



Isoperimetric inequalities in convex cylinders and cylindrically bounded convex bodies

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Abstract In this paper we consider the isoperimetric profile of convex cylinders $K \times \mathbb{R}^q$, where K is an m -dimensional convex body, and of cylindrically bounded convex sets, i.e., those with a relatively compact orthogonal projection over some hyperplane of \mathbb{R}^{n+1} , asymptotic to a right convex cylinder of the form $K \times \mathbb{R}$, with $K \subset \mathbb{R}^n$. Results concerning the concavity of the isoperimetric profile, existence of isoperimetric regions, and geometric descriptions of isoperimetric regions for small and large volumes are obtained.

Mathematics Subject Classification 49Q10 · 49Q20 · 52B60

1 Introduction

In these notes we consider the *isoperimetric problem* of minimizing perimeter under a given volume constraint inside a *cylindrically bounded convex body*, an unbounded closed convex set $C \subset \mathbb{R}^{n+1}$ with interior points and relatively compact projection onto the hyperplane $x_{n+1} = 0$. The perimeter considered here will be the one relative to the interior of C . A way to deal with this isoperimetric problem is to consider the *isoperimetric profile* of C , i.e., the function assigning to each $v > 0$ the infimum of the perimeter of the sets inside C of volume v . If this infimum is achieved for some set, this will be called an *isoperimetric region*. The isoperimetric profile can be understood as an optimal isoperimetric inequality on C .

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A cylindrically bounded convex set is always included and asymptotic, in a sense to be precised later, to a *convex right cylinder*, a set of the form $K \times \mathbb{R}$, where $K \subset \mathbb{R}^n$ is a convex body. Here we have identified \mathbb{R}^n with the hyperplane $x_{n+1} = 0$ of \mathbb{R}^{n+1} . In this work we first consider the more general convex cylinders of the form $C = K \times \mathbb{R}^q$, where $K \subset \mathbb{R}^m$ is an arbitrary convex body with interior points, and $\mathbb{R}^m \times \mathbb{R}^q = \mathbb{R}^{n+1}$, and prove a number of results for their isoperimetric profiles. No assumption on the regularity of ∂C will be made. Existence of isoperimetric regions is obtained in Proposition 3.2 following the scheme of proof by Galli and Ritoré [4], which essentially needs a uniform local relative isoperimetric inequality [17], a doubling property on $K \times \mathbb{R}^q$ given in Lemma 3.1, an upper bound for the isoperimetric profile of C given in (2.6), and a well-known deformation controlling the perimeter in terms of the volume. A proof of existence of isoperimetric regions in Riemannian manifolds with compact quotient under their isometry groups was previously given by Morgan [13]. Regularity results in the interior follow from Gonzalez et al. [5] and Morgan [12], but no boundary regularity result is known for general convex bodies. We also prove in Proposition 3.5 that the isoperimetric profile I of a convex cylinder, as well as its power $I^{(n+1)/n}$, are concave functions of the volume, a strong result that implies the connectedness of isoperimetric regions. Further assuming $C^{2,\alpha}$ regularity of the boundary of C , we prove in Theorem 3.6 that, for an isoperimetric region $E \subset C$, either the closure of $\partial E \cap \text{int}(C)$ is connected, or $E \subset K \times \mathbb{R}$ is a slab. This follows from the connectedness of isoperimetric regions and from the results by Stredulinsky and Ziemer [21]. Next we consider small and large volumes. For small volumes, following Ritoré and Vernadakis [17], we show in Theorem 3.7 that the isoperimetric profile of a convex cylinder for small volumes is asymptotic to the one of its narrowest tangent cone. As a consequence, we completely characterize the isoperimetric regions of small volumes in a convex prism, i.e., a cylinder $P \times \mathbb{R}^q$ based on a convex polytope $P \subset \mathbb{R}^m$. Indeed, we show in Theorem 3.8 that the only isoperimetric regions of sufficiently small volume inside a convex prism are geodesic balls centered at the vertices with tangent cone of the smallest possible solid angle. For large volumes, we shall assume that C is a *right* convex cylinder, i.e., $q = 1$. Adapting an argument by Duzaar and Stephen [2] to the case when ∂K is not smooth, we prove in Theorem 3.9 that for large volumes the only isoperimetric regions in $K \times \mathbb{R}$ are the slabs $K \times I$, where $I \subset \mathbb{R}$ is a compact interval. The case $K \times \mathbb{R}^q$, with $q > 1$, is more involved and will be treated in a different paper (see [18] for a proof for the Riemannian product $M \times \mathbb{R}^k$, where M is a compact Riemannian manifold without boundary).

In the second part of this paper we apply the previous results for right convex cylinders to obtain properties of the isoperimetric profile of cylindrically bounded convex bodies. In Theorem 4.1 we show that the isoperimetric profile of a cylindrically bounded convex body C approaches, when the volume grows, that of its asymptotic half-cylinder. We also show the continuity of the isoperimetric profile in Proposition 4.4. Further assuming $C^{2,\alpha}$ regularity of both the cylindrically bounded convex body C and of its asymptotic cylinder, we prove the concavity of $I_C^{(n+1)/n}$ and existence of isoperimetric regions of large volume in Proposition 4.5. Our final result, Theorem 4.13, implies that translations of isoperimetric regions of unbounded volume converge in Hausdorff distance to a half-slab in the asymptotic half-cylinder. The same convergence result holds for their free boundaries, that converge in Hausdorff distance to a flat $K \times \{t\}$, $t \in \mathbb{R}^+$. Theorem 4.13 is obtained from a clearing-out result for isoperimetric regions of large volume proven in Theorem 4.9 and its main consequence, lower density estimates for isoperimetric regions of large volume given in Proposition 4.10. Such lower density bounds provide an alternative proof of Theorem 3.9, given in Corollary 4.12.

We have organized this paper into four sections. The next one contains basic preliminaries, while Sect. 3 and 4 cover the already mentioned results for cylinders and cylindrically bounded sets, respectively.

2 Preliminaries

A *convex body* is a compact convex set with non-empty interior. If compact is replaced by closed and unbounded, we get an *unbounded convex body*. We refer to Schneider’s monograph [19] for background on convex sets.

The s -dimensional Hausdorff measure in \mathbb{R}^{n+1} will be denoted by H^s , for any $s \in \mathbb{N}$. For $E \subset C$, the relative boundary of E in the interior of C is $\partial_C E = \partial E \cap \text{int } C$. The $(n + 1)$ -dimensional Hausdorff measure of E , $H^{n+1}(E)$ will be denoted by $|E|$ and referred to as the *volume* of E . Moreover, for every $x \in C$ and $r > 0$ we shall define the intrinsic open ball $B_C(x, r) = B(x, r) \cap \text{int } C$, where $B(x, r)$ denotes the open Euclidean geodesic ball centered at x of radius r . The closure of a set $E \subset \mathbb{R}^{n+1}$ will be denoted by $\text{cl}(E)$.

We also define the relative *perimeter* of E in the interior of C by

$$P_C(E) = \sup \left\{ \int_E \text{div } \xi \, dH^{n+1}, \quad \xi \in \Gamma_0(C), \quad |\xi| \leq 1 \right\},$$

where $\Gamma_0(C)$ is the set of smooth vector fields with compact support in $\text{int } C$. Observe that we are only computing the H^n -measure of ∂E inside the interior of C . We shall say that E has finite perimeter in the interior of C , or simply that $E \subset C$ has finite perimeter, if $P_C(E) < \infty$. We refer the reader to Maggi’s monograph [10] for background on finite perimeter sets.

If $C, C' \subset \mathbb{R}^{n+1}$ are convex bodies (possibly unbounded) and $f : C \rightarrow C'$ is a Lipschitz map, then, for every $s > 0$ and $E \subset C$, we get $H^s(f(E)) \leq \text{Lip}(f)^s H^s(E)$. Furthermore, $f(\partial_C E) = \partial_{f(C)}(f(E))$. Thus we obtain

Lemma 2.1 *Let $C, C' \subset \mathbb{R}^{n+1}$ be (possibly unbounded) convex bodies, and $f : C \rightarrow C'$ a bilipschitz map. Then we have*

$$\begin{aligned} \text{Lip}(f^{-1})^{-n} P_C(E) &\leq P_{f(C)}(f(E)) \leq \text{Lip}(f)^n P_C(E) \\ \text{Lip}(f^{-1})^{-(n+1)} |E| &\leq |f(E)| \leq \text{Lip}(f)^{n+1} |E|. \end{aligned} \tag{2.1}$$

Remark 2.2 If $M_i, i = 1, 2, 3$ are metric spaces and $f_i : M_i \rightarrow M_{i+1}, i = 1, 2$ are lipschitz maps, then $\text{Lip}(f_2 \circ f_1) \leq \text{Lip}(f_1) \text{Lip}(f_2)$. Consequently if $g : M_1 \rightarrow M_2$ is a bilipschitz map, then $1 \leq \text{Lip}(g) \text{Lip}(g^{-1})$.

Given a (possibly unbounded) convex body, we define the *isoperimetric profile* of C by

$$I_C(v) = \inf \{ P_C(E) : E \subset C, |E| = v \}. \tag{2.2}$$

We shall say that $E \subset C$ is an *isoperimetric region* if $P_C(E) = I_C(|E|)$. The *renormalized isoperimetric profile* of C is given by

$$I_C^{(n+1)/n}. \tag{2.3}$$

Lower semicontinuity of perimeter and standard compactness results for finite perimeter sets imply that isoperimetric regions exist in a fixed bounded subset of Euclidean space.

The known results on the regularity of isoperimetric regions are summarized in the following Lemma. One can always assume that a representative of an isoperimetric region is chosen so that it is closed, includes its points of density one and does not contain the points of density zero.

Lemma 2.3 ([5,6], [20, Thm. 2.1]) *Let $C \subset \mathbb{R}^{n+1}$ be a (possible unbounded) convex body and $E \subset C$ an isoperimetric region. Then $\partial_C E = S_0 \cup S$, where $S_0 \cap S = \emptyset$ and*

- (i) *S is an embedded C^∞ hypersurface of constant mean curvature.*
- (ii) *S_0 is closed and $H^s(S_0) = 0$ for any $s > n - 7$.*

Moreover, if the boundary of C is of class $C^{2,\alpha}$ then $\text{cl}(\partial E \cap \text{int}(C)) = S \cup S_0$, where

- (iii) *S is an embedded $C^{2,\alpha}$ hypersurface of constant mean curvature.*
- (iv) *S_0 is closed and $H^s(S_0) = 0$ for any $s > n - 7$.*
- (v) *At points of $S \cap \partial C$, S meets ∂C orthogonally.*

The concavity of I_C and $I_C^{(n+1)/n}$ for a convex body, [7], [11, Cor. 6.11], [17, Cor. 4.2], imply

Lemma 2.4 ([17, Lemma 4.9]) *Let $C \subset \mathbb{R}^{n+1}$ be a convex body and $0 < v_0 < |C|$. Then*

$$I_C(v) \geq \frac{I_C(v_0)}{v_0} v \quad \text{and} \quad I_C(v) \geq \frac{I_C(v_0)}{v_0^{n/(n+1)}} v^{n/(n+1)}, \tag{2.4}$$

for all $0 \leq v \leq v_0$.

We also have the following uniform relative isoperimetric inequality and bounds on the volume of relative balls in convex cylinders.

Proposition 2.5 *Let $C = K \times \mathbb{R}^q$, where K is an m -dimensional convex body. Given $r_0 > 0$, there exist positive constants M, ℓ_1 , only depending on r_0 and C , and a universal positive constant ℓ_2 so that*

$$P_{\bar{B}_C(x,r)}(v) \geq M \min\{v, |\bar{B}_C(x,r) - v\}^{n/(n+1)}, \tag{2.5}$$

for all $x \in C, 0 < r \leq r_0$, and $0 < v < |\bar{B}_C(x,r)|$, and

$$\ell_1 r^{n+1} \leq |\bar{B}_C(x,r)| \leq \ell_2 r^{n+1}, \tag{2.6}$$

for any $x \in C, 0 < r \leq r_0$.

Proof Since the quotient of C by its isometry group is compact, the proof is reduced to that of [17, Thm. 4.12]. □

Let $K \subset \mathbb{R}^{n+1}$ be a closed convex cone with vertex p . Let $\alpha(K) = H^n(\partial B(p, 1) \cap \text{int}(K))$ be the *solid angle* of K . It is known that intrinsic geodesic balls (Euclidean balls intersected with K) centered at the vertex are isoperimetric regions in K , [9,16], and that they are the only ones [3] for general convex cones, without any regularity assumption on the boundary. The isoperimetric profile of K is given by

$$I_K(v) = \alpha(K)^{1/(n+1)} (n+1)^{n/(n+1)} v^{n/(n+1)}. \tag{2.7}$$

Consequently the isoperimetric profile of a convex cone is completely determined by its solid angle.

We define the tangent cone C_p of a convex body C at a given boundary point $p \in \partial C$ as the closure of the set

$$\bigcup_{\lambda > 0} h_{p,\lambda}(C),$$

where $h_{p,\lambda}$ is the dilation of center p and factor λ . Since the quotient of the cylinder $C = K \times \mathbb{R}^q$ by its isometry group is compact, then adapting [17, Lemma 6.1] we get the existence

of points in ∂C whose tangent cones are minima of the solid angle function. By (2.7), the isoperimetric profiles of tangent cones which are minima of the solid angle function coincide. The common profile will be denoted by $I_{C_{\min}}$.

Proposition 2.6 ([17, Proposition 6.2]) *Let $C \subset \mathbb{R}^{n+1}$ be a convex body (possibly unbounded), $p \in C$ and let $H \subset \mathbb{R}^{n+1}$ denote the closed half-space, then*

$$I_C(v) \leq I_{C_p}(v) \leq I_H(v), \tag{2.8}$$

for all $0 \leq v \leq |C|$. Moreover $I_C \leq I_{C_{\min}}$.

Remark 2.7 Proposition 2.6 implies that $\overline{E} \cap \partial C \neq \emptyset$ when $E \subset C$ is an isoperimetric region since, in case $\overline{E} \cap \partial C$ is empty, then \overline{E} is an Euclidean ball.

Let $C \subset \mathbb{R}^{n+1}$ be a closed unbounded convex set. Assume there exists a hyperplane Π such that the orthogonal projection $\pi : \mathbb{R}^{n+1} \rightarrow \Pi$ takes C to a bounded set, and let K be the bounded convex body defined as the closure of $\pi(C)$. Then C is contained in the right convex cylinder $\text{cyl}(K)$ of base K . Since C is unbounded, it contains a half-line which is necessarily parallel to the axis of $\text{cyl}(K)$. If C contains a complete line, then $C = \text{cyl}(K)$ is a cylinder [19, Lemma 1.4.2]. This implies that C is either a cylinder, or is contained in a half-cylinder. We shall say that the unbounded convex body C is *cylindrically bounded* if it is contained in a convex cylinder of bounded base, and it is not a cylinder itself.

3 Isoperimetric regions in cylinders

In this Section we consider the isoperimetric problem when the ambient space is a convex cylinder $K \times \mathbb{R}^q$, where $K \subset \mathbb{R}^m$ is a convex body. We shall assume that $m + q = n + 1$. Existence of isoperimetric regions in $K \times \mathbb{R}^q$ can be obtained following the strategy of Galli and Ritoré for contact sub-Riemannian manifolds [4] with compact quotient under their contact isometry group. One of the basic ingredients in this strategy is the relative isoperimetric inequality in Proposition 2.5. A second one is the property that any unbounded convex body C is a doubling metric space.

Lemma 3.1 *Let $C \subset \mathbb{R}^{n+1}$ be an unbounded convex body. Then*

$$|B_C(x, 2r)| \leq (2^{n+1} + 1)|B_C(x, r)|, \tag{3.1}$$

for any $x \in C$ and any $r > 0$.

Proof Let $x \in C, r > 0$ and let K denote the closed cone with vertex x subtended by the closure of $\partial B_C(x, r)$. Then

$$\begin{aligned} |B_C(x, 2r)| &= |B_C(x, 2r) \setminus B_C(x, r)| + |B_C(x, r)| \\ &\leq |B_K(x, 2r) \setminus B_K(x, r)| + |B_C(x, r)| \\ &\leq |B_K(x, 2r)| + |B_C(x, r)| \\ &= {}^{n+1}|B_K(x, r)| + |B_C(x, r)| \\ &\leq (2^{n+1} + 1)|B_C(x, r)|, \end{aligned}$$

as we claimed. Inequality $|B_C(x, 2r) \setminus B_C(x, r)| \leq |B_K(x, 2r) \setminus B_K(x, r)|$ follows since, by convexity and the definition of $K, B_C(x, 2r) \setminus B_C(x, r) \subset B_K(x, 2r) \setminus B_K(x, r)$. \square

Using Lemma 3.1 and Proposition 2.6 we can show

Proposition 3.2 *Consider the convex cylinder $C = K \times \mathbb{R}^q$, where $K \subset \mathbb{R}^m$ is a convex body. Then isoperimetric regions exist in $K \times \mathbb{R}^q$ for all volumes and they are bounded.*

Proof To follow the strategy of Galli and Ritoré [4] (see Morgan [13] for a slightly different proof for smooth Riemannian manifolds), we only need a relative isoperimetric inequality (4.16) for balls $\overline{B}_C(x, r)$ of small radius with a uniform constant; the doubling property (3.1); inequality (2.8) giving an upper bound of the isoperimetric profile; and a deformation of isoperimetric sets E by finite perimeter sets E_t satisfying

$$|H^n(\partial E_t \cap \text{int}(C)) - H^n(\partial E \cap \text{int}(C))| \leq M ||E_t| - |E||,$$

for small $|t|$ and some constant $M > 0$ not depending in t , which can be obtained by deforming the regular part of the boundary of E using the flow associated to a vector field with compact support.

Using all these ingredients, the proof of Theorem 6.1 in [4] applies to prove existence of isoperimetric regions in $K \times \mathbb{R}^q$. □

Let us prove now the concavity of the isoperimetric profile of the cylinder and of its power $\frac{n+1}{n}$. We start by proving its continuity.

Proposition 3.3 *Let $C = K \times \mathbb{R}^q$, where K is an m -dimensional convex body. Then I_C is non-decreasing and continuous.*

Proof Given $t > 0$, the smooth map $\varphi_t : C \rightarrow C$ defined by $\varphi_t(x, y) = (x, ty)$, $x \in K$, $y \in \mathbb{R}^q$, satisfies $|\varphi_t(E)| = t^q |E|$. When $t \leq 1$, we also have $P_C(\varphi_t(E)) \leq t^{q-1} P_C(E)$. This implies that the isoperimetric profile is a non-decreasing function: let $v_1 < v_2$, and $E \subset C$ an isoperimetric region of volume v_2 . Let $0 < t < 1$ so that $|\varphi_t(E)| = v_1$. We have

$$I_C(v_1) \leq P(\varphi_t(E)) \leq P(E) = I_C(v_2).$$

This shows that I_C is non-decreasing.

Let us prove now the right-continuity of I_C at v . Consider an isoperimetric region E of volume v . We can take a smooth vector field Z with support in the regular part of the boundary of E such that $\int_E \text{div } Z \neq 0$. The flow $\{\varphi_t\}_{t \in \mathbb{R}}$ of Z satisfies $(d/dt)|_{t=0} |\varphi_t(E)| \neq 0$. Using the Inverse Function Theorem we obtain a smooth family $\{E_w\}$, for w near v , with $|E_w| = w$ and $E_v = E$. The function $f(w) = P(E_w)$ satisfies $f \geq I_C$ and $I_C(v) = f(v)$. This implies that I_C is right-continuous at v since, for $v_i \downarrow v$, we have

$$I_C(v) = f(v) = \lim_{i \rightarrow \infty} f(v_i) \geq \lim_{i \rightarrow \infty} I_C(v_i) \geq I_C(v),$$

by the monotonicity of I_C .

To prove the left-continuity of I_C at v we take a sequence of isoperimetric regions E_i with $v_i = |E_i| \uparrow v$ and we consider balls B_i disjoint from E_i so that $|E_i \cup B_i| = |E_i| + |B_i|$. Then $I_C(v) \leq P(E_i \cup B_i) = I_C(v_i) + P(B_i) \leq I_C(v) + P(B_i)$ by the monotonicity of I_C , and the left-continuity follows by taking limits since $\lim_{i \rightarrow \infty} P(B_i) = 0$. □

Lemma 3.4 *Let $\{K_i\}_{i \in \mathbb{N}}$ be a sequence of m -dimensional convex bodies converging to a convex body K in Hausdorff distance. Then $\{K_i \times \mathbb{R}^q\}_{i \in \mathbb{N}}$ converges to $K \times \mathbb{R}^q$ in lipschitz distance.*

Proof By [17, Theorem 3.4], there exists a sequence of bilipschitz maps $f_i : K_i \rightarrow K$ such that $\text{Lip}(f_i), \text{Lip}(f_i^{-1}) \rightarrow 1$ as $i \rightarrow \infty$. For every $i \in \mathbb{N}$, define $F_i : K_i \times \mathbb{R}^q \rightarrow K \times \mathbb{R}^q$ by

$$F_i(x, y) = (f_i(x), y), \quad (x, y) \in K_i \times \mathbb{R}^q. \tag{3.2}$$

Take now $(x_1, y_1), (x_2, y_2) \in K_i \times \mathbb{R}^q$. We have

$$\begin{aligned} |F_i(x_1, y_1) - F_i(x_2, y_2)|^2 &= |f_i(x_1) - f_i(x_2)|^2 + |y_1 - y_2|^2 \\ &\leq \max\{\text{Lip}(f_i)^2, 1\} (|x_1 - x_2|^2 + |y_1 - y_2|^2) \\ &= \max\{\text{Lip}(f_i)^2, 1\} |(x_1, y_1) - (x_2, y_2)|^2, \end{aligned} \tag{3.3}$$

where $|\cdot|$ is the Euclidean norm in the suitable Euclidean space. Hence we get

$$\limsup_{i \rightarrow \infty} \text{Lip}(F_i) \leq 1$$

since $\lim_{i \rightarrow \infty} \text{Lip}(f_i) = 1$. In a similar way we find $\limsup_{i \rightarrow \infty} \text{Lip}(F_i^{-1}) \leq 1$. By Remark 2.2, we get $\text{Lip}(F_i^{-1}) \text{Lip}(F_i) \geq 1$ and the proof follows. \square

Proposition 3.5 *Let $K \subset \mathbb{R}^m$ be a convex body and $C = K \times \mathbb{R}^q$. Then $I_C^{(n+1)/n}$ is a concave function. This implies that I_C is concave and every isoperimetric set in C is connected.*

Proof When the boundary of a convex cylinder C is smooth, its isoperimetric profile I_C and its power $I_C^{(n+1)/n}$ are known to be concave using a suitable deformation of an isoperimetric region and the first and second variations of perimeter and volume, as in Kuwert [7].

By approximation [19], there exists a sequence $\{K_i\}_{i \in \mathbb{N}}$ of convex bodies in \mathbb{R}^m with C^∞ boundary such that $K_i \rightarrow K$ in Hausdorff distance. Set $C_i = K_i \times \mathbb{R}^q$. By Lemma 3.4, $C_i \rightarrow C$ in Lipschitz distance. Fix now some $v > 0$. By Proposition 3.2, there is a sequence of isoperimetric sets $E_i \subset C_i$ of volume v . Thus arguing as in [17, Theorem 4.1], using the continuity of the isoperimetric profile I_C , we get

$$I_C(v) \leq \liminf_{i \rightarrow \infty} I_{C_i}(v).$$

Again by Proposition 3.2 there exists an isoperimetric set $E \subset C$ of volume v . Arguing again as in [17, Theorem 4.1], we obtain

$$I_C(v) \geq \limsup_{i \rightarrow \infty} I_{C_i}(v).$$

Combining both inequalities we get

$$I_C(v) = \lim_{i \rightarrow \infty} I_{C_i}(v).$$

So $I_C^{(n+1)/n}, I_C$ are concave functions as they are pointwise limits of concave functions.

Connectedness of isoperimetric regions is a consequence of the concavity of $I_C^{(n+1)/n}$ as in [17, Theorem 4.6]. \square

Assume now that the cylinder $C = K \times \mathbb{R}^q$ has $C^{2,\alpha}$ boundary. By Theorem 2.6 in Stredulinsky and Ziemer [21], a local minimizer of perimeter under a volume constraint has the property that either $\text{cl}(\partial E \cap \text{int}(C))$, the closure of $\partial E \cap \text{int}(C)$, is either connected or it consists of a union of parallel (totally geodesic) components meeting ∂C orthogonally with the part of C lying between any two of such components consisting of a *right* cylinder. By

the connectedness of isoperimetric regions proven in Proposition 3.5, E must be a slab in $K \times \mathbb{R}$. So we have proven the following

Theorem 3.6 *Let $C = K \times \mathbb{R}^q$ be a convex cylinder with $C^{2,\alpha}$ boundary, and $E \subset C$ an isoperimetric region. Then either the closure of $\partial E \cap \text{int}(C)$ is connected or E is an slab in $K \times \mathbb{R}$.*

Let us consider now the isoperimetric profile for small volumes. The following is inspired by [17, Theorem 6.6], although we have simplified the proof.

Theorem 3.7 *Let $C = K \times \mathbb{R}^q$, where $K \subset \mathbb{R}^m$ is a convex body. Then isoperimetric regions of small volume are close to points with the narrowest tangent cone. Furthermore,*

$$\lim_{v \rightarrow 0} \frac{I_C(v)}{I_{C_{\min}}(v)} = 1. \tag{3.4}$$

Proof To prove (3.4), consider a sequence $\{E_i\}_{i \in \mathbb{N}} \subset C$ of isoperimetric regions of volumes $v_i \rightarrow 0$. By Proposition 3.5, the sets E_i are connected. The key of the proof is to show

$$\text{diam}(E_i) \rightarrow 0. \tag{3.5}$$

To accomplish this we consider $\lambda_i \rightarrow \infty$ so that the isoperimetric regions $\lambda_i E \subset \lambda_i C$ have volume 1. Then we argue exactly as in [17, Theorem 6.6]. We first produce an elimination Lemma as in [17, Theorem 5.5], with $\varepsilon > 0$ independent of λ_i , that yields a perimeter lower density bound [17, Corollary 5.8] independent of λ_i . Hence the sequence $\{\text{diam}(\lambda_i E_i)\}_{i \in \mathbb{N}}$ must be bounded, since otherwise applying the perimeter lower density bound we would get $P_{\lambda_i C}(\lambda_i E_i) \rightarrow \infty$, contradicting Proposition 2.6. Since $\{\text{diam}(\lambda_i E_i)\}_{i \in \mathbb{N}}$ is bounded, (3.5) follows.

Translating each set of the sequence $\{E_i\}_{i \in \mathbb{N}}$, and eventually C , we may assume that E_i converges to $0 \in \partial K \times \mathbb{R}^k$ in Hausdorff distance. Taking $r_i = (\text{diam}(E_i))^{1/2}$ we have $\text{diam}(r_i^{-1} E_i) \rightarrow 0$ and so

$$r_i^{-1} E_i \rightarrow 0 \text{ in Hausdorff distance.} \tag{3.6}$$

Let $q \in \text{int}(K \cap \bar{D}(0, 1))$ and let D_q be an m -dimensional closed ball centered at q and contained in $\text{int}(K \cap \bar{D}(0, 1))$. As the sequence $r_i^{-1} K \cap \bar{D}(0, 1)$ converges to $K_0 \cap \bar{D}(0, 1)$ in Hausdorff distance, we construct, using [17, Thm. 3.4], a family of bilipschitz maps $f_i : r_i^{-1} K \cap \bar{D}(0, 1) \rightarrow K_0 \cap \bar{B}(0, 1)$ with $\text{Lip}(f_i), \text{Lip}(f_i^{-1}) \rightarrow 1$, where f_i is the identity on D_q and is extended linearly along the segments leaving from q . We define, as in Lemma 3.4, the maps $F_i : (r_i^{-1} K \cap \bar{D}(0, 1)) \times \mathbb{R}^k \rightarrow (K_0 \cap \bar{D}(0, 1)) \times \mathbb{R}^k$ by $F_i(x, y) = (f_i(x), y)$. These maps satisfy $\text{Lip}(F_i), \text{Lip}(F_i^{-1}) \rightarrow 1$. Since (3.6) holds, the maps F_i have the additional property

$$P_{C_0}(F_i(r_i^{-1} E_i)) = P_{C_0 \cap \bar{B}(0, 1)}(F_i(r_i^{-1} E_i)), \text{ for large } i \in \mathbb{N}. \tag{3.7}$$

Thus by Lemma 2.1 and (2.7) we get

$$\begin{aligned}
 \frac{P_C(E_i)}{|E_i|^{n/(n+1)}} &= \frac{P_{r_i^{-1}C}(r_i^{-1}E_i)}{|r_i^{-1}E_i|^{n/(n+1)}} \\
 &\geq \frac{P_{C_0}(F_i(r_i^{-1}E_i))}{|F_i(r_i^{-1}E_i)|^{n/(n+1)}} \left(\text{Lip}(F_i) \text{Lip}(F_i^{-1})\right)^{-n} \\
 &\geq \alpha(C_0)^{1/(n+1)} (n+1)^{n/(n+1)} \left(\text{Lip}(F_i) \text{Lip}(F_i^{-1})\right)^{-n} \tag{3.8}
 \end{aligned}$$

Since E_i are isoperimetric regions of volumes v_i , passing to the limit we get

$$\liminf_{i \rightarrow \infty} \frac{I_C(v_i)}{v_i^{n/(n+1)}} \geq \alpha(C_0)^{1/(n+1)} (n+1)^{n/(n+1)}.$$

From (2.7) we obtain,

$$\liminf_{i \rightarrow \infty} \frac{I_C(v_i)}{I_{C_0}(v_i)} \geq 1.$$

Combining this with (2.8) and the minimal property of $I_{C_{\min}}$ we deduce

$$\limsup_{i \rightarrow \infty} \frac{I_C(v_i)}{I_{C_0}(v_i)} \leq \limsup_{i \rightarrow \infty} \frac{I_C(v_i)}{I_{C_{\min}}(v_i)} \leq 1 \leq \liminf_{i \rightarrow \infty} \frac{I_C(v_i)}{I_{C_0}(v_i)}.$$

Thus

$$\lim_{i \rightarrow \infty} \frac{I_C(v_i)}{I_{C_{\min}}(v_i)} = 1. \tag{3.9}$$

By (2.7), we conclude that C_0 has minimum solid angle. □

A convex prism Π is a set of the form $P \times \mathbb{R}^q$ where $P \subset \mathbb{R}^m$ is a polytope. For convex prisms we are able to characterize the isoperimetric regions for small volumes.

Theorem 3.8 *Let $\Pi \subset \mathbb{R}^{n+1}$ be a convex prism. For small volumes the isoperimetric regions in Π are geodesic balls centered at vertices with the smallest solid angle.*

Proof Let $\{E_i\}_{i \in \mathbb{N}}$ be a sequence of isoperimetric regions in Π with $|E_i| \rightarrow 0$. By Theorem 3.7, after translation, a subsequence of E_i is close to some vertex x in Π . Since $\text{diam}(E_i) \rightarrow 0$ we can assume that the sets E_i are also subsets of the tangent cone Π_x and they are isoperimetric regions in Π_x . By [3] the only isoperimetric regions in this cone are, after translation, the geodesic balls centered at x . These geodesic balls are also subsets of Π . □

To end this section, let us characterize the isoperimetric regions for large volume in the right cylinder $K \times \mathbb{R}$. We closely follow the proof by Duzaar and Steffen [2], which is slightly simplified by the use of Steiner symmetrization. The case of the cylinder $K \times \mathbb{R}^q$, with $q > 1$, is more involved and will be treated in a different paper.

We shall say that a set $E \subset K \times \mathbb{R}$ is *normalized* if, for every $x \in K$, the intersection $E \cap (\{x\} \times \mathbb{R})$ is a segment with midpoint $(x, 0)$.

Theorem 3.9 *Let $C = K \times \mathbb{R}$, where $K \subset \mathbb{R}^n$ is a convex body. Then there is a constant $v_0 > 0$ so that the slabs $K \times I$, where $I \subset \mathbb{R}$ is a compact interval, are the only isoperimetric regions of volume larger than or equal to v_0 . In particular, $I_C(v) = 2H^n(K)$ for all $v \geq v_0$.*

Proof The proof is modeled on [2, Prop 2.11]. By comparison with slabs we have $I_C(v) \leq 2 H^n(K)$ for all $v > v_0$.

Let us assume first that $E \subset K \times \mathbb{R}$ is a normalized set of finite volume and $H^n(\partial_C E) \leq 2 H^n(K)$, and let E^* be its orthogonal projection over $K_0 = K \times \{0\}$. We claim that, if $H^n(K_0 \setminus E^*) > 0$, then there is a constant $c > 0$ so that

$$H^n(\partial_C E) \geq c|E|. \tag{3.10}$$

For $t \in \mathbb{R}$, we define $E_t = E \cap (K \times \{t\})$. As E is normalized, we can choose $\tau > 0$ so that $H^n(E_t) \leq H^n(K)/2$ for $t \geq \tau$ and $H^n(E_t) > H^n(K)/2$ for $0 < t < \tau$.

For $t \geq \tau$ we apply the coarea formula and Lemma 2.4 to get

$$\begin{aligned} H^n(\partial_C E) &\geq H^n(\partial_C E \cap (K \times [t, \infty))) \\ &\geq \int_{\tau}^{+\infty} H^{n-1}(\partial_C E_s) ds \geq c_1 \int_{\tau}^{+\infty} H^n(E_s) ds \geq c_1|E \cap (K \times [\tau, +\infty))|, \end{aligned} \tag{3.11}$$

where c_1 is a constant only depending on $H^n(K)/2$.

Let $S_t = K \times \{t\}$. For $0 < t < \tau$ we have

$$H^n(S_t \setminus E_t) \geq H^n(\partial_C E \cap (K \times (0, t))), \tag{3.12}$$

since otherwise

$$\begin{aligned} H^n(K) &= H^n(S_t \setminus E_t) + H^n(E_t) \\ &< H^n(\partial_C E \cap (K \times (0, t))) + H^n(\partial_C E \cap (K \times [t, +\infty))) \\ &\leq H^n(\partial_C E)/2, \end{aligned}$$

and we should get a contradiction to our assumption $H^n(\partial_C E) \leq 2 H^n(K)$, what proves (3.12). So we obtain from (3.12) and Lemma 2.4

$$\begin{aligned} H^n(S_t \setminus E_t) &\geq H^n(\partial_C E \cap (K \times (0, t))) \\ &\geq \int_0^t H^{n-1}(\partial_C E \cap S_t) dt \\ &\geq c_2 \int_0^{\tau} H^n(S_t \setminus E_t)^{(n-1)/n} dt, \end{aligned} \tag{3.13}$$

where c_2 is a constant only depending on $H^n(K)/2$. Letting $y(t) = H^n(S_t \setminus E_t)$, inequality (3.13) can be rewritten as the integral inequality

$$y(t) \geq c_2 \int_0^t y(s)^{(n-1)/n} ds.$$

Since $H^n(K_0 \setminus E^*) > 0$ by assumption and E is normalized, we have $y(t) > 0$ for all $t > 0$, and so

$$2 H^n(K) \geq H^n(S_{\tau} \setminus E_{\tau}) = y(\tau) \geq \frac{c_2^n}{n^n} \tau^n,$$

what implies

$$\tau \leq \frac{n}{c_2 (2 H^n(K))^{1/n}}. \tag{3.14}$$

We finally estimate

$$|E \cap (K \times [0, \tau])| = \int_0^{\tau} H^n(E_t) dt \leq 2 H^n(E_0) \tau \leq \frac{n}{c_2 (2 H^n(K))^{1/n}} H^n(\partial_C E). \tag{3.15}$$

Combining (3.11) and (3.15), we get (3.10). This proves the claim.

Let now $E \subset K \times \mathbb{R}$ be an isoperimetric region of large enough volume v . Following Talenti [22] or Maggi [10], we may consider its Steiner symmetrized $\text{sym } E$. The set $\text{sym } E$ is normalized and we have $|E| = |\text{sym } E|$ and $P_C(\text{sym } E) \leq P_C(E)$. Of course, since E is an isoperimetric region we have $P_C(\text{sym } E) = P_C(E)$. If $H^n(K_0 \setminus E^*) > 0$, then (3.10) implies

$$P_C(E) = P_C(\text{sym } E) = H^n(\partial_C(\text{sym } E)) \geq c |\text{sym } E| = c |E|,$$

providing a contradiction since $I_C \leq 2 H^n(K)$.

We conclude that $H^n(K_0 \setminus E^*) = 0$ and that E is the intersection of the subgraph of a function $u : K \rightarrow \mathbb{R}$ and the epigraph of a function $v : K \rightarrow \mathbb{R}$. The perimeter of E is then given by

$$P_C(E) = \int_K \sqrt{1 + |\nabla u|^2} dH^n + \int_K \sqrt{1 + |\nabla v|^2} dH^n \geq 2H^n(K),$$

with equality if and only if $\nabla u = \nabla v = 0$. Hence u, v are constant functions and E is a slab. □

As a consequence we have

Corollary 3.10 *Let $K \subset \mathbb{R}^n$ be a convex body and $C = K \times [0, \infty)$. Then there is a constant $v_0 > 0$ such that any isoperimetric region in M with volume $v \geq v_0$ is the slab $K \times [0, b]$, where $b = v/H^n(K)$. In particular, $I_C(v) = H^n(K)$ for $v \geq v_0$.*

Proof Just reflect with respect to the plane $x_{n+1} = 0$ and apply Theorem 3.9. Alternatively, the proof of Theorem 3.9 can also be adapted to handle this case. □

4 Cylindrically bounded convex sets

Given a cylindrically bounded convex body $C \subset \mathbb{R}^n \times \mathbb{R}$ so that K is the closure of the orthogonal projection of C over $\mathbb{R}^n \times \{0\}$, we shall say that $C_\infty = K \times \mathbb{R}$ is the *asymptotic cylinder* of C . Recall that, from our definition, C is different from a cylinder. Assuming C is unbounded in the positive vertical direction, the asymptotic cylinder can be obtained as a Hausdorff limit of downward translations of C . Another property of C_∞ is the following: given $t \in \mathbb{R}$, define

$$C_t = C \cap (\mathbb{R}^n \times \{t\}). \tag{4.1}$$

Then the orthogonal projection of C_t to $\mathbb{R}^n \times \{0\}$ converges in Hausdorff distance to the basis K of the asymptotic cylinder when $t \uparrow +\infty$ by [19, Thm. 1.8.16]. In particular, this implies

$$\lim_{t \rightarrow +\infty} H^n(C_t) = H^n(K).$$

Let us prove now that the isoperimetric profile of C is asymptotic to the one of the half-cylinder.

Theorem 4.1 *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex body with asymptotic cylinder $C_\infty = K \times \mathbb{R}$. Then*

$$\lim_{v \rightarrow \infty} I_C(v) = H^n(K). \tag{4.2}$$

Proof We assume that C is unbounded in the positive x_{n+1} -direction and consider the sets $\Omega(v) = C \cap (\mathbb{R}^n \times (-\infty, t(v)])$, where $t(v)$ is chosen so that $|\Omega(v)| = v$. Then

$$I_C(v) \leq P_C(\Omega(v)) \leq H^n(K),$$

and taking limits we get

$$\limsup_{v \rightarrow \infty} I_C(v) \leq H^n(K).$$

Let us prove now that

$$H^n(K) \leq \liminf_{v \rightarrow \infty} I_C(v). \tag{4.3}$$

Fix $\varepsilon > 0$. We consider a sequence of volumes $v_i \rightarrow \infty$ and a sequence $E_i \subset C$ of finite perimeter sets of volume v_i with smooth boundary, so that

$$P_C(E_i) \leq I_C(v_i) + \varepsilon. \tag{4.4}$$

We shall consider two cases. Recall that $(E_i)_t = E_i \cap (\mathbb{R}^n \times \{t\})$.

Case 1. $\liminf_{i \rightarrow \infty} \left(\sup_{t > 0} H^n((E_i)_t) \right) = H^n(K)$.

This is an easy case. Since the projection over the horizontal hyperplane does not increase perimeter we get

$$I_C(v_i) + \varepsilon \geq P_C(E_i) \geq \sup_{t > 0} H^n((E_i)_t).$$

Taking inferior limit, we get (4.3) since $\varepsilon > 0$ is arbitrary.

Case 2. $\liminf_{i \rightarrow \infty} \left(\sup_{t > 0} H^n((E_i)_t) \right) < H^n(K)$.

In this case, passing to a subsequence, there exists $v_0 < H^n(K)$ such that $H^n((E_i)_t) \leq v_0$ for all t . By [19, Thm. 1.8.16] we have $H^n(C_t) \rightarrow H^n(K)$. Hence there exists $t_0 > 0$ such that $v_0 < H^n(C_t)$ for $t \geq t_0$. By Lemma 2.4, for $c_t = I_{C_t}(v_0)/v_0$, we get

$$I_{C_t}(v) \geq c_t v, \quad \text{for all } v \leq v_0, t \geq t_0.$$

Furthermore, as $I_{C_t}(v_0) \rightarrow I_K(v_0) > 0$ and $I_K(v_0) > 0$, we obtain the existence of $c > 0$ such that $c_t > c$ for t large enough. Taking t_0 larger if necessary we may assume $c_t > c$ holds when $t \geq t_0$. Thus for large $i \in \mathbb{N}$ we obtain

$$\begin{aligned} |E_i| &= \int_0^\infty H^n((E_i)_t) dt \leq b + \int_{t_0}^\infty H^n((E_i)_t) dt \\ &\leq b + \int_{t_0}^\infty c_t^{-1} H^{n-1}((\partial E_i)_t) dt \\ &\leq b + c^{-1} \int_0^\infty H^{n-1}((\partial E_i)_t) dt \leq b + c^{-1} P_C(E_i), \end{aligned}$$

where $b = t_0 H^n(K)$. So $P_C(E_i) \rightarrow \infty$ when $|E_i| \rightarrow \infty$. From (4.4) and $I_C \leq H^n(K)$ we get a contradiction. This proves that Case 2 cannot hold and so (4.3) is proven. \square

Let us show now that the isoperimetric profile of C is continuous and, when the boundary of C is smooth enough, that the isoperimetric profile I_C and its normalization $I_C^{(n+1)/n}$ are both concave non-decreasing functions. We shall need first some preliminary results.

Proposition 4.2 *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex set, and $C_\infty = K \times \mathbb{R}$ its asymptotic cylinder. Consider a diverging sequence of finite perimeter sets $\{E_i\}_{i \in \mathbb{N}} \subset C$ such that $v = \lim_{i \rightarrow \infty} |E_i|$. Then*

$$\liminf_{i \rightarrow \infty} P_C(E_i) \geq I_{C_\infty}(v).$$

Proof Without loss of generality we assume $E_i \subset C \cap \{x_{n+1} \geq i\}$. Let $r > 0$ and $t_0 > 0$ so that the half-cylinder $B(0, r) \times [t_0, +\infty)$ is contained in $C \cap \{x_{n+1} \geq t_0\}$. Consider the horizontal sections $C_t = C \cap \{x_{n+1} = t\}$, $(C_\infty)_t = C_\infty \cap \{x_{n+1} = t\}$. We define a map $F : C \cap \{x_{n+1} \geq t_0\} \rightarrow C_\infty \cap \{x_{n+1} \geq t_0\}$ by

$$F(x, t) = (f_t(x), t),$$

where $f_t : C_t \rightarrow (C_\infty)_t$ is defined as in (3.6) in [17]. For $i \in \mathbb{N}$, let $F_i = F|_{C \cap \{x_{n+1} \geq i\}}$. We will check that $\max \left\{ \text{Lip}(F_i), \text{Lip} \left(F_i^{-1} \right) \right\} \rightarrow 1$ when $i \rightarrow \infty$.

Take now $(x, t), (y, s) \in C \cap \{x_{n+1} \geq i\}$, and assume $t \geq s, i \geq t_0$. Then we have

$$\begin{aligned} |F(x, t) - F(y, s)| &= (|f_t(x) - f_s(y)|^2 + |t - s|^2)^{1/2} \\ &= (|f_t(x) - f_t(y) + f_t(y) - f_s(y)|^2 + |t - s|^2)^{1/2} \\ &= (|f_t(x) - f_t(y)|^2 + |f_t(y) - f_s(y)|^2 \\ &\quad + 2|f_t(x) - f_t(y)||f_t(y) - f_s(y)| + |t - s|^2)^{1/2} \end{aligned} \tag{4.5}$$

We have $|f_t(x) - f_t(y)| \leq \text{Lip}(f_t)|x - y|$. By [17, Theorem 3.4], we can write $\text{Lip}(f_t) < (1 + \varepsilon_i)$ for $t \geq i$, where $\varepsilon_i \rightarrow 0$ when $i \rightarrow \infty$. Hence

$$|(f_t(x) - f_t(y))| \leq (1 + \varepsilon_i)|x - y|, \quad \text{for } t \geq i. \tag{4.6}$$

We estimate now $|f_t(y) - f_s(y)|$. In case $|y| \leq r$, we trivially have $|f_t(y) - f_s(y)| = 0$. So we assume $|y| \geq r$. For $u \in \mathbb{S}^{n-1}$, consider the functions $\rho_t(u) = \rho(C_t, u)$, $\rho(u) = \rho(K, u)$. Observe that, for every $u \in \mathbb{S}^n$ orthogonal to $\partial/\partial x_{n+1}$, the two-dimensional half-plane defined by u and $\partial/\partial x_{n+1}$ intersected with C is a two-dimensional convex set, and the function $t \mapsto \rho_t(u)$ is concave with a horizontal asymptotic line at height $\rho(u)$. So we have, taking $u = y/|y|$,

$$\frac{|f_t(y) - f_s(y)|}{|t - s|} = \frac{(|y| - r)}{|t - s|} \left| \frac{\rho_t(u) - r}{\rho(u) - r} - \frac{\rho_s(u) - r}{\rho(u) - r} \right| \leq \frac{|\rho_t(u) - \rho_s(u)|}{|t - s|},$$

since $|y| - r \geq \rho(u) - r$. Using the concavity of $t \mapsto \rho_t(u)$ we get

$$\frac{|\rho_t(u) - \rho_s(u)|}{|t - s|} \leq |\rho_i(u) - \rho_{i-1}(u)|, \quad \text{for } t, s \geq i.$$

Letting $\ell_i = \sup_{u \in \mathbb{S}^{n-1}} |\rho_i(u) - \rho_{i-1}(u)|$, we get

$$|f_t(y) - f_s(y)| \leq \ell_i |t - s|. \tag{4.7}$$

As C_∞ is the asymptotic cylinder of C we conclude that $\ell_i \rightarrow 0$ when $i \rightarrow \infty$.

From (4.5) to (4.7), and trivial estimates, we obtain

$$|F_i(x, t) - F_i(y, s)| \leq ((1 + \varepsilon_i)^2 + \ell_i^2 + (1 + \varepsilon_i)\ell_i)^{1/2} |(x, t) - (y, s)| \tag{4.8}$$

Now $\varepsilon_i \rightarrow 0$ and $\ell_i \rightarrow 0$ as $i \rightarrow \infty$. Thus inequality (4.8) yields

$$\limsup_{i \rightarrow \infty} \text{Lip}(F_i) \leq 1.$$

Similarly we find $\limsup_{i \rightarrow \infty} \text{Lip}(F_i^{-1}) \leq 1$ and since $\text{Lip}(F_i^{-1}) \text{Lip}(F_i) \geq 1$ by Remark 2.2, we finally get $\max\{\text{Lip}(F_i), \text{Lip}(F_i^{-1})\} \rightarrow 1$ when $i \rightarrow \infty$.

Thus we have

$$\begin{aligned} v &= \lim_{i \rightarrow \infty} |E_i| = \lim_{i \rightarrow \infty} |F_i(E_i)|, \\ \liminf_{i \rightarrow \infty} P_C(E_i) &= \liminf_{i \rightarrow \infty} P_{C_\infty}(F_i(E_i)). \end{aligned} \tag{4.9}$$

Now from (4.9) and the continuity of I_{C_∞} we get

$$\liminf_{i \rightarrow \infty} P_C(E_i) = \liminf_{i \rightarrow \infty} P_{C_\infty}(F_i(E_i)) \geq I_{C_\infty}(v).$$

□

Lemma 4.3 *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex set and $C_\infty = K \times \mathbb{R}$ its asymptotic cylinder. Let $E_\infty \subset C_\infty$ a bounded set of finite perimeter. Then there exists a sequence $\{E_i\}_{i \in \mathbb{N}} \subset C$ of finite perimeter sets such that $|E_i| = |E_\infty|$ and $\lim_{i \rightarrow \infty} P_C(E_i) = P_{C_\infty}(E_\infty)$.*

Proof Let $e_{n+1} = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. We consider the truncated downward translations of C defined by

$$C_i = (-i e_{n+1} + C) \cap \{t \geq 0\}, \quad i \in \mathbb{N}.$$

These convex bodies have the same asymptotic cylinder and

$$\bigcup_{i \in \mathbb{N}} C_i = C_\infty \cap [0, \infty). \tag{4.10}$$

Translating E_∞ along the vertical direction if necessary we assume $E_\infty \subset \{t > 0\}$. Consider the sets $G_i = E_\infty \cap C_i$. For large indices G_i is not empty by (4.10).

By the monotonicity of the Hausdorff measure we have $|G_i| \uparrow |E_\infty|$, and $H^n(\partial G_i \cap \text{int}(C_i)) \uparrow H^n(\partial E_\infty \cap \text{int}(C_\infty))$. As E_∞ is bounded, for large i we can find Euclidean geodesic balls $B_i \subset \text{int}(C_i)$, disjoint from G_i , such that $|B_i| = |E_\infty| - |G_i|$. Obviously the volume and the perimeter of these balls go to zero when i goes to infinity. Then $E_i = G_i \cup B_i$ are the desired sets. □

Proposition 4.4 *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex body. Then I_C is continuous.*

Proof Let $C_\infty = K \times \mathbb{R}$ be the asymptotic cylinder of C . The continuity of the isoperimetric profile I_C at $v = 0$ is proven by comparison with geodesic balls intersected with C .

Fix $v > 0$ and let $\{v_i\}_{i \in \mathbb{N}}$ be a sequence of positive numbers converging to v . Let us prove first the lower semicontinuity of I_C . By the definition of isoperimetric profile, given $\varepsilon > 0$, there is a finite perimeter set E_i of volume v_i so that $I_C(v_i) \leq P_C(E_i) \leq I_C(v_i) + \frac{1}{i}$, for every $i \in \mathbb{N}$. Reasoning as in [16, Thm. 2.1], we can decompose $E_i = E_i^c \cup E_i^d$ into convergent and diverging pieces, and there is a finite perimeter set $E \subset C$, eventually empty, so that

$$\begin{aligned} |E_i| &= |E_i^c| + |E_i^d|, \\ P_C(E_i) &= P_C(E_i^c) + P_C(E_i^d), \\ |E_i^c| &\rightarrow |E|, \\ P_C(E) &\leq \liminf_{i \rightarrow \infty} P_C(E_i^c). \end{aligned} \tag{4.11}$$

Let $w_1 = |E|$. By Proposition 3.2, there exists an isoperimetric region $E_\infty \subset C_\infty$ of volume $|E_\infty| = w_2 = v - w_1$. By Proposition 4.2 we have $P_{C_\infty}(E_\infty) \leq \liminf_{i \rightarrow \infty} P_C(E_i^d)$. Hence

$$\begin{aligned} I_C(v) &\leq I_C(w_1) + I_{C_\infty}(w_2) \leq P_C(E) + P_{C_\infty}(E_\infty) \\ &\leq \liminf_{i \rightarrow \infty} P_C(E_i^c) + \liminf_{i \rightarrow \infty} P_C(E_i^d) \\ &\leq \liminf_{i \rightarrow \infty} P_C(E_i) \\ &= \liminf_{i \rightarrow \infty} I_C(v_i). \end{aligned}$$

To prove the upper semicontinuity of I_C we shall use a variational argument. We can find bounded sets $E_0 \subset C$, $E_\infty \subset C_\infty$ such that $|E_0| + |E_\infty| = v$ and $I_C(v) = P_C(E_0) + P_{C_\infty}(E_\infty)$. Deforming these sets and using Lemma 4.3, we easily show the existence of subsets $E_s \subset C$, with $|E_s| = v + s$, $s \in (-\delta, \delta)$, such that

$$|H^n(\partial_C E_s) - I_C(v)| \leq Ms,$$

for some uniform constant $M > 0$. Hence

$$I_C(v + s) \leq H^n(\partial_C E_s) \leq I_C(v) + Ms.$$

Taking a sequence $v_i \rightarrow v$ we get $\limsup_{i \rightarrow \infty} I_C(v_i) \leq I_C(v)$ and we obtain the upper semicontinuity of I_C . □

Proposition 4.5 *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex body with asymptotic cylinder $C_\infty = K \times \mathbb{R}$. Assume that both C and C_∞ have smooth boundary. Then isoperimetric regions exist on C for large volumes and have connected boundary. Moreover $I_C^{(n+1)/n}$ and so I_C are concave non-decreasing functions.*

Proof Fix $v > 0$. By [16, Thm. 2.1] there exists an isoperimetric region $E \subset C$ (eventually empty) of volume $|E| = v_1 \leq v$, and a diverging sequence $\{E_i\}_{i \in \mathbb{N}}$ of finite perimeter sets of volume $v_2 = v - v_1$, such that

$$I_C(v) = P_C(E) + \lim_{i \rightarrow \infty} P_C(E_i) \tag{4.12}$$

By Proposition 3.2, there is an isoperimetric region $E_\infty \subset C_\infty$ of volume v_2 . We claim

$$\lim_{i \rightarrow \infty} P_C(E_i) = P_{C_\infty}(E_\infty). \tag{4.13}$$

If (4.13) does not hold, then Proposition 4.2 implies $\liminf_{i \rightarrow \infty} P_C(E_i) > I_{C_\infty}(v_2)$, and Lemma 4.3 provides a sequence of finite perimeter sets in C , of volume v_2 , approaching E_∞ . This way we can build a minimizing sequence of sets of volume v whose perimeters converge to some quantity strictly smaller than $I_C(v)$, a contradiction that proves (4.13). From (4.12) to (4.13) we get

$$I_C(v) = P_C(E) + P_{C_\infty}(E_\infty). \tag{4.14}$$

Reasoning as in the proof of Theorem 2.8 in [15], the configuration $E \cup E_\infty$ in the disjoint union of the sets C, C_∞ must be stationary and stable, since otherwise we could slightly perturb $E \cup E_\infty$, keeping constant the total volume, to get a set $E' \cup E'_\infty$ such that

$$P_C(E') + P_{C_\infty}(E'_\infty) < P_C(E) + P_{C_\infty}(E_\infty),$$

contradicting (4.14).

Now as C, C_∞ are convex and have smooth boundary, we can use a stability argument similar to that in [1, Proposition 3.9] to conclude that one of the sets E or E_∞ must be empty and the remaining one must have connected boundary. A third possibility, that $\partial_C E \cup \partial_{C_\infty} E_\infty$ consists of a finite number of hyperplanes intersecting orthogonally both C and C_∞ , can be discarded since in this case E_∞ would be a slab with $P_{C_\infty}(E_\infty) = 2H^n(K) > I_C$.

If v is large enough so that isoperimetric regions in C_∞ are slabs, then the above argument shows existence of isoperimetric regions of volume v in C .

As I_C is always realized by an isoperimetric set in C or C_∞ , the arguments in [1, Theorem 3.2] imply that the second lower derivative of $I_C^{(n+1)/n}$ is non-negative. As $I_C^{(n+1)/n}$ is continuous by Proposition 4.4, Lemma 3.2 in [14] implies that $I_C^{(n+1)/n}$ is concave and hence non-decreasing. Then I_C is also concave as a composition of $I_C^{(n+1)/n}$ with the concave non-increasing function $x \mapsto x^{n/(n+1)}$.

The connectedness of the isoperimetric regions in C follows easily as an application of the concavity of $I_C^{(n+1)/n}$, as in [17, Theorem 4.6]. □

The concavity of $I_C^{(n+1)/n}$ also implies the following Lemma. The proof in [17, Lemma 4.9] for convex bodies also holds in our setting.

Lemma 4.6 *Let C be a cylindrically bounded convex body with asymptotic cylinder C_∞ . Assume that both C and C_∞ have smooth boundary. Let $\lambda \geq 1$. Then*

$$I_{\lambda C}(v) \geq I_C(v) \tag{4.15}$$

for all $0 \leq v \leq |C|$.

Our aim now is to get a density estimate for isoperimetric regions of large volume in Theorem 4.9. This estimate would imply the convergence of the free boundaries of large isoperimetric regions to hyperplanes in Hausdorff distance given in Theorem 4.13.

Proposition 4.7 *Let C be cylindrically bounded convex body with asymptotic cylinder C_∞ . Given $r_0 > 0$, there exist positive constants M, ℓ_1 , only depending on r_0 and C, C_∞ , and a universal positive constant ℓ_2 so that*

$$P_{\overline{B}_C(x,r)}(v) \geq M \min\{v, |\overline{B}_C(x,r) - v\}^{n/(n+1)}, \tag{4.16}$$

for all $x \in C, 0 < r \leq r_0$, and $0 < v < |\overline{B}_C(x,r)|$. Moreover

$$\ell_1 r^{n+1} \leq |\overline{B}_C(x,r)| \leq \ell_2 r^{n+1}, \tag{4.17}$$

for any $x \in C, 0 < r \leq r_0$.

Proof Reasoning as in [17, Theorem 4.12], it is enough to show

$$\Lambda_0 = \inf_{x \in C} \text{inr}(\overline{B}_C(x, r_0)) > 0.$$

To see this consider a sequence $\{x_i\}_{i \in \mathbb{N}}$ so that $\text{inr}(\overline{B}_C(x_i, r_0))$ converges to Λ_0 . If $\{x_i\}_{i \in \mathbb{N}}$ contains a bounded subsequence then we can extract a convergent subsequence to some point $x_0 \in C$ so that $\Lambda_0 = \text{inr}(\overline{B}_C(x_0, r_0)) > 0$. If $\{x_i\}_{i \in \mathbb{N}}$ is unbounded, we translate vertically the balls $\overline{B}_C(x_i, r_0)$ so that the new centers x'_i lie in the hyperplane $x_{n+1} = 0$. Passing to

a subsequence we may assume that x'_i converges to some point $x_0 \in C_\infty$. By the proof of Proposition 4.2, we have Hausdorff convergence of the translated balls to $\overline{B}_{C_\infty}(x_0, r_0)$ and so $\Lambda_0 = \text{inr}(\overline{B}_{C_\infty}(x_0, r_0)) > 0$. □

The next Lemma appeared in [17, Lemma 5.4]. We recall the proof here for completeness.

Lemma 4.8 *For any $v > 0$, consider the function $f_v : [0, v] \rightarrow \mathbb{R}$ defined by*

$$f_v(s) = s^{-n/(n+1)} \left(\left(\frac{v-s}{v} \right)^{n/(n+1)} - 1 \right).$$

Then there is a constant $0 < c_2 < 1$ that does not depends on v so that $f_v(s) \geq -(1/2)v^{-n/(n+1)}$ for all $0 \leq s \leq c_2 v$.

Proof By continuity, $f_v(0) = 0$. Observe that $f_v(v) = -v^{-n/(n+1)}$ and that, for $s \in [0, 1]$, we have $f_v(sv) = f_1(s)v^{-n/(n+1)}$. The derivative of f_1 in the interval $(0, 1)$ is given by

$$f'_1(s) = \frac{n}{n+1} \frac{(s-1) + (1-s)^{n/(n+1)}}{s-1} s^{-1-n/(n+1)},$$

which is strictly negative and so f_1 is strictly decreasing. Hence there exists $0 < c_2 < 1$ such that $f_1(s) \geq -1/2$ for all $s \in [0, c_2]$. This implies $f_v(s) = f_1(s/v)v^{-n/(n+1)} \geq -(1/2)v^{-n/(n+1)}$ for all $s \in [0, c_2v]$. □

Theorem 4.9 *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex body with asymptotic cylinder $C_\infty = K \times \mathbb{R}$. Assume that C, C_∞ have smooth boundary. Let $E \subset C$ an isoperimetric region of volume $v > 1$. Choose ε so that*

$$0 < \varepsilon < \left\{ \ell_2^{-1}, c_2, \frac{\ell_2^n}{8^{n+1}}, \ell_2^{-1} \left(\frac{I_C(1)}{4} \right)^{n+1} \right\}, \tag{4.18}$$

where c_2 is the constant in Lemma 4.8., and ℓ_1, ℓ_2 the constants in Proposition 4.7.

Then, for any $x \in C$ and $R \leq 1$ so that $h(x, R) \leq \varepsilon$, we get

$$h(x, R/2) = 0. \tag{4.19}$$

Moreover, in case $h(x, R) = |E \cap B_C(x, R)||B_C(x, R)|^{-1}$, we get $|E \cap B_C(x, R/2)| = 0$ and, in case $h(x, R) = |B_C(x, R) \setminus E||B_C(x, R)|^{-1}$, we have $|B_C(x, R/2) \setminus E| = 0$.

Proof From the concavity of $I_C^{(n+1)/n}$ and the fact that $I_C(0) = 0$ we get, as in Lemma 4.9 in [17], the following inequality

$$I_C(w) \geq c_1 w^{n/(n+1)}, \quad c_1 = I_C(1), \tag{4.20}$$

for all $0 \leq w \leq 1$.

Assume first that

$$h(x, R) = \frac{|E \cap B_C(x, R)|}{|B_C(x, R)|}.$$

Define $m(t) = |E \cap B_C(x, t)|, 0 < t \leq R$. Thus $m(t)$ is a non-decreasing function. For $t \leq R \leq 1$ we get

$$m(t) \leq m(R) = |E \cap B_C(x, R)| = h(x, R) |B_C(x, R)| \leq h(x, R) \ell_2 R^{n+1} \leq \varepsilon \ell_2 < 1, \tag{4.21}$$

by (4.18). Since $v > 1$, we get $v - m(t) > 0$.

By the coarea formula, when $m'(t)$ exists, we obtain

$$m'(t) = \frac{d}{dt} \int_0^t H^n(E \cap \partial_C B(x, s)) ds = H^n(E \cap \partial_C B(x, t)). \tag{4.22}$$

Define

$$\lambda(t) = \frac{v^{1/(n+1)}}{(v - m(t))^{1/(n+1)}}, \quad E(t) = \lambda(t)(E \setminus B_C(x, t)). \tag{4.23}$$

Then $E(t) \subset \lambda(t)C$ and $|E(t)| = |E| = v$. By Lemma 4.6, we get $I_{\lambda(t)C} \geq I_C$ since $\lambda(t) \geq 1$. Combining this with [23, Cor. 5.5.3], Equ. (4.22), and elementary properties of the perimeter functional, we have

$$\begin{aligned} I_C(v) &\leq I_{\lambda(t)C}(v) \leq P_{\lambda(t)C}(E(t)) = \lambda^n(t) P_C(E \setminus B_C(x, t)) \\ &\leq \lambda^n(t) (P_C(E) - P(E, B_C(x, t)) + H^n(E \cap \partial B_C(x, t))) \\ &\leq \lambda^n(t) (P_C(E) - P_C(E \cap B_C(x, t)) + 2H^n(E \cap \partial B_C(x, t))) \\ &\leq \lambda^n(t) (I_C(v) - c_1 m(t)^{n/(n+1)} + 2m'(t)), \end{aligned} \tag{4.24}$$

where c_1 is the constant in (4.20). Multiplying both sides by $I_C(v)^{-1} \lambda(t)^{-n}$ we find

$$\lambda(t)^{-n} - 1 + \frac{c_1}{I_C(v)} m(t)^{n/(n+1)} \leq \frac{2}{I_C(v)} m'(t). \tag{4.25}$$

As we have $I_C \leq H^n(K)$, and I_C is concave by Proposition 4.5, there exists a constant $\alpha > 0$ such that $I_C \geq \alpha$ for sufficient large volumes. Set

$$a = \frac{2}{\alpha} \geq \frac{2}{I_C(v)}, \quad \text{and} \quad b = \frac{c_1}{H^n(K)} \leq \frac{c_1}{I_C(v)}. \tag{4.26}$$

From the definition (4.23) of $\lambda(t)$ we get

$$f(m(t)) \leq am'(t) \quad H^1\text{-a.e.}, \tag{4.27}$$

where

$$\frac{f(s)}{s^{n/(n+1)}} = b + \frac{\left(\frac{v-s}{v}\right)^{n/(n+1)} - 1}{s^{n/(n+1)}}. \tag{4.28}$$

By Lemma 4.8, there exists a universal constant $0 < c_2 < 1$, not depending on v , so that

$$\frac{f(s)}{s^{n/n+1}} \geq b/2 \quad \text{whenever} \quad 0 < s \leq c_2. \tag{4.29}$$

Since $\varepsilon \leq c_2$ by (4.18), Eq. (4.29) holds in the interval $[0, \varepsilon]$. If there were $t \in [R/2, R]$ such that $m(t) = 0$ then, by monotonicity of $m(t)$, we would conclude $m(R/2) = 0$ as well. So we assume $m(t) > 0$ in $[R/2, R]$. Then by (4.27) and (4.29), we get

$$b/2a \leq \frac{m'(t)}{m(t)^{n/n+1}}, \quad H^1\text{-a.e.}$$

Integrating between $R/2$ and R we get by (4.21)

$$bR/4a \leq \left(m(R)^{1/(n+1)} - m(R/2)^{1/(n+1)}\right) \leq m(R)^{1/(n+1)} \leq (\varepsilon \ell_2)^{1/(n+1)} R.$$

This is a contradiction, since $\varepsilon \ell_2 < (b/4a)^{n+1} = I_C(v)^{n+1} / (8^{n+1} v^n) \leq \ell_2^{n+1} / 8^{n+1}$ by (4.18) and Proposition 2.6. So the proof in case $h(x, R) = |E \cap B_C(x, R)| / |B_C(x, R)|^{-1}$

is completed. For the remaining case, when $h(x, R) = |B_C(x, R)|^{-1}|B_C(x, R) \setminus E|$, we use Lemma 2.4 and the fact that I_C is non-decreasing proven in Proposition 4.5. Then we argue as in Case 1 in Lemma 4.2 of [8] to get

$$c_1/4 \leq (\varepsilon \ell_2)^{1/(n+1)}.$$

This is a contradiction, since $\varepsilon \ell_2 < (c_1/4)^{n+1}$ by assumption (4.18) □

Proposition 4.10 *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex body and C_∞ its asymptotic cylinder. Assume that both C and C_∞ have smooth boundary. Then there exists a constant $c > 0$ such that, for each isoperimetric region E of volume $v > 1$,*

$$P(E, B_C(x, r)) \geq cr^n, \tag{4.30}$$

for $r \leq 1$ and $x \in \partial_C E$.

Proof Let $E \subset C$ be an isoperimetric region of volume larger than 1. Choose $\varepsilon > 0$ satisfying (4.18). Since $x \in \partial_C E$ we have $\lim_{r \rightarrow 0} h(x, r) \neq 0$ and, by Theorem 4.9, $h(x, r) \geq \varepsilon$ for $0 < r \leq 1$. So we get

$$\begin{aligned} P(E, B_C(x, r)) &\geq M \min \{|E \cap B_C(x, r)|, |B_C(x, r) \setminus E|\}^{n/(n+1)} \\ &= M (|B_C(x, r)| h(x, r))^{n/(n+1)} \geq M (|B_C(x, r)| \varepsilon)^{n/(n+1)} \\ &\geq M (\ell_1 \varepsilon)^{n/(n+1)} r^n. \end{aligned}$$

Inequality (4.30) follows by taking $c = M(\ell_1 \varepsilon)^{n/(n+1)}$, which is independent of v . □

Remark 4.11 Theorem 4.9 and Proposition 4.10 also hold if C is a convex cylinder.

As a Corollary we obtain a new proof of Theorem 3.9

Corollary 4.12 *Let $C = K \times \mathbb{R}$, where $K \subset \mathbb{R}^n$ is a convex body. Then there is a constant $v_0 > 0$ so that $I_C(v) = 2H^n(K)$ for all $v \geq v_0$. Moreover, the slabs $K \times [t_1, t_2]$ are the only isoperimetric regions of volume larger than or equal to v_0 .*

Proof Let E be an isoperimetric region with volume

$$|E| > 2mr_0H^n(K), \tag{4.31}$$

where $r_0, c > 0$, are the constants in Proposition 4.10 (see also Remark 4.11), and $m > 0$ is chosen so that

$$mcr_0^n > 2H^n(K). \tag{4.32}$$

By results of Talenti on Steiner symmetrization for finite perimeter sets [22], we can assume that the boundary of E is the union of two graphs, symmetric with respect to a horizontal hyperplane, over a subset $K^* \subset K$. If $K^* = K$ then $P_C(E) \geq 2H^n(K)$, since the orthogonal projection over $K \times \{0\}$ is perimeter non-increasing. This implies $P_C(E) = 2H^n(K)$ and it follows, as in the proof of Theorem 3.9, that E is a slab.

So assume that K^* is a proper subset of K . Since $|E| > 2mr_0H^n(K)$, E cannot be contained in the slab $K \times [-r_0m, r_0m]$. Then as $\partial_C E$ is a union of two graphs over K^* we can find $x_j \in \partial_C E$, $1 \leq j \leq m$, so that the balls centered at these points are disjoint. Then by the lower density bound (4.30) we get

$$P_C(E) \geq \sum_{j=1}^m P(E, B_C(x_j, r_0)) \geq mcr_0^n > 2H^n(K), \tag{4.33}$$

a contradiction since $I_C \leq 2H^n(K)$. □

Recall that, in Corollary 3.10, we showed that, given a half-cylinder $K \times [0, \infty)$, there exists $v_0 > 0$ so that every isoperimetric region in $K \times [0, \infty)$ of volume larger than or equal to v_0 is a slab $K \times [0, b]$, where $b = v/H^n(K)$. We can use this result to obtain

Theorem 4.13 *Let $C \subset \mathbb{R}^{n+1}$ be a cylindrically bounded convex body, $C_\infty = K \times \mathbb{R}$ its asymptotic cylinder and $C_\infty^+ = K \times [0, \infty)$. Let $\{E_i\}_{i \in \mathbb{N}}$ be a sequence of isoperimetric regions with $\lim_{i \rightarrow \infty} |E_i| = \infty$.*

Then truncated downward translations of E_i converge in Hausdorff distance to a half-slab $K \times [0, b]$ in C_∞^+ . The same convergence result holds for their free boundaries.

Proof By Corollary 3.10, we can choose $v_0 > 0$ such that each isoperimetric region with volume $v \geq v_0$ in C_∞^+ is a half-slab $K \times [0, b(v)]$ of perimeter $H^n(K)$, where $b(v) = v/H^n(K)$.

Since $|E_i| \rightarrow \infty$, we can find vertical vectors y_i , with $|y_i| \rightarrow \infty$, so that $\Omega_i = (-y_i + E_i) \cap \{x_{n+1} \geq 0\}$ has volume v_0 for large enough $i \in \mathbb{N}$. We observe also that, by Proposition 4.10 and the fact that $I_C \leq H^n(K)$, the sets ∂E_i have uniformly bounded diameter.

Consider the convex bodies

$$C_i = (-y_i + C) \cap \{x_{n+1} \geq 0\}, \tag{4.34}$$

for $i \in \mathbb{N}$. The sets C_i have the same asymptotic cylinder C_∞ and we have

$$\bigcup_{i \in \mathbb{N}} C_i = C_\infty^+. \tag{4.35}$$

By construction

$$P_{C_i}(\Omega_i) \leq P_C(E_i) \leq H^n(K). \tag{4.36}$$

Since ∂E_i are uniformly bounded and $|\Omega_i| = v_0$, there exists a Euclidean geodesic ball B such that $\Omega_i \subset B$ for all $i \in \mathbb{N}$. By (4.35) the sequence of convex bodies $\{C_i \cap B\}_{i \in \mathbb{N}}$ converges to $C_\infty^+ \cap B$ in Hausdorff distance and, by [17, Theorem 3.4], in lipschitz distance.

Hence, by the proof of [17, Theorem 3.4] and [17, Lemma 2.3], we conclude there exists a finite perimeter set $\Omega \subset C_\infty^+$, such that

$$\Omega_i \xrightarrow{L^1} \Omega \quad \text{and} \quad P_{C_\infty^+}(\Omega) \leq \liminf_{i \rightarrow \infty} P_{C_i}(\Omega_i). \tag{4.37}$$

So we obtain from (4.36) and (4.37),

$$H^n(K) = I_{C_\infty^+}(v_0) \leq P_{C_\infty^+}(\Omega) \leq \liminf_{i \rightarrow \infty} P_{C_i}(\Omega_i) \leq \liminf_{i \rightarrow \infty} P_C(E_i) \leq H^n(K), \tag{4.38}$$

what implies that Ω is an isoperimetric region of volume v_0 in C_∞^+ and so it is a slab.

Furthermore, the arguments of [17, Theorem 5.11] and [17, Theorem 5.13] can be applied here to improve the L^1 convergence to Hausdorff convergence, both for the sets Ω_i and for their free boundaries. □

Remark 4.14 The proof of Theorem 4.13 implies $\lim_{v \rightarrow \infty} I_C(v) = H^n(K)$. This way we get a different proof of Theorem 4.1.

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