Calc. Var. (2015) 54:251–273
DOI 10.1007/s00526-014-0785-y

Calculus of Variations

Traveling wave solutions to some reaction diffusion equations with fractional Laplacians

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Received: 15 March 2014 / Accepted: 30 August 2014 / Published online: 22 October 2014
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Abstract We show the nonexistence of traveling wave solutions in the combustion model with fractional Laplacian $(-\Delta)^s$ when $s \in (0, 1/2]$. Our method can be used to give a direct and simple proof of the nonexistence of traveling fronts for the usual Fisher-KPP nonlinearity. Also we prove the existence and nonexistence of traveling wave solutions for different ranges of the fractional power s for the generalized Fisher–KPP type model.

Mathematics Subject Classification primary 35B32 · 35C07 · 35J20 · 35R09 · 35R11 · 45G05 · 47G10

1 Introduction

The paper is concerned with the traveling fronts of the reaction diffusion equation:

$$u_t + (-\Delta)^s u = f(u), \quad \text{in } \mathbb{R} \times \mathbb{R},$$

for $f \in C^1(\mathbb{R})$, namely the solution to the following equation:

$$\begin{cases} (-\Delta)^s u(x) + \mu u'(x) = f(u(x)), \quad \forall x \in \mathbb{R} \\ \lim\limits_{x \to -\infty} u(x) = 0, \quad \lim\limits_{x \to \infty} u(x) = 1 \end{cases} \tag{1.1}$$

where μ is the speed of propagation of the front and the operator $(-\Delta)^s$ denotes the fractional power of the Laplacian in one dimension with $0 < s < 1$. Recall the fractional Laplacian is defined as follows:

$$(-\Delta)^s u(x) = C_{1,s}(\text{P.V.}) \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2s}} dy,$$

Communicated by A. Malchiodi.

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where (P.V.) stands for Cauchy principal value and $C_{1,s} = \dfrac{2^{2s} s\Gamma((1+2s)/2)}{\pi^{1/2}\Gamma(1-s)}$, see for example [8]. The original models with the standard Laplacian $(-\Delta)$ arise in applied sciences such as population genetics, combustion, and nerve pulse propagation, etc. The detailed formulations of the models were discussed by Fisher in [5], Kolmogorov et al. in [7] and Aronson and Weinberger in [1], etc. The classical results of the existence and nonexistence of traveling fronts for the models can be found therein.

By a compactness argument, we know that if (1.1) has a solution $u(x)$ then

$$\lim_{|x|\to\infty} u'(x) = 0 \quad \text{and} \quad f(0) = f(1) = 0 \tag{1.2}$$

Multiplying $u'(x)$ on both sides in (1.1) and integrating over $\mathbb{R}$, we can get the Hamiltonian identity as in [6]:

$$\mu \int_{\mathbb{R}} |u'(x)|^2\, dx = \int_0^1 f(u)\, du \tag{1.3}$$

Roquejoffre et al. [9] studied the combustion model, i.e, there exists some $\theta \in (0, 1)$ such that $f \in C^1(\mathbb{R})$ satisfies

$$f(u) = f(1) = 0, \quad \forall u \in [0, \theta], \qquad f(u) > 0, \quad \forall u \in (\theta, 1), \qquad \text{and} \qquad f'(1) < 0. \tag{1.4}$$

They have shown that when $s \in (1/2, 1)$ and f satisfies (1.4), there exists an unique (μ, u) with $\mu > 0$ to (1.1).

In this paper, we will show that when $s \in (0, 1/2]$ and f satisfies (1.4), there is no traveling wave solution for the combustion model, i.e., (1.1) has no solution. In fact, we shall show the following:

Theorem 1.1 *Suppose that there exists some $\theta \in (0, 1)$ such that $f \in C^1(\mathbb{R})$ satisfies*

$$\int_0^1 f(u)\, du > 0, \qquad \text{and} \qquad f'(u) \geq 0, \quad \forall u \in (0, \theta]. \tag{1.5}$$

Then there is no solution to (1.1) *if* $0 < s \leq \frac{1}{2}$.

Obviously this theorem applies to the combustion model. For the Fisher-KPP model, i.e, $f \in C^1(\mathbb{R})$ satisfies

$$f(u) > 0 = f(0) = f(1), \quad \forall u \in (0, 1), \qquad f'(0) > 0, \qquad \text{and} \qquad f'(1) < 0, \tag{1.6}$$

Theorem 1.1 implies that if $0 < s \leq 1/2$, (1.1) has no solution.

We shall also study the generalized Fisher-KPP model and prove nonexistence and existence of solutions to (1.1) for different ranges of $s \in (0, 1)$. We shall point out that there is a delicate balance between the diffusion factor s and the reaction power p in order to have a traveling wave solution. In fact, we shall prove the following theorem.

Theorem 1.2 *Assume there exist some $\theta \in (0, 1)$, $0 < p < \infty$, $A_1 > 0$ and $A_2 > 0$ such that*

$$\begin{cases} f(u) > 0 = f(0) = f(1), \quad \forall u \in (0, 1), \quad f'(1) < 0, \\ A_1 u^p \leq f(u) \leq A_2 u^p, \quad \forall u \in [0, \theta], \\ f'(u) \geq A_1 u^{p-1}, \quad \forall u \in (0, \theta). \end{cases} \tag{1.7}$$

Then (1.1) *has a solution if and only if $p > 2$ and $s \geq \frac{p}{2(p-1)}$.*

We note that the condition $A_1 u^p \leq f(u) \leq A_2 u^p$, $\forall u \in [0, \theta]$ in the above theorem is not needed for the nonexistence result. We include it in (1.7) for the simplicity of the statement.

We also obtain the asymptotics of solutions as $x \to \pm\infty$ when they exist. Indeed we show the following asymptotic behaviors.

Theorem 1.3 *Assume that f satisfies (1.7), let (μ, u) be a solution to (1.1) with $\mu > 0$. Then there exists some constant $C > 0$ such that*

$$\frac{1}{C|x|^{1+2s}} \leq u'(x) \leq \frac{C}{|x|^{1+2s}}, \quad \text{and} \quad \frac{1}{C|x|^{2s}} \leq 1 - u(x) \leq \frac{C}{|x|^{2s}}, \qquad \forall x \geq 1.$$

$$\frac{1}{C|x|^{2s}} \leq u'(x), \quad \text{and} \quad \frac{1}{C|x|^{2s-1}} \leq u(x) \leq \frac{C}{|x|^{2s-1}}, \qquad \forall x \leq -1.$$

Note in Theroem 1.3, s is always bigger than $1/2$ by Theorem 1.2.

We would also like to point out that Cabré and Roquejoffre in [4] already proved that when $0 < s < 1$, there is no traveling wave solution for the Fisher-KPP model by studying the exponential speed of the front propagation. Theorem 1.3, in particular, shows directly that (1.1) has no solution for the Fisher-KPP model.

The plan of the paper is the following. In Sect. 2, we prove Theorem 1.1 by considering the s-harmonic extension of the fractional Laplacian given in [3]. The key ingredient is to show certain asymptotic rates of solutions at $-\infty$. Section 3 is devoted to prove Theorem 1.2. The proof of nonexistence follows the same idea as in Sect. 2 by using s-harmonic extension of the fractional Laplacian. We use an iterative argument to obtain an accurate asymptotic behavior of possible solutions. The proof of existence relies on the truncation of domain, asymptotic behavior of solutions and a sliding argument as in [9]. In Sect. 4, detailed asymptotical behavior of solutions will be given.

2 Nonexistence in the combustion and Fisher–KPP models when $0 < s \leq 1/2$

In this section, we assume $0 < s \leq 1/2$ and $f \in C^1(\mathbb{R})$ satisfies condition (1.5). We prove Theorem 1.1 by contradiction. Assume that (μ, u) is a solution to (1.1). By (1.3) and (1.5), we have $\mu > 0$. Let $\overline{u}$ be the s-harmonic extension of u on $\mathbb{R}^2_+$, that is,

$$\overline{u}(x, y) = P_y * u(x), \quad \forall (x, y) \in \mathbb{R}^2_+ \tag{2.1}$$

with

$$P_y(x) = \frac{a_s y^{2s}}{[y^2 + x^2]^{\frac{1+2s}{2}}}, \quad \forall (x, y) \in \mathbb{R}^2_+ \quad \text{and} \quad a_s = \frac{\Gamma\left(\frac{1+2s}{2}\right)}{\pi^{\frac{1}{2}}\Gamma(s)}. \tag{2.2}$$

Let $v(x, y) = \overline{u}_x(x, y) = P_y * u'(x)$ for all $(x, y) \in \mathbb{R}^2_+$, Caffarelli and Silvestre [3] proved that v satisfies

$$\begin{cases} \operatorname{div}[y^{1-2s}\nabla v(x, y)] = 0, \quad \forall (x, y) \in \mathbb{R}^2_+, \\ \lim_{y \searrow 0} -d_s y^{1-2s} v_y(x, y) = (-\Delta)^s u'(x), \quad \forall x \in \mathbb{R}, \\ v(x, 0) = u'(x), \quad \forall x \in \mathbb{R}. \end{cases}$$

where $d_s = \dfrac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$.

By the standard maximal principle arguments, it is easy to see that $u'(x) > 0$ for all $x \in \mathbb{R}$ and $\lim_{|x|\to\infty} u'(x) = 0$ (see, e.g., [6,9]). Then we know that

$$v(x,y) > 0,\ \ \forall (x,y) \in \overline{\mathbb{R}^2_+}, \qquad \text{and} \qquad \lim_{|(x,y)|\to\infty} v(x,y) = 0.$$

By (1.1), without loss of generality, we can assume $u(-1) = \theta$. Since $u'(x) > 0$ for all $x \in \mathbb{R}$, we have $f'(u(x)) \geq 0$ for all $x \leq -1$. In summary, we know that v satisfies

$$\begin{cases} \operatorname{div}[y^{1-2s}\nabla v(x,y)] = 0, \quad \forall (x,y) \in \mathbb{R}^2_+, \\ \lim_{y\searrow 0} -d_s y^{1-2s} v_y(x,y) + \mu v_x(x,0) = f'(u(x))u'(x) \geq 0, \quad \forall x \leq -1, \\ v(x,y) > 0,\ \ \forall (x,y) \in \overline{\mathbb{R}^2_+} \quad \text{and} \quad \lim_{|(x,y)|\to\infty} v(x,y) = 0. \end{cases} \tag{2.3}$$

Define the auxiliary function

$$\varphi(x,y) = \frac{y^{2s}}{[x^2+y^2]^{\frac{1+2s}{2}}} + \frac{sd_s}{\mu} \cdot \frac{1}{[x^2+y^2]^{\frac{1}{2}}} \quad \forall x \leq -1,\ y \geq 0.$$

Direct computations tell us that for all $x \leq -1$ and all $y \geq 0$, we have

$$\begin{aligned} \frac{sd_s}{\mu} \cdot \frac{1}{|(x,y)|} &\leq \varphi(x,y) \leq \left(1 + \frac{sd_s}{\mu}\right) \cdot \frac{1}{|(x,y)|}, \\ \operatorname{div}[y^{1-2s}\nabla\varphi(x,y)] &= \frac{2s^2 d_s}{\mu} \cdot \frac{y^{1-2s}}{[x^2+y^2]^{\frac{3}{2}}} \geq 0, \\ \lim_{y\searrow 0} -d_s y^{1-2s}\varphi_y(x,y) &= d_s \lim_{y\searrow 0} \left[\frac{y^2 - 2sx^2}{[x^2+y^2]^{\frac{3}{2}+s}} + \frac{sd_s}{\mu} \cdot \frac{y^{2-2s}}{[x^2+y^2]^{\frac{3}{2}}} \right] = -\frac{2sd_s}{|x|^{1+2s}}, \\ \varphi_x(x,0) &= \frac{sd_s}{\mu} \cdot \frac{1}{|x|^2}. \end{aligned}$$

Since $0 < s \leq \frac{1}{2}$, we have $\frac{1}{|x|^2} \leq \frac{1}{|x|^{1+2s}}$ for all $x \leq -1$. Hence for all $x \leq -1$, we have

$$\begin{aligned} \lim_{y\searrow 0} -d_s y^{1-2s}\varphi_y(x,y) + \mu\varphi_x(x,0) &= -\frac{2sd_s}{|x|^{1+2s}} + \frac{sd_s}{|x|^2} \\ &\leq -\frac{2sd_s}{|x|^{1+2s}} + \frac{sd_s}{|x|^{1+2s}} = -\frac{sd_s}{|x|^{1+2s}} < 0. \end{aligned}$$

For any $\delta > 0$, let $w_\delta(x,y) = v(x,y) - \delta\varphi(x,y)$ for all $x \leq -1$ and all $y \geq 0$, then w_δ satisfies

$$\begin{cases} \operatorname{div}[y^{1-2s}\nabla w_\delta(x,y)] \leq 0, \quad \forall x \leq -1,\ y > 0, \\ \lim_{y\searrow 0} -d_s y^{1-2s} D_y w_\delta(x,y) + \mu D_x w_\delta(x,0) \geq 0, \quad \forall x \leq -1, \\ \lim_{|(x,y)|\to\infty} w_\delta(x,y) = 0. \end{cases} \tag{2.4}$$

Lemma 2.1 *There exists some $\delta_0 > 0$ such that $w_{\delta_0}(-1,y) \geq 0$ for all $y \geq 0$.*

Proof First we see that

$$\lim_{y\to\infty} \frac{\varphi(-1,y)}{\frac{y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}}} = \lim_{y\to\infty} \frac{\frac{y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}} + \frac{sd_s}{\mu} \cdot \frac{1}{[1+y^2]^{\frac{1}{2}}}}{\frac{y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}}} = 1 + \frac{sd_s}{\mu} < \infty.$$

Since $u'(x) > 0$ for all $x \in \mathbb{R}$, then $u(0) > u(-1)$, which implies that there exists some constant $B_1 > 0$ such that

$$a_s[u(0) - u(-1)] \cdot \frac{y^{2s}}{[1 + y^2]^{\frac{1+2s}{2}}} \geq B_1\varphi(-1, y), \quad \forall y \geq 1. \tag{2.5}$$

Since $v(x, y) = P_y * u'(x)$ for all $(x, y) \in \mathbb{R}^2_+$, by (2.2), for all $y \geq 1$ we have

$$\begin{aligned} v(-1, y) &= \int_{\mathbb{R}} \frac{a_s y^{2s}}{[(-1 - x)^2 + y^2]^{\frac{1+2s}{2}}} \cdot u'(x)\, dx \\ &\geq \frac{a_s y^{2s}}{[1 + y^2]^{\frac{1+2s}{2}}} \int_{-1}^{0} u'(x)\, dx \\ &= a_s[u(0) - u(-1)] \cdot \frac{y^{2s}}{[1 + y^2]^{\frac{1+2s}{2}}} \\ &\geq B_1\varphi(-1, y). \end{aligned}$$

On the other hand, since $v(x, y) > 0$ for all $(x, y) \in \overline{\mathbb{R}^2_+}$, there exists some $B_2 > 0$ such that

$$\inf_{0\leq y\leq 1} v(-1, y) \geq B_2 \cdot \sup_{0\leq y\leq 1} \varphi(-1, y).$$

Let $\delta_0 = \min\{B_1, B_2\} > 0$, we know that

$$w_{\delta_0}(-1, y) \geq 0, \quad \forall y \geq 0.$$

□

Lemma 2.2 *For the above δ_0 in Lemma 2.1, there holds*

$$w_{\delta_0}(x, y) \geq 0, \quad \forall x \leq -1, \ y \geq 0.$$

Proof Assume $w_{\delta_0}(x_0, y_0) < 0$ for some $x_0 \leq -1$ and some $y_0 \geq 0$. Since $w_{\delta_0}(x, y) \to 0$, as $|(x, y)| \to \infty$, by Lemma 2.1, we know that there exists some $x_1 < -1$ and some $y_1 \geq 0$ such that

$$w_{\delta_0}(x_1, y_1) = \inf_{x\leq -1,\ y\geq 0} w_{\delta_0}(x, y) < 0.$$

By the strong maximum principle for uniformly elliptic equations, we know that $y_1 = 0$. Applying Hopf lemma as in [2], we have

$$\lim_{y\searrow 0} -d_s y^{1-2s} D_y w_{\delta_0}(x_1, y) < 0.$$

Since x_1 is an interior minimum of $w_{\delta_0}(x, 0)$ in $x < -1$, then we have $D_x w_{\delta_0}(x_1, 0) = 0$. By (2.4), we get

$$\lim_{y\searrow 0} -d_s y^{1-2s} D_y w_{\delta_0}(x_1, y) = \lim_{y\searrow 0} -d_s y^{1-2s} D_y w_{\delta_0}(x_1, y) + \mu D_x w_{\delta_0}(x_1, 0) \geq 0.$$

We get a contradiction. Therefore

$$w_{\delta_0}(x, y) \geq 0, \quad \forall x \leq -1, \ y \geq 0.$$

□

Proof of Theorem 1.1 Assume (μ, u) is a solution to (1.1). By Lemma 2.2, we know that

$$w_{\delta_0}(x,0) \geq 0, \quad \forall x \leq -1.$$

Since $\varphi(x,0) \geq \frac{sd_s}{\mu} \cdot \frac{1}{|x|}$ for all $x \leq -1$, we know that

$$u'(x) \geq \frac{\delta_0 s d_s}{\mu} \cdot \frac{1}{|x|}, \quad \forall x \leq -1.$$

On the other hand, we know that $\int_{\mathbb{R}} u'(x)\,dx = 1$. This is a contradiction which implies that there is no solution to (1.1). □

3 Generalized Fisher-KPP model when 1/2 < s < 1

In this section, we assume that $\frac{1}{2} < s < 1$ and $f \in C^1(\mathbb{R})$ satisfies condition (1.7). One example for (1.7) is the following:

$$f(u) = u^p(1-u), \quad \forall u \in \mathbb{R},$$

where $p > 0$ is the reaction power.

Our goal is to find the critical exponent $s = s(p)$ such that a solution of (1.1) exists if and only if $1 > s \geq s(p)$. In this section, we provide the proof of nonexistence of solutions for (1.1) when $s < s(p)$ by studying the asymptotics of solutions related to (1.1). By Theorem 1.1, it is readily seen that the solution to (1.1) does not exist when $0 < s \leq 1/2$. Later, we shall discuss the existence of solutions to (1.1) by a similar argument as in [9].

3.1 Nonexistence results

The following lemma is important in the proof of nonexistence, and has already been proven in [9]. For completeness, we list the proof here.

Lemma 3.1 *Let* $\frac{1}{2} < s < 1$ *and* $u \in C^2(\mathbb{R})$ *such that* $\lim_{|x|\to\infty} u'(x) = 0$ *and* $\lim_{x\to\pm\infty} u(x) = L^{\pm}$ *for some* $L^-, L^+ \in \mathbb{R}$*, then we have*

$$\lim_{R\to\infty} \int_{-R}^{R} (-\Delta)^s u(y)\,dy = 0.$$

Proof For any $R > 0$, we have

$$\begin{aligned}\int_{-R}^{R} (-\Delta)^s u(y)\,dy &= C_{1,s}\left[\int_{-R}^{R}\int_{|w|\geq 1} \frac{u(y)-u(y+w)}{|w|^{1+2s}}\,dwdy\right.\\ &\qquad\left. + \int_{-R}^{R} (\text{P.V.})\int_{|w|<1} \frac{u(y)-u(y+w)}{|w|^{1+2s}}\,dwdy\right]\\ &= C_{1,s}\left[\int_{-R}^{R}\int_{|w|\geq 1} \frac{u(y)-u(y+w)}{|w|^{1+2s}}\,dwdy\right.\\ &\qquad\left. - \int_{-R}^{R}\int_{|w|<1} \frac{u(y+w)-u(y)-u'(y)w}{|w|^{1+2s}}\,dwdy\right].\end{aligned}$$

For $\int_{-R}^{R}\int_{|w|\geq 1}\frac{u(y)-u(y+w)}{|w|^{1+2s}}\,dwdy$, since $\frac{1}{2}<s$, by Fubini–Tonelli's theorem and the dominated convergence theorem, we know that

$$\begin{aligned}
&\int_{-R}^{R}\int_{|w|\geq 1}\frac{u(y)-u(y+w)}{|w|^{1+2s}}\,dwdy\\
&\quad=-\int_{-R}^{R}\int_{|w|\geq 1}\int_{0}^{1}\frac{u'(y+tw)\cdot w}{|w|^{1+2s}}\,dtdwdy\\
&\quad=-\int_{|w|\geq 1}\frac{w}{|w|^{1+2s}}\int_{0}^{1}\int_{-R}^{R}u'(y+tw)\,dydtdw\\
&\quad=-\int_{|w|\geq 1}\frac{w}{|w|^{1+2s}}\int_{0}^{1}[u(R+tw)-u(-R+tw)]\,dtdw\\
&\quad\to\int_{|w|\geq 1}\frac{w}{|w|^{1+2s}}\cdot(L^{-}-L^{+})\,dtdw=0,\quad \text{as } R\to\infty
\end{aligned}$$

For $\int_{-R}^{R}\int_{|w|<1}\frac{u(y+w)-u(y)-u'(y)w}{|w|^{1+2s}}\,dwdy$, since $s<1$, by Fubini–Tonelli's theorem and the dominated convergence theorem, we know that

$$\begin{aligned}
&\int_{-R}^{R}\int_{|w|<1}\frac{u(y+w)-u(y)-u'(y)w}{|w|^{1+2s}}\,dwdy\\
&=\int_{-R}^{R}\int_{|w|<1}\int_{0}^{1}\int_{0}^{1}\frac{1}{|w|^{2s-1}}\cdot u''(y+rtw)\,drdtdwdy\\
&=\int_{|w|\leq 1}\frac{1}{|w|^{2s-1}}\int_{0}^{1}\int_{0}^{1}\int_{-R}^{R}u''(y+rtw)\,dydrdtdw\\
&=\int_{|w|\leq 1}\frac{1}{|w|^{2s-1}}\int_{0}^{1}\int_{0}^{1}[u'(R+rtw)-u'(-R+rtw)]\,drdtdw\\
&\to 0,\quad \text{as } R\to\infty.
\end{aligned}$$

Therefore, we can conclude that $\int_{-R}^{R}(-\Delta)^{s}u(y)\,dy\to 0$, as $R\to\infty$. □

Remark 3.1 If (μ,u) is a solution to (1.1), since $u'\in L^{1}(\mathbb{R})$, by Lemma 3.1 and $f(u)\geq 0$ for all $u\in[0,1]$ we know that $f(u)\in L^{1}(\mathbb{R})$. In particular, if we know that there exists some constants $C>0$ and $r>0$ such that

$$u'(x)\geq\frac{C}{|x|^{r}},\quad \forall x\leq -1,$$

then we have $r>1$ by the integrability of u'. On the other hand, by (1.7), we know that $f(u(x))\geq A\left(\frac{C}{r-1}\cdot\frac{1}{|x|^{r-1}}\right)^{p}$ for all $x\leq -1$. Hence it necessarily holds that $(r-1)p>1$, i.e., $r>\frac{p+1}{p}$.

In the following, we assume that (μ,u) is a solution to (1.1) with $\mu>0$ and $u(-1)=\theta$. Let $\overline{u}$ be the s-harmonic extension of u on $\mathbb{R}^{2}_{+}$ and $v(x,y)=\overline{u}_{x}(x,y)=P_{y}*u'(x)$ for all $(x,y)\in\mathbb{R}^{2}_{+}$, by the same discussion as in Sect. 2, we know that v satisfies

$$\begin{cases} \operatorname{div}[y^{1-2s}\nabla v(x,y)] = 0, \quad \forall (x,y) \in \mathbb{R}^2_+, \\ \lim\limits_{y\searrow 0} -d_s y^{1-2s} v_y(x,y) + \mu v_x(x,0) = f'(u(x))u'(x), \quad \forall x \in \mathbb{R}, \\ v(x,y) > 0, \ \forall (x,y) \in \overline{\mathbb{R}^2_+}, \qquad \text{and} \qquad \lim\limits_{|(x,y)|\to\infty} v(x,y) = 0. \end{cases} \tag{3.1}$$

For any $\alpha \in [1, 2s]$ and $\beta > 0$, we consider the auxiliary functions

$$\varphi_{\alpha,\beta}(x,y) = \frac{y^{2s}}{[x^2+y^2]^{\frac{1+2s}{2}}} + \frac{2\beta s d_s}{\alpha\mu} \cdot \frac{1}{[x^2+y^2]^{\frac{\alpha}{2}}}, \qquad \forall x \le -1, \ y \ge 0.$$

By direct computations, for all $x \le -1$ and all $y \ge 0$ we know that

$$\begin{aligned} \frac{2\beta s d_s}{2s\mu} \cdot \frac{1}{[x^2+y^2]^{\frac{\alpha}{2}}} &\le \varphi_{\alpha,\beta}(x,y) \le \left(1 + \frac{2\beta s d_s}{\alpha\mu}\right) \cdot \frac{1}{|(x,y)|}, \\ \operatorname{div}[y^{1-2s}\nabla \varphi_{\alpha,\beta}(x,y)] &= \frac{2\beta s d_s}{\mu} \cdot \frac{(2s-1+\alpha)y^{1-2s}}{[x^2+y^2]^{\frac{\alpha+2}{2}}} \ge 0, \\ \lim_{y\searrow 0} -d_s y^{1-2s} D_y \varphi_{\alpha,\beta}(x,y) &= d_s \lim_{y\searrow 0} \left[\frac{y^2 - 2sx^2}{[x^2+y^2]^{\frac{3}{2}+s}} + \frac{2\beta s d_s}{\mu} \cdot \frac{y^{2-2s}}{[x^2+y^2]^{\frac{\alpha+2}{2}}} \right] \\ &= -\frac{2sd_s}{|x|^{1+2s}}, \text{ and} \\ D_x \varphi_{\alpha,\beta}(x,0) &= \frac{2\beta s d_s}{\mu} \cdot \frac{1}{|x|^{1+\alpha}}. \end{aligned}$$

Hence for all $x \le -1$, we have

$$\lim_{y\searrow 0} -d_s y^{1-2s} D_y \varphi_{\alpha,\beta}(x,y) + \mu D_x \varphi_{\alpha,\beta}(x,0) = -\frac{2sd_s}{|x|^{1+2s}} + \frac{2\beta s d_s}{|x|^{1+\alpha}}.$$

For any $\delta \in (0, 1)$, let

$$w_{\delta,\alpha,\beta}(x,y) = v(x,y) - \delta\varphi_{\alpha,\beta}(x,y), \quad \forall x \le -1, \ y \ge 0.$$

Then $w_{\delta,\alpha,\beta}$ satisfies

$$\begin{cases} \operatorname{div}[y^{1-2s}\nabla w_{\delta,\alpha,\beta}(x,y)] \le 0, \quad \forall x \le -1, \ y > 0, \\ \lim\limits_{y\searrow 0} -d_s y^{1-2s} D_y w_{\delta,\alpha,\beta}(x,y) + \mu D_x w_{\delta,\alpha,\beta}(x,0) \\ \qquad = f'(u(x))u'(x) - \frac{2\delta\beta s d_s}{|x|^{1+\alpha}} + \frac{2\delta s d_s}{|x|^{1+2s}}, \quad \forall x \le -1, \\ \lim\limits_{|(x,y)|\to\infty} w_{\delta,\alpha,\beta}(x,y) = 0. \end{cases} \tag{3.2}$$

Lemma 3.2 *For any fixed $\alpha \in [1, 2s]$ and $\beta > 0$, for all $\delta \in (0, 1]$, if we have*

$$f'(u(x))u'(x) - \frac{2\delta\beta s d_s}{|x|^{1+\alpha}} + \frac{2\delta s d_s}{|x|^{1+2s}} \ge 0, \quad \forall x \le -1,$$

then there exists some constant $C > 0$ such that

$$u'(x) \ge \frac{C}{|x|^{\alpha}}, \quad \text{and} \quad u(x) \ge \frac{C}{|x|^{\alpha-1}}, \qquad \forall x \le -1.$$

Proof Since $\alpha \geq 1$, we know that $\dfrac{1}{[1+y^2]^{\frac{\alpha}{2}}} \leq \dfrac{1}{[1+y^2]^{\frac{1}{2}}}$ for all $y \geq 0$. By taking the limit of the ratio, one can get

$$\lim_{y\to\infty} \frac{\varphi_{\alpha,\beta}(-1,y)}{\frac{y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}}} \leq \lim_{y\to\infty} \frac{\frac{y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}} + \frac{2\alpha\beta s d_s}{\mu}\cdot\frac{1}{[1+y^2]^{\frac{1}{2}}}}{\frac{y^{2s}}{[1+y^2]^{\frac{1+2s}{2}}}} = 1 + \frac{2\alpha\beta s d_s}{\mu} > 0.$$

By the same arguments as in Lemma 2.1 and Lemma 2.2, we know that there exists some small $\delta_0 > 0$ such that

$$w_{\delta_0,\alpha,\beta}(x,y) \geq 0, \quad \forall x \leq -1,\ y \geq 0.$$

Since $\varphi_{\alpha,\beta}(x,0) \geq \dfrac{2\beta s d_s}{\mu}\cdot\dfrac{1}{|x|^\alpha}$ for all $x \leq -1$, we have

$$u'(x) = v(x,0) \geq \frac{2\delta_0\beta s d_s}{\mu}\cdot\frac{1}{|x|^\alpha}, \quad \forall x \leq -1.$$

□

Lemma 3.3 (Initial Asymptotic Rate) *There exists some constant $C_0 > 0$ such that*

$$u'(x) \geq \frac{C_0}{|x|^{2s}}, \quad \text{and} \quad u(x) \geq \frac{C_0}{|x|^{2s-1}}, \qquad \forall x \leq -1.$$

Proof Let $\alpha = 2s$ and $\beta = 1$ in Lemma 3.2. Observe that

$$\begin{aligned} f'(u(x))u'(x) - \frac{2\delta\beta s d_s}{|x|^{1+\alpha}} + \frac{2\delta s d_s}{|x|^{1+2s}} &= f'(u(x))u'(x) - \frac{2\delta s d_s}{|x|^{1+2s}} + \frac{2\delta s d_s}{|x|^{1+2s}} \\ &= f'(u(x))u'(x) \geq 0, \quad \forall x \leq -1. \end{aligned}$$

Then Lemma 3.2 leads to the conclusion. □

Remark 3.2 Lemma 3.3 provides an alternative proof of Proposition 4.2 in [9].

As an immediate consequence of Lemma 3.3 and Remark 3.1, we have the following

Theorem 3.1 *Let $\dfrac{1}{2} < s \leq \dfrac{p+1}{2p}$, then there is no solution to (1.1). In particular, for all $0 < p \leq 1$ and $\dfrac{1}{2} < s < 1$, there is no solution to (1.1).*

Lemma 3.4 (Asymptotic Rate Lifting) *Let $\frac{p+1}{2p} < s < 1$ and $r \in (\frac{p+1}{p}, 2s]$, we assume there exists some constant $B_0 > 0$ such that*

$$u'(x) \geq \frac{B_0}{|x|^r}, \quad \text{and} \quad u(x) \geq \frac{B_0}{|x|^{r-1}}, \qquad \forall x \leq -1.$$

Let $\alpha \in [1, 2s]$ be such that $\alpha \geq p(r-1)$, then there exists some constant $C > 0$ such that

$$u'(x) \geq \frac{C}{|x|^\alpha}, \quad \text{and} \quad u(x) \geq \frac{C}{|x|^{\alpha-1}}, \qquad \forall x \leq -1.$$

Proof By the assumption and (1.7), for all $\beta > 0$, all $\delta \in (0, 1]$ and all $x \le -1$, we know that

$$\begin{aligned} f'(u(x))u'(x) - \frac{2\delta\beta s d_s}{|x|^{1+\alpha}} + \frac{2\delta s d_s}{|x|^{1+2s}} &\ge A_1|u(x)|^{p-1}u'(x) - \frac{2\delta\beta s d_s}{|x|^{1+\alpha}} \\ &\ge A_1\left(\frac{B_0}{|x|^{r-1}}\right)^{p-1}\cdot\frac{B_0}{|x|^r} - \frac{2\delta\beta s d_s}{|x|^{1+\alpha}} \\ &= \frac{A_1 B_0^p}{|x|^{r+(p-1)(r-1)}} - \frac{2\delta\beta s d_s}{|x|^{1+\alpha}} \\ &\ge \frac{A_1 B_0^p - 2\delta\beta s d_s}{|x|^{1+\alpha}}. \end{aligned}$$

Let $\beta = \dfrac{A_1 B_0^p}{2\delta s d_s} > 0$, by Lemma 3.2, we have completed the proof. □

Remark 3.3 If $\frac{p+1}{p} < r < \frac{p}{p-1}$, by letting $\rho(r) := p(r-1)$, we know that

$$1 < \rho(r) < \frac{r}{r-1}(r-1) = r.$$

We shall show the following theorem.

Theorem 3.1 *Let $p > 1$ and $\frac{1}{2} < s < \min\left\{1, \frac{p}{2(p-1)}\right\}$, then (1.1) has no solution.*

Proof By Lemma 3.4, we have the following
Claim : if

$$u'(x) \ge \frac{B_0}{|x|^r}, \quad \text{and} \quad u(x) \ge \frac{B_0}{|x|^r}, \qquad \forall x \le -1.$$

for some $r \in \left(\frac{p+1}{p}, 2s\right]$, then

$$u'(x) \ge \frac{C}{|x|^\alpha}, \quad \text{and} \quad u(x) \ge \frac{C}{|x|^\alpha}, \qquad \forall x \le -1$$

with $\alpha \in \left(\frac{p+1}{p}, p(r-1)\right)$. This is a consequence of the fact that the function $\rho(r) = p(r-1)$ has a unique fixed point at $r = \frac{p}{p-1}$ and $\rho(r) < r$ for $r < \frac{p}{p-1}$, which implies that, for α and r as above, there holds $r < 2s < \frac{p}{p-1}$ and then $\alpha < r \le 2s$. The claim then follows from Lemma 3.4. Now, one can apply recursively the claim, starting with $r = 2s < \frac{p}{p-1}$ and after a finite number of steps, get $\alpha = \frac{p+1}{p}$, because $\rho^{(n)}(r) := \rho \circ \rho \cdots \rho(r) = \frac{p^n[p(r-1)-r]+p}{p-1} \to -\infty$ as $n \to \infty$. This is a contradiction to Remark 3.1. □

Note that $\frac{p}{2(p-1)} \ge 1$ if $1 < p \le 2$, and $\frac{p}{2(p-1)} < 1$ if $2 < p$. Therefore, there is no solution to (1.1) for all $s \in (0, 1)$ if $p \le 2$.

3.2 Existence results

In this subsection, we assume that f satisfies (1.7), $p > 2$ and $\frac{p}{2(p-1)} \le s < 1$, we will show that a solution to (1.1) exists. Mellet et al. [9] have shown the existence of traveling fronts for the non local combustion model when $\frac{1}{2} < s < 1$. The proof for the generalized Fisher-KPP model follows a similar argument to that in [9]. For any $\mu \in \mathbb{R}$ and $b > 0$, we first consider the following truncated problem:

$$\begin{cases} (-\Delta)^s u(x) + \mu u'(x) = f(u(x)), & \forall x \in (-b, b), \\ u(x) = 0, \quad \forall x \leq -b, \\ u(x) = 1, \quad \forall x \geq b. \end{cases} \tag{3.3}$$

Proposition 3.1 *Assume* $s \geq \dfrac{p}{2(p-1)}$ *and* f *satisfies* (1.7)*. Then there exists a constant* M *such that if* $b > M$ *the truncated problem* 3.3 *has a solution* (u_b, μ_b)*. Furthermore, the following properties hold:*

(1) *There exists* K *independent of* b *such that* $-K \leq \mu_b \leq K$*;*
(2) u_b *is non-decreasing with respect to* x *and satisfies* $0 < u_b(x) < 1$ *for all* $x \in (-b, b)$*.*

To prove this Proposition, we need the construction of sub- and super-solutions. The construction is based on the following lemmas, same as in [9]. We would like to present the proof of the following second lemma, and especially elaborate on the sliding method mentioned in [9].

Lemma 3.5 *For any* $\mu \in \mathbb{R}$ *and* $b > 0$*,* (3.3) *has a solution* $u_{\mu,b}$ *such that* $0 \leq u_{\mu,b}(x) \leq 1$ *in* $\mathbb{R}$*,* $u_{\mu,b}$ *is non-decreasing in* $\mathbb{R}$ *and* $\mu \to u_{\mu,b}$ *is continuous.*

Proof The proof is the same as the proof of Lemma 2.4 in [9]. □

Lemma 3.6 *There exists some constants* $M, K > 0$ *such that for all* $b > M$*, we have*

a. *If* $\mu > K$*, then* $u_{\mu,b}(0) < \theta$*;*
b. *If* $\mu < -K$*, then* $u_{\mu,b}(0) > \theta$*.*

Together with Lemma 3.5, Lemma 3.6 implies that there exists $\mu_b \in [-K, K]$ such that $u_{\mu,b}(0) = \theta$.

Proof Consider the function

$$\varphi(x) = \begin{cases} \frac{1}{|x|^{2s-1}}, & \forall x \leq -1, \\ 1, \quad \forall x > -1. \end{cases}$$

Since $2s > 1$, by Lemma 2.2 in [9], we have

$$(-\Delta)^s \varphi(x) + \mu \varphi'(x) = -\frac{C_{1,s}}{2s|x|^{2s}} + \frac{\mu(2s-1)}{|x|^{2s}} + O\left(\frac{1}{|x|^{4s-1}}\right), \quad \text{as } x \to -\infty.$$

Moreover, by (1.7), we get

$$f(\varphi(x)) \leq A_2 |\varphi(x)|^p \leq \frac{A_2}{|x|^{(2s-1)p}}, \quad \forall x \leq -1.$$

Since $\frac{p}{p-1} \leq 2s$, we have $(2s-1)p \geq 2s$, which implies that for all $\mu \geq \dfrac{C_{1,s}}{2s(2s-1)} + \dfrac{A_2 + 1}{2s - 1}$,

$$(-\Delta)^s \varphi(x) + \mu \varphi'(x) - f(\varphi(x)) \geq \frac{1}{|x|^{2s}} + O\left(\frac{1}{|x|^{4s-1}}\right), \quad \text{as } x \to -\infty.$$

Since $4s-1>2s$, we know that there exists some large $A>0$, which is independent of μ, such that for all $\mu \geq \frac{C_{1,s}}{2s(2s-1)}+\frac{A_2+1}{2s-1}$, we have

$$(-\Delta)^s\varphi(x)+\mu\varphi'(x)\geq f(\varphi(x)), \qquad x\leq -A.$$

For $-A<x<-1$, we know that $(-\Delta)^s\varphi(x)$ is bounded, but $\varphi'(x)=\frac{2s-1}{|x|^{2s}}\geq\frac{2s-1}{A^{2s}}$. So there exists some $K>0$ such that for all $\mu\geq K$,

$$(-\Delta)^s\varphi(x)+\mu\varphi'(x)\geq \sup_{x\in[-A,-1]} f(\varphi(x)).$$

Hence for all $\mu\geq K$, we have

$$(-\Delta)^s\varphi(x)+\mu\varphi'(x)\geq f(\varphi(x)), \quad \forall x\leq -1.$$

On the other hand, by the definition of $\varphi(x)$ and (1.2), we know that for all $x\geq -1$, $(-\Delta)^s\varphi(x)>0$, $\varphi'(x)=0$ and $f(\varphi(x))=0$. In summary, for all $\mu\geq K$, we have $\varphi(x)$ is a super-solution for (3.3). Now fix some large $M>0$ such that $\varphi(-M)=\frac{1}{M^{2s-1}}<\theta$. Claim: For all $\mu\geq K$ and all $b\geq M$, we have $u_{\mu,b}(x)\leq\varphi(x-M)$ for all $x\in\mathbb{R}$, in particular, $u_{\mu,b}(0)<\theta$.

Let $\phi(x)=\varphi(x-M)$ and define

$$\Psi_t(x)=\phi(x+t)-u_\mu(x), \qquad x\in\mathbb{R}.$$

Let

$$\mathcal{O}=\{t\geq 0:\Psi_t(x)=\phi(x+t)-u_\mu(x)\geq 0, \qquad x\in\mathbb{R}\},$$

then $\mathcal{O}$ is nonempty since $\{t\geq 2b\}\subset\mathcal{O}$. $\mathcal{O}$ is clearly closed. Take a convergent sequence $\{t_n\}\subset\mathcal{O}$, $t_n\to t$ as $n\to\infty$, then

$$\lim_{n\to\infty}\Psi_{t_n}(x)=\lim_{n\to\infty}\phi(x+t_n)-u_\mu(x)\geq 0, \qquad x\in\mathbb{R}.$$

Therefore $t\in\mathcal{O}$.

Next we show that for any $t\in\mathcal{O}$,

$$\Psi_t(x)=\phi(x+t)-u_\mu(x)>0 \text{ for all } x\in(-b,b).$$

In fact, if there exists $x_0\in(-b,b)$ such that $\Psi_t(x_0)=\phi(x_0+t)-u_\mu(x_0)=0$, then

$$0>(-\Delta)^s\Psi_t(x_0)+\mu\Psi_t'(x_0)\geq f(\phi(x_0+t))-f(u_\mu(x_0))=0.$$

This is a contradiction.

It follows that $\mathcal{O}$ is open. Together with the fact that $\mathcal{O}$ is closed, we get $\mathcal{O}=[0,\infty)$. By the above sliding argument we know

$$u_\mu(0)\leq\varphi(-M)<\theta.$$

Similarly, for a lower bound we define $\varphi_1(x)=1-\varphi(-x)$. Then if $\mu\leq -K$, $x>1$,

$$(-\Delta)^s\varphi_1(x)+\mu\varphi_1'(x)=-[(-\Delta)^s\varphi(-x)-\mu\varphi'(x)]\leq 0\leq f(\varphi_1).$$

Moreover $\varphi_1(x)=0$ for $x\leq 1$. Take M so that $\varphi_1(-M)=1-t_0$, then $\varphi_1(x)>\theta$ for $x\geq M$. Define $\varphi_{1,M}(x)=\varphi_1(x+M)$, then $\varphi_{1,M}$ is a sub-solution to 3.3. Therefore by the same argument as above $u_\mu(0)\geq\varphi_{1,M}(0)>\theta$ for $\mu<-K$. □

Theorem 3.2 *Under the conditions of Proposition 3.1, there exists a subsequence $b_n \to \infty$ such that $u_{b_n} \to u_0$ and $\mu_{b_n} \to \mu_0$. Furthermore, $\mu_0 \in (0, K]$ and u_0 is a monotone increasing solution of (1.1).*

Proof By Lemma 3.6, $\mu_b \in [-K, K]$ we have the elliptic estimate for u_b:

$$\|u_b\|_{C^{2,\alpha}} \le C$$

for some $\alpha \in (0, 1)$. Thus there exists a subsequence $b_n \to \infty$ such that

$$\mu_n := \mu_{b_n} \to \mu_0 \in [-K, K]$$
$$u_n := u_{b_n} \to u_0, \quad \text{as } n \to \infty.$$

Thus u_0 satisfies $(-\Delta)^s u_0 + \mu_0 u_0' = f(u_0)$. Also we know u_0 is monotone increasing, $u_0(0) = \theta$ and u_0 is bounded. By a compactness argument, there exist γ_0, γ_1 such that $\lim_{x\to-\infty} u_0(x) = \gamma_0$ and $\lim_{x\to\infty} u_0(x) = \gamma_1$ with

$$0 \le \gamma_0 \le \theta \le \gamma_1 \le 1.$$

We know both γ_0 and γ_1 satisfy $f(\gamma_0) = 0$ and $f(\gamma_1) = 0$ which implies $\gamma_0 = 0,\ \gamma_1 = 1$. Moreover, by integrating $(-\Delta)^s u_0 + \mu_0 u_0' = f(u_0)$ over $\mathbb{R}$, together with Lemma 3.1, we know

$$\mu_0 = \int_{\mathbb{R}} f(u_0(x))dx > 0.$$

□

4 Asymptotic rate at ±∞

In this section, we will study asymptotic behaviors of solutions to (1.1) when $x \to \pm\infty$. Let $f \in C^1(\mathbb{R})$ satisfy (1.7) and (μ, u) be a solution to (1.1). First we investigate the asymptotic behavior of u when $x \to \infty$. Let $M = \|f\|_{C^1([0,1])} > 0$, by (3.1), we know that

$$\begin{cases} \operatorname{div}[y^{1-2s}\nabla v(x, y)] = 0, \quad \forall (x, y) \in \mathbb{R}^2_+, \\ \lim_{y\searrow 0} -d_s y^{1-2s} v_y(x, y) + \mu v_x(x, 0) + Mv(x, 0) \\ \qquad = [M + f'(u(x))]u'(x) \ge 0, \quad \forall x \in \mathbb{R}, \\ v(x, y) > 0, \ \ \forall (x, y) \in \overline{\mathbb{R}^2_+}, \qquad \text{and} \qquad \lim_{|(x,y)|\to\infty} v(x, y) = 0. \end{cases} \tag{4.1}$$

We consider the auxiliary function

$$\varphi(x, y) = \frac{y^{2s}}{[x^2 + y^2]^{\frac{1+2s}{2}}} + \frac{2sd_s}{M} \cdot \frac{1}{[x^2 + y^2]^{\frac{1+2s}{2}}}, \qquad \forall x \ge 1,\ y \ge 0.$$

By direct computations, for all $x \ge 1$ and all $y \ge 0$, we know that

$$\frac{2sd_s}{M} \cdot \frac{1}{[x^2 + y^2]^{\frac{1+2s}{2}}} \le \varphi(x, y) \le \left(1 + \frac{2sd_s}{M}\right) \cdot \frac{1}{|(x, y)|},$$
$$\operatorname{div}[y^{1-2s}\nabla\varphi(x, y)] = \frac{2sd_s}{M} \cdot \frac{(4s)(1 + 2s)y^{1-2s}}{[x^2 + y^2]^{\frac{2s+3}{2}}} \ge 0,$$

$$\lim_{y\searrow 0} -d_s y^{1-2s}\varphi_y(x,y) = d_s \lim_{y\searrow 0}\left[\frac{y^2-2sx^2}{[x^2+y^2]^{\frac{3}{2}+s}} + \frac{2sd_s}{M}\cdot\frac{y^{2-2s}}{[x^2+y^2]^{\frac{\alpha+2}{2}}}\right]$$
$$= -\frac{2sd_s}{|x|^{1+2s}}, \quad \text{and}$$
$$D_x\varphi(x,0) = -\frac{2sd_s}{M}\cdot\frac{2s}{|x|^{2+2s}}.$$

Hence for all $x \geq 1$, we have

$$\lim_{y\searrow 0} -d_s y^{1-2s}\varphi_y(x,y) + \mu\varphi_x(x,0) + M\varphi(x,0)$$
$$= -\frac{2sd_s}{|x|^{1+2s}} - \frac{2\mu sd_s}{M}\cdot\frac{2s}{|x|^{2+2s}} + M\cdot\frac{2sd_s}{M}\cdot\frac{1}{|x|^{1+2s}},$$
$$= -\frac{2\mu sd_s}{M}\cdot\frac{2s}{|x|^{2+2s}} \leq 0.$$

For any $\delta > 0$, let

$$w_\delta(x,y) = v(x,y) - \delta\varphi(x,y), \quad \forall x \geq 1,\ y \geq 0.$$

Then w_δ satisfies

$$\begin{cases} \operatorname{div}[y^{1-2s}\nabla w_\delta(x,y)] \leq 0, \quad \forall x \geq 1,\ y > 0, \\ \lim_{y\searrow 0} -d_s y^{1-2s} D_y w_\delta(x,y) + \mu D_x w_\delta(x,0) + M w_\delta(x,0) \geq 0, \quad \forall x \geq 1, \\ \lim_{|(x,y)|\to\infty} w_\delta(x,y) = 0. \end{cases} \tag{4.2}$$

We have the following

Proposition 4.1 *There exists some constant $C > 0$ such that*

$$u'(x) \geq \frac{C}{|x|^{1+2s}}, \quad \forall x \geq 1.$$

Proof By the same argument as in Lemma 2.2, we know that there is a positive constant δ_0 such that

$$v(x,y) \geq \delta_0\varphi(x,y), \quad \forall x \geq 1,\ y \geq 0.$$

In particular, we know that

$$u'(x) = v(x,0) \geq \delta_0\varphi(x,0) = \frac{2\delta_0 sd_s}{|x|^{1+2s}}, \quad \forall x \geq 0.$$

□

Lemma 4.1 *Let $\beta > 0$, we consider the function*

$$\psi_\beta(x) = \begin{cases} \frac{1}{|x|^\beta}, \quad \forall x < -1, \\ 0, \quad \forall x \geq -1. \end{cases}$$

Then

a. *If $0 < \beta < 1$, we have*

$$(-\Delta)^s\psi_\beta(x) = -\frac{C_{1,s}\cdot B(2s+\beta, 1-\beta)}{x^{2s+\beta}} + o\left(\frac{1}{x^{2s+\beta}}\right), \quad \text{as } x \to \infty.$$

b. *If* $\beta > 1$, *we have*

$$(-\Delta)^s \psi_\beta(x) = -\frac{C_{1,s}}{\beta - 1} \cdot \frac{1}{x^{1+2s}} + o\left(\frac{1}{x^{1+2s}}\right), \quad as\ x \to \infty.$$

c. *If* $\beta = 1$, *we have*

$$(-\Delta)^s \psi_1(x) = -\frac{C_{1,s} \ln x}{x^{2s+1}} + o\left(\frac{\ln x}{x^{2s+1}}\right), \quad as\ x \to \infty.$$

Proof In fact, for all $x \geq 2$, by changing of variables, we know that

$$\begin{aligned}(-\Delta)^s \psi_\beta(x) = C_{1,s}\Bigg[&\int_{-\infty}^{-x-1} \frac{\psi_\beta(x) - \psi_\beta(x+y)}{|y|^{1+2s}}\,dy \\ &+(\text{P.V.})\int_{-x-1}^{\infty} \frac{\psi_\beta(x) - \psi_\beta(x+y)}{|y|^{1+2s}}\,dy\Bigg] \\ = C_{1,s}&\int_{-\infty}^{-x-1} \frac{-1}{|x+y|^\beta |y|^{1+2s}}\,dy = -\frac{C_{1,s}}{x^{2s+\beta}}\int_{-\infty}^{-1-\frac{1}{x}} \frac{1}{|z+1|^\beta |z|^{1+2s}}\,dz.\end{aligned}$$

a. When $0 < \beta < 1$, we have

$$\int_{-2}^{-1} \frac{1}{|z+1|^\beta}\,dz < \infty.$$

By the dominated convergence theorem, we know that

$$\int_{-\infty}^{-1-\frac{1}{x}} \frac{1}{|z+1|^\beta |z|^{1+2s}}\,dz \to \int_{-\infty}^{-1} \frac{1}{|z+1|^\beta |z|^{1+2s}}\,dz, \quad \text{as } x \to \infty.$$

On the other hand, we know that

$$\begin{aligned}\int_{-\infty}^{-1} \frac{1}{|z+1|^\beta |z|^{1+2s}}\,dz &= \int_0^1 \frac{y^{1+2s}}{\left|-\frac{1}{y}+1\right|^\beta} \cdot \frac{1}{y^2}\,dy \quad \left(\text{by letting } z = -\frac{1}{y}\right) \\ &= \int_0^1 y^{2s+\beta-1}(1-y)^{-\beta}\,dy = B(2s+\beta, 1-\beta) > 0.\end{aligned}$$

So we know that

$$(-\Delta)^s \psi_\beta(x) = -\frac{C_{1,s} \cdot B(2s+\beta, 1-\beta)}{x^{2s+\beta}} + o\left(\frac{1}{x^{2s+\beta}}\right), \quad \text{as } x \to \infty.$$

b. When $\beta > 1$, we know that $\int_{-2}^{-1} \frac{1}{|z+1|^\beta}\,dz = \infty$, which implies that

$$\int_{-\infty}^{-1-\frac{1}{x}} \frac{1}{|z+1|^\beta |z|^{1+2s}}\,dz \to \infty, \quad \text{as } x \to \infty.$$

By L'Hospital rule, we have

$$\lim_{x\to\infty} \frac{\int_{-\infty}^{-1-\frac{1}{x}} \frac{1}{|z+1|^\beta |z|^{1+2s}}\,dz}{x^{\beta-1}} = \lim_{x\to\infty} \frac{x^\beta \cdot \left|1+\frac{1}{x}\right|^{-1-2s} \cdot \frac{1}{x^2}}{(\beta-1)x^{\beta-2}} = \frac{1}{\beta-1}.$$

So we derive

$$(-\Delta)^s \psi_\beta(x) = -\frac{C_{1,s}}{\beta - 1} \cdot \frac{1}{x^{1+2s}} + o\left(\frac{1}{x^{1+2s}}\right), \quad \text{as } x \to \infty.$$

c. When $\beta = 1$, we know that $\int_{-2}^{-1} \frac{1}{|z+1|}\, dz = \infty$, which implies that

$$\int_{-\infty}^{-1-\frac{1}{x}} \frac{1}{|z+1||z|^{1+2s}}\, dz \to \infty, \quad \text{as } x \to \infty.$$

By L'Hospital rule, we know that

$$\lim_{x\to\infty} \frac{\int_{-\infty}^{-1-\frac{1}{x}} \frac{1}{|z+1||z|^{1+2s}}\, dz}{\ln x} = \lim_{x\to\infty} \frac{|x| \cdot \left|1+\frac{1}{x}\right|^{-1-2s} \cdot \frac{1}{x^2}}{\frac{1}{x}} = 1.$$

Therefore we have

$$(-\Delta)^s \psi_1(x) = -\frac{C_{1,s} \ln x}{x^{2s+1}} + o\left(\frac{\ln x}{x^{2s+1}}\right), \quad \text{as } x \to \infty.$$

□

Lemma 4.2 *Let $\beta > 0$, $\psi_\beta(x)$ be defined as in Lemma 4.1, then we have the following estimates:*

a. *If $0 < \beta < 1$, there holds that*

$$(-\Delta)^s \psi_\beta(x) = -\frac{C_{1,s} \cdot A(s,\beta)}{|x|^{2s+\beta}} + o\left(\frac{1}{|x|^{2s+\beta}}\right), \quad \textit{as } x \to -\infty;$$

where

$$A(s,\beta) = \int_1^\infty \frac{1}{|z|^{1+2s}|z+1|^\beta}\, dz - \frac{1}{s} + \int_0^1 \frac{\frac{1}{|z-1|^\beta} + \frac{1}{|z+1|^\beta} - 2}{|z|^{1+2s}}\, dz;$$

b. *If $\beta > 1$, we have*

$$(-\Delta)^s \psi_\beta(x) = -\frac{C_{1,s}}{\beta - 1} \cdot \frac{1}{|x|^{1+2s}} + o\left(\frac{1}{|x|^{2s+1}}\right), \quad \textit{as } x \to -\infty;$$

c. *If $\beta = 1$, we have*

$$(-\Delta)^s \psi_1(x) = -\frac{C_{1,s} \ln|x|}{|x|^{2s+1}} + o\left(\frac{\ln|x|}{|x|^{2s+1}}\right), \quad \textit{as } x \to -\infty.$$

Proof For all $x < -2$, we know that $x + 1 < -x - 1$ and

$$\begin{aligned}
(-\Delta)^s \psi_\beta(x) &= -\frac{C_{1,s}}{2} \int_{\mathbb{R}} \frac{\psi_\beta(x+y) + \psi_\beta(x-y) - 2\psi_\beta(x)}{|y|^{1+2s}}\, dy \\
&= -\frac{C_{1,s}}{2} \Bigg[\int_{-\infty}^{x+1} \frac{\frac{1}{|x+y|^\beta} - \frac{2}{|x|^\beta}}{|y|^{1+2s}}\, dy + \int_{x+1}^{-x-1} \frac{\frac{1}{|x+y|^\beta} + \frac{1}{|x-y|^\beta} - \frac{2}{|x|^\beta}}{|y|^{1+2s}}\, dy \\
&\quad + \int_{-x-1}^{\infty} \frac{\frac{1}{|x-y|^\beta} - \frac{2}{|x|^\beta}}{|y|^{1+2s}}\, dy \Bigg] \\
&= -\frac{C_{1,s}}{2|x|^{2s+\beta}} \Bigg[\int_{-\infty}^{-1-\frac{1}{x}} \frac{\frac{1}{|z-1|^\beta} - 2}{|z|^{1+2s}}\, dz + \int_{-1-\frac{1}{x}}^{1+\frac{1}{x}} \frac{\frac{1}{|z-1|^\beta} + \frac{1}{|z+1|^\beta} - 2}{|z|^{1+2s}}\, dz
\end{aligned}$$

$$+\int_{1+\frac{1}{x}}^{\infty}\frac{\frac{1}{|z+1|^{\beta}}-2}{|z|^{1+2s}}\,dz\Bigg]\quad \text{Let} y=-xz$$

$$=-\frac{C_{1,s}}{|x|^{2s+\beta}}\left[\int_{1+\frac{1}{x}}^{\infty}\frac{\frac{1}{|z+1|^{\beta}}-2}{|z|^{1+2s}}\,dz+\int_{0}^{1+\frac{1}{x}}\frac{\frac{1}{|z-1|^{\beta}}+\frac{1}{|z+1|^{\beta}}-2}{|z|^{1+2s}}\,dz\right]$$

For the first term inside the bracket, we know that

$$\lim_{x\to-\infty}\int_{1+\frac{1}{x}}^{\infty}\frac{\frac{1}{|z+1|^{\beta}}-2}{|z|^{1+2s}}\,dz=\int_{1}^{\infty}\frac{1}{|z|^{1+2s}|z+1|^{\beta}}\,dz-\frac{1}{s}.$$

a. Since $\beta\in(0,1)$, we know that $\int_0^1\frac{1}{|z-1|^{\beta}}\,dz<\infty$, which implies that

$$\lim_{x\to-\infty}\int_{0}^{1+\frac{1}{x}}\frac{\frac{1}{|z-1|^{\beta}}+\frac{1}{|z+1|^{\beta}}-2}{|z|^{1+2s}}\,dz=\int_{0}^{1}\frac{\frac{1}{|z-1|^{\beta}}+\frac{1}{|z+1|^{\beta}}-2}{|z|^{1+2s}}\,dz.$$

Let

$$A(s,\beta)=\int_{1}^{\infty}\frac{1}{|z|^{1+2s}|z+1|^{\beta}}\,dz-\frac{1}{s}+\int_{0}^{1}\frac{\frac{1}{|z-1|^{\beta}}+\frac{1}{|z+1|^{\beta}}-2}{|z|^{1+2s}}\,dz,$$

then we have

$$(-\Delta)^s\psi_\beta(x)=-\frac{C_{1,s}\cdot A(s,\beta)}{|x|^{2s+\beta}}+o\left(\frac{1}{|x|^{2s+\beta}}\right),\quad \text{as } x\to-\infty.$$

b. Since $\beta>1$, we know that $\int_0^1\frac{1}{|z-1|^{\beta}}\,dz=\infty$, which implies that

$$\int_{0}^{1+\frac{1}{x}}\frac{\frac{1}{|z-1|^{\beta}}+\frac{1}{|z+1|^{\beta}}-2}{|z|^{1+2s}}\,dz\to\infty,\quad \text{as } x\to-\infty.$$

By L'Hospital rule, we know that

$$\lim_{x\to-\infty}\frac{\int_0^{1+\frac{1}{x}}\frac{\frac{1}{|z-1|^{\beta}}+\frac{1}{|z+1|^{\beta}}-2}{|z|^{1+2s}}\,dz}{(-x)^{\beta-1}}=\lim_{x\to-\infty}\frac{\left[|x|^{\beta}+\frac{1}{2^{\beta}}-2\right]\cdot\left(-\frac{1}{x^2}\right)}{-(\beta-1)(-x)^{\beta-2}}=\frac{1}{\beta-1}.$$

Hence we have

$$(-\Delta)^s\psi_\beta(x)=-\frac{C_{1,s}}{\beta-1}\cdot\frac{1}{|x|^{2s+1}}+o\left(\frac{1}{|x|^{1+2s}}\right),\quad \text{as } x\to-\infty.$$

c. Since $\beta=1$, we know that $\int_0^1\frac{1}{|z-1|}\,dz=\infty$, which implies that

$$\int_{0}^{1+\frac{1}{x}}\frac{\frac{1}{|z-1|^{\beta}}+\frac{1}{|z+1|^{\beta}}-2}{|z|^{1+2s}}\,dz\to\infty,\quad \text{as } x\to-\infty.$$

By L'Hospital rule, we know that

$$\lim_{x\to-\infty}\frac{\int_0^{1+\frac{1}{x}}\frac{\frac{1}{|z-1|}+\frac{1}{|z+1|}-2}{|z|^{1+2s}}\,dz}{\ln(-x)}=\lim_{x\to-\infty}\frac{\left[|x|+\frac{1}{2}-2\right]\cdot\left(-\frac{1}{x^2}\right)}{\frac{1}{x}}=1.$$

Hence we have

$$(-\Delta)^s \psi_1(x) = -\frac{C_{1,s} \ln|x|}{|x|^{2s+1}} + o\left(\frac{\ln|x|}{|x|^{2s+1}}\right), \quad \text{as } x \to -\infty.$$

□

Lemma 4.3 *Consider the function*

$$\phi(x) = \begin{cases} 1, & \forall x \leq -1, \\ 0, & \forall x > -1. \end{cases}$$

Then

$$(-\Delta)^s \phi(x) = -\frac{C_{1,s}}{2s} \cdot \frac{1}{|x|^{2s}} + o\left(\frac{1}{|x|^{2s}}\right), \quad \textit{as } x \to \infty, \quad \textit{and}$$

$$(-\Delta)^s \phi(x) = \frac{C_{1,s}}{2s} \cdot \frac{1}{|x|^{2s}} + o\left(\frac{1}{|x|^{2s}}\right), \quad \textit{as } x \to -\infty.$$

Proof a. In fact, for all $x \geq 2$, we have

$$\begin{aligned}
(-\Delta)^s \phi(x) &= C_{1,s} \left[\int_{-\infty}^{-x-1} \frac{\phi(x) - \phi(x+y)}{|y|^{1+2s}} \, dy + (\text{P.V.}) \int_{-x-1}^{\infty} \frac{\phi(x) - \phi(x+y)}{|y|^{1+2s}} \, dy \right] \\
&= -C_{1,s} \int_{-\infty}^{-x-1} \frac{1}{|y|^{1+2s}} \, dy \\
&= -\frac{C_{1,s}}{2s} \cdot \frac{1}{|x+1|^{2s}} \\
&= -\frac{C_{1,s}}{2s} \cdot \frac{1}{|x|^{2s}} + o\left(\frac{1}{|x|^{2s}}\right), \quad \text{as } x \to \infty.
\end{aligned}$$

b. If $x \leq -2$, we have

$$\begin{aligned}
(-\Delta)^s \phi(x) &= C_{1,s} \left[(\text{P.V.}) \int_{-\infty}^{-x-1} \frac{\phi(x) - \phi(x+y)}{|y|^{1+2s}} \, dy + \int_{-x-1}^{\infty} \frac{\phi(x) - \phi(x+y)}{|y|^{1+2s}} \, dy \right] \\
&= C_{1,s} \int_{-x-1}^{\infty} \frac{1}{|y|^{1+2s}} \, dy \\
&= \frac{C_{1,s}}{2s} \cdot \frac{1}{|x+1|^{2s}} \\
&= \frac{C_{1,s}}{2s} \cdot \frac{1}{|x|^{2s}} + o\left(\frac{1}{|x|^{2s}}\right), \quad \text{as } x \to -\infty.
\end{aligned}$$

□

Below we show a form of the maximal principle which is a slight variation of those in [2,6].

Lemma 4.4 (The Maximum Principle) *Let H be a nonempty open subset of $\mathbb{R}$, assume $d(x) \geq 0$ for all $x \in H$ and $w \in C^1(\overline{H})$ satisfies*

$$\begin{cases} (-\Delta)^s w(x) + \mu w'(x) + d(x) w(x) \geq 0, & \forall x \in H, \\ \lim\limits_{|x| \to \infty} w(x) = 0, \\ w(x) \geq 0, & \forall x \notin H. \end{cases}$$

Then $w(x) \geq 0$ for all x in $\mathbb{R}$.

Proof Assume $w(x_0) < 0$ for some $x_0 \in \mathbb{R}$, since $w(x) \geq 0$ for all $x \notin H$, $\lim_{|x|\to\infty} w(x) = 0$, and $w \in C^1(\overline{H})$, then there exists some $x_1 \in H$ such that

$$w(x_1) = \inf_{x\in\mathbb{R}} w(x) < 0.$$

Since x_1 is a global minimum of w in $\mathbb{R}$, $x_1 \in H$ and $w \in C^1(H)$, then

$$(-\Delta)^s w(x_1) < 0, \quad \text{and} \quad w'(x_1) = 0.$$

Since $d(x) \geq 0$ for all $x \in H$, and $x_1 \in H$, so we have

$$(-\Delta)^s w(x_1) + \mu w'(x_1) + d(x_1)w(x_1) < 0,$$

which contradicts with the assumption. □

The following two propositions give suitable lower and upper bounds of the asymptotic decay rates of u' and $1 - u$ at ∞, which are expected to be a power of $1 + 2s$ and $2s$, respectively.

Proposition 4.2 *Let* $\frac{1}{2} < s < 1$ *and* (μ, u) *be a solution to* (1.1) *with* $\mu > 0$. *Assume that* $f'(1) < 0$, *then there exists some constant* $C > 0$ *such that*

$$u'(x) \leq \frac{C}{|x|^{2s}}, \quad \text{and} \quad 1 - u(x) \leq \frac{C}{|x|^{2s}}, \qquad x \geq 1.$$

Proof Since $f'(1) < 0$, there exists some $m > 0$ and $\theta_0 \in (0, 1)$ such that $f'(u) \leq -m$ for all $u \in [\theta_0, 1]$. Let $\epsilon > 0$ be such that $-\frac{C_{1,s}}{2s} + m\epsilon^{-2s} = \frac{m}{2}\epsilon^{-2s}$, that is, $\epsilon = \left(\frac{sm}{C_{1,s}}\right)^{\frac{1}{2s}}$. Consider

$$\Psi(x) = \phi\left(x - \epsilon^{-1} - 1\right) + \psi_{2s}(-\epsilon x) \quad \forall x \in \mathbb{R}.$$

we know that

$$\Psi(x) = \epsilon^{-2s} \cdot \frac{1}{|x|^{2s}}, \quad \text{and} \quad \Psi'(x) = -2s\epsilon^{-2s} \cdot \frac{1}{|x|^{1+2s}}, \qquad \forall x > \frac{1}{\epsilon}.$$

By Lemma 4.3 and Lemma 4.2, we know that

$$(-\Delta)^s \Psi(x) = -\frac{C_{1,s}}{2s} \cdot \frac{1}{|x|^{2s}} + o\left(\frac{1}{|x|^{2s}}\right), \quad \text{as } x \to \infty.$$

Hence we have

$$\begin{aligned}(-\Delta)^s \Psi(x) + \mu\Psi'(x) + m\Psi(x) &= \left[-\frac{C_{1,s}}{2s} + m\epsilon^{-2s}\right] \cdot \frac{1}{|x|^{2s}} + o\left(\frac{1}{|x|^{2s}}\right) \\ &= \frac{m}{2}\epsilon^{-2s} \cdot \frac{1}{|x|^{2s}} + o\left(\frac{1}{|x|^{2s}}\right), \quad \text{as } x \to \infty.\end{aligned}$$

So there exists some large $R > 0$ such that

$$(-\Delta)^s \Psi(x) + \mu\Psi'(x) + m\Psi(x) \geq 0, \quad \forall x \geq R.$$

Up to a translation, without loss of generality, we assume $u(0) = \theta_0$. Notice that $v(x) = u'(x) > 0$ in $\mathbb{R}$ satisfies

$$(-\Delta)^s v(x) + \mu v'(x) + mv(x) = [m + f'(u(x))]v(x) \leq 0, \quad \forall x \geq R.$$

Since $\Psi(x) > 0$ for all $x \in \mathbb{R}$, there exists some $C > 0$ such that

$$C > \|v\|_{C(\mathbb{R})} \quad \text{and} \quad C \inf_{x\in[\epsilon^{-1},R]} \Psi(x) \geq \|v\|_{C(\mathbb{R})}.$$

Since $\Psi(x) = \phi\left(x - \epsilon^{-1} - 1\right) = 1$ for all $x \leq \epsilon^{-1}$, we get $C\Psi(x) = C \geq \|v\|_{C(\mathbb{R})}$ for all $x \leq \epsilon^{-1}$. In summary, we know that

$$C\Psi(x) \geq v(x), \quad \forall x \leq R.$$

Let $w(x) = C\Psi(x) - v(x)$ for all $x \in \mathbb{R}$, we have

$$\begin{cases} (-\Delta)^s w(x) + \mu w'(x) + m w(x) \geq 0, & \forall x \geq R, \\ \lim_{x\to\infty} w(x) = 0, \\ w(x) \geq 0, \quad \forall x \leq R. \end{cases}$$

By Lemma 4.4, we have $w(x) \geq 0$ in $\mathbb{R}$, which implies that

$$\frac{C}{|x|^{2s}} \geq v(x) = u'(x), \quad \forall x \geq 1.$$

□

Proposition 4.3 *Let* $\frac{1}{2} < s < 1$, *assume that* $f'(1) < 0$, *let* (μ, u) *be a solution to* (1.1) *with* $\mu > 0$. *Then there exists some constant* $C > 0$ *such that*

$$u'(x) \leq \frac{C}{|x|^{1+2s}}, \quad \text{and} \quad 1 - u(x) \leq \frac{C}{|x|^{2s}}, \qquad x \geq 1.$$

Proof Since $f'(1) < 0$, there exists some $m > 0$ and $\theta_0 \in (0, 1)$ such that $f'(u) \leq -m$ for all $u \in [\theta_0, 1]$. Let $\epsilon > 0$ be such that

$$-\epsilon^{-1} \cdot \frac{C_{1,s}}{2s-1} - \epsilon^{-1} \cdot \frac{C_{1,s}}{2s} + m\epsilon^{-1-2s} = \frac{m}{2} \cdot \epsilon^{-1-2s}.$$

That is, we have

$$\frac{\epsilon^{2s}}{2s} + \frac{\epsilon^{2s}}{2s-1} = \frac{m}{2C_{1,s}}.$$

By considering the function $\Psi(x) = \psi_{2s}(\epsilon x - 2) + \psi_{1+2s}(-\epsilon x)$ for all $x \in \mathbb{R}$, we know that

$$\Psi(x) = \epsilon^{-1-2s} \cdot \frac{1}{|x|^{1+2s}}, \quad \text{and} \quad \Psi'(x) = -\epsilon^{-1-2s} \cdot \frac{1+2s}{|x|^{2+2s}}, \quad \forall x > \epsilon^{-1}.$$

By Lemma 4.1 and Lemma 4.2, we know that

$$(-\Delta)^s\Psi(x) = -\epsilon^{-1} \cdot \frac{C_{1,s}}{2s-1} \cdot \frac{1}{|x|^{1+2s}} - \epsilon^{-1} \cdot \frac{C_{1,s}}{2s} \cdot \frac{1}{|x|^{1+2s}} + o\left(\frac{1}{|x|^{1+2s}}\right), \quad \text{as } x \to \infty.$$

So we get

$$(-\Delta)^s\Psi(x) + \mu\Psi'(x) + m\Psi(x) = \left[-\epsilon^{-1} \cdot \frac{C_{1,s}}{2s-1} - \epsilon^{-1} \cdot \frac{C_{1,s}}{2s} + m\epsilon^{-1-2s}\right] \cdot \frac{1}{|x|^{1+2s}}$$

$$+o\left(\frac{1}{|x|^{1+2s}}\right)$$
$$=\frac{m}{2}\cdot\epsilon^{-1-2s}\cdot\frac{1}{|x|^{1+2s}}+o\left(\frac{1}{|x|^{1+2s}}\right),\quad \text{as } x\to\infty.$$

Hence there exists some large $R>0$ such that

$$(-\Delta)^s\Psi(x)+\mu\Psi'(x)+m\Psi(x)\geq 0,\quad \forall x\geq R.$$

Without loss of generality, we assume $u\left(\epsilon^{-1}\right)=\theta_0$, we know that $v=u'$ satisfies

$$(-\Delta)^s v(x)+\mu v'(x)+mv(x)=[m+f'(u(x))]v(x)\leq 0,\quad \forall x\geq\epsilon^{-1}.$$

For all $x\leq\epsilon^{-1}$, we have $\epsilon x-2\leq -1$ and $-\epsilon x\geq -1$, which implies that

$$\Psi(x)=\psi_{2s}(\epsilon x-2)=\frac{1}{|\epsilon x-2|^{2s}}.$$

By Proposition (4.2), we know that there exists some constant $C_1>0$ such that

$$u'(x)=v(x)\leq C_1\Psi(x),\quad \forall x\leq\epsilon^{-1}.$$

Notice that for all $x\geq\epsilon^{-1}$, $\Psi(x)\geq\psi_{1+2s}(-\epsilon x)>0$, which implies that there exists some $C_2>0$ such that

$$C_2\inf_{x\in[\frac{1}{\epsilon},R]}\Psi(x)\geq\sup_{x\in[\epsilon^{-1},R]}v(x).$$

Let $C=\max\{C_1,C_2\}>0$ and $w(x)=C\Psi(x)-v(x)$ for all $x\in\mathbb{R}$, then

$$\begin{cases}(-\Delta)^s w(x)+\mu w'(x)\geq 0, & \forall x\geq R,\\ \lim\limits_{x\to\infty} w(x)=0,\\ w(x)\geq 0, & \forall x\leq R.\end{cases}$$

By Lemma 4.4, we know that $w(x)\geq 0$ in $\mathbb{R}$, which implies

$$\frac{C}{|x|^{1+2s}}\geq v(x)=u'(x),\quad \forall x\geq 1.$$

The inequality for $1-u(x)$ follows immediately. □

Proposition 4.4 *Let* $\frac{1}{2}<s<1$,*let* (μ,u) *be a solution to* (1.1) *with* $\mu>0$ *in Theorem* 3.2. *Then there exists some constant* $C>0$ *such that*

$$\frac{1}{C|x|^{2s}}\leq u'(x),\quad\text{and}\quad\frac{1}{C|x|^{2s-1}}\leq u(x)\leq\frac{C}{|x|^{2s-1}},\qquad\forall x\leq -1.$$

Proof We have shown in the proof of Theorem 3.2 that there exists some constant $C>0$ such that

$$u(x)\leq\frac{C}{|x|^{2s-1}},\qquad\forall x\leq -1.$$

Now it suffices to show that there exists some constant $C>0$ such that

$$\frac{1}{C|x|^{2s}}\leq u'(x),\qquad\forall x\leq -1.$$

Let $\epsilon > 0$ be such that $-\dfrac{C_{1,s}}{2s-1} + 2s\mu\epsilon^{1-2s} = -\dfrac{C_{1,s}}{2(2s-1)}$, that is,

$$\epsilon^{1-2s} = \frac{C_{1,s}}{4s\mu(2s-1)}.$$

Let $\Phi(x) = \psi_{2s}(\epsilon x)$ in $\mathbb{R}$, then

$$\Phi(x) = \epsilon^{-2s} \cdot \frac{1}{|x|^{2s}}, \quad \text{and} \quad \Phi'(x) = \epsilon^{-2s} \cdot \frac{2s}{|x|^{1+2s}}, \qquad \forall x \le -\epsilon^{-1}.$$

By Lemma 4.2, we have

$$(-\Delta)^s \Phi(x) = -\frac{C_{1,s}}{2s-1} \cdot \frac{\epsilon^{-1}}{|x|^{1+2s}} + o\left(\frac{1}{|x|^{1+2s}}\right), \quad \text{as } x \to -\infty.$$

So we get

$$\begin{aligned}(-\Delta)^s \Phi(x) + \mu\Phi'(x) &= -\frac{C_{1,s}}{2s-1} \cdot \frac{\epsilon^{-1}}{|x|^{1+2s}} + \mu\epsilon^{-2s} \cdot \frac{2s}{|x|^{1+2s}} + o\left(\frac{1}{|x|^{1+2s}}\right) \\ &= \left[-\frac{C_{1,s}}{2s-1} + 2s\mu\epsilon^{1-2s}\right] \frac{\epsilon^{-1}}{|x|^{2s}} + o\left(\frac{1}{|x|^{1+2s}}\right) \\ &= -\frac{C_{1,s}}{2(2s-1)} \cdot \frac{\epsilon^{-1}}{|x|^{1+2s}} + o\left(\frac{1}{|x|^{1+2s}}\right), \quad \text{as } x \to -\infty.\end{aligned}$$

Therefore there exists some large $R > 0$ such that

$$(-\Delta)^s \Phi(x) + \mu\Phi'(x) \le 0, \quad \forall x \le -R.$$

Since $f'(t) \ge 0$ for all $t \in [0, \theta_0]$, without loss of generality, we may assume $u(-\epsilon^{-1}) = \theta_0$. Notice that $v(x) = u'(x) > 0$ in $\mathbb{R}$ satisfies

$$(-\Delta)^s v(x) + \mu v'(x) = f'(u(x))(x) \ge 0, \quad \forall x \le -\epsilon^{-1}.$$

Since $\Phi(x) = 0$ for all $x \ge -\epsilon^{-1}$, we get $\Phi(x) \le v(x)$ for all $x \ge -\epsilon^{-1}$. Since $v(x) > 0$ in $\mathbb{R}$, there exists some $C > 1$ such that

$$C \inf_{x\in[-R,-\epsilon^{-1}]} v(x) \ge \sup_{x\in[-R,-\epsilon^{-1}} \Phi(x).$$

Let $w(x) = Cv(x) - \Phi(x)$ for all $x \in \mathbb{R}$, we have

$$\begin{cases} (-\Delta)^s w(x) + \mu w'(x) \ge 0, \quad \forall x \le -R, \\ \lim\limits_{x\to-\infty} w(x) = 0, \\ w(x) \ge 0, \quad \forall x \ge -R. \end{cases}$$

By Lemma 4.4, we have $w(x) \ge 0$ in $\mathbb{R}$, which implies that

$$\frac{C}{|x|^{2s}} \le v(x) = u'(x), \quad \forall x \le -1.$$

□

Acknowledgments This work was partially supported by a grant from the Simons Foundation (Award # 199305) and a NSF IPA award. The authors would also like to thank the anonymous referee for helpful suggesions for the revision of the manuscript.

References

1. Aronson, D., Weinberger, H.: Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, partial differential equations and related topics, lecture notes in mathematics, pp. 4–49. Springer, Berlin (1975)
2. Cabré, X., Sire, Y.: Nonlinear equations for fractional Laplacians I: regularity, maximum principles, and hamiltonian estimates. ArXiv (2010)
3. Caffarelli, L., Silvestre, L.: An extension problem related to the fractional Laplacian. Commun. Partial Differ. Equ. **32**, 1245–1260 (2007)
4. Cabré, X., Roquejoffre, J.: The influence of fractional diffusion in Fisher–KPP equations. Comm. Math. Phys. **320**(3), 679–722 (2013)
5. Fisher, R.A.: The wave of advance of advantageous genes. Ann. Eugen. **7**, 353–369 (1937)
6. Gui, C., Zhao, M.: Traveling Wave Solutions of Allen–Cahn Equation with a Fractional Laplacian, Ann. I. H. Poincaré-AN (2014). doi:10.1016/j.anihpc.2014.03.005
7. Kolmogorov, A., Petrovskii, I., Piskunov, N.: A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem. Bull. Moscow Univ. Math. Ser. A **1**, 1–25 (1937)
8. Landkof, N.S.: Foundations of Modern Potential Theory. In: Doohovskoy, A.P., (ed) Die Grundlehren der mathematischen Wissenschaften, Band 180. Translated from the Russian. Springer, New York, (1972)
9. Mellet, A., Roquejoffre, J., Sire, Y.: Existence and asymptotics of fronts in non local combustion models. Arxiv (2010)