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Calculus of Variations

Positive solutions of quasi-linear elliptic equations with dependence on the gradient

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Abstract In the present paper we prove a multiplicity theorem for a quasi-linear elliptic problem with dependence on the gradient ensuring the existence of a positive solution and of a negative solution. In addition, we show the existence of the extremal constant-sign solutions: the smallest positive solution and the biggest negative solution. Our approach relies on extremal solutions for an auxiliary parametric problem. Other basic tools used in our paper are sub-supersolution techniques, Schaefer's fixed point theorem, regularity results and strong maximum principle. In our hypotheses we only require a general growth condition with respect to the solution and its gradient, and an assumption near zero involving the first eigenvalue of the negative p-Laplacian operator.

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1 Introduction and statement of main results

There is a wide literature on semilinear and quasilinear elliptic equations with gradient dependence on the nonlinear term of the following type

$$(P) \quad \begin{cases} -\Delta_p u = f(x, u, \nabla u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

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where Ω is a smooth bounded domain in $\mathbb{R}^N$, $1 < p < N$, Δ_p is the p-Laplacian operator and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function.

In this setting, the classical variational methods cannot be applied. This kind of problems are usually studied by means of topological degree, method of sub-supersolutions, fixed point theory and approximation techniques. For instance, in [2] the authors, assuming that f is a C^1 function with growth given by

$$|f(x, s, \xi)| \leq a(s)(1 + |\xi|^2),$$

for some increasing function a, obtain a solution of (P) in an ordered interval of sub-supersolutions. In this respect, we also mention [14] where existence results for (P), when $p = 2$, are obtained via sub-supersolutions in the Sobolev space $W^{2,q}(\Omega)$ with $q > N$, provided $f(x, s, \xi)$ is Lipschitzian with respect to ξ. For the general theory of sub-supersolutions for nonlinear elliptic problems depending on the gradient we refer to [4]. In [15], by combining Krasnoselskii's fixed point theorem in cones with blow up techniques, the existence of a positive solution of (P) is proved when $f(x, s, \xi)$ is a non negative function and has a suitable growth with respect to s and ξ. Recently, in [5] and [16], under different growth in the gradient, the existence of a positive solution is achieved via an approximation on finite dimensional subspaces. An approximation approach in a general functional setting with a p-sublinear growth condition in the gradient can be found in [3]. A different approach is proposed in [7] where the authors prove the existence of a positive and a negative solution for (P), when $p = 2$, through an iterative method involving Mountain Pass technique assuming that $f(x, s, \xi)$ satisfies Lipschitz conditions on s and ξ in a neighborhood of zero and has a growth like

$$|f(x, s, \xi)| \leq a_1(1 + |s|^q), \quad \text{with} \quad 1 < q < 2^* - 1.$$

It is shown in [11, Theorem 4.3] that if there exist a subsolution $\underline{u}$ and a supersolution $\overline{u}$ belonging to $C^1(\overline{\Omega})$ with $\underline{u} \leq \overline{u}$, then the growth condition of Bernstein-Nagumo type

$$|f(x, s, \xi)| \leq a(x) + b|\xi|^p, \quad \text{with} \quad a \in L^{\frac{p}{p-1}}(\Omega) \quad \text{and} \quad b > 0,$$

for a.a. $x \in \Omega$, all $s \in [\underline{u}(x), \overline{u}(x)]$, all $\xi \in \mathbb{R}^N$, implies the existence of a solution $u \in W_0^{1,p}(\Omega)$ of (P) satisfying $\underline{u} \leq u \leq \overline{u}$. Different other results based on the sub- supersolution method can be found in [6].

In the present paper, we will prove the existence of a positive and a negative solution for problem (P) by combining sub-supersolution techniques with Schaefer's fixed point theorem. More precisely, we will consider an auxiliary problem which can be studied through operator theory and sub-supersolutions method and we will obtain solutions of extremal type. This allows us to construct a map whose fixed points are exactly the solutions of our problem (P). Finally, we will show the existence of a fixed point for the constructed map. Moreover, under the same assumptions, we will prove the existence of the smallest positive and of the biggest negative solution of (P). Notice that our assumptions imply that zero is a solution of our problem.

We believe that our result gives a natural approach to the theory of quasilinear elliptic problems with gradient dependence. Furthermore, the hypotheses we assume on the nonlinear reaction term are general and verifiable. An example is provided at the end of our work.

Let us introduce our main results. In the sequel $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function that is, $f(\cdot, s, \xi)$ is measurable for every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, $f(x, \cdot, \cdot)$ is continuous for almost every $x \in \Omega$. Let p' stand for the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. As usual, λ_1 denotes the first eigenvalue of the negative p-Laplacian operator on $W_0^{1,p}(\Omega)$. For a later use, we recall that the cone of nonnegative functions $C_0^1(\overline{\Omega})_+ = \{u \in C_0^1(\overline{\Omega}) : u \geq 0 \quad \text{in} \quad \Omega\}$ has a nonempty interior in the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u = 0 \quad \text{on} \quad \partial\Omega\}$ given by

$$\operatorname{int}(C_0^1(\overline{\Omega})_+) = \left\{u \in C_0^1(\overline{\Omega}) : u > 0 \ \text{ in } \ \Omega, \frac{\partial u}{\partial \nu} < 0 \ \text{ on } \ \partial\Omega\right\},$$

where ν stands for the outward normal unit vector to $\partial\Omega$.

Our assumptions are:

(f_1) for every $M > 0$, there exist constants $k_M > 0$ and $0 < \theta_M < \lambda_1$ such that

$$|f(x, s, \xi)| \leq k_M + \theta_M |s|^{p-1}$$

for a.e. $x \in \Omega$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$ with $|\xi| \leq M$;

$(f_2)_+$ for every $M > 0$ there exists a constant $\eta_M > \lambda_1$ such that

$$\liminf_{s \to 0^+} \frac{f(x, s, \xi)}{s^{p-1}} \geq \eta_M > \lambda_1$$

uniformly for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$ with $|\xi| \leq M$;

$(f_3)_+$ for every $M > 0$ there exists a constant $\zeta_M > 0$ such that

$$\limsup_{s \to 0^+} \frac{f(x, s, \xi)}{s^{p-1}} \leq \zeta_M$$

uniformly for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$ with $|\xi| \leq M$.

Assuming (f_1), $(f_2)_+$, $(f_3)_+$, for every $w \in C_0^1(\overline{\Omega})$, the Dirichlet problem

$$(P_w) \qquad \begin{cases} -\Delta_p u = f(x, u, \nabla w) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

has a smallest positive solution $u_w \in C_0^1(\overline{\Omega})$. Then we introduce the map

$$T : C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega}), \ w \mapsto u_w,$$

which is continuous and compact. We notice that the fixed points of T coincide with the solutions of (P). Later on, to apply Schaefer's fixed point theorem, we need to strengthen the growth condition (f_1). Namely, we will need

$(\widetilde{f_1})$ there exist positive constants k_0, θ_0, θ_1 with $\theta_0 + \theta_1 \lambda_1^{1/p'} < \lambda_1$ such that

$$|f(x, s, \xi)| \leq k_0 + \theta_0 |s|^{p-1} + \theta_1 |\xi|^{p-1}$$

for a.a. $x \in \Omega$, all $s \in \mathbb{R}$, and $\xi \in \mathbb{R}^N$.

Our first result reads as follows:

Theorem 1.1 *Assume* $(\widetilde{f_1}), (f_2)_+, (f_3)_+$. *Then, problem* (P) *has a solution* $u \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

Let us state now the counterpart of the previous theorem on the negative half-line. We formulate in a symmetric way the corresponding hypotheses:

$(f_2)_-$ for every $M > 0$ there exists a constant $\eta_M > \lambda_1$ such that

$$\liminf_{s \to 0^-} \frac{f(x, s, \xi)}{|s|^{p-2} s} \geq \eta_M > \lambda_1$$

uniformly for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$ with $|\xi| \leq M$;

$(f_3)_-$ for every $M > 0$ there exists a constant $\zeta_M > 0$ such that

$$\limsup_{s\to 0^-} \frac{f(x,s,\xi)}{|s|^{p-2}s} \le \zeta_M$$

uniformly for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$ with $|\xi| \le M$.

On the pattern of Theorem 1.1 we can state

Theorem 1.2 *Assume* $(\widetilde{f}_1), (f_2)_-, (f_3)_-$. *Then, problem* (P) *has a solution* $v \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$.

By combining Theorems 1.1 and 1.2 we obtain our main multiplicity result.

Theorem 1.3 *Assume* $(\widetilde{f}_1), (f_2)_\pm, (f_3)_\pm$. *Then, problem* (P) *has at least two solutions* u *and* v, *with* $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ *and* $v \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$.

Corollary 1.1 *Under the same assumptions as in Theorem 1.3, problem* (P) *has the smallest positive solution and the biggest negative solution.*

Remark 1.1 Notice that if u is a positive solution of (P), then it is valid the estimate

$$f(x,u(x),\nabla u(x)) \le \lambda_1 u^{p-1}(x)$$

on a set of positive measure in Ω. Indeed, if not, u solves the problem

$$\begin{cases} -\Delta_p u = m(x)u^{p-1} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

for a.e. $x \in \Omega$, where $m(x) = \frac{f(x,u(x),\nabla u(x))}{u^{p-1}(x)} > \lambda_1$. Then the function u has to change sign (see Remark 2.1 below), which contradicts that u is positive. An analogous remark holds for a negative solution v of (P).

2 Auxiliary results

We split the present section in two parts. The first one deals with the sub-supersolution method for an auxiliary problem with fixed gradient in the right-hand side of the elliptic equation. These preliminary results will be used to construct the map T on which relies our fixed point approach for investigating problem (P). The properties of the map T will be studied in the second part of this section.

2.1 Sub-supersolution method

Let us first recall some well known results that are needed in the sequel. In what follows we endow the Sobolev space $W_0^{1,p}(\Omega)$ with the standard norm $\|u\| = \left(\int_\Omega |\nabla u|^p \, dx\right)^{\frac{1}{p}}$. As usual, we set $u^+ = \max\{u,0\}$ and $u^- = \max\{0,-u\}$. It is well known if $u \in W_0^{1,p}(\Omega)$, then $u^+, u^- \in W_0^{1,p}(\Omega)$.

Given $m \in L^\infty(\Omega)_+$, $m \neq 0$, consider the nonlinear weighted eigenvalue problem

$$\begin{cases} -\Delta_p u = \hat{\lambda} m(x)|u|^{p-2}u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}.$$

The least number $\hat{\lambda} > 0$, denoted by $\hat{\lambda}_1(m)$, such that the above problem admits a nontrivial solution is called the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega), m)$. It is well known that $\hat{\lambda}_1(m)$ is positive, isolated, simple and the following variational characterization holds

$$\hat{\lambda}_1(m) = \min\left\{\frac{\|u\|^p}{\int_\Omega m|u|^p\,dx} : u \in W_0^{1,p}(\Omega), u \neq 0\right\}.$$

We denote by $\phi_{1,m}$ the positive eigenfunction normalized as $\|\phi_{1,m}\|_p = 1$, which is associated to $\hat{\lambda}_1(m)$. One has $\phi_{1,m} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$.

As usual, if $m \equiv 1$, set $\lambda_1 = \hat{\lambda}_1(m)$ and $\phi_1 = \phi_{1,m}$. The next remark contains useful information on the weighted eigenvalue problems (for the proof and further details we refer to [1]).

Remark 2.1 (1) If $m_1, m_2 \in L^\infty(\Omega)_+\setminus\{0\}$ satisfy $m_1 \leq m_2$ a.e. in Ω, then one has $\hat{\lambda}_1(m_2) \leq \hat{\lambda}_1(m_1)$. If in addition $m_1 \neq m_2$, then, $\hat{\lambda}_1(m_2) < \hat{\lambda}_1(m_1)$.
(2) If u is an eigenfunction corresponding to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1(m)$, then $u \in C_0^1(\overline{\Omega})$ changes sign.

Remark 2.2 Because of assumptions $(f_2)_+, (f_3)_+$, we get that $f(x, 0, \xi) = 0$ for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^N$. So, in particular, $u = 0$ is a solution of (P).

Since in the construction below we will deal with positive solutions, without loss of generality we may assume that $f(x, s, \xi) = 0$ for a.e. $x \in \Omega$, $s \leq 0$, $\xi \in \mathbb{R}^N$.

We are going to consider an auxiliary problem and to prove existence of solutions for it. Namely, for every $w \in C_0^1(\overline{\Omega})$, let us state the Dirichlet problem

$$(P_w)\begin{cases} -\Delta_p u = f(x, u, \nabla w) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We recall that, for fixed $w \in C_0^1(\overline{\Omega})$, a function $\overline{u}_w \in W^{1,p}(\Omega)$, with $\overline{u}_w \geq 0$ on $\partial\Omega$ (in the sense of trace), is a *supersolution* for problem (P_w) if

$$\int_\Omega |\nabla\overline{u}_w|^{p-2}\nabla\overline{u}_w\nabla v\,dx \geq \int_\Omega f(x, \overline{u}_w, \nabla w)v\,dx$$

for all $v \in W_0^{1,p}(\Omega)$, $v \geq 0$ a.e. in Ω. A function $\underline{u}_w \in W^{1,p}(\Omega)$, with $\underline{u}_w \leq 0$ on $\partial\Omega$ (in the sense of trace), is a *subsolution* for problem (P_w) if

$$\int_\Omega |\nabla\underline{u}_w|^{p-2}\nabla\underline{u}_w\nabla v\,dx \leq \int_\Omega f(x, \underline{u}_w, \nabla w)v\,dx$$

for all $v \in W_0^{1,p}(\Omega)$, $v \geq 0$ a.e. in Ω.

By a solution of problem (P_w) we mean a weak solution, i.e. a function $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx = \int_\Omega f(x, u, \nabla w)v\,dx$$

for all $v \in W_0^{1,p}(\Omega)$.

Before stating the theorem guaranteeing existence of solutions of (P_w), some auxiliary lemmas are required.

Lemma 2.1 *Assume* (f_1). *Then, for every* $w \in C_0^1(\overline{\Omega})$ *there exists* $\overline{u}_w \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ *supersolution of* (P_w).

Proof Let us fix $w \in C_0^1(\overline{\Omega})$ and set $M = \|w\|_{C^1}$. From assumption (f_1), we have that

$$|f(x, s, \nabla w(x))| \leq k_M + \theta_M |s|^{p-1} \tag{2.1}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, where $k_M > 0$ and $0 < \theta_M < \lambda_1$.

Consider the following Dirichlet problem

$$\begin{cases} -\Delta_p u = k_M + \theta_M |u|^{p-1} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Since the embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact, the superposition operator $K_M : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by $K_M(u(\cdot)) = k_M + \theta_M |u(\cdot)|^{p-1}$ is completely continuous. Since the p-Laplacian operator is strictly monotone, continuous and bounded (see [4]), we have that $-\Delta_p - K_M$ is pseudomonotone and bounded. Thanks to the estimate

$$\frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \int_\Omega \left[k_M |u| + \frac{1}{p}\theta_M |u|^p \right] dx \geq \frac{1}{p}\left(1 - \frac{\theta_M}{\lambda_1}\right) \|u\|^p - k_M |\Omega|^{\frac{1}{p'}} \lambda_1^{-\frac{1}{p}} \|u\|,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω, and using $\theta_M < \lambda_1$ (see (f_1)), it follows that $-\Delta_p - K_M$ is coercive, hence surjective. Therefore, there exists a function $\overline{u}_w \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta_p \overline{u}_w = k_M + \theta_M |\overline{u}_w|^{p-1} & \text{in} \quad \Omega \\ \overline{u}_w = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Let us prove that $\overline{u}_w \geq 0$. Acting as test function with $-\overline{u}_w^-$, we get

$$\int_\Omega |\nabla \overline{u}_w^-|^p \, dx = -\int_\Omega |\nabla \overline{u}_w|^{p-2} \nabla \overline{u}_w \nabla \overline{u}_w^- \, dx = -\int_\Omega (k_M + \theta_M |\overline{u}_w|^{p-1}) \overline{u}_w^- \, dx \leq 0,$$

which implies that $\overline{u}_w^- = 0$, so $\overline{u}_w \geq 0$. Notice that $\overline{u}_w \neq 0$. By classical regularity results (see [9,10,17]), we have that $\overline{u}_w \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Then from (2.1) we infer that $\overline{u}_w$ is a supersolution of (P_w), which completes the proof. □

Lemma 2.2 *Assume $(f_2)_+$. Then, for every $w \in C_0^1(\overline{\Omega})$, there exists $\delta = \delta(w) > 0$ such that if $0 < \varepsilon < \delta$, then $\varepsilon\phi_1$ is a subsolution of (P_w).*

Proof Let us fix $w \in C_0^1(\overline{\Omega})$ and set $M = \|w\|_{C^1}$. From assumption $(f_2)_+$, for $\sigma > 0$ so small that $\eta_M - \sigma > \lambda_1$, there exists $\gamma = \gamma(w) > 0$ such that

$$f(x, s, \nabla w(x)) \geq (\eta_M - \sigma) s^{p-1} > \lambda_1 s^{p-1} \tag{2.2}$$

for a.e. $x \in \Omega$ and all $0 < s < \gamma$. Set $\delta = \gamma \|\phi_1\|_{L^\infty}^{-1}$. Then, for every $0 < \varepsilon < \delta$ and all $\varphi \in W^{1,p}(\Omega)$, $\varphi \geq 0$, we find that

$$\int_\Omega |\nabla(\varepsilon\phi_1)|^{p-2} \nabla(\varepsilon\phi_1) \nabla\varphi \, dx = \lambda_1 \int_\Omega (\varepsilon\phi_1)^{p-1} \varphi \, dx \leq \int_\Omega f(x, \varepsilon\phi_1, \nabla w(x)) \varphi \, dx,$$

that is, $\varepsilon\phi_1$ is a subsolution of (P_w). □

Now we are ready to prove the main result of this subsection.

Theorem 2.1 *Assume (f_1), $(f_2)_+$, $(f_3)_+$. Then, for every $w \in C_0^1(\overline{\Omega})$ there exists $u_w \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ which is the smallest positive solution of (P_w).*

Proof Let us fix $w \in C_0^1(\overline{\Omega})$ and set $M = \|w\|_{C^1}$. From Lemmas 2.1 and 2.2, we get that there exist a supersolution $\overline{u}_w$ and for $0 < \varepsilon < \delta(w)$ a subsolution $\varepsilon\phi_1$ of (P_w). Since $\overline{u}_w$ and $\varepsilon\phi_1$ belong to $\operatorname{int}(C_0^1(\overline{\Omega})_+)$, it is possible to choose ε small enough in order that $\overline{u}_w - \varepsilon\phi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Consider now the following truncation of f:

$$f_+(x,s) = \begin{cases} f(x, \varepsilon\phi_1(x), \nabla w(x)) & \text{if} \quad s < \varepsilon\phi_1(x) \\ f(x, s, \nabla w(x)) & \text{if} \quad \varepsilon\phi_1(x) \le s \le \overline{u}_w(x) \\ f(x, \overline{u}_w(x), \nabla w(x)) & \text{if} \quad s > \overline{u}_w(x). \end{cases}$$

Denote by $\mathcal{E}_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ the associated energy functional, that is

$$\mathcal{E}_+(u) = \frac{1}{p}\|u\|^p - \int_\Omega \int_0^{u(x)} f_+(x,t)\,dt\,dx \quad \text{for all} \quad u \in W_0^{1,p}(\Omega).$$

Clearly, $\mathcal{E}_+$ is sequentially weakly lower semicontinuous, coercive, and continuously differentiable. Hence, it has a global minimum $\tilde{u}_w^\varepsilon$ which is a critical point of $\mathcal{E}_+$, thus a weak solution of

$$\begin{cases} -\Delta_p u = f_+(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, by a standard comparison argument one can show that

$$\varepsilon\phi_1 \le \tilde{u}_w^\varepsilon \le \overline{u}_w,$$

a.e. in Ω, so that $\tilde{u}_w^\varepsilon$ is a solution of (P_w). Also the strong maximum principle entails $\tilde{u}_w^\varepsilon \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Denote by S_ε the set of C_0^1-solutions of (P_w) which lie in the ordered interval $[\varepsilon\phi_1, \overline{u}_w]$. As seen from above, $S_\varepsilon \neq \emptyset$. If we consider in S_ε the pointwise order, then S_ε is downward directed and, as can be noticed through Zorn's Lemma, it has a minimal element u_w^ε (see [12] for more details). Let us prove that u_w^ε is the smallest solution of (P_w) in S_ε. To this end, take $v \in S_\varepsilon$. The function $\min\{v, u_w^\varepsilon\}$ is a supersolution of (P_w) and clearly, $\min\{v, u_w^\varepsilon\} \ge \varepsilon\phi_1$. Then, there exists a solution z of (P_w) such that $\varepsilon\phi_1 \le z \le \min\{v, u_w^\varepsilon\}$. In particular, it turns out that $z \in S_\varepsilon$, $z \le u_w^\varepsilon$, and from the minimality of u_w^ε in S_ε, we conclude that $z = u_w^\varepsilon$. So $u_w^\varepsilon \le v$, as we wished.

Fix now a decreasing sequence $\{\varepsilon_n\}_n$ of positive numbers such that $\varepsilon_n \to 0$. For every $n \in \mathbb{N}$ there exists $u_w^{\varepsilon_n}$ which is the smallest solution of (P_w) in the ordered interval $[\varepsilon_n\phi_1, \overline{u}_w]$. The sequence $\{u_w^{\varepsilon_n}\}_n$ is bounded in $W_0^{1,p}(\Omega)$, so there exists $u_w \in W_0^{1,p}(\Omega)$ such that $u_w^{\varepsilon_n} \rightharpoonup u_w$ in $W_0^{1,p}(\Omega)$. In particular, $u_w^{\varepsilon_n} \to u_w$ in $L^p(\Omega)$ and $u_w^{\varepsilon_n}(x) \to u_w(x)$ for a.e. $x \in \Omega$. Notice that $\{u_w^{\varepsilon_n}\}_n$ is decreasing by construction, so that the convergence $u_w^{\varepsilon_n} \to u_w$ is uniform.

We want to prove that $u_w \neq 0$. Assume by contradiction that $u_w = 0$ and set $z_n = \frac{u_w^{\varepsilon_n}}{\|u_w^{\varepsilon_n}\|}$. Hence $z_n \in W_0^{1,p}(\Omega)$ and $\|z_n\| = 1$ for every $n \in \mathbb{N}$. So, $\{z_n\}_n$ converges weakly in $W_0^{1,p}(\Omega)$ to some $z \in W_0^{1,p}(\Omega)$. On account of $z_n \ge 0$ a.e. in Ω, we get $z \ge 0$ a.e. in Ω. Denote

$$h_n(x) = \frac{f(x, u_w^{\varepsilon_n}(x), \nabla w(x))}{(u_w^{\varepsilon_n}(x))^{p-1}}.$$

The fact that $u_w^{\varepsilon_n}$ is a solution of (P_w) reads as

$$-\Delta_p u_w^{\varepsilon_n} = f(x, u_w^{\varepsilon_n}(x), \nabla w(x)),$$

so we get

$$-\Delta_p z_n = h_n(x) z_n^{p-1}. \tag{2.3}$$

Notice also that by invoking assumptions $(f_2)_+$ and $(f_3)_+$ and using the uniform convergence of $\{u_w^{\varepsilon_n}\}_n$ to zero, we deduce (for a possibly larger ζ_M)

$$\lambda_1 < \eta_M - \sigma \leq \frac{f(x, u_w^{\varepsilon_n}(x), \nabla w(x))}{(u_w^{\varepsilon_n}(x))^{p-1}} \leq \zeta_M,$$

that is

$$\lambda_1 < \eta_M - \sigma \leq h_n(x) \leq \zeta_M$$

a.e. $x \in \Omega$, whenever n is sufficiently large. From the above inequality we get that h_n is bounded in $L^\infty(\Omega)$. Consequently, there exists a function $h \in L^{p'}(\Omega)$ such that $h_n \rightharpoonup h$ in $L^{p'}(\Omega)$. By Mazur's Lemma (see [8, Chapter II]) we derive that

$$\lambda_1 < \eta_M - \sigma \leq h(x) \leq \zeta_M \tag{2.4}$$

a.e. $x \in \Omega$. On the other hand, (2.3) implies that $\{z_n\}_n$ strongly converges to some z in $W_0^{1,p}(\Omega)$. Here we use that $\{h_n\}_n$ is bounded in $L^\infty(\Omega)$ and the (S_+) property of the p-Laplacian operator. The strong convergence ensures that $\|z\| = 1$, so $z \neq 0$. Passing to the limit in (2.3) yields that z verifies

$$\begin{cases} -\Delta_p z = hz^{p-1} & \text{in} \quad \Omega \\ z = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Therefore 1 is an eigenvalue of this weighted eigenvalue problem. From (2.4) and Remark 2.1, we have that $1 = \hat{\lambda}_1(\lambda_1) > \hat{\lambda}_1(h)$ which, again from Remark 2.1, implies that z changes sign, a contradiction. So, we have proved that $u_w \neq 0$.

Classical regularity results enable us to derive that $u_w \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. In order to conclude the proof it remains to show that u_w is the smallest positive solution of (P_w). It is enough to prove that it is the smallest positive solution in the ordered interval $[0, \overline{u}_w]$. To this end, fix a solution v of (P_w) in $[0, \overline{u}_w]$. In particular, v is a super solution of (P_w) and we can choose $\varepsilon_n > 0$ such that $v - \varepsilon_n \phi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Then we infer that $\varepsilon_n \phi_1 \leq u_w^{\varepsilon_n} \leq v \leq \overline{u}_w$ for $n \in \mathbb{N}$ large enough. Letting $n \to \infty$, we are led to $u_w \leq v$ as we wished. □

2.2 Existence result via Schaefer's fixed point theorem

Let us recall the well known (see, e.g., [13, Theorem 4.27])

Theorem 2.2 *(Schaefer's fixed point theorem) Let X be a Banach space and let $T : X \longrightarrow X$ be a continuous and compact map. Assume that the set*

$$\{u \in X : \ u = \lambda T(u) \ \ \textit{for some} \quad \lambda \in [0, 1]\}$$

is bounded. Then T has a fixed point.

Throughout the rest of the paper, assumptions (f_1), $(f_2)_\pm$, $(f_3)_\pm$ hold.

In view of Theorem 2.1, it is well defined the map $T : C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ given by

$$T(w) = u_w,$$

where u_w is the smallest positive solution of (P_w) as guaranteed by Theorem 2.1.

It is clear that a fixed point for T will provide a positive solution to the original problem (P). In order to prove the continuity and the compactness of the map T we will need some preliminary results. For any $w \in C_0^1(\overline{\Omega})$, denote by S_w the set of all functions $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ that are solutions of problem (P_w).

Lemma 2.3 *If $\{w_n\}_n$ is a bounded sequence in $C_0^1(\overline{\Omega})$ and $\{u_n\}_n$ is a sequence in $C_0^1(\overline{\Omega})$ with $u_n \in S_{w_n}$ for all n, then $\{u_n\}_n$ is relatively compact in $C_0^1(\overline{\Omega})$.*

Proof Let $M > 0$ satisfy $\|w_n\|_{C^1} \leq M$ for all n. We claim that there exists a subsequence of $\{u_n\}_n$ converging in $C_0^1(\overline{\Omega})$. From assumption (f_1) it follows that

$$\begin{aligned}\|u_n\|^p &= \int_\Omega f(x, u_n, \nabla w_n) u_n \, dx \\ &\leq k_M |\Omega|^{\frac{1}{p'}} \|u_n\|_p + \theta_M \lambda_1^{-1} \|u_n\|^p \\ &\leq k_M |\Omega|^{\frac{1}{p'}} \lambda_1^{-\frac{1}{p}} \|u_n\| + \theta_M \lambda_1^{-1} \|u_n\|^p,\end{aligned}$$

which implies that $\{u_n\}_n$ is bounded in $W_0^{1,p}(\Omega)$ because $\theta_M < \lambda_1$. On the basis of classical regularity results, we get that $\{u_n\}_n$ is bounded in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ independent of n (due to the assumption that $\{w_n\}_n$ is bounded in $C_0^1(\overline{\Omega})$). Since $C^{1,\alpha}(\overline{\Omega})$ is compactly embedded into $C^1(\overline{\Omega})$, we achieve our claim. □

Lemma 2.4 *If $\{w_n\}_n$ is a sequence in $C_0^1(\overline{\Omega})$ such that $w_n \to w$ in $C_0^1(\overline{\Omega})$ and if $\{u_n\}_n$ is a sequence in $C_0^1(\overline{\Omega})$ with $u_n \in S_{w_n}$ for all n, then there exist a subsequence $\{u_{n_k}\}_k$ and an element $u \in S_w$ such that $u_{n_k} \to u$ in $C_0^1(\overline{\Omega})$.*

Proof From Lemma 2.3, there exist a subsequence $\{u_{n_k}\}_k$ and some $u \in C_0^1(\overline{\Omega})$ such that $u_{n_k} \longrightarrow u$ in $C_0^1(\overline{\Omega})$. We wish to prove that $u \in S_w$. We note that

$$\begin{cases} -\Delta_p u_{n_k} = f(x, u_{n_k}, \nabla w_{n_k}) & \text{in} \quad \Omega \\ u_{n_k} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

for all $k \in \mathbb{N}$. So, for every $\varphi \in W_0^{1,p}(\Omega)$ and $k \in \mathbb{N}$ we obtain

$$\int_\Omega |\nabla u_{n_k}|^{p-2} \nabla u_{n_k} \nabla \varphi \, dx = \int_\Omega f(x, u_{n_k}, \nabla w_{n_k}) \varphi \, dx.$$

Passing to the limit as $k \to \infty$ we deduce

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_\Omega f(x, u, \nabla w) \varphi \, dx,$$

i.e. u is a solution of (P_w). In order to prove that $u \neq 0$ it is enough to make use of the same argument as in the corresponding part of the proof of Theorem 2.1. Then classical regularity results and maximum principle imply $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, which completes the proof. □

The following is the key Lemma in our construction.

Lemma 2.5 *If $\{w_n\}_n$ is a sequence in $C_0^1(\overline{\Omega})$ such that $w_n \to w$ in $C_0^1(\overline{\Omega})$. Then, for any $v \in S_w$ there exist $v_n \in S_{w_n}$ such that*

$$v_n \to v \quad \textit{in } C_0^1(\overline{\Omega}).$$

Proof Fix $n \in \mathbb{N}$, and let z_n^0 be the unique solution of the problem

$$\begin{cases} -\Delta_p u = f(x, v, \nabla w_n) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Exploiting assumption $(f_2)_+$ and bearing in mind that $v \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ observe that $f(\cdot, v(\cdot), \nabla w_n(\cdot)) \not\equiv 0$, which leads to $z_n^0 \neq 0$.

Let us prove that z_n^0 lies in $\operatorname{int}(C_0^1(\overline{\Omega})_+)$. First, it is straightforward to establish that $\{z_n^0\}_n$ is bounded in $W_0^{1,p}(\Omega)$, thus it is bounded in $L^\infty(\Omega)$ (see [9]), and therefore in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ (see [10]). Since $C^{1,\alpha}(\overline{\Omega})$ is compactly embedded in $C^1(\overline{\Omega})$, there exists a subsequence $\{z_{n_p}^0\}_p$ strongly convergent in $C_0^1(\overline{\Omega})$ to a solution of the problem

$$\begin{cases} -\Delta_p u = f(x, v, \nabla w) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Taking into account that v is the unique solution of the above problem, we get

$$\lim_{p \to \infty} z_{n_p}^0 = v \quad \text{in } C_0^1(\overline{\Omega}).$$

Actually, as readily seen, the strong convergence is true for the whole sequence

$$\lim_{n \to \infty} z_n^0 = v \quad \text{in } C_0^1(\overline{\Omega}),$$

which implies the desired assertion.

Now, let us consider the unique positive solution z_n^1 of the following problem

$$\begin{cases} -\Delta_p u = f(x, z_n^0, \nabla w_n) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

As before we infer that

$$\lim_{n \to \infty} z_n^1 = v \quad \text{in } C_0^1(\overline{\Omega}).$$

Inductively, we can define, for each $n \in \mathbb{N}$, z_n^k in $C_0^1(\overline{\Omega})$ as the unique solution of the problem

$$(P_n^k) \begin{cases} -\Delta_p u = f(x, z_n^{k-1}, \nabla w_n) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and for each $k \in \mathbb{N}$ it follows that

$$\lim_{n \to \infty} z_n^k = v \quad \text{in } C_0^1(\overline{\Omega}).$$

Let us now prove that there exists a constant $c > 0$ such that $\|z_n^k\| \leq c$ for all $n, k \in \mathbb{N}$. Indeed, setting $M = \max\{\sup_n \|w_n\|_{C^1}, \|w\|_{C^1}\}$, by hypothesis (f_1) we have that

$$\|z_n^k\|^p = \int_\Omega f(x, z_n^{k-1}, \nabla w_n) z_n^k \, dx \leq \left[k_M |\Omega|^{\frac{1}{p'}} + \theta_M \left(\int_\Omega (z_n^{k-1})^p \, dx \right)^{\frac{p-1}{p}} \right] \lambda_1^{-\frac{1}{p}} \|z_n^k\|,$$

so

$$\|z_n^k\|^{p-1} \leq \lambda_1^{-\frac{1}{p}} \left[k_M |\Omega|^{\frac{1}{p'}} + \theta_M \|z_n^{k-1}\|_{L^p}^{p-1} \right] \leq \lambda_1^{-\frac{1}{p}} k_M |\Omega|^{\frac{1}{p'}} + \lambda_1^{-1} \theta_M \|z_n^{k-1}\|^{p-1}.$$

If we set $c_1 = \lambda_1^{-\frac{1}{p}} k_M |\Omega|^{\frac{1}{p'}}$ and $c_2 = \lambda_1^{-1}\theta_M$, it is easy to see by induction that

$$\|z_n^k\|^{p-1} \le c_1 \left(\sum_{j=0}^{k-1} c_2^j \right) + c_2^k \|z_n^0\|^{p-1}$$

for every $k \in \mathbb{N}$. Since $c_2 < 1$ (in view of assumption (f_1)) and $\{z_n^0\}_n$ is bounded in $W_0^{1,p}(\Omega)$, we deduce that, for every $n, k \in \mathbb{N}$, there holds

$$\|z_n^k\|^{p-1} \le c_1 \left(\sum_{j=0}^{\infty} c_2^j \right) + c_3,$$

with a constant $c_3 > 0$. This entails our claim.

By standard arguments as already used before, we have that the net $\{z_n^k\}_{n,k}$ is relatively compact in $C_0^1(\overline{\Omega})$. Then up to a subnet, we can assume that there exists $u \in C_0^1(\overline{\Omega})$ such that

$$\lim_{n,k\to\infty} z_n^k = u,$$

that is, there exists $N \in \mathbb{N}$ such that

$$\|z_n^k - u\|_{C^1} < \varepsilon \tag{2.5}$$

for all $n, k > N$. At this point, for every $k > N$, we obtain that $\limsup_{n\to\infty} \|z_n^k - u\|_{C^1} \le \varepsilon$. We are thus able to select a subsequence $\{z_{n_p}\}_p$ with the property

$$\lim_{p\to\infty} \|z_{n_p}^k - u\|_{C^1} = \|\lim_{p\to\infty} z_{n_p}^k - u\|_{C^1} = \|v - u\|_{C^1} \le \varepsilon,$$

thereby $u = v$.

Consequently, according to (2.5), for all $n, k > N$ we have

$$\|z_n^k - v\|_{C^1} < \varepsilon. \tag{2.6}$$

Now, for each $n \in \mathbb{N}$ there exist $\{k_s = k_s(n)\}_s \subseteq \mathbb{N}$ and $v_n \in C_0^1(\overline{\Omega})$ satisfying

$$\lim_{s\to\infty} z_n^{k_s} = v_n \text{ in } C_0^1(\overline{\Omega}).$$

Clearly, v_n is a solution of (P_{w_n}). Let us prove that

$$\lim_{n\to\infty} v_n = v \text{ in } C_0^1(\overline{\Omega}).$$

If not, we can construct a subsequence $\{n_p\}_p$ with $n_p > N$ such that $\|v_{n_p} - v\|_{C^1} > \varepsilon > 0$ for all $p \in \mathbb{N}$ and some $\varepsilon > 0$. This amounts to saying that

$$\varepsilon < \|v_{n_p} - v\|_{C^1} = \|\lim_{s\to\infty} z_{n_p}^{k_s} - v\|_{C^1} = \lim_{s\to\infty} \|z_{n_p}^{k_s} - v\|_{C^1}.$$

In particular, for any $p \in \mathbb{N}$ there exists $s_p = s_p(n_p) \in \mathbb{N}$ such that $k_{s_p} > N$ and

$$\|z_{n_p}^{k_{s_p}} - v\|_{C^1} > \varepsilon,$$

against (2.6).

Recall that $v \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. Since the convergence of $\{v_n\}_n$ to v is uniform on compact subsets of Ω, it follows that $v_n > 0$ in Ω whenever n is sufficiently large. Also, since $\{\frac{\partial v_n}{\partial \nu}\}_n$ converges uniformly to $\frac{\partial v}{\partial \nu}$ on $\partial\Omega$ we obtain that $v_n \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ for n large enough. This completes the proof. □

3 Proofs of main results

In the present section, in order to apply Schaefer's theorem, we replace (f_1) with the stronger assumption $(\widetilde{f}_1)$.

Proof of Theorem 1.1 We wish to prove that the map T introduced in the previous section is continuous and compact. First, let us note that the map T is compact, i.e. for any sequence $\{w_n\}_n$ bounded in $C_0^1(\overline{\Omega})$, $\{T(w_n)\}_n$ is relatively compact in $C_0^1(\overline{\Omega})$. This follows readily from Lemma 2.3 applied to $u_n = T(w_n) \in S_{w_n}$.

Let us prove now that T is continuous. Let $\{w_n\}_n$ be a sequence in $C_0^1(\overline{\Omega})$ such that

$$\lim_{n\to\infty} w_n = w \quad \text{in } C_0^1(\overline{\Omega})$$

and, for every $n \in \mathbb{N}$, set $u_n = T(w_n)$. By Lemma 2.4, there exist a subsequence $\{u_{n_k}\}_k$ and $u \in S_w$ that fulfill

$$\lim_{k\to\infty} u_{n_k} = u \quad \text{in } C_0^1(\overline{\Omega}).$$

We need to check that u is the smallest positive solution of (P_w). Fix a positive solution v of (P_w). By Lemma 2.5, there exists $v_n \in \text{int}(C_0^1(\overline{\Omega})_+)$ (positive) solution of (P_{w_n}) such that

$$\lim_{n\to\infty} v_n = v \quad \text{in } C_0^1(\overline{\Omega}).$$

Since u_{n_k} is the smallest positive solution of $(P_{w_{n_k}})$, we have that

$$u_{n_k} \le v_{n_k} \quad \text{for all} \quad k \in \mathbb{N}.$$

Passing to the limit yields $u \le v$. This means that

$$\lim_{k\to\infty} T(w_{n_k}) = T(w) \quad \text{in } C_0^1(\overline{\Omega}).$$

In fact, the whole sequence $T(w_n)$ converges to $T(w)$. If not, there exists a subsequence $\{w_{n_p}\}_p$ such that $\|T(w_{n_p}) - T(w)\|_{C^1} \ge \varepsilon$ for every $p \in \mathbb{N}$ and for some $\varepsilon > 0$. Arguing as above, we find a subsequence $u_{n_{p_r}} = T(w_{n_{p_r}})$ converging to $T(w)$, which gives rise to a contradiction. It is thus proven that T is continuous.

Now we check that the set

$$\{w \in C_0^1(\overline{\Omega}) : \ w = \lambda T(w) \quad \text{for some} \quad \lambda \in [0, 1]\} \tag{3.1}$$

is bounded in $C_0^1(\overline{\Omega})$. So let $w \in C_0^1(\overline{\Omega})$ and $\lambda \in [0, 1]$ be such that $w = \lambda T(w)$. We may assume that $\lambda > 0$ (otherwise, $w = 0$). Then, w is a solution of the problem

$$\begin{cases} -\Delta_p w = \tilde{f}_\lambda(x, w, \nabla w) & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\tilde{f}_\lambda(x, s, \xi) = \lambda^{p-1} f(x, \lambda^{-1}s, \xi)$. By $(\widetilde{f}_1)$, the Carathéodory function $\tilde{f}_\lambda$ satisfies the growth condition

$$|\tilde{f}_\lambda(x, s, \xi)| \le k_0 + \theta_0|s|^{p-1} + \theta_1|\xi|^{p-1} \tag{3.2}$$

for a.a. $x \in \Omega$, all $s \in \mathbb{R}$, and all $\xi \in \mathbb{R}^N$. Notice that the coefficients in (3.2) are independent of $\lambda \in (0, 1]$. We claim that there is $M > 0$ independent of w and λ such that

$$\|w\| \le M. \tag{3.3}$$

To see this, acting with the test function w and using (3.2), we obtain the following estimate

$$\|w\|^p = \int_\Omega \tilde{f}_\lambda(x, w, \nabla w) w \, dx \le \int_\Omega \left(k_0 w + \theta_0 |w|^p + \theta_1 |\nabla w|^{p-1} |w|\right) dx$$
$$\le k_0 |\Omega|^{\frac{1}{p'}} \lambda_1^{-\frac{1}{p}} \|w\| + \theta_0 \lambda_1^{-1} \|w\|^p + \theta_1 \lambda_1^{-\frac{1}{p}} \|w\|^p.$$

Since $1 > \theta_0 \lambda_1^{-1} + \theta_1 \lambda_1^{-\frac{1}{p}}$ (see $(\widetilde{f}_1)$) and $p > 1$, we get (3.3). In view of (3.2), (3.3), classical regularity theory (cf. [9,10]) yields $M' > 0$ independent of w and λ such that $\|w\|_{C^1} \le M'$. This establishes the boundedness of the set in (3.1).

Observe that any fixed point of the map T is a solution of problem (P) in the interior of the cone of nonnegative functions by construction, the thesis follows by applying Schaefer's fixed point theorem (Theorem 2.2). □

Mutatis mutandis we can show that the map assigning to every $w \in C_0^1(\overline{\Omega})$ the biggest negative solution of (P_w) (its existence can be proved similarly to that of the smallest positive solution) is also continuous and compact. Finally, we can conclude as in the case of positive solutions to achieve the results in Theorems 1.2 and 1.3.

Proof of Corollary 1.1 From Theorem 1.1 we know that there exists a solution $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ which can be regarded as a supersolution of (P).

By virtue of $(f_2)_+$ we can find $\varepsilon_0 > 0$ such that

$$-\Delta_p(\varepsilon\phi_1) = \lambda_1(\varepsilon\phi_1)^{p-1} \le f(x, \varepsilon\phi_1, \varepsilon\nabla\phi_1)$$

provided $0 < \varepsilon \le \varepsilon_0$. This expresses that $\varepsilon\phi_1$ is a subsolution of (P) for all $0 < \varepsilon \le \varepsilon_0$. In addition, since $u, \phi_1 \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, we may choose ε_0 so small that $u - \varepsilon\phi_1 \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ for every $0 < \varepsilon \le \varepsilon_0$.

We are thus in the position to apply [4, Theorem 3.22], which yields the existence of the smallest positive solution u_ε of problem (P) in the ordered interval $[\varepsilon\phi_1, u]$ for all $0 < \varepsilon \le \varepsilon_0$. Set $u_n := u_{\frac{1}{n}}$. Thanks to the choice of u_n, we note that the sequence $\{u_n\}_n$ is decreasing. So, there exists $u_0 \in C_0^1(\overline{\Omega})_+$ such that $u_n \to u_0$ in $C_0^1(\overline{\Omega})$ and u_0 is a solution of (P). On the basis of hypotheses $(f_2)_+$ and $(f_3)_+$ we also infer that $u_0 \not\equiv 0$ (see the proof of Theorem 2.1). This enables us to apply the strong maximum principle to obtain that $u_0 \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$.

Now we show that u_0 is the smallest positive solution. To this end, let v be a positive solution of problem (P). The nonlinear regularity theory and strong maximum principle ensure that $v \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$. It follows that

$$\frac{1}{n}\phi_1 \le \min\{u, v\} \le u$$

whenever n is sufficiently large. By [4, Theorem 3.22] we see that there exists a solution v_n of (P) with $v_n \in [\frac{1}{n}\phi_1, \min\{u, v\}]$ because $\min\{u, v\}$ is a supersolution. Then the minimality property of u_n entails that $u_n \le v_n \le v$. Letting $n \to \infty$ leads to $u_0 \le v$.

In an analogous way we can proceed to justify the existence of the biggest negative solution of problem (P). □

Finally, we provide a simple example of nonlinearity $f(x, s, \xi)$ which fulfills our hypotheses.

Example 3.1 Let $g : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ be a continuous, positive function. Then, for any continuous function $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfying the growth condition $(\widetilde{f_1})$ and

$$f(x, s, \xi) = |s|^{p-2}s(\lambda_1 + g(x, \xi)) \quad \text{for } |s| \text{ small},$$

Theorem 1.3 applies. Indeed, it is readily seen that hypotheses $(f_2)_\pm$, $(f_3)_\pm$ hold true.

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