L^2_t(L^2)} + \|\Gamma\|^2_{L^\infty_t(L^2)} + \|\widetilde{\nabla}\Gamma\|^2_{L^2_t(L^2)}\right),
\end{aligned}$$

from which and (2.3), we infer

$$\|\nabla u\|_{L_t^\infty(L^2)}^2+\left\|\frac{u^r}{r}\right\|_{L_t^\infty(L^2)}^2+\|\partial_t u\|_{L_t^2(L^2)}^2+\|u\|_{L_t^2(\dot H^2)}^2+\|\Gamma\|_{L_t^2(L^2)}^2+\|\nabla\Pi\|_{L_t^2(L^2)}^2$$
$$\leq C\exp\left(C\|u_0\|_{L^2}^2\left(1+\|u_0\|_{L^2}^6\right)\right)\left(\|u_0\|_{H^1}^2+\left\|\frac{u_0^r}{r}\right\|_{L^2}^2+\|\Gamma\|_{L_t^\infty(L^2)}^2+\|\nabla\Gamma\|_{L_t^2(L^2)}^2\right). \tag{2.24}$$

• <u>The estimate of Γ</u>

Let $a\overset{\text{def}}{=}1/\rho-1$. Then we get, by taking L^2 inner product of (1.6) with Γ and using integrating by parts, that

$$\begin{aligned}&\frac12\frac{d}{dt}\|\Gamma(t)\|_{L^2}^2+\int_{\mathbb{R}_+^2}\frac1\rho|\widetilde\nabla\Gamma|^2r\,dr\,dz-2\int_{\mathbb{R}_+^2}\partial_r\left(\frac{\Gamma}{\rho}\right)\Gamma\,dr\,dz\\&=\int_{\mathbb{R}_+^2}a\left(\partial_r\Pi\partial_z\Gamma-\partial_z\Pi\partial_r\Gamma\right)\,dr\,dz\\&\leq\left\|\frac ar\right\|_{L^\infty}\|\widetilde\nabla\Pi\|_{L^2}\|\widetilde\nabla\Gamma\|_{L^2}.\end{aligned}$$

Note that $a(t,0,z)=0$, by using integration by parts, one has

$$\begin{aligned}-2\int_{\mathbb{R}_+^2}\partial_r\left(\frac{\Gamma}{\rho}\right)\Gamma\,dr\,dz&=-2\int_{\mathbb{R}_+^2}\partial_r\Gamma\Gamma\,dr\,dz-2\int_{\mathbb{R}_+^2}\partial_r(a\Gamma)\Gamma\,dr\,dz\\&=\int_{\mathbb{R}}\Gamma^2(t,0,z)\,dz+2\int_{\mathbb{R}_+^2}a\Gamma\partial_r\Gamma\,dr\,dz\\&\geq-C\left\|\frac ar\right\|_{L^\infty}^2\|\Gamma\|_{L^2}^2-\frac14\left\|\frac{\partial_r\Gamma}{\sqrt\rho}\right\|_{L^2}^2.\end{aligned}$$

Therefore due to (2.2), we infer

$$\frac{d}{dt}\|\Gamma(t)\|_{L^2}^2+\frac1m\|\widetilde\nabla\Gamma\|_{L^2}^2\leq C\left\|\frac ar\right\|_{L^\infty}^2\left(\|\widetilde\nabla\Pi\|_{L^2}^2+\|\Gamma\|_{L^2}^2\right). \tag{2.25}$$

On the other hand, it follows from the transport equation of (1.4) that

$$\begin{aligned}&\partial_t a+u^r\partial_r a+u^z\partial_z a=0\quad\text{and}\\&\partial_t\frac ar+u^r\partial_r\frac ar+u^z\partial_z\frac ar+\frac{u^r}{r}\frac ar=0,\end{aligned}$$

which yields

$$\left\|\frac ar(t)\right\|_{L^\infty}\leq\left\|\frac{a_0}{r}\right\|_{L^\infty}\exp\left(\left\|\frac{u^r}{r}\right\|_{L_t^1(L^\infty)}\right). \tag{2.26}$$

While note from [2,8] that

$$\left\|\frac{u^r}{r}\right\|_{L_t^1(L^\infty)}\lesssim\|\Gamma\|_{L_t^1(L^{3,1})}\lesssim t^{\frac34}\|\Gamma\|_{L_t^\infty(L^2)}^{\frac12}\|\nabla\Gamma\|_{L_t^2(L^2)}^{\frac12}.$$

So that by integrating (2.25) over $[0, t]$, we obtain

$$\|\Gamma\|^2_{L^\infty_t(L^2)} + \|\nabla\Gamma\|^2_{L^2_t(L^2)} \leq \|\Gamma_0\|^2_{L^2} + C\left\|\frac{a_0}{r}\right\|^2_{L^\infty} \exp\left(Ct^{\frac{3}{4}}\|\Gamma\|^{\frac{1}{2}}_{L^\infty_t(L^2)}\|\nabla\Gamma\|^{\frac{1}{2}}_{L^2_t(L^2)}\right)\left(\|\nabla\Pi\|^2_{L^2_t(L^2)} + \|\Gamma\|^2_{L^2_t(L^2)}\right).$$

Resuming the Estimate (2.24) into the above inequality leads to

$$\|\Gamma\|^2_{L^\infty_t(L^2)} + \|\nabla\Gamma\|^2_{L^2_t(L^2)} \leq \|\Gamma_0\|^2_{L^2} + C\left\|\frac{a_0}{r}\right\|^2_{L^\infty} \exp\left(Ct^{\frac{3}{4}}\|\Gamma\|^{\frac{1}{2}}_{L^\infty_t(L^2)}\|\nabla\Gamma\|^{\frac{1}{2}}_{L^2_t(L^2)}\right) \times \exp\left(C\|u_0\|^2_{L^2}\left(1+\|u_0\|^6_{L^2}\right)\right)\left(\|u_0\|^2_{H^1} + \left\|\frac{u_0^r}{r}\right\|^2_{L^2} + \|\Gamma\|^2_{L^\infty_t(L^2)} + \|\nabla\Gamma\|^2_{L^2_t(L^2)}\right). \tag{2.27}$$

Proposition 2.1 *Let* $(\rho, u, \nabla\Pi)$ *be a smooth enough solution of* (1.4) *on* $[0, T^*)$, *which satisfies* (2.2). *Let* $\mathcal{G}_0$ *be given by* (1.11) *and*

$$t_1 \stackrel{def}{=} \left(\frac{1}{2C\|\Gamma_0\|_{L^2}} \ln\left(\frac{\|\Gamma_0\|^2_{L^2}}{2C\,\|\frac{a_0}{r}\|^2_{L^\infty}\,\mathcal{G}_0}\right)\right)^{\frac{4}{3}}. \tag{2.28}$$

Then under the assumption of (1.10), *one has* $T^* \geq t_1$ *and there holds*

$$\|\Gamma\|^2_{L^\infty_{t_1}(L^2)} + \|\nabla\Gamma\|^2_{L^2_{t_1}(L^2)} \leq 2\|\Gamma_0\|^2_{L^2}, \tag{2.29}$$

$$\|\nabla u\|^2_{L^\infty_{t_1}(L^2)} + \left\|\frac{u^r}{r}\right\|^2_{L^\infty_{t_1}(L^2)} + \|\partial_t u\|^2_{L^2_{t_1}(L^2)} + \|u\|^2_{L^2_{t_1}(\dot{H}^2)} + \|\nabla\Pi\|^2_{L^2_t(L^2)} \leq C\mathcal{G}_0. \tag{2.30}$$

Proof Indeed if $\|\frac{a_0}{r}\|_{L^\infty}$ is sufficiently small, we deduce from (2.27) and (2.28) that

$$\|\Gamma\|^2_{L^\infty_{t_1}(L^2)} + \|\nabla\Gamma\|^2_{L^2_{t_1}(L^2)} \leq \frac{3}{2}\|\Gamma_0\|^2_{L^2}.$$

Substituting the above estimate into (2.24) gives rise to (2.30). (2.30) together with the blow-up criteria in [14] implies that $T^* \geq t_1$. □

2.2 The global in time H^1 estimate

The goal of this subsection is to present the global in time H^1 estimate for the velocity field. Toward this, we first prove such a estimate for small solutions of (1.1), which does not use the axisymmetric structure of the solutions.

Lemma 2.2 *Let* $(\rho, u, \nabla\Pi)$ *be a smooth enough solution of* (1.1) *on* $[0, T^*)$, *which satisfies* (2.2). *Then there exist positive constants* η_1 *and* η_2, *which depend only on* $\|u_0\|_{L^2}$, *so that there holds*

$$\|\nabla u(t)\|^2_{L^2} + \int_{t_0}^t \left(m\|\partial_t u(t')\|^2_{L^2} + \eta_2\left(\|\nabla^2 u(t')\|^2_{L^2} + \|\nabla\Pi(t')\|^2_{L^2}\right)\right)dt' \leq \|\nabla u(t_0)\|^2_{L^2} \tag{2.31}$$

provided that $\|\nabla u(t_0)\|_{L^2} \leq \eta_1$.

Proof We first get, by taking the L^2 inner product of the momentum equations of (1.1) with $\partial_t u$ and using integration by parts, that

$$\begin{aligned}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 &= -(\rho u\cdot\nabla u \mid \partial_t u)_{L^2}\\ &\le \|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^3}\|\nabla u\|_{L^6}\|\sqrt{\rho}\partial_t u\|_{L^2}\\ &\le C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^2 + \frac{1}{4}\|\sqrt{\rho}\partial_t u\|_{L^2}^2,\end{aligned}$$

which gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \frac{3}{4}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 \le C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^2.$$

On the other hand, it follows from the classical estimates on linear Stokes operator and

$$\begin{cases}-\Delta u + \nabla\Pi = \rho\partial_t u - \rho u\cdot\nabla u,\\ \operatorname{div} u = 0,\end{cases} \tag{2.32}$$

that

$$\begin{aligned}\|\nabla^2 u\|_{L^2}^2 + \|\nabla\Pi\|_{L^2}^2 &\le C\left(\|\rho\partial_t u\|_{L^2}^2 + \|\rho u\cdot\nabla u\|_{L^2}^2\right)\\ &\le C\left(\|\sqrt{\rho}\partial_t u\|_{L^2}^2 + \|\rho\|_{L^\infty}\|u\|_{L^3}^2\|\nabla u\|_{L^6}^2\right)\\ &\le C\left(\|\sqrt{\rho}\partial_t u\|_{L^2}^2 + \|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^2\right),\end{aligned}$$

so that we obtain for any $\eta_2 > 0$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \left(\frac{3m}{4} - C\eta_2\right)\|\partial_t u\|_{L^2}^2\\ &\quad + \left(\eta_2 - C\|u_0\|_{L^2}\|\nabla u\|_{L^2}\right)\left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla\Pi\|_{L^2}^2\right) \le 0.\end{aligned} \tag{2.33}$$

We denote

$$\tau^* \overset{\text{def}}{=} \sup\left\{ t\in[t_0, T^*) \,\middle|\, \|\nabla u(t)\|_{L^2} \le 2\eta_1\right\}. \tag{2.34}$$

We claim that $\tau^* = T^*$ provided that η_1 is sufficiently small. Indeed if $\tau^* < T^*$, taking $\eta_2 = \frac{m}{4C}$ and $\eta_1 \le \frac{\eta_2}{2C\|u_0\|_{L^2}}$, we deduce from (2.33) that

$$\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + m\|\partial_t u\|_{L^2}^2 + \eta_2\left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla\Pi\|_{L^2}^2\right) \le 0 \qquad \text{for all } t\in[t_0,\tau^*),$$

which implies

$$\|\nabla u(t)\|_{L^2}^2 + \int_{t_0}^{\tau^*}\left(m\|\partial_t u(t')\|_{L^2}^2 + \eta_2(\|\nabla^2 u(t')\|_{L^2}^2 + \|\nabla\Pi(t')\|_{L^2}^2)\right)dt' \le \|\nabla u(t_0)\|_{L^2}^2 \le \eta_1^2.$$

This contradict with (2.34), and thus $\tau^* = T^*$. This concludes the proof of the lemma. □

Proposition 2.2 *Let* $(\rho, u, \nabla\Pi)$ *be the local unique smooth solution of* (1.4) *on* $[0, T^*)$, *which satisfies* (2.2). *Then* $T^* = \infty$ *and there holds* (1.11) *provided that* ε_0 *in* (1.10) *is sufficiently small.*

Proof It follows from the derivation of (2.3) that

$$\frac{1}{2}\|\sqrt{\rho}u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(t')\|_{L^2}^2\,dt' = \frac{1}{2}\|\sqrt{\rho_0}u_0\|_{L^2}^2, \tag{2.35}$$

which ensures that for any positive integer N, there holds

$$\sum_{k=0}^{N-1}\int_k^{k+1}\|\nabla u(t')\|_{L^2}^2\,dt'\leq\frac{1}{2}\|\sqrt{\rho_0}u_0\|_{L^2}^2.$$

Thus there exists $0\leq k_0\leq N-1$ and some $t_0\in(k_0,k_0+1)$ such that

$$\int_{k_0}^{k_0+1}\|\nabla u\|_{L^2}^2\,d\tau\leq\frac{1}{2N}\|\sqrt{\rho_0}u_0\|_{L^2}^2\quad\text{and}\quad\|\nabla u(t_0)\|_{L^2}^2\leq\frac{1}{2N}\|\sqrt{\rho_0}u_0\|_{L^2}^2.$$

For η_1 given by Lemma 2.2, taking N so large that

$$\|\nabla u(t_0)\|_{L^2}^2\leq\frac{1}{2N}\|\sqrt{\rho_0}u_0\|_{L^2}^2\leq\eta_1^2.$$

Then we deduce from Lemma 2.2 that there holds (2.31).

On the other hand, in view of (2.28), we can take $\|\frac{a_0}{r}\|_{L^\infty}$ to be so small that $t_1\geq t_0$. Thus by summing up (2.30) and (2.31), we obtain for any $t<T^*$,

$$\begin{aligned}&\|\nabla u\|_{L_t^\infty(L^2)}^2+\|\partial_t u\|_{L_t^2(L^2)}^2+\|\nabla^2u\|_{L_t^2(L^2)}^2+\|\nabla\Pi\|_{L_t^2(L^2)}^2\\&\leq\|\nabla u\|_{L^\infty(0,t_0;L^2)}^2+\|\partial_t u\|_{L^2(0,t_0;L^2)}^2+\|\nabla^2u\|_{L^2(0,t_0;L^2)}^2+\|\nabla\Pi\|_{L^2(0,t_0;L^2)}^2\\&\quad+\|\nabla u\|_{L^\infty(t_0,t;L^2)}^2+\|\partial_t u\|_{L^2(t_0,t;L^2)}^2+\|\nabla^2u\|_{L^2(t_0,t;L^2)}^2+\|\nabla\Pi\|_{L^2(t_0,t;L^2)}^2\\&\leq C\mathcal{G}_0+\eta_1,\end{aligned}\tag{2.36}$$

for $\mathcal{G}_0$ given by (1.11) and η_1 being determined by Lemma 2.2. Then thanks to (2.36) and the blow-up criteria in [14], we conclude that $T^*=\infty$. Moreover, by summing up (2.3) and (2.36), we achieve (1.11). This finishes the proof of Proposition 2.2. □

3 Decay estimates of the global solutions of (1.1)

The purpose of this section is to present the decay estimates (1.13) for any global smooth solutions of (1.1), which does not use the particular axisymmetric structure of the solutions.

Lemma 3.1 *Let* $(\rho,u,\nabla\Pi)$ *be a smooth enough solution of* (1.1) *on* $[0,T^*)$, *which satisfies* (2.2). *Then for* $t<T^*$, *one has*

$$\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2+\|\sqrt{\rho}u_t(t)\|_{L^2}^2+\|u(t)\|_{\dot H^2}^2+\|\nabla\Pi(t)\|_{L^2}^2\leq C\|\nabla u(t)\|_{H^1}^2\|\nabla u(t)\|_{L^2}^2,\tag{3.1}$$

and

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho}u_t(t)\|_{L^2}^2+\|\nabla u_t(t)\|_{L^2}^2\\&\quad\leq C\left(\|\nabla u(t)\|_{H^1}^2+\|u(t)\|_{\dot H^1}^4\right)\left(\|\sqrt{\rho}u_t(t)\|_{L^2}^2+\|\nabla u(t)\|_{L^2}^4\right).\end{aligned}\tag{3.2}$$

Proof We first get, by a similar derivation of (2.33), that

$$\begin{aligned}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2+\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|u\|_{\dot H^2}^2+\|\nabla\Pi\|_{L^2}^2\right)&\leq C\|\sqrt{\rho}u\cdot\nabla u\|_{L^2}^2\\&\leq CM\|u\|_{L^6}^2\|\nabla u\|_{L^3}^2\\&\leq CM\|\nabla u\|_{\dot H^{\frac12}}^2\|\nabla u\|_{L^2}^2,\end{aligned}$$

which gives (3.1).

On the other hand, by taking ∂_t to the momentum equation of (1.1), we write

$$\rho\,(\partial_t u_t + u\cdot\nabla u_t) - \Delta u_t + \nabla\Pi_t = -\rho_t u_t - (\rho u)_t\cdot\nabla u.$$

Taking L^2 inner product of the above equation with u_t and using the transport equation of (1.1), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 = &-\int_{\mathbb{R}^3}\rho_t|u_t|^2\,dx - \int_{\mathbb{R}^3}\rho_t u\cdot\nabla u\mid u_t\,dx\\ &-\int_{\mathbb{R}^3}\rho u_t\cdot\nabla u\mid u_t\,dx.\end{aligned}\tag{3.3}$$

By using the transport equation of (1.1) and integration by parts, one has

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\rho_t|u_t|^2\,dx\right| &= \left|\int_{\mathbb{R}^3}\mathrm{div}(\rho u)|u_t|^2\,dx\right|\\ &= 2\left|\int_{\mathbb{R}^3}\rho u\cdot\nabla u_t\mid u_t\,dx\right|\\ &\le 2\sqrt{M}\|u\|_{L^\infty}\|\sqrt{\rho}u_t\|_{L^2}\|\nabla u_t\|_{L^2},\end{aligned}$$

which together with the 3-D interpolation inequality that

$$\|u\|_{L^\infty} \le C\|u\|_{\dot{H}^1}^{\frac{1}{2}}\|u\|_{\dot{H}^2}^{\frac{1}{2}},\tag{3.4}$$

implies

$$\left|\int_{\mathbb{R}^3}\rho_t|u_t|^2\,dx\right| \le CM\|u\|_{\dot{H}^1}\|u\|_{\dot{H}^2}\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{1}{6}\|\nabla u_t\|_{L^2}^2.$$

Along the same line, we have

$$\begin{aligned}&\int_{\mathbb{R}^3}\rho_t u\cdot\nabla u\mid u_t\,dx = -\int_{\mathbb{R}^3}\mathrm{div}(\rho u)u\cdot\nabla u\mid u_t\,dx\\ &= \sum_{i,j,k=1}^{3}\left(\int_{\mathbb{R}^3}\rho u^i\partial_i u^j\partial_j u^k u_t^k\,dx + \int_{\mathbb{R}^3}\rho u^i u^j\partial_i\partial_j u^k u_t^k\,dx + \int_{\mathbb{R}^3}\rho u^i u^j\partial_j u^k\partial_i u_t^k\,dx\right).\end{aligned}$$

Applying Hölder's inequality gives

$$\begin{aligned}\sum_{i,j,k=1}^{3}\left|\int_{\mathbb{R}^3}\rho u^i\partial_i u^j\partial_j u^k u_t^k\,dx\right| &\le \sqrt{M}\|u\|_{L^\infty}\|\nabla u\|_{L^3}\|\nabla u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^2}\\ &\le C\left(\|u\|_{L^\infty}^2\|u\|_{\dot{H}^2}^2 + \|\nabla u\|_{\dot{H}^{\frac{1}{2}}}^2\|\sqrt{\rho}u_t\|_{L^2}^2\right),\end{aligned}$$

and

$$\begin{aligned}\sum_{i,j,k=1}^{3}\left|\int_{\mathbb{R}^3}\rho u^i u^j\partial_i\partial_j u^k u_t^k\,dx\right| &\le \sqrt{M}\|u\|_{L^\infty}^2\|\nabla^2 u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}\\ &\le C\|u\|_{L^\infty}^2\left(\|u\|_{\dot{H}^2}^2 + \|\sqrt{\rho}u_t\|_{L^2}^2\right),\end{aligned}$$

and

$$\sum_{i,j,k=1}^{3}\left|\int_{\mathbb{R}^3}\rho u^i u^j \partial_j u^k \partial_i u_t^k\,dx\right| \le M\|u\|_{L^6}^2\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}$$
$$\le C\|\nabla u\|_{L^2}^4\|u\|_{\dot H^2}^2+\frac{1}{6}\|\nabla u_t\|_{L^2}^2.$$

This yields

$$\left|\int_{\mathbb{R}^3}\rho_t u\cdot\nabla u\mid u_t\,dx\right| \le C\left(\|u\|_{L^\infty}^2+\|\nabla u\|_{\dot H^{\frac12}}^2+\|\nabla u\|_{L^2}^4\right)\left(\|u\|_{\dot H^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2\right)$$
$$+\frac{1}{6}\|\nabla u_t\|_{L^2}^2.$$

Finally it is easy to observe that

$$\left|\int_{\mathbb{R}^3}\rho u_t\cdot\nabla u\mid u_t\,dx\right| \le \sqrt{M}\|u_t\|_{L^6}\|\nabla u\|_{L^3}\|\sqrt{\rho}u_t\|_{L^2}$$
$$\le C\|\nabla u\|_{\dot H^{\frac12}}^2\|\sqrt{\rho}u_t\|_{L^2}^2+\frac{1}{6}\|\nabla u_t\|_{L^2}^2.$$

Resuming the above estimates into (3.3) and using (3.4) results in

$$\frac{d}{dt}\|\sqrt{\rho}u_t(t)\|_{L^2}^2+\|\nabla u_t(t)\|_{L^2}^2$$
$$\le C\left(\|\nabla u(t)\|_{H^1}^2+\|u(t)\|_{\dot H^1}^4\right)\left(\|u(t)\|_{\dot H^2}^2+\|\sqrt{\rho}u_t(t)\|_{L^2}^2\right). \tag{3.5}$$

Whereas it follows from the classical estimates on linear Stokes operator and (2.32) that

$$\|u\|_{\dot H^2}+\|\nabla\Pi\|_{L^2} \le C\left(\|\rho u_t\|_{L^2}+\|\rho u\cdot\nabla u\|_{L^2}\right)$$
$$\le C\left(\sqrt{M}\|\sqrt{\rho}u_t\|_{L^2}+M\|u\|_{L^6}\|\nabla u\|_{L^3}\right)$$
$$\le C\left(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla u\|_{L^2}^2\right)+\frac{1}{2}\|u\|_{\dot H^2},$$

which yields

$$\|u\|_{\dot H^2}+\|\nabla\Pi\|_{L^2}\le C\left(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla u\|_{L^2}^2\right). \tag{3.6}$$

Substituting (3.6) into (3.5) leads to (3.2). This finishes the proof of the Lemma. □

Corollary 3.1 *Under the assumptions of Lemma 3.1 and that*

$$\|u\|_{L^\infty(0,T^*;H^1)}^2+\|\nabla u\|_{L^2(0,T^*;H^1)}^2\le C_0, \tag{3.7}$$

one has for any $t<T^*$,

$$\langle t\rangle\|\nabla u(t)\|_{L^2}^2+\int_0^t\langle t'\rangle\left(\|u_t(t')\|_{L^2}^2+\|u(t')\|_{\dot H^2}^2+\|\nabla\Pi(t')\|_{L^2}^2\right)dt'$$
$$\le C\exp(CC_0)\|u_0\|_{H^1}^2\overset{def}{=}C_1, \tag{3.8}$$

and

$$t\langle t\rangle \Big(\|u_t(t)\|_{L^2}^2 + \|u(t)\|_{\dot H^2}^2 + \|\nabla\Pi(t)\|_{L^2}^2\Big) + \int_0^t t'\langle t'\rangle \|\nabla u_t(t')\|_{L^2}^2\,dt'$$
$$\le CC_1(1+C_1)\exp\left(CC_0(1+C_0)\right) \stackrel{def}{=} C_2. \tag{3.9}$$

Proof We first get, by multiplying (3.1) by $\langle t\rangle$, that

$$\frac{d}{dt}\left(\langle t\rangle\|\nabla u(t)\|_{L^2}^2\right) + \langle t\rangle\Big(\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \|u(t)\|_{\dot H^2}^2 + \|\nabla\Pi(t)\|_{L^2}^2\Big)$$
$$\le \|\nabla u(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{H^1}^2\langle t\rangle\|\nabla u(t)\|_{L^2}^2.$$

Applying Gronwall's inequality and using (2.35), (3.7) gives rise to (3.8).

While multiplying (3.2) by $t\langle t\rangle$ results in

$$\frac{d}{dt}\left(t\langle t\rangle\|\sqrt{\rho}u_t(t)\|_{L^2}^2\right) + t\langle t\rangle\|\nabla u_t(t)\|_{L^2}^2 \le 2\langle t\rangle\|\sqrt{\rho}u_t(t)\|_{L^2}^2$$
$$+C\Big(\|\nabla u(t)\|_{H^1}^2 + \|u(t)\|_{\dot H^1}^4\Big)t\langle t\rangle\left(\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^4\right).$$

Applying Gronwall's inequality leads to

$$t\langle t\rangle\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \int_0^t t'\langle t'\rangle\|\nabla u_t(t')\|_{L^2}^2\,dt'$$
$$\le C\exp\Big(C\Big(\|\nabla u\|_{L^2_t(H^1)}^2 + \|u\|_{L^\infty_t(\dot H^1)}^2\|u\|_{L^2_t(\dot H^1)}^2\Big)\Big)\Big(\int_0^t\langle t'\rangle\|\sqrt{\rho}u_t(t')\|_{L^2}^2\,dt'$$
$$+\big\|\langle t'\rangle\|\nabla u(t')\|_{L^2}^2\big\|_{L^\infty_t}^2\Big(\|\nabla u\|_{L^2_t(H^1)}^2 + \|u\|_{L^\infty_t(\dot H^1)}^2\|u\|_{L^2_t(\dot H^1)}^2\Big)\Big),$$

from which, (3.6–3.8), we conclude the proof of (3.9). □

Proposition 3.1 *Let $p\in[1,2)$ and $\beta(p)\stackrel{def}{=}\frac{3}{4}(\frac{2}{p}-1)$. Then under the assumptions of Corollary 3.1, if we assume further that $a_0\stackrel{def}{=}\frac{1}{\rho_0}-1\in L^2(\mathbb{R}^3)$ and $u_0\in L^p(\mathbb{R}^3)$, there holds*

$$\|u(t)\|_{L^2} \le \begin{cases} C\langle t\rangle^{-\beta(p)} & \text{if } \ 1<p<2,\\ C\langle t\rangle^{-\left(\frac{3}{4}\right)_-} & \text{if } \ p=1,\end{cases} \tag{3.10}$$

for any $t<T^*$, where the constant C depends on $\|a_0\|_{L^2}$, C_0, C_1 and C_2 given by Corollary 3.1.

Proof Motivated by [4], in order to use Schonbek's strategy in [21], we split the phase-space $\mathbb{R}^3$ into two time-dependent regions so that

$$\|\nabla u(t)\|_{L^2}^2 = \int_{S(t)}|\xi|^2|\hat u(t,\xi)|^2\,d\xi + \int_{S(t)^c}|\xi|^2|\hat u(t,\xi)|^2 d\xi,$$

where $S(t)\stackrel{\text{def}}{=}\{\xi : |\xi|\le\sqrt{\frac{M}{2}}\,g(t)\}$ and $g(t)$ satisfies $g(t)\sim\langle t\rangle^{-\frac{1}{2}}$, which will be chosen later on. Then due to the energy law (2.35) of (1.1), one has

$$\frac{d}{dt}\|\sqrt{\rho}u(t)\|_{L^2}^2 + g^2(t)\|\sqrt{\rho}\,u(t)\|_{L^2}^2 \le Mg^2(t)\int_{S(t)}|\hat u(t,\xi)|^2\,d\xi \tag{3.11}$$

To deal with the low frequency part of u on the right-hand side of (3.11), we rewrite the momentum equations of (1.1) as

$$u(t)=e^{t\Delta}u_0+\int_0^t e^{(t-t')\Delta}\mathbb{P}\left(-\nabla\cdot(u\otimes u)+a(\Delta u-\nabla\Pi)\right)(t')\,dt'.$$

where $a\overset{\text{def}}{=}\frac{1}{\rho}-1$ and $\mathbb{P}\overset{\text{def}}{=}Id-\nabla\Delta^{-1}\text{div}$ denotes the Leray projection operator. Taking Fourier transform with respect to x variables leads to

$$|\hat{u}(t,\xi)|\lesssim e^{-t|\xi|^2}|\widehat{u}_0(\xi)|+\int_0^t e^{-(t-t')|\xi|^2}\left(|\xi||\mathcal{F}_x(u\otimes u)|+|\mathcal{F}_x(a(\Delta u-\nabla\Pi))|\right)(t')\,dt',$$

which implies that

$$\int_{S(t)}|\hat{u}(t,\xi)|^2d\xi\lesssim\int_{S(t)}e^{-2t|\xi|^2}|\widehat{u}_0(\xi)|^2d\xi+g^5(t)\left(\int_0^t\|\mathcal{F}_x(u\otimes u)(t')\|_{L^\infty_\xi}\,dt'\right)^2$$
$$+g^3(t)\left(\int_0^t\|\mathcal{F}_x(a(\Delta u-\nabla\Pi))(t')\|_{L^\infty_\xi}\,dt'\right)^2. \tag{3.12}$$

Thanks to (3.9), we have

$$\left(\int_0^t\|\mathcal{F}_x(a(\Delta u-\nabla\Pi))(t')\|_{L^\infty_\xi}\,dt'\right)^2\leq\|a\|^2_{L^\infty_t(L^2)}\left(\int_0^t\|(\Delta u-\nabla\Pi)(t')\|_{L^2}\,dt'\right)^2$$
$$\lesssim\|a_0\|^2_{L^2}\left(\int_0^t(t')^{-\frac12}\langle t'\rangle^{-\frac12}\,dt'\right)^2\lesssim\ln^2\langle t\rangle. \tag{3.13}$$

While it is easy to observe that

$$\left(\int_0^t\|\mathcal{F}_x(u\otimes u)(t')\|_{L^\infty_\xi}\,dt'\right)^2\leq\left(\int_0^t\|u(t')\|^2_{L^2}\,dt'\right)^2\lesssim t^2\|u_0\|^2_{L^2}.$$

Note that for $u_0\in L^p(\mathbb{R}^3)$, let $\frac1q\overset{\text{def}}{=}\frac43\beta(p)=\frac2p-1$ and $\frac1p+\frac1{p'}=1$, one has

$$\int_{S(t)}e^{-2t|\xi|^2}|\widehat{u}_0(\xi)|^2\,d\xi\lesssim\left(\int_{S(t)}e^{-2qt|\xi|^2}\,d\xi\right)^{\frac1q}\|\widehat{u}_0\|^2_{L^{p'}}$$
$$\lesssim\|u_0\|^2_{L^p}\langle t\rangle^{-2\beta(p)}, \tag{3.14}$$

where we used the Hausdörff–Young inequality in the last line so that $\|\widehat{u}_0\|_{L^{p'}}\leq C\|u_0\|_{L^p}$. Then since $g(t)\lesssim\langle t\rangle^{-\frac12}$, we deduce from (3.12) that

$$\int_{S(t)}|\hat{u}(t,\xi)|^2\,d\xi\lesssim\langle t\rangle^{-2\beta(p)}+\langle t\rangle^{-\frac12}\lesssim\begin{cases}\langle t\rangle^{-\frac12} & \text{if}\ \ 1\leq p<\frac32,\\ \langle t\rangle^{-2\beta(p)} & \text{if}\ \ \frac32\leq p<2,\end{cases} \tag{3.15}$$

In the case when $\frac32\leq p<2$, by substituting (3.15) into (3.11), we obtain

$$\frac{d}{dt}\|\sqrt{\rho}u(t)\|^2_{L^2}+g^2(t)\|\sqrt{\rho}\,u(t)\|^2_{L^2}\lesssim g^2(t)\langle t\rangle^{-2\beta(p)}\lesssim\langle t\rangle^{-1-2\beta(p)},$$

from which, we infer

$$e^{\int_0^t g^2(t')\,dt'}\|\sqrt{\rho}u(t)\|^2_{L^2}\lesssim\|\sqrt{\rho_0}u_0\|^2_{L^2}+\int_0^t e^{\int_0^{t'}g^2(t')dt'}\langle t'\rangle^{-1-2\beta(p)}\,dt'.$$

Taking $\alpha > 2\beta(p)$ and $g^2(t) = \alpha\langle t\rangle^{-1}$ in the above inequality leads to

$$\|\sqrt{\rho}u(t)\|_{L^2}^2\langle t\rangle^{\alpha} \lesssim 1 + \int_0^t \langle t'\rangle^{\alpha-1-2\beta(p)}\,dt' \lesssim 1 + \langle t\rangle^{\alpha-2\beta(p)},$$

which yields (3.10) for $p \in [3/2, 2)$.

In the case when $1 \leq p < \frac{3}{2}$, by substituting the Estimate (3.15) into (3.11), one has

$$\frac{d}{dt}\|\sqrt{\rho}u(t)\|_{L^2}^2 + g^2(t)\|\sqrt{\rho}\,u(t)\|_{L^2}^2 \lesssim g^2(t)\langle t\rangle^{-\frac{1}{2}} \lesssim \langle t\rangle^{-\frac{3}{2}},$$

which implies

$$e^{\int_0^t g^2(t')\,dt'}\|\sqrt{\rho}u(t)\|_{L^2}^2 \lesssim \|\sqrt{\rho_0}u_0\|_{L^2}^2 + \int_0^t e^{\int_0^{t'} g^2(t')dt'}\langle t'\rangle^{-\frac{3}{2}}\,dt'.$$

Taking $\alpha > \frac{1}{2}$ and $g^2(t) \overset{\text{def}}{=} \alpha\langle t\rangle^{-1}$ in the above inequality results in

$$\|\sqrt{\rho}u(t)\|_{L^2}^2\langle t\rangle^{\alpha} \lesssim 1 + \int_0^t \langle t'\rangle^{\alpha-\frac{3}{2}}\,dt' \lesssim 1 + \langle t\rangle^{\alpha-\frac{1}{2}},$$

which gives

$$\|u(t)\|_{L^2} \lesssim \langle t\rangle^{-\frac{1}{4}}. \tag{3.16}$$

Then by virtue of (3.16), we write

$$\left(\int_0^t \|\mathcal{F}_x(u\otimes u)(t')\|_{L^\infty_\xi}\,dt'\right)^2 \leq \left(\int_0^t \|u(t')\|_{L^2}^2\,dt'\right)^2 \lesssim \left(\int_0^t \langle t'\rangle^{-\frac{1}{2}}\,dt'\right)^2 \lesssim \langle t\rangle. \tag{3.17}$$

Resuming the Estimates (3.13), (3.14) and (3.17) into (3.12) results in

$$\int_{\bar{S}(t)} |\hat{u}(t,\xi)|^2 d\xi \lesssim \langle t\rangle^{-2\beta(p)} + \langle t\rangle^{-\left(\frac{3}{2}\right)_-} \lesssim \begin{cases} \langle t\rangle^{-2\beta(p)} & \text{if} \quad 1 < p < \frac{3}{2}, \\ \langle t\rangle^{-\left(\frac{3}{2}\right)_-} & \text{if} \quad p = 1. \end{cases} \tag{3.18}$$

With (3.18), we can repeat the previous argument to prove (3.10) for the remaining case when $p \in [1, 3/2)$. This completes the proof of the proposition. □

Proposition 3.2 *Under the assumptions of Proposition* 3.1*, there holds* (1.13) *for any* $t < T^*$.

Proof With Proposition 3.1, we shall use a similar argument for the classical Navier–Stokes system to derive the decay estimates for the derivatives of the velocity (see [12] for instance). In fact, for any $s < t < T^*$, we deduce from the energy equality of (1.1) that

$$\frac{1}{2}\|\sqrt{\rho}u(t)\|_{L^2}^2 + \int_s^t \|\nabla u(t')\|_{L^2}^2\,dt' = \frac{1}{2}\|\sqrt{\rho}u(s)\|_{L^2}^2. \tag{3.19}$$

While multiplying (3.1) by $(t-s)$ leads to

$$\begin{aligned} &\frac{d}{dt}\left((t-s)\|\nabla u(t)\|_{L^2}^2\right) + (t-s)\left(\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla\Pi(t)\|_{L^2}^2\right) \\ &\quad \leq \|\nabla u(t)\|_{L^2}^2 + C\|\nabla u(t)\|_{H^1}^2(t-s)\|\nabla u(t)\|_{L^2}^2, \end{aligned}$$

Applying Gronwall's inequality and using (3.19) results in

$$(t-s)\|\nabla u(t)\|_{L^2}^2 \leq \exp\left(C\|\nabla u\|_{L_t^2(H^1)}^2\right)\int_s^t \|\nabla u(t')\|_{L^2}^2\,dt' \\ \leq \frac{\exp(CC_0)}{2}\|\sqrt{\rho}u(s)\|_{L^2}^2.$$

In particular, taking $s=\frac{t}{2}$ gives

$$\|\nabla u(t)\|_{L^2}^2 \leq C\langle t\rangle^{-1}\|u(t/2)\|_{L^2}^2,$$

from which and (3.10), we infer for any $t<T^*$

$$\|\nabla u(t)\|_{L^2}^2 \leq C\begin{cases}\langle t\rangle^{-1-2\beta(p)} & \text{if } \ 1<p<2,\\ \langle t\rangle^{-\left(\frac{5}{2}\right)_-} & \text{if } \ p=1,\end{cases} \tag{3.20}$$

Similarly by applying Gronwall's lemma to (3.1) over $[s,t]$, we write

$$\begin{aligned}&\|\nabla u(t)\|_{L^2}^2+\int_s^t\left(\|\sqrt{\rho}u_t(t')\|_{L^2}^2+\|\nabla^2u(t')\|_{L^2}^2+\|\nabla\Pi(t')\|_{L^2}^2\right)dt'\\ &\leq \exp\left(C\|\nabla u\|_{L_t^2(H^1)}^2\right)\|\nabla u(s)\|_{L^2}^2\\ &\leq \exp(CC_0)\,\|\nabla u(s)\|_{L^2}^2.\end{aligned} \tag{3.21}$$

Whereas by multiplying (3.2) by $(t-s)$ and applying Gronwall's lemma to resulting inequality, we get

$$\begin{aligned}(t-s)\|\sqrt{\rho}u_t(t)\|_{L^2}^2 &\leq \left(\int_s^t\|\sqrt{\rho}u_t(t')\|_{L^2}^2\,dt'+\|\nabla u\|_{L^\infty(s,t;L^2)}^4\right)\\ &\quad\times\exp\left(C\left(\|\nabla u\|_{L_t^2(H^1)}^2+\|u\|_{L_t^\infty(\dot H^1)}^2\|u\|_{L_t^2(\dot H^1)}^2\right)\right)\\ &\leq \exp(CC_0(1+C_0))\left(\|\nabla u(s)\|_{L^2}^2+\|\nabla u\|_{L^\infty(s,t;L^2)}^4\right).\end{aligned}$$

Taking $s=\frac{t}{2}$ in the above inequality and using (3.20), we obtain

$$\|u_t(t)\|_{L^2}^2 \leq C\begin{cases}t^{-1}\langle t\rangle^{-1-2\beta(p)} & \text{if } \ 1<p<2,\\ t^{-1}\langle t\rangle^{-\left(\frac{5}{2}\right)_-} & \text{if } \ p=1.\end{cases}$$

which together with (3.6) and (3.20) ensures that

$$\|u_t(t)\|_{L^2}^2+\|u(t)\|_{\dot H^2}^2+\|\nabla\Pi(t)\|_{L^2}^2 \leq C\begin{cases}Ct^{-1}\langle t\rangle^{-1-2\beta(p)} & \text{if } \ 1<p<2,\\ t^{-1}\langle t\rangle^{-\left(\frac{5}{2}\right)_-} & \text{if } \ p=1,\end{cases} \tag{3.22}$$

for any $t<T^*$.

With (3.20) and (3.22), it remains to prove (1.13) for $p=1$. As a matter of fact, we first deduce from (3.22) that

$$\begin{aligned}\left(\int_0^t\|\mathcal{F}_x(a(\Delta u-\nabla\Pi))(t')\|_{L_\xi^\infty}\,dt'\right)^2 &\leq \|a\|_{L_t^\infty(L^2)}^2\left(\int_0^t\|(\Delta u-\nabla\Pi)(t')\|_{L^2}\,dt'\right)^2\\ &\lesssim \|a_0\|_{L^2}^2\left(\int_0^t (t')^{-\frac{1}{2}}\langle t'\rangle^{-\left(\frac{5}{4}\right)_-}\,dt'\right)^2 \leq C.\end{aligned}$$

With (3.13) being replaced by the above inequality, by repeating the proof of Proposition 3.1, we can prove the first inequality of (1.13) for $p = 1$. Then repeating the proof of (3.22), we conclude the proof of the remaining two inequalities in (1.13) for $p = 1$. This finishes the proof of Proposition 3.2. □

4 The proof of Theorem 1.1

The goal of this section is to complete the proof of Theorem 1.1. In order to do so, we first prove the following globally in time Lipschitz estimate for the convection velocity field, which will be used to prove the propagation of the size for $\|\frac{a_0}{r}\|_{L^\infty}$.

Lemma 4.1 *Let $(\rho, u, \nabla\Pi)$ be a c smooth enough axisymmetric solution of (1.1) on $[0, T^*)$. Then under the assumptions (1.7) and (1.10), we have $T^* = \infty$, and there holds*

$$\|\nabla u\|_{L^1(\mathbb{R}_+;L^\infty)} \le C, \tag{4.1}$$

for some positive constant depending on m, M and $\|u_0\|_{H^1}$.

Proof Under the assumptions of (1.7) and (1.10), we deduce from Proposition 2.2 that $T^* = \infty$ and moreover Corollary 3.1 ensures that

$$\begin{aligned}&\sup_{t\in[0,\infty)} \langle t\rangle\|\nabla u(t)\|_{L^2}^2 + \int_0^\infty \langle t\rangle \left(\|u_t(t)\|_{L^2}^2 + \|u(t)\|_{\dot H^2}^2 + \|\nabla\Pi(t)\|_{L^2}^2\right) dt \le C_1,\\ &\sup_{t\in[0,\infty)} t\langle t\rangle \left(\|u_t(t)\|_{L^2}^2 + \|u(t)\|_{\dot H^2}^2 + \|\nabla\Pi(t)\|_{L^2}^2\right) + \int_0^\infty t\langle t\rangle\|\nabla u_t(t)\|_{L^2}^2\, dt \le C_2,\end{aligned} \tag{4.2}$$

where C_1 and C_2 given by (3.8) and (3.9) respectively. In particular, by using Sobolev imbedding theorem, we obtain

$$\int_0^\infty t\langle t\rangle\|u_t(t)\|_{L^6}^2\, dt \le C_2. \tag{4.3}$$

On the other hand, in view of (2.32), we deduce from the classical estimates of linear Stokes operator that

$$\|\nabla^2 u(t)\|_{L^6} + \|\nabla\Pi(t)\|_{L^6} \le C\left(\|u_t(t)\|_{L^6} + \|u(t)\|_{L^\infty}\|\nabla u(t)\|_{L^6}\right),$$

which together with (3.4) yields

$$\begin{aligned}&\int_0^\infty t\langle t\rangle \left(\|\nabla^2 u(t)\|_{L^6}^2 + \|\nabla\Pi(t)\|_{L^6}^2\right) dt\\ &\quad\le C\left(\int_0^\infty t\langle t\rangle\|u_t(t)\|_{L^6}^2\, dt + \int_0^\infty t\langle t\rangle\|u(t)\|_{\dot H^1}\|u(t)\|_{\dot H^2}^3\, dt\right).\end{aligned}$$

Yet it follows from (4.2) that

$$\int_0^\infty t\langle t\rangle\|u(t)\|_{\dot H^1}\|u(t)\|_{\dot H^2}^3\, dt \le C\sqrt{C_1}C_2^{\frac32}\int_0^\infty t^{-\frac12}\langle t\rangle^{-1}\, dt \le C\sqrt{C_1}C_2^{\frac32},$$

which together with (4.3) ensures that

$$\int_0^\infty t\langle t\rangle \left(\|\nabla^2 u(t)\|_{L^6}^2 + \|\nabla\Pi(t)\|_{L^6}^2\right) dt \le CC_2\left(1 + \sqrt{C_1C_2}\right) \stackrel{\text{def}}{=} C_3. \tag{4.4}$$

By virtue of (4.2) and (4.4), we infer

$$\begin{aligned}\int_0^\infty \|\nabla u(t)\|_{L^\infty}\,dt &\le C\int_0^\infty \|u(t)\|_{\dot H^2}^{\frac12}\|\nabla^2 u(t)\|_{L^6}^{\frac12}\,dt\\ &\le CC_2^{\frac14}\int_0^\infty t^{-\frac12}\langle t\rangle^{-\frac12}\left(t\langle t\rangle\|\nabla^2 u(t)\|_{L^6}^2\right)^{\frac14}\,dt\\ &\le CC_2^{\frac14}\left(\int_0^\infty t^{-\frac23}\langle t\rangle^{-\frac23}\,dt\right)^{\frac34}\left(\int_0^\infty t\langle t\rangle\|\nabla^2 u(t)\|_{L^6}^2\,dt\right)^{\frac14}\\ &\le CC_2^{\frac14}C_3^{\frac14}.\end{aligned}$$

This gives rise to (4.1). □

Now we are in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1 The general strategy to prove the existence result to a nonlinear partial differential equation is first to construct an appropriate approximate solutions, and then perform the uniform estimates to these approximate solution sequence, and finally the existence result follows from a compactness argument. For simplicity, here we just present the a priori estimates to smooth enough solutions of (1.4).

Given axisymmetric initial data (ρ_0, u_0) with ρ_0 satisfying (1.7) and $a_0 \in L^2\cap L^\infty$, $\frac{a_0}{r}\in L^\infty$, $u_0\in H^1$, we deduce from (2.23) and (2.25) that there exists a maximal positive time T^* so that (1.4) has a solution on $[0, T^*)$ which satisfies for any $T < T^*$,

$$\|u\|_{L^\infty_T(H^1)}+\|\nabla u\|_{L^2_T(H^1)}+\|\partial_t u\|_{L^2_T(L^2)}+\|\nabla\Pi\|_{L^2_T(L^2)}+\|\Gamma\|_{L^\infty_T(L^2)}+\|\nabla\Gamma\|_{L^2_T(L^2)}\le C,$$

from which and Corollary 3.1, we deduce that there holds (1.8). And hence the uniqueness part of Theorem 1.1 follows from the uniqueness result in [20].

Now if $T^* < \infty$ and there holds

$$\lim_{t\to T^*}\left\|\frac{a(t)}{r}\right\|_{L^\infty} = C_* < \infty.$$

Let us take δ so small that

$$2mCC_* \le \frac12.$$

Then we get, by summing up (2.23) and $2m\delta\times$ (2.25), that

$$\begin{aligned}&\frac{d}{dt}\left(\|\widetilde\nabla u(t)\|_{L^2}^2+\left\|\frac{u^r(t)}{r}\right\|_{L^2}^2+2m\delta\|\Gamma(t)\|_{L^2}^2\right)\\ &\quad+\|\partial_t u\|_{L^2}^2+\|u\|_{\dot H^2}^2+\frac12\left(\|\widetilde\nabla\Pi\|_{L^2}^2+\|\Gamma\|_{L^2}^2\right)+\delta\|\widetilde\nabla\Gamma\|_{L^2}^2\\ &\le C_\delta\left((1+\|u\|_{L^2}^6)\left(\|\widetilde\nabla u\|_{L^2}^2+\left\|\frac{u^r}{r}\right\|_{L^2}^2\right)\left(\|\widetilde\nabla u\|_{L^2}^2+\|\Gamma\|_{L^2}^2\right)+(1+\|u^z\|_{L^2}^4)\|\partial_z u\|_{L^2}^2\right).\end{aligned}$$

Applying Gronwall's inequality and using (2.3) leads to

$$\begin{aligned}&\|\widetilde\nabla u\|_{L^\infty_T(L^2)}^2+\left\|\frac{u^r}{r}\right\|_{L^\infty_T(L^2)}^2+\|\Gamma\|_{L^\infty_T(L^2)}^2+\|\partial_t u\|_{L^2_T(L^2)}^2+\|\widetilde\nabla\Pi\|_{L^2_T(L^2)}^2\\ &\quad+\|u\|_{L^2_T(\dot H^2)}^2+\|\Gamma\|_{L^2_T(L^2)}^2+\|\widetilde\nabla\Gamma\|_{L^2_T(L^2)}^2\le C,\end{aligned}$$

for any $T < T^*$. Therefore we can extend the solution beyond the time T^*, which contradicts with the maximality of T^*. Hence there holds (1.9).

Under the assumption of (1.10), we deduce from Proposition 2.2 that $T^* = \infty$ and there holds (1.11). Moreover, Lemma 4.1 ensures that

$$\|\nabla u\|_{L^1(\mathbb{R}^+;L^\infty)} \le C,$$

which together with (2.26) and

$$\left\|\frac{u^r}{r}\right\|_{L^1(\mathbb{R}^+;L^\infty)} \le \|\nabla u\|_{L^1(\mathbb{R}^+;L^\infty)}$$

gives rise to (1.12).

Finally with additional assumption that $u_0 \in L^p$ for some $p \in [1, 2)$, we deduce from Proposition 3.2 that there holds the decay estimate (1.13). This finishes the proof of Theorem 1.1.

Acknowledgments We would like to thank Raphaël Danchin and Guilong Gui for profitable discussions on this topic. Part of this work was done when we were visiting Morningside Center of Mathematics, CAS, in the summer of 2014. We appreciate the hospitality and the financial support from the Center. P. Zhang is partially supported by NSF of China under Grant 11371037, the fellowship from Chinese Academy of Sciences and innovation grant from National Center for Mathematics and Interdisciplinary Sciences.

References

1. Abidi, H.: Résultats de régularité de solutions axisymétriques pour le système de Navier-Stokes. Bull. Sci. Math. **132**, 592–624 (2008)
2. Abidi, H., Hmidi, T., Keraani, S.: On the global well-posedness for the axisymmetric Euler equations. Math. Ann. **347**, 15–41 (2010)
3. Abidi, H., Hmidi, T., Keraani, S.: On the global regularity of axisymmetric Navier-Stokes-Boussinesq system. Discrete Contin. Dyn. Syst. **29**, 737–756 (2011)
4. Abidi, H., Gui, G., Zhang, P.: Stability to the global large solutions of the 3-D inhomogeneous Navier–Stokes equations. Commun. Pure. Appl. Math. **64**, 832–881 (2011)
5. Abidi, H., Gui, G., Zhang, P.: Well-posedness of 3-D inhomogeneous Navier–Stokes equations with highly oscillatory initial velocity field. J. Math. Pures Appl. (9) **100**, 166–203 (2013)
6. Badiale, M., Tarantello, G.: A Sobolev-Hardy inequality with applications to a nonlinear elliptic equation arising in astrophysics. Arch. Ration. Mech. Anal. **163**, 259–293 (2002)
7. Chae, D., Lee, J.: On the regularity of the axisymmetric solutions of the Navier–Stokes equations. Math. Z. **239**, 645–671 (2002)
8. Danchin, R.: Axisymmetric incompressible flows with bounded vorticity. Russ. Math. Survey **62**, 73–94 (2007)
9. Danchin, R., Mucha, P.B.: A Lagrangian approach for the incompressible Navier–Stokes equations with variable density. Commun. Pure. Appl. Math. **65**, 1458–1480 (2012)
10. Danchin, R., Zhang, P.: Inhomogeneous Navier–Stokes equations in the half-space, with only bounded density. J. Funct. Anal. **267**, 2371–2436 (2014)
11. Grafakos, L.: Classical and Modern Fourier Analysis. Pearson Education Inc, Upper Saddle River (2004)
12. He, C., Miyakawa, T.: On two-dimensional Navier–Stokes flows with rotational symmetries. Funkcial. Ekvac. **49**, 163–192 (2006)
13. Hmidi, T., Rousset, F.: Global well-posedness for the Euler–Boussinesq system with axisymmetric data. J. Funct. Anal. **260**, 745–796 (2011)
14. Kim, H.: A blow-up criterion for the nonhomogeneous incompressible Navier–Stokes equations. SIAM J. Math. Anal. **37**, 1417–1434 (2006)
15. Ladyženskaja, O.A.: Unique global solvability of the three-dimensional Cauchy problem for the Navier-Stokes equations in the presence of axial symmetry (Russian). Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **7**, 155–177 (1968)
16. Ladyzhenskaya, O., Solonnikov, V.: The unique solvability of an initial-boundary value problem for viscous incompressible inhomogeneous fluids. J. Sov. Math. **9**, 697–749 (1978)

17. Leonardi, S., Málek, J., Nečas, J., Pokorny, M.: On axially symmetric flows in $\mathbb{R}^3$. Z. Anal. Anwendungen **18**, 639–649 (1999)
18. Lions, P.L.: Mathematical topics in fluid mechanics, vol. 1. Incompressible models, Oxford Lecture Series in Mathematics and its Applications, 3. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1996)
19. Miao, C., Zheng, X.: On the global well-posedness for the Boussinesq system with horizontal dissipation. Commun. Math. Phys. **321**, 33–67 (2013)
20. Paicu, M., Zhang, P., Zhang, Z.: Global well-posedness of inhomogeneous Navier–Stokes equations with bounded density. Commun. Partial Differ. Equ. **38**, 1208–1234 (2013)
21. Schonbek, M.: Large time behavior of solutions to Navier–Stokes equations. Commun. Partial Differ. Equ. **11**, 733–763 (1986)
22. Simon, J.: Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure. SIAM J. Math. Anal. **21**, 1093–1117 (1990)
23. Ukhovskii, M.R., Iudovich, V.I.: Axially symmetric flows of ideal and viscous fluids filling the whole space. J. Appl. Math. Mech. **32**, 52–61 (1968)
24. Zhang, P., Zhang, T.: Global axisymmetric solutions to three-dimensional Navier–Stokes system. Int. Math. Res. Not., IMRN2014, no. 3, pp. 610–642