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Calculus of Variations

Existence and multiplicity of solutions for Schrödinger–Poisson equations with sign-changing potential

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Abstract In this paper, we study the existence and multiplicity of solutions for the Schrödinger–Poisson equations

$$\begin{cases} -\Delta u + \lambda V(x)u + K(x)\phi u = f(x,u) & \text{in}\mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ is a parameter, the potential V may change sign and f is either superlinear or sublinear in u as $|u| \to \infty$.

Mathematics Subject Classification 35J47 · 35J50

1 Introduction and main results

Consider the following Schödinger–Poisson equations:

$$\begin{cases} -\Delta u + \lambda V(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \qquad (SP)_\lambda$$

where $\lambda \geq 1$ is a parameter, $V \in C(\mathbb{R}^3, \mathbb{R})$ and $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$.

Problem $(SP)_\lambda$ (also called Schrödinger–Maxwell equation) arises in applications from mathematical physics, such as in quantum electrodynamics, to describe the interaction of a charged particle with the electromagnetic field, and also in semiconductor theory, in nonlinear optics and in plasma physics. For more details in physical aspects, we refer to [9,12].

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There has been a vast literature on the study of existence and multiplicity of solutions of system $(SP)_\lambda$ under various hypotheses on the potential $V(x)$ and the nonlinearity $f(x,u)$, see [1–3,5,9–14,18,19,21,22,24–28,31,34–37] and the references therein. Most of them dealt with the situation where $V(x)$ is a positive constant or being radially symmetric and $f(x,u)=|u|^{p-1}u$, $1<p<5$. In [25] the case $p=5/3$ was studied. The authors applied a minimization procedure in an appropriate manifold to find a positive solution (possibly non-radial) for system $(SP)_1$ (i.e. $(SP)_\lambda$ with $\lambda=1$). In [11,12], a radial positive solution of $(SP)_1$ was obtained for $3\le p<5$, by taking advantage of the mountain pass theorem due to Ambrosetti and Rabinowitz [4]. In [13], a related Pohozǎev identity was found, and with this in hand, the authors proved that problem $(SP)_1$ has no nontrivial solutions for $p\le 1$ or $p>5$. This result was completed in [24], where Ruiz showed that if $p\le 2$, problem $(SP)_1$ does not admit any nontrivial solution, and if $2<p<5$, there exists a positive radial solution of $(SP)_1$. Ambrosetti and Ruiz [2] and Ambrosetti [3] considered problem $(SP)_1$ with a parameter, i.e.,

$$\begin{cases} -\Delta u+u+\lambda\phi u=|u|^{p-1}u & \text{in } \mathbb{R}^3,\\ -\Delta\phi=u^2 & \text{in } \mathbb{R}^3. \end{cases} \qquad (A)_\lambda$$

Using variational methods, they constructed the existence of infinitely many pairs of radial solutions of problem $(A)_\lambda$, where $2<p<5$, for all $\lambda>0$, and also multiple solutions (but not infinitely many) of $(A)_\lambda$, where $1<p\le 2$, for $\lambda>0$ small sufficiently. In addition, the existence of infinitely many non-radial solutions of system $(SP)_1$ was constructed in d'Avenia et al. [14], when $1<p<5$ and $K(x)$ is a positive radial function decaying at infinity. See also [5,19,34,37] for the critical case.

The case of positive and non-radial potential V has been discussed in [10,22,26,28,31,35]. In particular, supposing that $V(x)$ satifies:

- (V_1) $V\in C(\mathbb{R}^3,\mathbb{R})$ and $\inf_{x\in\mathbb{R}^3}V(x)\ge a>0$, where a is a positive constant;
- (V_2) For any $b>0$, meas$\{x\in\mathbb{R}^3 : V(x)\le b\}<+\infty$, where meas denotes the Lebesgue measure in $\mathbb{R}^3$;

[10,22,31] established the existence of infinitely many high-energy solutions of problem $(SP)_1$, where f is 4-superlinear at infinity, while the existence of infinitely many small-energy solutions was proved in Sun [26] with sublinear nonlinearity. The proofs in [10,22,31] were based on the (variant) fountain theorem. It is worth mentioning that conditions (V_1)–(V_2) were first introduced by Bartsch and Wang [8] to guarantee the compact embedding of the functional space (see [8, Remark 3.5]). If replacing (V_2) by a more general assumption:

- (V_3) There is $b>0$ such that meas$\{x\in\mathbb{R}^3 : V(x)\le b\}<+\infty$,

the compactness of the embedding fails and this situation becomes more delicated.

Recently, [32,35] considered this case. Yang et al. [32] investigated the semiclassical solutions of the Schrödinger–Poisson equations

$$\begin{cases} -\varepsilon^2\Delta u+V(x)u+\phi u=f(x,u) & \text{in} \mathbb{R}^3,\\ -\Delta\phi=4\pi u^2 & \text{in } \mathbb{R}^3. \end{cases} \qquad (B)_\varepsilon$$

They assumed that (V_3) holds, $V(0)=\min V=0$ and $f(x,u)$ satisfies:

- (g_1) $f(x,u)=o(u)$ as $u\to 0$ uniformly in x;
- (g_2) There are $c_0>0$ and $q<6$ such that $|f(x,u)|\le c_0(1+|u|^{q-1})$ for all (x,u);
- (g_3) There are $a_0>0$, $p>4$ and $\mu>4$ such that $F(x,u)\ge a_0|u|^p$ and $\mu F(x,u)\le f(x,u)u$ for all (x,u), where $F(x,u):=\int_0^u f(x,s)ds$.

They showed that for any $\sigma > 0$ there exists $\varepsilon_\sigma > 0$ such that $(B)_\varepsilon$ has at least one solution when $\varepsilon \le \varepsilon_\sigma$; and if additionally $f(x, u)$ is odd in u, then given any $\varepsilon > 0$ small enough $(B)_\varepsilon$ has at least m pairs of solutions. Zhao et al. [35] studied the existence of nontrivial solution and concentration results (as $\lambda \to +\infty$) of $(SP)_\lambda$, provided that V satisfies (V_3) and

(V_4) $V \in C(\mathbb{R}^3, \mathbb{R})$ and V is bounded below,
(V_5) $\Omega = int V^{-1}(0)$ is nonempty and has smooth boundary and $\overline{\Omega} = V^{-1}(0)$,

and $f(x, u) = |u|^{p-2}u$ $(2 < p < 6)$.

We also note that if $K \equiv 0$, $(SP)_\lambda$ reduces to the Schödinger equation

$$-\Delta u + \lambda V(x)u = f(x, u), \qquad x \in \mathbb{R}^N, \qquad (C)_\lambda$$

which has been the object of interest for many authors, see e.g. [15,16,29] and their references. In [16], Ding and Szulkin studied the existence and the number of decaying solutions of problem $(C)_\lambda$ when V may change sign, satisfies (V_4) and

(V_6) There exists $b > 0$ such that the set $\{x \in \mathbb{R}^N : V(x) < b\}$ is nonempty and has finite measure;

and f is either asymptotically linear or superlinear (but subcritical) in u as $|u| \to \infty$. Wang and Zhou [29] dealt with the ground states of problem $(C)_\lambda$, where $V(x)$ changes sign and may vanish at infinity, $f(x, u) = K_1(x)g(u)$ and g is either of the form $g(u) = |u|^{p-1}u$ with $1 < p < \frac{N+2}{N-2}$ or asymptotically linear.

Motivated by the works mentioned above, in the present paper, we are mostly interested in sign-changing potentials though in a few cases we need to have $V \ge 0$. Under (V_3)–(V_4) and some more generic 4-superlinear conditions on $f(x, u)$, we prove the existence and multiplicity of solutions of problem $(SP)_\lambda$ when $\lambda > 0$ large, using variational method. Furthermore, we investigate the situation where the nonlinearity $f(x, u)$ is sublinear with mild assumptions different from those studied previously. Infinitely many small-energy solutions are obtained for problem $(SP)_1$ by applying a new version of symmetric mountain pass lemma developed by Kajikiya. The main results are the following theorems.

First, we handle the 4-superlinear case, and hence make the following assumptions:

(f_1) $F(x, u) \ge 0$ for all (x, u) and $f(x, u) = o(u)$ uniformly in x as $u \to 0$.
(f_2) $F(x, u)/u^4 \to +\infty$ as $|u| \to \infty$ uniformly in x.
(f_3) $\mathcal{F}(x, u) := \frac{1}{4} f(x, u)u - F(x, u) \ge 0$ for all $(x, u) \in \mathbb{R}^3 \times \mathbb{R}$.
(f_4) There exist $a_1, L_1 > 0$ and $\tau \in (3/2, 2)$ such that

$$|f(x, u)|^\tau \le a_1 \mathcal{F}(x, u)|u|^\tau, \qquad \forall x \in \mathbb{R}^3, \ \ |u| \ge L_1.$$

(K) $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ and $K(x) \ge 0$ for all $x \in \mathbb{R}^3$.

Remark 1.1 It follows from (f_2) and (f_4) that $|f(x, u)|^\tau \le \frac{a_1}{4}|f(x, u)||u|^{\tau+1}$ for large u. Thus, by (f_1), for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|f(x, u)| \le \varepsilon|u| + C_\varepsilon|u|^{p-1}, \qquad \forall (x, u) \in \mathbb{R}^3 \times \mathbb{R} \tag{1.1}$$

and

$$|F(x, u)| \le \varepsilon u^2 + C_\varepsilon|u|^p, \qquad \forall (x, u) \in \mathbb{R}^3 \times \mathbb{R},$$

where $p = 2\tau/(\tau - 1) \in (4, 2^*)$, $2^* = 6$ is the critical exponent for the Sobolev embedding in dimension 3.

Theorem 1.1 (Superlinear) *Assume that* (V_3)–(V_4), (K) *and* (f_1)–(f_4) *are satisfied.*

(i) *If* $V(x) < 0$ *for some* $x \in \mathbb{R}^3$, *then for each* $k \in \mathbb{N}$, *there exist* $\lambda_k > k$ *and* $b_k > 0$ *such that problem* $(SP)_\lambda$ *has a nontrivial solution* $(u_\lambda, \phi_\lambda) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ *for every* $\lambda = \lambda_k$ *and* $|K|_2 < b_k$ *(or* $|K|_\infty < b_k$*).*

(ii) *If* $V^{-1}(0)$ *has nonempty interior, then there exist* $\Lambda > 0$ *and* $b_\lambda > 0$ *such that problem* $(SP)_\lambda$ *has a nontrivial solution* $(u_\lambda, \phi_\lambda) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ *for every* $\lambda > \Lambda$ *and* $|K|_2 < b_\lambda$ *(or* $|K|_\infty < b_\lambda$*).*

Remark 1.2 Theorem 1.1 (ii) generalizes [35, Theorem 1.1], which is the special case of Theorem 1.1 (ii) corresponding to $f(x,u) = |u|^{p-2}u$ $(4 < p < 6)$.

If $V \geq 0$, the restriction on the norm of K can be removed and we have the following theorem.

Theorem 1.2 (Superlinear) *Assume that* $V \geq 0$, (V_3)–(V_4), (K) *and* (f_1)–(f_4) *are satisfied, and* $V^{-1}(0)$ *has nonempty interior* Ω. *Then there exist* $\Lambda_* > 0$ *such that problem* $(SP)_\lambda$ *has at least one nontrivial solution* $(u_\lambda, \phi_\lambda) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ *whenever* $\lambda > \Lambda_*$. *Moreover, if* f *is odd in* t, *then for each* $k \geq 1$ *there exists* $\Lambda_k > 0$ *such that problem* $(SP)_\lambda$ *has at least* k *pairs of nontrivial solutions whenever* $\lambda > \Lambda_k$.

Remark 1.3 Theorem 1.2 can be viewed as an improvement of the results in Yang et al. [32] and Zhao et al. [35]. Comparing with [32, Theorems 1.1 and 1.2], our hypotheses on f are much weaker. Indeed, assumption (g_3) implies

$$0 < \mu F(x,u) \leq f(x,u)u \quad \text{for some } \mu > 4 \text{ and all } (x,u) \text{ with } u \neq 0.$$

So, if f satisfies (g_1) and (g_3), it is easy to see that (f_2)–(f_3) hold, and it will be showed as in the proof of [16, Lemma 2.2 (i)] that so does (f_4). As for [35], we consider a larger class of nonlinearities and discuss the multiplicity result.

Remark 1.4 There are functions f which match conditions (f_1)–(f_4) but not satisfying the results in [32,35]. For example, let

$$f(x,t) = h(x)t^3 \left(2\ln(1+t^2) + \frac{t^2}{1+t^2} \right), \qquad \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$

where h is a continuous bounded function with $\inf_{x \in \mathbb{R}^3} h(x) > 0$.

Next, we treat the sublinear case. Assume that:

(f_5) There exist constants $\sigma, \gamma \in (1,2)$ and functions $m \in L^{2/(2-\sigma)}(\mathbb{R}^3, \mathbb{R}^+)$, $h \in L^{2/(2-\gamma)}(\mathbb{R}^3, \mathbb{R}^+)$ such that

$$|f(x,u)| \leq m(x)|u|^{\sigma-1} + h(x)|u|^{\gamma-1}, \qquad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

(f_6) There exist $x_0 \in \mathbb{R}^3$, two sequences $\{\varepsilon_n\}$, $\{M_n\}$ and constants $a_2, \varepsilon, \delta > 0$ such that $\varepsilon_n > 0$, $M_n > 0$ and

$$\begin{aligned} &\lim_{n\to\infty} \varepsilon_n = 0, \qquad \lim_{n\to\infty} M_n = +\infty, \\ &\varepsilon_n^{-2} F(x,u) \geq M_n \qquad \text{for } |x - x_0| \leq \delta \text{ and } |u| = \varepsilon_n, \\ &F(x,u) \geq -a_2 u^2 \qquad \text{for } |x - x_0| \leq \delta \text{ and } |u| \leq \varepsilon. \end{aligned} \tag{1.2}$$

Theorem 1.3 (Sublinear) *Assume that* $V \geq 0$, (V_3), (K) *and* (f_5)–(f_6) *are satisfied and that* $f(x,u)$ *is odd in* u. *Then problem* $(SP)_1$ *possesses infinitely many nontrivial solutions* $\{(u_k, \phi_k)\}$ *such that*

$$\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_k|^2 + V(x)u_k^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_k u_k^2 dx - \int_{\mathbb{R}^3} F(x,u_k)dx \to 0^- \quad as \;\; k \to \infty.$$

Remark 1.5 In Sun [26], the existence of infinitely many small-energy solutions was obtained for $(SP)_1$, where $K \equiv 1$, under assumptions (V_1)–(V_2) and:

(f') $f(x,u) = b(x)|u|^{\sigma-1}$, where $b : \mathbb{R}^3 \to \mathbb{R}^+$ is a positive continuous function such that $b \in L^{2/(2-\sigma)}(\mathbb{R}^3, \mathbb{R})$ and $1 < \sigma < 2$ is a constant.

Observing (f') implies that there is an open set $J \subset \mathbb{R}^3$ such that

$$F(x,t)/t^2 \to +\infty \qquad \text{as } t \to 0 \text{ uniformly for } x \in J,$$

it is stronger than (f_5)–(f_6). Hence Theorem 1.3 improves [26, Theorem 1.1] by weakening hypotheses on V, K and f. There are functions V, K and f which match our setting but not satisfying the results in [21,26]. For example, let

$$V \equiv c(>0), \qquad K(x) = |x|^{-4},$$

and

$$f(x,u) = \begin{cases} |x|e^{-|x|^2}\left[\sigma|u|^{\sigma-2}u\sin^2\left(\frac{1}{|u|^{\varrho}}\right) - \varrho|u|^{\sigma-\varrho-2}\sin\left(\frac{2}{|u|^{\varrho}}\right)\right], & t \neq 0,\\ 0, & t = 0, \end{cases}$$

where $\varrho > 0$ small enough and $\sigma \in (1+\varrho, 2)$. Simple calculation shows that

$$F(x,u) = \begin{cases} |x|e^{-|x|^2}|u|^{\sigma}\sin^2\left(\frac{1}{|u|^{\varrho}}\right), & t \neq 0,\\ 0, & t = 0. \end{cases}$$

It is easy to check that (V_3)–(V_4), (K) and (f_5)–(f_6) are satisfied with $\varepsilon_n = \left(\frac{2}{(2n+1)\pi}\right)^{1/\varrho}$. However, in this case, (V_2) and (f') fail.

The paper is organized as follows. In Sect. 2 we introduce the variational setting and recall some related preliminaries. Section 3 is concerned with the 4-superlinear case and Sect. 4 with the sublinear case. In Sect. 5, concentration of solutions to problem $(SP)_\lambda$ on the set $V^{-1}(0)$ as $\lambda \to +\infty$ is discussed.

Notation

- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard scalar and norm

$$(u,v)_{H^1} = \int_{\mathbb{R}^3}(\nabla u \cdot \nabla v + uv)dx; \qquad \|u\|_{H^1} = (u,u)_{H^1}^{1/2}.$$

- $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|^2_{\mathcal{D}^{1,2}} := \int_{\mathbb{R}^3}|\nabla u|^2 dx$.
- $L^s(\Omega)$, $1 \leq s \leq +\infty$, $\Omega \subset \mathbb{R}^3$, denotes a Lebesgue space; the norm in $L^s(\Omega)$ is denoted by $|u|_{s,\Omega}$, where Ω is a proper subset of $\mathbb{R}^3$, by $|\cdot|_s$ when $\Omega = \mathbb{R}^3$.
- $\bar{S}$ is the best Sobolev constant for the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, i.e.,

$$\bar{S} = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|_{\mathcal{D}^{1,2}}}{|u|_6}.$$

- For any $r > 0$ and $z \in \mathbb{R}^3$, $B_r(z)$ denotes the ball of radius r centered at z.
- The letter c will be used to denote various positive constants which may vary from line to line and are not essential to the problem.

2 Variational setting and preliminaries

Let

$$E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V^+(x)u^2 dx < +\infty \right\},$$

where $V^{\pm}(x) = \max\{\pm V(x), 0\}$. Then E is a Hilbert space with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V^+(x)uv)dx, \qquad \|u\| = (u, u)^{1/2}.$$

We also need the following inner product

$$(u, v)_\lambda = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \lambda V^+(x)uv)dx,$$

and the corresponding norm is denoted by $\|u\|_\lambda = (u, u)_\lambda^{1/2}$ (so $\|\cdot\| = \|\cdot\|_1$). Set $E_\lambda = (E, \|\cdot\|_\lambda)$. It follows from (V_3), (V_4) and the Poincaré inequality that the embedding $E_\lambda \hookrightarrow H^1(\mathbb{R}^3)$ is continuous, and hence, for $s \in [2, 2^*]$, there exists $\nu_s > 0$ (independent of λ) such that

$$|u|_s \leq \nu_s \|u\|_\lambda, \qquad \forall u \in E_\lambda. \tag{2.1}$$

Let

$$F_\lambda := \left\{ u \in E_\lambda : \mathrm{supp} u \subset V^{-1}([0, +\infty)) \right\},$$

and $F_\lambda^\perp$ denote the orthogonal complement of F_λ in E_λ. Clearly, $F_\lambda = E_\lambda$ if $V \geq 0$, otherwise $F_\lambda^\perp \neq \{0\}$. Define

$$A_\lambda := -\Delta + \lambda V,$$

then A_λ is formally self-adjoint in $L^2(\mathbb{R}^3)$ and the associated bilinear form

$$a_\lambda(u, v) := \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \lambda V(x)uv)dx$$

is continuous in E_λ. As in [16], we consider the eigenvalue problem

$$-\Delta u + \lambda V^+(x)u = \mu\lambda V^-(x)u, \qquad u \in F_\lambda^\perp. \tag{2.2}$$

In view of (V_3)–(V_4), the functional $I(u) = \int_{\mathbb{R}^3} V^-(x)u^2 dx$ for $u \in F_\lambda^\perp$ is weakly continuous. Hence, as a result of [30, Theorems 4.45 and 4.46], we deduce the following proposition, which is the spectral theorem for compact self-adjoint operators jointly with the Courant-Fischer minimax characterization of eigenvalues.

Proposition 2.1 *Assume that* (V_3)–(V_4) *hold, then for any fixed* $\lambda > 0$*, problem* (2.2) *has a sequence of positive eigenvalues* $\{\mu_j(\lambda)\}_{j=1}^{\infty}$*, which may be characterized by*

$$\mu_j(\lambda) = \inf_{dim M \geq j, M \subset F_\lambda^\perp} \sup \left\{ \|u\|_\lambda^2 : u \in M, \int_{\mathbb{R}^3} \lambda V^-(x) u^2 dx = 1 \right\}, \qquad j = 1, 2, \ldots.$$

Furthermore, $\mu_1(\lambda) \leq \mu_2(\lambda) \leq \cdots \leq \mu_j(\lambda) \xrightarrow{j} +\infty$ *and the corresponding eigenfunctions* $\{e_j(\lambda)\}_{j=1}^{\infty}$*, which may be chosen so that* $(e_i(\lambda), e_j(\lambda))_\lambda = \delta_{ij}$*, are a basis of* $F_\lambda^\perp$*.*

For the eigenvalues $\{\mu_j(\lambda)\}$ defined above, we have the following properties.

Proposition 2.2 (see Lemma 2.1 in [16]) *Assume that* (V_3)–(V_4) *hold and* $V^- \not\equiv 0$*. Then, for each fixed* $j \in \mathbb{N}$*,*

(i) $\mu_j(\lambda) \to 0$ *as* $\lambda \to +\infty$*.*
(ii) $\mu_j(\lambda)$ *is a non-increasing continuous function of* λ*.*

Remark 2.1 By Proposition 2.2 (i), there exists $\Lambda_0 > 0$ such that $\mu_1(\lambda) \leq 1$ for all $\lambda > \Lambda_0$. Take

$$E_\lambda^- := \text{span}\left\{e_j(\lambda) : \mu_j(\lambda) \leq 1\right\} \quad \text{and} \quad E_\lambda^+ := \text{span}\left\{e_j(\lambda) : \mu_j(\lambda) > 1\right\}.$$

Then we have the following orthogonal decomposition:

$$E_\lambda = E_\lambda^- \bigoplus E_\lambda^+ \bigoplus F_\lambda.$$

From Remark 2.1, we have that $\dim E_\lambda^- \geq 1$ when $\lambda > \Lambda_0$. Moreover, $\dim E_\lambda^- < +\infty$ for every fixed $\lambda > 0$ since $\mu_j(\lambda) \xrightarrow{j} +\infty$.

It is well known that problem $(SP)_\lambda$ can be transformed into a Schrödinger equation with a nonlocal term (see e.g. [24]). Indeed, the Lax-Milgram theorem implies that for all $u \in E_\lambda$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$, which can be expressed as $\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|} dy$, satisfying

$$-\Delta \phi_u = K(x) u^2. \tag{2.3}$$

If $K \in L^\infty(\mathbb{R}^3)$, by Hölder and Sobolev inequality, we get

$$\|\phi_u\|_{\mathcal{D}^{1,2}}^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \leq \bar{S}^{-2} \nu_{12/5}^4 |K|_\infty^2 \|u\|_\lambda^4.$$

Similarly, if $K \in L^2(\mathbb{R}^3)$,

$$\|\phi_u\|_{\mathcal{D}^{1,2}}^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \leq \bar{S}^{-2} \nu_6^4 |K|_2^2 \|u\|_\lambda^4.$$

Thus, there exists $C_0 > 0$ such that

$$\|\phi_u\|_{\mathcal{D}^{1,2}}^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \leq C_0 \|u\|_\lambda^4, \qquad \forall K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3). \tag{2.4}$$

Take

$$N(u) = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx = \frac{1}{4\pi} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{K(x)K(y)u^2(x)u^2(y)}{|x-y|} dxdy$$

We recall some important properties of the functional N.

Lemma 2.1 *Let $K \in L^\infty(\mathbb{R}^3) \cup L^2(\mathbb{R}^3)$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^3$, then*

(i) $\phi_{u_n} \rightharpoonup \phi_u$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, and $N(u) \le \liminf_{n\to\infty} N(u_n)$;
(ii) $N(u_n - u) = N(u_n) - N(u) + o(1)$;
(iii) $N'(u_n - u) = N'(u_n) - N'(u) + o(1)$ in $H^{-1}(\mathbb{R}^3)$.

Proof A straightforward adaption of [37, Lemma 2.1] shows that (i) holds. If $K \equiv 1$, the proofs of (ii) and (iii) have been given in [36], and it is easy to see that the conclusions remain valid if $K \in L^\infty(\mathbb{R}^3)$. Hence we only consider the case $K \in L^2(\mathbb{R}^3)$.

We claim that

$$\int_{\mathbb{R}^3} (K(x)\phi_{u_n} u_n^2 - K(x)\phi_u u^2) dx \xrightarrow{n} 0 \tag{2.5}$$

and

$$\int_{\mathbb{R}^3} (K(x)\phi_{u_n} u_n \psi - K(x)\phi_u u\psi) dx \xrightarrow{n} 0 \tag{2.6}$$

uniformly for $\psi \in H^1(\mathbb{R}^3)$ with $\|\psi\|_{H^1} \le 1$. It follows from (i) and Hölder's inequality that

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{R}^3} (K(x)\phi_{u_n} u_n^2 - K(x)\phi_u u^2) dx \\ &\le \lim_{n\to\infty} \int_{\mathbb{R}^3} \left[K(x)\phi_{u_n}(u_n^2 - u^2) + K(x)(\phi_{u_n} - \phi_u)u^2\right] dx \\ &\le \lim_{n\to\infty} |\phi_{u_n}|_6 |u_n + u|_6 |K(x)(u_n - u)|_{3/2} \\ &\quad + \lim_{n\to\infty} \int_{\mathbb{R}^3} K(x)u^2(\phi_{u_n} - \phi_u) dx. \end{aligned} \tag{2.7}$$

The first limit on the right is 0 by the fact $K^{3/2} \in L^{4/3}(\mathbb{R}^3)$ and $(u_n - u)^{3/2} \rightharpoonup 0$ in $L^4(\mathbb{R}^3)$, and so is the second limit because $(\phi_{u_n} - \phi_u) \rightharpoonup 0$ in $L^6(\mathbb{R}^3)$ and $K(x)u^2 \in L^{6/5}(\mathbb{R}^3)$. Thus (2.5) holds. Moreover, observing that $|K(x)u|^{6/5} \in L^{5/4}(\mathbb{R}^3)$ and $(\phi_{u_n} - \phi_u)^{6/5} \rightharpoonup 0$ in $L^5(\mathbb{R}^3)$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3} (K(x)\phi_{u_n} u_n \psi - K(x)\phi_u u\psi) dx \\ &\le \int_{\mathbb{R}^3} \left[K(x)\phi_{u_n}(u_n - u)\psi + K(x)(\phi_{u_n} - \phi_u)u\psi\right] dx \\ &\le |\phi_{u_n}|_6 |\psi|_6 |K(x)(u_n - u)|_{3/2} + |\psi|_6 |K(x)u(\phi_{u_n} - \phi_u)|_{6/5} \\ &\le c|K(x)(u_n - u)|_{3/2} + c|K(x)u(\phi_{u_n} - \phi_u)|_{6/5} \\ &\to 0 \end{aligned}$$

uniformly with respect to ψ, i.e., (2.6) is satisfied. Now (ii) and (iii) follow from (2.5) and (2.6), respectively. □

By (1.1) and the above lemma, the functional $\varphi_\lambda : E_\lambda \to \mathbb{R}$,

$$\varphi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V(x)u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx,$$

is of class C^1 with derivative

$$\langle \varphi'_\lambda(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \lambda V(x)uv + K(x)\phi_u uv - f(x,u)v)dx$$

for all $u, v \in E_\lambda$. It can be proved that the pair $(u, \phi) \in E_\lambda \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of problem $(SP)_\lambda$ if and only if $u \in E_\lambda$ is a critical point of φ_λ and $\phi = \phi_u$ (see [9]).

To conclude this section, we state the following propositions, which will be applied to prove Theorems 1.1–1.3. Recall that a C^1 functional I satisfies Cerami condition at level c ($(C)_c$ condition for short) if any sequence $(u_n) \subset E$ such that $I(u_n) \to c$ and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$ has a converging subsequence; such a sequence is then called a $(C)_c$ sequence.

Proposition 2.3 (see [17]) *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfying*

$$\max\{I(0), I(e)\} \le a < b \le \inf_{\|u\|=\rho} I(u)$$

for some $a < b$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \ge b$ be characterized by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths jointing 0 and e, then I possesses a $(C)_c$ sequence.

If $V(x)$ is sign-changing, we need the following linking theorem.

Proposition 2.4 (see [23]) *Let $E = X \bigoplus Y$ be a Banach space with* $\dim Y < +\infty$, *$I \in C^1(E, \mathbb{R})$. If there exist $R > \rho > 0$, $\alpha > 0$ and $e_0 \in X$ such that*

$$\alpha := \inf I(X \cap S_\rho) > \sup I(\partial Q)$$

where $S_\rho = \{u \in E : \|u\| = \rho\}$, $Q = \{u = v + te_0 : v \in Y, t \ge 0, \|u\| \le R\}$. Then I has a $(C)_c$ sequence with $c \in [\alpha, \sup I(Q)]$.

Proposition 2.5 (see [6]) *Suppose that $I \in C^1(E, \mathbb{R})$ is even, $I(0) = 0$ and there exist closed subspaces E_1, E_2 such that* $\operatorname{codim} E_1 < +\infty$, $\inf I(E_1 \cap S_\rho) \ge \alpha$ *for some $\rho, \alpha > 0$ and* $\sup I(E_2) < +\infty$. *If I satisfies the $(C)_c$-condition for all $c \in [\alpha, \sup I(E_2)]$, then I has at least* $\dim E_2 - \operatorname{codim} E_1$ *pairs of critical points with corresponding critical values in* $[\alpha, \sup I(E_2)]$.

To establish the existence of infinitely many solutions in the sublinear case, we require the new version of symmetric mountain pass lemma of Kajikiya (see [20]). Let E be a Banach space and

$$\Gamma := \{A \subset E\backslash\{0\} : A \text{ is closed and symmetric with respect to the origin}\}.$$

We define

$$\Gamma_k := \{A \in \Gamma : \gamma(A) \geq k\},$$

where $\gamma(A) := \inf\{m \in \mathbb{N} : \exists h \in C(A, \mathbb{R}^m \setminus \{0\}), -h(x) = h(-x)\}$. If there is no such mapping h for any $m \in \mathbb{N}$, we set $\gamma(A) = +\infty$.

Proposition 2.6 (Symmetric mountain pass lemma) *Let E be an infinite dimensional Banach space and $I \in C^1(E, \mathbb{R})$ be even, $I(0) = 0$ and satisfies the following conditions:*

(i) I is bounded from below and satisfies the Palais-Smale condition (PS), i.e., $(u_n) \subset E$ has a converging subsequence whenever $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$.
(ii) For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} I(u) < 0$.

Then either (1) or (2) holds.

(1) There exists a sequence $\{u_k\}$ such that $I'(u_k) = 0$, $I(u_k) < 0$ and $\{u_k\}$ converges to zero.
(2) There exist two sequence $\{u_k\}$ and $\{v_k\}$ such that $I'(u_k) = 0$, $I(u_k) = 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$, $I'(v_k) = 0$, $I(v_k) < 0$, $\lim_{k\to\infty} I(v_k) = 0$ and $\{v_k\}$ converges to a non-zero limit.

Remark 2.2 From Proposition 2.6, we deduce a sequence $\{u_k\}$ of critical points such that $I(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.

3 Proofs of Theorems 1.1–1.2

We first discuss the $(C)_c$ sequence. We only consider the case $K \in L^2(\mathbb{R}^3)$, the other case $K \in L^\infty(\mathbb{R}^3)$ is similar.

Lemma 3.1 *Let (V_3)–(V_4), (K), (f_1)–(f_4) be satisfied. Then each $(C)_c$-sequence ($c \in \mathbb{R}$) of φ_λ is bounded in E_λ.*

Proof Let $(u_n) \subset E_\lambda$ be a $(C)_c$ sequence of φ_λ. Arguing indirectly, we can assume that

$$\varphi_\lambda(u_n) \to c, \quad \|\varphi'_\lambda(u_n)\|(1 + \|u_n\|_\lambda) \to 0, \quad \|u_n\|_\lambda \to \infty \tag{3.1}$$

as $n \to \infty$ after passing to a subsequence. Take $w_n := u_n/\|u_n\|_\lambda$. Then $\|w_n\|_\lambda = 1$, $w_n \rightharpoonup w$ in E_λ and $w_n(x) \to w(x)$ a.e. $x \in \mathbb{R}^3$ after passing to a subsequence.

We first consider the case $w = 0$. Combining this with (3.1), (f_3) and the fact $w_n \to 0$ in $L^2(\{x \in \mathbb{R}^3 : V(x) < 0\})$, we obtain

$$\begin{aligned} o(1) &= \frac{1}{\|u_n\|_\lambda^2}\left(\varphi_\lambda(u_n) - \frac{1}{4}\langle \varphi'_\lambda(u_n), u_n\rangle\right) \\ &\geq \frac{1}{4}\|w_n\|_\lambda^2 - \frac{\lambda}{4}\int_{\mathbb{R}^3} V^-(x) w_n^2 dx + \frac{1}{\|u_n\|_\lambda^2}\int_{\mathbb{R}^3} \mathcal{F}(x, u) dx \\ &\geq \frac{1}{4} - \frac{\lambda}{4}|V^-|_\infty \int_{suppV^-} w_n^2 dx \\ &= \frac{1}{4} + o(1), \end{aligned}$$

a contradiction.

If $w \neq 0$, then the set $\Omega_1 = \{x \in \mathbb{R}^3 : w(x) \neq 0\}$ has positive Lebesgue measure. For $x \in \Omega_1$, one has $|u_n(x)| \to \infty$ as $n \to \infty$, and then, by (f_2),

$$\frac{F(x, u_n(x))}{u_n^4(x)} w_n^4(x) \to +\infty \quad \text{as } n \to \infty,$$

which, jointly with Fatou's lemma (see [33]), shows that

$$\int_{\Omega_1} \frac{F(x, u_n)}{u_n^4} w_n^4 dx \to +\infty \quad \text{as } n \to \infty. \tag{3.2}$$

We see from (f_1), (2.4), (3.2) and the first limit of (3.1) that

$$\frac{C_0}{4} \geq \limsup_{n\to\infty} \int_{\mathbb{R}^3} \frac{F(x, u_n)}{\|u_n\|_\lambda^4} dx \geq \limsup_{n\to\infty} \int_{\Omega_1} \frac{F(x, u_n)}{u_n^4} w_n^4 dx = +\infty.$$

This is impossible.

In any case, we deduce a contradiction. Hence (u_n) is bounded in E_λ. □

Lemma 3.2 *Suppose that* (V_3)–(V_4), (K) *and* (1.1) *are satisfied. If* $u_n \rightharpoonup u$ *in* E_λ, $u_n(x) \to u(x)$ *a.e. in* $\mathbb{R}^3$, *and we denote* $w_n := u_n - u$, *then*

$$\varphi_\lambda(u_n) = \varphi_\lambda(w_n) + \varphi_\lambda(u) + o(1) \tag{3.3}$$

and

$$\varphi'_\lambda(u_n) = \varphi'_\lambda(w_n) + \varphi'_\lambda(u) + o(1) \tag{3.4}$$

as $n \to \infty$. *In particular, if* $\varphi_\lambda(u_n) \to d$ $(\in \mathbb{R})$ *and* $\varphi'_\lambda(u_n) \to 0$ *in* E_λ^* *(the dual space of* E_λ*), then* $\varphi'_\lambda(u) = 0$, *and*

$$\varphi_\lambda(w_n) \to d - \varphi_\lambda(u), \qquad \varphi'_\lambda(w_n) \to 0 \tag{3.5}$$

after passing to a subsequence.

Proof Since $u_n \rightharpoonup u$ in E_λ, one has $(u_n - u, u)_\lambda \to 0$ as $n \to \infty$, which implies that

$$\|u_n\|_\lambda^2 = (w_n + u, w_n + u)_\lambda = \|w_n\|_\lambda^2 + \|u\|_\lambda^2 + o(1). \tag{3.6}$$

Recall (V_3) and $w_n \rightharpoonup 0$, we have

$$\left| \int_{\mathbb{R}^3} V^-(x) w_n u dx \right| = \left| \int_{suppV^-} V^-(x) w_n u dx \right| \leq |V^-|_\infty \left(\int_{suppV^-} w_n^2 dx \right)^{1/2} |u|_2 \xrightarrow{n} 0$$

by the Hölder inequality. Thus

$$\int_{\mathbb{R}^3} V^-(x) u_n^2 dx = \int_{\mathbb{R}^3} V^-(x) w_n^2 dx + \int_{\mathbb{R}^3} V^-(x) u^2 dx + o(1).$$

Combining this with (3.6) and Lemma 2.1 (ii), we obtain

$$\frac{1}{2} a_\lambda(u_n, u_n) + \frac{1}{4} N(u_n) = \left(\frac{1}{2} a_\lambda(w_n, w_n) + \frac{1}{4} N(w_n) \right) + \left(\frac{1}{2} a_\lambda(u, u) + \frac{1}{4} N(u) \right) + o(1).$$

Similarly, by Lemma 2.1 (iii),

$$a_\lambda(u_n,h)+\int\limits_{\mathbb{R}^3}K(x)\phi_{u_n}u_nhdx=\left(a_\lambda(w_n,h)+\int\limits_{\mathbb{R}^3}K(x)\phi_{w_n}w_nhdx\right)+\left(a_\lambda(u,h)+\int\limits_{\mathbb{R}^3}K(x)\phi_u uhdx\right)+o(1),\qquad \forall h\in E_\lambda.$$

Therefore, to obtain (3.3) and (3.4), it suffices to check that

$$\int\limits_{\mathbb{R}^3}(F(x,u_n)-F(x,w_n)-F(x,u))dx=o(1) \tag{3.7}$$

and

$$\sup_{\|h\|_\lambda=1}\int\limits_{\mathbb{R}^3}(f(x,u_n)-f(x,w_n)-f(x,u))hdx=o(1). \tag{3.8}$$

Here, we only prove (3.8), the verification of (3.7) is similar. Inspired by [1], we take $\lim_{n\to\infty}\sup_{\|h\|_\lambda=1}\left|\int\limits_{\mathbb{R}^3}(f(x,u_n)-f(x,w_n)-f(x,u))hdx\right|=A$. If $A>0$, then, there is $h_0\in E_\lambda$ with $\|h_0\|_\lambda=1$ such that

$$\left|\int\limits_{\mathbb{R}^3}(f(x,u_n)-f(x,w_n)-f(x,u))h_0dx\right|\geq\frac{A}{2} \tag{3.9}$$

for n large enough. It follows form (1.1) and the Young inequality that

$$\begin{aligned}|(f(x,u_n)-f(x,w_n))h_0|&\leq\varepsilon(|w_n+u|+|w_n|)|h_0|+C_\varepsilon(|w_n+u|^{p-1}+|w_n|^{p-1})|h_0|\\&\leq c(\varepsilon|w_n||h_0|+\varepsilon|u||h_0|+C_\varepsilon|w_n|^{p-1}|h_0|+C_\varepsilon|u|^{p-1}|h_0|)\\&\leq c(\varepsilon w_n^2+\varepsilon h_0^2+\varepsilon u^2+\varepsilon|w_n|^p+C_{\varepsilon,1}|u|^p+C_{\varepsilon,2}|h_0|^p)\end{aligned}$$

for all n. Hence

$$|(f(x,u_n)-f(x,w_n)-f(x,u))h_0|\leq c(\varepsilon w_n^2+\varepsilon h_0^2+\varepsilon u^2+\varepsilon|w_n|^p+C_{\varepsilon,1}|u|^p+C_{\varepsilon,2}|h_0|^p).$$

Letting

$$g_n(x):=\max\left\{|(f(x,u_n)-f(x,w_n)-f(x,u))h_0|-c\varepsilon(w_n^2+|w_n|^p),0\right\},$$

we have

$$0\leq g_n(x)\leq c(\varepsilon h_0^2+\varepsilon u^2+C_{\varepsilon,1}|u|^p+C_{\varepsilon,2}|h_0|^p)\in L^1(\mathbb{R}^3),$$

which implies that

$$\int\limits_{\mathbb{R}^3}g_n(x)dx\to 0\quad\text{ as }\ n\to\infty \tag{3.10}$$

because of the Lebesgue dominated convergence theorem and the fact $w_n\to 0$ a.e. in $\mathbb{R}^3$. The definition of $g_n(x)$ implies that

$$|(f(x,u_n)-f(x,w_n)-f(x,u))h_0|\leq g_n(x)+c\varepsilon(w_n^2+|w_n|^p),$$

which, together with (3.10) and (2.1), shows that

$$\left|\int_{\mathbb{R}^3}(f(x,u_n)-f(x,w_n)-f(x,u))h_0dx\right|\le c\varepsilon$$

for n sufficiently large. This contradicts (3.9). Hence $A=0$ and (3.8) holds.

If moreover $\varphi'_\lambda(u_n)\to 0$ as $n\to\infty$, then $\varphi'_\lambda(u)=0$. Indeed, for each $\psi\in C_0^\infty(\mathbb{R}^3)$, we have

$$(u_n-u,\psi)_\lambda \xrightarrow{n} 0, \tag{3.11}$$

and

$$\left|\int_{\mathbb{R}^3}V^-(x)(u_n-u)\psi dx\right|\le |V^-|_\infty\left(\int_{supp\psi}(u_n-u)^2dx\right)^{1/2}|\psi|_2\xrightarrow{n}0, \tag{3.12}$$

since $u_n\to u$ in $L^2_{loc}(\mathbb{R}^3)$. By Lemma 2.1 (i), $u_n\rightharpoonup u$ in E_λ yields $\phi_{u_n}\rightharpoonup\phi_u$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. So

$$\phi_{u_n}\rightharpoonup\phi_u \quad \text{in } L^6(\mathbb{R}^3),$$

and hence

$$\int_{\mathbb{R}^3}K(x)(\phi_{u_n}-\phi_u)u\psi dx\to 0$$

since $K(x)u\psi\in L^{6/5}(\mathbb{R}^3)$. Combining this with Hölder's inequality, we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}(K(x)\phi_{u_n}u_n\psi-K(x)\phi_u u\psi)dx\right|\\ &\le\int_{\mathbb{R}^3}|K(x)\phi_{u_n}(u_n-u)\psi|dx+\int_{\mathbb{R}^3}\left|K(x)(\phi_{u_n}-\phi_u)u\psi\right|dx\\ &\le|\psi|_\infty|K|_2|\phi_{u_n}|_6|u_n-u|_{3,supp\psi}+\int_{\mathbb{R}^3}|K(x)(\phi_{u_n}-\phi_u)u\psi|dx\\ &=o(1).\end{aligned} \tag{3.13}$$

Furthermore, it follows from (1.1) and the dominated convergence theorem that

$$\int_{\mathbb{R}^3}(f(x,u_n)-f(x,u))\psi dx=\int_{supp\psi}(f(x,u_n)-f(x,u))\psi dx=o(1).$$

This, jointly with (3.13), (3.12) and (3.11), shows that

$$\langle\varphi'_\lambda(u),\psi\rangle=\lim_{n\to\infty}\langle\varphi'_\lambda(u_n),\psi\rangle=0,\qquad\forall\psi\in C_0^\infty(\mathbb{R}^3).$$

Consequently, $\varphi'_\lambda(u)=0$ and (3.5) follows from (3.3)–(3.4). The proof is complete. □

Lemma 3.3 *Let $V\ge 0$, (V_3)–(V_4), (K), (f_1)–(f_4) be satisfied. Then, for any $M>0$, there exists $\Lambda=\Lambda(M)>0$ such that φ_λ satisfies $(C)_c$ condition for all $c<M$ and $\lambda>\Lambda$.*

Proof Let $(u_n) \subset E_\lambda$ be a $(C)_c$ sequence with $c < M$. According to Lemma 3.1, (u_n) is bounded. Hence we may assume that

$$u_n \rightharpoonup u \text{ in } E_\lambda, \quad u_n \to u \text{ in } L^s_{loc}(\mathbb{R}^3) \ (2 \le s < 2^*), \quad u_n(x) \to u(x) \text{ a.e. } x \in \mathbb{R}^3 \quad (3.14)$$

after passing to a subsequence. Denote $w_n := u_n - u$, we claim that $w_n \to 0$ in E_λ for $\lambda > 0$ large. In fact, Lemma 3.2 yields that $\varphi'_\lambda(u) = 0$, and

$$\varphi_\lambda(w_n) \to c - \varphi_\lambda(u), \quad \varphi'_\lambda(w_n) \to 0 \qquad \text{as } n \to \infty. \tag{3.15}$$

Noting $V \ge 0$ and using (f_3), we get

$$\varphi_\lambda(u) = \varphi_\lambda(u) - \frac{1}{4}\langle \varphi'_\lambda(u), u\rangle = \frac{1}{4}\|u\|^2_\lambda + \int_{\mathbb{R}^3} \mathcal{F}(x, u_n)dx \ge 0,$$

and then, by (3.15),

$$\int_{\mathbb{R}^3} \mathcal{F}(x, w_n)dx \le \varphi_\lambda(w_n) - \frac{1}{4}\langle \varphi'_\lambda(w_n), w_n\rangle = c - \varphi_\lambda(u) + o(1) \le M + o(1). \tag{3.16}$$

Since $V(x) < b$ on a set of finite measure and $w_n \rightharpoonup 0$,

$$|w_n|^2_2 \le \frac{1}{\lambda b} \int_{V \ge b} \lambda V^+(x) w_n^2 dx + \int_{V<b} w_n^2 dx \le \frac{1}{\lambda b}\|w_n\|^2_\lambda + o(1). \tag{3.17}$$

For $2 < s < 2^*$, by (3.17) and the Hölder and Sobolev inequality, we obtain

$$\begin{aligned} |w_n|^s_s &\le \left(\int_{\mathbb{R}^3} w_n^2 dx\right)^{\frac{2^*-s}{2^*-2}} \left(\int_{\mathbb{R}^3} w_n^{2^*} dx\right)^{\frac{s-2}{2^*-2}} \\ &\le \left(\frac{1}{\lambda b}\|w_n\|^2_\lambda\right)^{\frac{2^*-s}{2^*-2}} \bar{S}^{-\frac{2^*(s-2)}{2^*-2}} \left(\int_{\mathbb{R}^3} |\nabla w_n|^2 dx\right)^{\frac{2^*(s-2)}{2(2^*-2)}} + o(1) \\ &\le \bar{S}^{-\frac{2^*(s-2)}{2^*-2}} \left(\frac{1}{\lambda b}\right)^{\frac{2^*-s}{2^*-2}} \|w_n\|^s_\lambda + o(1). \end{aligned} \tag{3.18}$$

By (f_1), for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|f(x,t)| \le \varepsilon|t|$ for all $x \in \mathbb{R}^3$ and $|t| \le \delta$, and (f_4) is satisfied for $|t| \ge \delta$ (with the same τ but possibly larger a_1). Hence we obtain

$$\int_{|w_n| \le \delta} f(x, w_n) w_n dx \le \varepsilon \int_{|w_n| \le \delta} w_n^2 dx \le \frac{\varepsilon}{\lambda b}\|w_n\|^2_\lambda + o(1), \tag{3.19}$$

and

$$\int\limits_{|w_n|\geq\delta} f(x,w_n)w_n dx \leq \left(\int_{|w_n|\geq\delta}\left|\frac{f(x,w_n)}{w_n}\right|^{\tau}dx\right)^{1/\tau}|w_n|_s^2$$
$$\leq \left(\int_{|w_n|\geq\delta} a_1\mathcal{F}(x,w_n)dx\right)^{1/\tau}|w_n|_s^2$$
$$\leq (a_1M)^{1/\tau}\bar{S}^{-\frac{2^*(2s-4)}{s(2^*-2)}}\left(\frac{1}{\lambda b}\right)^{\theta}\|w_n\|_\lambda^2+o(1) \tag{3.20}$$

by (f_4), (3.16), (3.18) with $s=2\tau/(\tau-1)$ and the Hölder inequality, where $\theta=\frac{2(2^*-s)}{s(2^*-2)}>0$. Therefore, using (3.20), (3.19) and the second limit of (3.15),

$$o(1)=\langle\varphi_\lambda'(w_n),w_n\rangle$$
$$\geq \|w_n\|_\lambda^2-\int\limits_{\mathbb{R}^3} f(x,w_n)w_n dx$$
$$\geq \left[1-\frac{\varepsilon}{\lambda b}-(a_1M)^{1/\tau}\bar{S}^{-\frac{2^*(2s-4)}{s(2^*-2)}}\left(\frac{1}{\lambda b}\right)^{\theta}\right]\|w_n\|_\lambda^2+o(1). \tag{3.21}$$

So, there exists $\Lambda=\Lambda(M)>0$ such that $w_n\to 0$ in E_λ when $\lambda>\Lambda$. Since $w_n=u_n-u$, it follows that $u_n\to u$ in E_λ. □

Lemma 3.4 *Suppose that* (V_3)–(V_4), (K), (f_1)–(f_4) *are satisfied, and* $(u_n)\subset E_\lambda$ *be a* $(C)_c$ *(*$c>0$*) sequence of* φ_λ *satisfying* $u_n\rightharpoonup u$ *as* $n\to\infty$*. Then, for any* $M>0$*, there exists* $\Lambda=\Lambda(M)>0$ *such that,* u *is a nontrivial critical point of* φ_λ *and* $\varphi_\lambda(u)\leq c$ *for all* $c<M$ *and* $\lambda>\Lambda$.

Proof By Lemma 3.2, we have $\varphi_\lambda'(u)=0$ and

$$\varphi_\lambda(w_n)\to c-\varphi_\lambda(u),\qquad \varphi_\lambda'(w_n)\to 0\qquad \text{as } n\to\infty. \tag{3.22}$$

Since V is allowed to be sign-changing, from

$$\varphi_\lambda(u)=\varphi_\lambda(u)-\frac{1}{4}\langle\varphi_\lambda'(u),u\rangle=\frac{1}{4}\|u\|_\lambda^2-\frac{\lambda}{4}\int\limits_{\mathbb{R}^3}V^-(x)u^2dx+\int\limits_{\mathbb{R}^3}\mathcal{F}(x,u)dx,$$

it cannot deduce $\varphi_\lambda(u)\geq 0$. We consider two possibilities:

(i) $\varphi_\lambda(u)<0$,
(ii) $\varphi_\lambda(u)\geq 0$.

If $\varphi_\lambda(u)<0$, then $u\neq 0$ and the proof is done. If $\varphi_\lambda(u)\geq 0$, following the same lines as the proof of Lemma 3.3, we can deduce $u_n\to u$ in E_λ. Indeed, using (V_2) and the fact $w_n\to 0$ in $L^2(\{x\in\mathbb{R}^3 : V(x)<b\})$, we have

$$\left|\int\limits_{\mathbb{R}^3}V^-(x)w_n^2dx\right|\leq |V^-|_\infty\int\limits_{suppV^-}w_n^2dx=o(1).$$

Combining this with (3.22), we obtain

$$\int_{\mathbb{R}^3} \mathcal{F}(x, w_n)dx = \varphi_\lambda(w_n) - \frac{1}{4}\langle \varphi'_\lambda(w_n), w_n\rangle + \frac{1}{4}\int_{\mathbb{R}^3} \lambda V^-(x)w_n^2 dx - \frac{1}{4}\|w_n\|_\lambda^2$$
$$\le c - \varphi_\lambda(u) + o(1)$$
$$\le M + o(1).$$

It follows that (3.20) and (3.21) remain valid. Hence $u_n \to u$ in E_λ and $\varphi_\lambda(u) = c\ (> 0)$. This completes the proof. □

Next, we give some preliminary results, i.e., Lemmas 3.5 to 3.8, which ensure that the functional φ_λ has the linking structure.

Lemma 3.5 *Suppose that* (V_3)–(V_4), (K) *and (1.1) with* $p \in (4, 2^*)$ *are satisfied. Then, for each* $\lambda > \Lambda_0$ *(*Λ_0 *is the constant given in Remark 2.1), there exist* $\alpha_\lambda, \rho_\lambda > 0$ *such that*

$$\varphi_\lambda(u) \ge \alpha_\lambda \quad \text{for all } u \in E_\lambda^+ \bigoplus F_\lambda \text{ with } \|u\|_\lambda = \rho_\lambda. \tag{3.23}$$

Furthermore, if $V \ge 0$, *we can choose* $\alpha, \rho > 0$ *independent of* λ.

Proof For any $u \in E_\lambda^+ \bigoplus F_\lambda$, writing $u = u_1 + u_2$ with $u_1 \in E_\lambda^+$ and $u_2 \in F_\lambda$. Clearly, $(u_1, u_2)_\lambda = 0$, and

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V(x)u^2)dx = \int_{\mathbb{R}^3} (|\nabla u|_1^2 + \lambda V(x)u_1^2)dx + \|u_2\|_\lambda^2. \tag{3.24}$$

For each fixed $\lambda > \Lambda_0$, noticing $\mu_j(\lambda) \overset{j}{\longrightarrow} +\infty$, there exists a positive integer n_λ such that $\mu_j(\lambda) \le 1$ for $j \le n_\lambda$ and $\mu_j(\lambda) > 1$ for $j \ge n_\lambda + 1$. For $u_1 \in E_\lambda^+$, we set $u_1 = \sum_{j=n_\lambda+1}^\infty a_j(\lambda)e_j(\lambda)$. Thus

$$\int_{\mathbb{R}^3} (|\nabla u_1|^2 + \lambda V(x)u_1^2)dx = \|u_1\|_\lambda^2 - \int_{\mathbb{R}^3} \lambda V^-(x)u_1^2 dx \ge \left(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\right)\|u_1\|_\lambda^2 \tag{3.25}$$

Now, using (3.24), (3.25) and (2.1), we obtain

$$\varphi_\lambda(u) \ge \frac{1}{2}\left(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\right)\|u\|_\lambda^2 - \varepsilon|u|_2^2 - C_\varepsilon|u|_p^p$$
$$\ge \left[\frac{1}{2}\left(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\right) - \varepsilon\nu_2^2\right]\|u\|_\lambda^2 - C_\varepsilon\nu_p^p\|u\|_\lambda^p,$$

consequently the conclusion follows because $p > 2$ and ε has been chosen arbitrarily.

If $V \ge 0$, since $E_\lambda = F_\lambda$, and

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V(x)u^2)dx = \|u\|_\lambda^2,$$

we can choose $\alpha, \rho > 0$ (independent of λ) such that (3.23) holds. □

Lemma 3.6 *Let* (V_3)–(V_4), (K), (f_1) *and* (f_2) *be satisfied. Then, for any finite dimensional subspace* $\widetilde{E}_\lambda \subset E_\lambda$, *there holds*

$$\varphi_\lambda(u) \to -\infty \qquad \text{as} \quad \|u\|_\lambda \to \infty, \ \ u \in \widetilde{E}_\lambda.$$

Proof Assuming the contrary, there is a sequence $(u_n) \subset \widetilde{E}_\lambda$ with $\|u_n\|_\lambda \to \infty$ such that

$$-\infty < \inf_n \varphi_\lambda(u_n). \tag{3.26}$$

Take $v_n := u_n/\|u_n\|_\lambda$. Since $\dim \widetilde{E}_\lambda < +\infty$, there exists $v \in \widetilde{E}_\lambda \setminus \{0\}$ such that

$$v_n \to v \text{ in } \widetilde{E}_\lambda, \qquad v_n(x) \to v(x) \text{ a.e. } x \in \mathbb{R}^3$$

after passing to a subsequence. If $v(x) \neq 0$, then $|u_n(x)| \overset{n}{\to} +\infty$, and hence by (f_2),

$$\frac{F(x, u_n(x))}{u_n^4(x)} v_n^4(x) \to +\infty \qquad \text{as } n \to \infty.$$

Combining this with (f_1), (2.4) and Fatou's lemma, we obtain

$$\begin{aligned}\frac{\varphi_\lambda(u_n)}{\|u_n\|_\lambda^4} &\leq \frac{1}{2\|u_n\|_\lambda^2} + \frac{C_0}{4} - \int_{\mathbb{R}^3} \frac{F(x, u_n)}{\|u_n\|_\lambda^4} dx \\ &= \frac{1}{2\|u_n\|_\lambda^2} + \frac{C_0}{4} - \left(\int_{v=0} + \int_{v \neq 0} \right) \frac{F(x, u_n)}{u_n^4} v_n^4 dx \\ &\leq \frac{1}{2\|u_n\|_\lambda^2} + \frac{C_0}{4} - \int_{v \neq 0} \frac{F(x, u_n)}{u_n^4} v_n^4 dx \\ &\to -\infty,\end{aligned}$$

a contradiction with (3.26). □

Lemma 3.7 *Suppose that* (V_3)–(V_4), (K), (f_1) *and* (f_2) *are satisfied. If* $V(x) < 0$ *for some* x, *then, for each* $k \in \mathbb{N}$, *there exist* $\lambda_k > k$, $w_k \in E^+_{\lambda_k} \bigoplus F_{\lambda_k}$, $R_{\lambda_k} > \rho_{\lambda_k}$ (ρ_{λ_k} *is the constant given in Lemma* 3.5) *and* $b_k > 0$ *such that, for* $|K|_2 < b_k$ (*or* $|K|_\infty < b_k$),

(a) $\sup \varphi_{\lambda_k}(\partial Q_k) \leq 0$,
(b) $\sup \varphi_{\lambda_k}(Q_k)$ *is bounded above by a constant independent of* λ_k,

where $Q_k := \left\{ u = v + t w_k : v \in E^-_{\lambda_k}, t \geq 0, \|u\| \leq R_{\lambda_k} \right\}$.

Proof We adapt an argument in Ding and Szulkin [16]. For each $k \in \mathbb{N}$, since $\mu_j(k) \to +\infty$ as $j \to \infty$, there is $j_k \in \mathbb{N}$ such that $\mu_{j_k}(k) > 1$. By Proposition 2.2, there is $\lambda_k > k$ such that

$$1 < \mu_{j_k}(\lambda_k) < 1 + \frac{1}{\lambda_k}.$$

Taking $w_k := e_{j_k}(\lambda_k)$ be an eigenvalue of $\mu_{j_k}(\lambda_k)$, then $w_k \in E^+_{\lambda_k}$ as $\mu_{j_k}(\lambda_k) > 1$. Since $\dim E^-_{\lambda_k} \bigoplus \mathbb{R} w_k < +\infty$, it follows directly from Lemma 3.6 that (a) holds with $R_{\lambda_k} > 0$ large.

By (f_2), for each $\eta > |V^-|_\infty$, there is $r_\eta > 0$ such that $F(x, t) \geq \frac{1}{2}\eta t^2$ if $|t| \geq r_\eta$. For $u = v + w \in E^-_{\lambda_k} \bigoplus \mathbb{R} w_k$, we get

$$\int_{\mathbb{R}^3} V^-(x) u^2 dx = \int_{\mathbb{R}^3} V^-(x) v^2 dx + \int_{\mathbb{R}^3} V^-(x) w^2 dx$$

by the orthogonality of $E^-_{\lambda_k}$ and $\mathbb{R}w_k$. Hence we obtain

$$
\begin{aligned}
\varphi_{\lambda_k}(u) &\le \frac{1}{2}\int\limits_{\mathbb{R}^3}(|\nabla w|^2+\lambda_k V(x)w^2)dx+\frac{1}{4}\int\limits_{\mathbb{R}^3}K(x)\phi_u u^2dx-\int\limits_{suppV^-}F(x,u)dx\\
&\le \frac{1}{2}\left(\mu_{j_k}(\lambda_k)-1\right)\lambda_k\int\limits_{\mathbb{R}^3}V^-(x)w^2dx-\int\limits_{suppV^-}\frac{1}{2}\eta u^2dx+\frac{1}{4}\bar{S}^{-2}\nu_6^4|K|_2^2\|u\|^4_{\lambda_k}\\
&\quad -\int\limits_{suppV^-,|u|\le r_\eta}\left(F(x,u)-\frac{1}{2}\eta u^2\right)dx\\
&\le \frac{1}{2}\int\limits_{\mathbb{R}^3}V^-(x)w^2dx-\frac{\eta}{2|V^-|_\infty}\int\limits_{\mathbb{R}^3}V^-(x)w^2dx+C_\eta+\frac{1}{4}\bar{S}^{-2}\nu_6^4|K|_2^2R^4_{\lambda_k}\\
&\le C_\eta+1
\end{aligned}
$$

for $u=v+w\in E^-_{\lambda_k}\bigoplus\mathbb{R}w_k$ with $\|u\|\le R_{\lambda_k}$ and $|K|_2<b_k:=2\bar{S}(\nu_6R_{\lambda_k})^{-2}$, where C_η depends on η but not λ. □

Lemma 3.8 *Suppose that (V_3)–(V_4), (K), (f_1) and (f_2) are satisfied. If $\Omega:=intV^{-1}(0)$ is nonempty, then, for each $\lambda>\Lambda_0$, there exist $w\in E^+_\lambda\bigoplus F_\lambda$, $R_\lambda>0$ and $b_\lambda>0$ such that for $|K|_2<b_\lambda$ (or $|K|_\infty<b_\lambda$),*

(a) $\sup\varphi_\lambda(\partial Q)\le 0$,

(b) $\sup\varphi_\lambda(Q)$ *is bounded above by a constant independent of* λ,

where $Q=\left\{u=v+tw: v\in E^-_\lambda, t\ge 0, \|u\|\le R_\lambda\right\}$.

Proof Choose $e_0\in C_0^\infty(\Omega)\setminus\{0\}$, then $e_0\in F_\lambda$. By Lemma 3.6, there is $R_\lambda>0$ large such that $\varphi_\lambda(u)\le 0$ whenever $u\in E^-_\lambda\bigoplus\mathbb{R}e_0$ and $\|u\|_\lambda\ge R_\lambda$.

For $u=v+w\in E^-_\lambda\bigoplus\mathbb{R}e_0$, we obtain

$$
\begin{aligned}
\varphi_\lambda(u) &\le \frac{1}{2}\int\limits_{\mathbb{R}^3}|\nabla w|^2dx+\frac{1}{4}\int\limits_{\mathbb{R}^3}K(x)\phi_u u^2dx-\int_\Omega F(x,u)dx\\
&\le \frac{1}{2}\int\limits_{\mathbb{R}^3}|\nabla w|^2dx-\frac{\eta}{2}\int\limits_{\Omega}u^2dx-\int\limits_{\Omega,|u|\le r_\eta}\left(F(x,u)-\frac{\eta}{2}u^2\right)dx+\frac{1}{4}\bar{S}^{-2}\nu_6^4|K|_2^2\|u\|^4_{\lambda_k}\\
&\le \frac{1}{2}\int\limits_{\mathbb{R}^3}|\nabla w|^2dx-\frac{\eta}{2}\int\limits_{\Omega}u^2dx+C_\eta+\frac{1}{4}\bar{S}^{-2}\nu_6^4|K|_2^2\|u\|^4_{\lambda_k}.
\end{aligned}\tag{3.27}
$$

Observing $w\in C_0^\infty(\Omega)$, one has

$$
\int\limits_{\mathbb{R}^3}|\nabla w|^2dx=\int\limits_{\Omega}(-\Delta w)udx\le|\Delta w|_2|u|_{2,\Omega}\le d_0|\nabla w|_2|u|_{2,\Omega}\le\frac{d_0^2}{2\eta}|\nabla w|_2^2+\frac{\eta}{2}|u|^2_{2,\Omega},\tag{3.28}
$$

where d_0 is a constant depending on e_0. Choosing $\eta\ge d_0^2$, we have $|\nabla w|_2^2\le\eta|u|^2_{2,\Omega}$, and it follows from (3.27) that

$$
\varphi_\lambda(u)\le C_\eta+\frac{1}{4}\bar{S}^{-2}\nu_6^4|K|_2^2\|u\|^4_{\lambda_k}\le C_\eta+1
$$

for all $u\in E^-_\lambda\bigoplus\mathbb{R}e_0$ with $\|u\|\le R_\lambda$ and $|K|_2<b_\lambda:=2\bar{S}(\nu_6R_\lambda)^{-2}$. □

Now we are in a position to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1 Case (i). It follows from Lemmas 3.5, 3.7 and Proposition 2.4 that, for $\lambda = \lambda_k$ and $|K|_2 \in (0, b_k)$, φ_{λ_k} has a $(C)_c$ sequence with $c \in [\alpha_{\lambda_k}, \sup \varphi_{\lambda_k}(Q_k)]$. Setting $M := \sup \varphi_{\lambda_k}(Q_k)$, then φ_{λ_k} has a nontrivial critical point according to Lemmas 3.1 and 3.4.

Case (ii). The conclusion follows from Lemmas 3.1, 3.4, 3.5, 3.8 and Proposition 2.4. □

Proof of Theorem 1.2 (Existence) Suppose $V \geq 0$. By Lemma 3.5, there exist constants α, $\rho > 0$ (independent of λ) such that

$$\varphi_\lambda(u) \geq \alpha \quad \text{for } u \in E_\lambda \text{ with } \|u\|_\lambda = \rho. \tag{3.29}$$

Take $e_0 \in C_0^\infty(\Omega)\setminus\{0\}$. Then, by (f_1), (f_2) and Fatou's lemma,

$$\frac{\varphi_\lambda(te_0)}{t^4} \leq \frac{1}{2t^2}\int_\Omega |\nabla e_0|^2 dx + \frac{1}{4}N(e_0) - \int_{\{x\in\Omega: e_0(x)\neq 0\}} \frac{F(x,te_0)}{(te_0)^4} e_0^4 dx \to -\infty$$

as $t \to +\infty$, which yields that $\varphi_\lambda(te_0) < 0$ for $t > 0$ large. Clearly, there is $C_1 > 0$ (independent of λ) such that

$$c_\lambda := \inf_{h\in\Gamma} \max_{t\in[0,1]} \varphi_\lambda(h(t)) \leq \sup_{t\geq 0} \varphi_\lambda(te_0) \leq C_1, \tag{3.30}$$

where $\Gamma = \{h \in C([0,1], E_\lambda) : h(0) = 0, \|h(1)\|_\lambda \geq \rho, \varphi_\lambda(h(1)) < 0\}$. By Proposition 2.3 and Lemma 3.3, we obtain a nontrivial critical point u_λ of φ_λ with $\varphi_\lambda(u_\lambda) \in [\alpha, C_1]$ for λ large.

(Multiplicity) For each $k \in \mathbb{N}$, we choose k functions $e_i \in C_0^\infty(\Omega)$ such that $\operatorname{supp} e_i \cap \operatorname{supp} e_j = \emptyset$ if $i \neq j$. Let

$$W_k = \operatorname{span}\{e_1, e_2, \ldots, e_k\}.$$

According to (3.29), Lemma 3.3 and Proposition 2.5, it suffices to show that $\sup \varphi_\lambda(W_k)$ is bounded above by a constant independent of λ.

For $u \in W_k$ and $\eta > 0$, we have [cf. (3.28)]

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx \leq \frac{d_k^2}{2\eta}|\nabla u|_2^2 + \frac{\eta}{2}|u|_{2,\Omega}^2$$

(d_k is a constant depending on W_k). It follows that

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx \leq \eta |u|_{2,\Omega}^2, \qquad \text{if } \eta \geq d_k^2. \tag{3.31}$$

Combining this with (2.4) and the Hölder inequality, we obtain

$$N(u) \leq C_0\|u\|_\lambda^4 = C_0\left(\int_\Omega |\nabla u|^2 dx\right)^2 \leq C_0\eta^2\left(\int_\Omega u^2 dx\right)^2 \leq C_0\eta^2|\Omega|\int_\Omega u^4 dx \text{ for all } u \in W_k. \tag{3.32}$$

By (f_2), for each $\eta > d_k^2$, there is $r_\eta > 0$ such that

$$F(x,t) \geq \frac{1}{2}\eta t^2 + \frac{1}{4}C_0\eta^2|\Omega|t^4, \qquad \forall x \in \mathbb{R}^3,\ |t| \geq r_\eta. \tag{3.33}$$

Hence we obtain, using (3.31)–(3.33),

$$
\begin{aligned}
\varphi_\lambda(u) &\le \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}N(u) - \int_{\Omega} F(x,u)dx \\
&\le \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}N(u) - \frac{\eta}{2}\int_{\Omega} u^2 dx - \frac{1}{4}C_0\eta^2|\Omega|\int_{\Omega} u^4 dx \\
&\quad - \int_{\Omega,|u|\le r_\eta} \left(F(x,u) - \frac{\eta}{2}u^2 - \frac{1}{4}C_0\eta^2|\Omega|u^4\right)dx \\
&\le C_\eta
\end{aligned}
$$

for all $u \in W_k$, where C_η is independent of λ. □

4 Proof of Theorem 1.3

In this section, we are concerned with problem $(SP)_1$ with sublinear nonlinearity. We consider the functional φ_1 (denoted by φ for simplicity) on $(E, \|\cdot\|)$:

$$
\varphi(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \psi(u),
$$

where $\psi(u) = \int_{\mathbb{R}^3} F(x,u)dx$. Since the constant ν_s given in (2.1) is independent of λ, it still holds

$$
|u|_s \le \nu_s\|u\|, \qquad \forall u \in E. \tag{4.1}
$$

It follows from (f_5) that

$$
|F(x,u)| \le m(x)|u|^\sigma + h(x)|u|^\gamma, \qquad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}, \tag{4.2}
$$

which, jointly with (4.1) and Hölder's inequality, shows that

$$
\begin{aligned}
\int_{\mathbb{R}^3} F(x,u)dx &\le \int_{\mathbb{R}^3} (m(x)|u|^\sigma + h(x)|u|^\gamma)dx \\
&\le |m|_{\frac{2}{2-\sigma}}|u|_2^\sigma + |h|_{\frac{2}{2-\gamma}}|u|_2^\gamma \\
&\le |m|_{\frac{2}{2-\sigma}}\nu_2^\sigma\|u\|^\sigma + |h|_{\frac{2}{2-\gamma}}\nu_2^\gamma\|u\|^\gamma \\
&< +\infty.
\end{aligned} \tag{4.3}
$$

Hence, ψ and φ are well defined. In addition, we have the following lemmas.

Lemma 4.1 *Assume that (V_3), (V_4) and (f_5) hold and $u_n \rightharpoonup u$ in E, then*

$$
f(x,u_n) \to f(x,u) \qquad in\ L^2(\mathbb{R}^3). \tag{4.4}
$$

Proof Since $u_n \rightharpoonup u$ in E, there is a constant $M > 0$ such that

$$
\|u_n\| \le M \quad \text{and} \quad \|u\| \le M, \quad \forall n \in \mathbb{N}. \tag{4.5}
$$

Up to a subsequence, we can assume that

$$\begin{aligned} &u_n \to u \quad \text{in } L^2_{loc}(\mathbb{R}^3), \\ &u_n(x) \to u(x) \quad \text{a.e. } x \in \mathbb{R}^3. \end{aligned} \tag{4.6}$$

By the properties of the functions m and h, we have, for every $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$\left(\int\limits_{|x|\ge T_\varepsilon} |m(x)|^{\frac{2}{2-\sigma}} dx \right)^{\frac{2-\sigma}{2}} < \sqrt{\varepsilon} \quad \text{and} \quad \left(\int\limits_{|x|\ge T_\varepsilon} |h(x)|^{\frac{2}{2-\gamma}} dx \right)^{\frac{2-\gamma}{2}} < \sqrt{\varepsilon}. \tag{4.7}$$

By (4.6), passing to a subsequence if necessary, we can assume that $\sum_{n=1}^{\infty} \int_{|x|\le T_\varepsilon} |u_n - u|^2 dx < +\infty$. Taking $w(x) = \sum_{n=1}^{\infty} |u_n(x) - u(x)|$ for $|x| \le T_\varepsilon$, then $\int_{|x|\le T_\varepsilon} w^2 dx < +\infty$. It follows from (f_5) that, for all $n \in \mathbb{N}$ and $|x| \le T_\varepsilon$,

$$\begin{aligned} |f(x,u_n) - f(x,u)|^2 &\le [m(x)(|u_n|^{\sigma-1} + |u|^{\sigma-1}) + h(x)(|u_n|^{\gamma-1} + |u|^{\gamma-1})]^2 \\ &\le 4m^2(x)(|u_n|^{2\sigma-2} + |u|^{2\sigma-2}) + 4h^2(x)(|u_n|^{2\gamma-2} + |u|^{2\gamma-2}) \\ &\le 2^{2\sigma+1} m^2(x)(|u_n - u|^{2\sigma-2} + |u|^{2\sigma-2}) \\ &\quad + 2^{2\gamma+1} h^2(x)(|u_n - u|^{2\gamma-2} + |u|^{2\gamma-2}) \\ &\le 2^{2\sigma+1} m^2(x)(|w|^{2\sigma-2} + |u|^{2\sigma-2}) \\ &\quad + 2^{2\gamma+1} h^2(x)(|w|^{2\gamma-2} + |u|^{2\gamma-2}), \end{aligned}$$

and, using Hölder's inequality,

$$\begin{aligned} &\int\limits_{|x|\le T_\varepsilon} \left[2^{2\sigma+1} m^2(x)(|w|^{2\sigma-2} + |u|^{2\sigma-2}) + 2^{2\gamma+1} h^2(x)(|w|^{2\gamma-2} + |u|^{2\gamma-2}) \right] dx \\ &\le 2^{2\sigma+1} |m|^2_{\frac{2}{2-\sigma}} \left[\left(\int\limits_{|x|\le T_\varepsilon} w^2 dx \right)^{\sigma-1} + \left(\int\limits_{|x|\le T_\varepsilon} u^2 dx \right)^{\sigma-1} \right] \\ &\quad + 2^{2\gamma+1} |h|^2_{\frac{2}{2-\gamma}} \left[\left(\int\limits_{|x|\le T_\varepsilon} w^2 dx \right)^{\gamma-1} + \left(\int\limits_{|x|\le T_\varepsilon} u^2 dx \right)^{\gamma-1} \right] \\ &< +\infty. \end{aligned}$$

Hence, by Lebesgue dominated convergence theorem, we obtain

$$\int\limits_{|x|\le T_\varepsilon} |f(x,u_n) - f(x,u)|^2 dx \to 0 \quad \text{as } n \to \infty. \tag{4.8}$$

On the other hand, using (f_5), (4.7), (4.5), (4.1) and the Hölder inequality, we have

$$\begin{aligned} &\int\limits_{|x|\ge T_\varepsilon} |f(x,u_n) - f(x,u)|^2 dx \\ &\quad \le \int\limits_{|x|\ge T_\varepsilon} [m(x)(|u_n|^{\sigma-1} + |u|^{\sigma-1}) + h(x)(|u_n|^{\gamma-1} + |u|^{\gamma-1})]^2 dx \end{aligned}$$

$$\leq 4 \int_{|x|\geq T_\varepsilon} m^2(x)(|u_n|^{2\sigma-2} + |u|^{2\sigma-2})dx$$
$$+4 \int_{|x|\geq T_\varepsilon} h^2(x)(|u_n|^{2\gamma-2} + |u|^{2\gamma-2})dx$$
$$\leq 4 \left(\int_{|x|\geq T_\varepsilon} |m|^{\frac{2}{2-\sigma}} dx \right)^{2-\sigma} (|u_n|_2^{2\sigma-2} + |u|_2^{2\sigma-2})$$
$$+4 \left(\int_{|x|\geq T_\varepsilon} |h|^{\frac{2}{2-\gamma}} dx \right)^{2-\gamma} (|u_n|_2^{2\gamma-2} + |u|_2^{2\gamma-2})$$
$$\leq 8\varepsilon \left(\nu_2^{2\sigma-2} M^{2\sigma-2} + \nu_2^{2\gamma-2} M^{2\gamma-2} \right).$$

This, together with (4.8), shows that (4.4) holds. This completes the proof. □

Lemma 4.2 *Assume that $V \geq 0$, (V_3), (K) and (f_5) hold. Then $\psi \in C^1(E, \mathbb{R})$ and $\psi' : E \to E^*$ (the dual space of E) is compact, and hence $\varphi \in C^1(E, \mathbb{R})$,*

$$\langle \psi'(u), v \rangle = \int_{\mathbb{R}^3} f(x, u) v dx, \tag{4.9}$$
$$\langle \varphi'(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv + K(x)\phi_u uv - f(x, u)v)\, dx$$

for all $u, v \in E$. If u is a critical point of φ, then the pair (u, ϕ_u) is a solution of problem $(SP)_1$.

Proof In view of Lemma 4.1 and (4.1), the proof is standard and we refer to [23]. □

Proof of Theorem 1.3 In view of Lemma 4.2 and the oddness of f, we know that $\varphi \in C^1(E, \mathbb{R})$ and $\varphi(-u) = \varphi(u)$. It remains to verify the conditions (i) and (ii) of Proposition 2.6. We follow an argument in [20].

Verification of (i). Since $V \geq 0$, we get $F_\lambda = E_\lambda$. It follows from (4.3) that

$$\varphi(u) \geq \frac{1}{2}\|u\|^2 - |m|_{\frac{2}{2-\sigma}} \nu_2^\sigma \|u\|^\sigma - |h|_{\frac{2}{2-\gamma}} \nu_2^\gamma \|u\|^\gamma, \qquad \forall u \in E.$$

Noting that $\sigma, \gamma \in (1, 2)$, we have

$$\varphi(u) \to +\infty \quad \text{as} \quad \|u\| \to \infty. \tag{4.10}$$

Thus φ is bounded from below.

Let $(u_n) \subset E$ be a (PS)-sequence of φ, i.e., $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$. By (4.10), (u_n) is bounded, and then $u_n \rightharpoonup u$ in E for some $u \in E$. Recall that

$$(xy)^{1/2}(x + y) \leq x^2 + y^2, \qquad \forall x, y \geq 0.$$

Hence we obtain, by (2.3) and Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^3} K(x)(\phi_{u_n}u_n u + \phi_u u_n u)dx \\
&\quad \le \left(\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2 dx\right)^{1/2}\left(\int_{\mathbb{R}^3} K(x)\phi_{u_n}u^2 dx\right)^{1/2} \\
&\qquad + \left(\int_{\mathbb{R}^3} K(x)\phi_u u_n^2 dx\right)^{1/2}\left(\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx\right)^{1/2} \\
&\quad = \left(\int_{\mathbb{R}^3} \nabla\phi_{u_n}\cdot\nabla\phi_u dx\right)^{1/2}\left(\|\phi_{u_n}\|_{\mathcal{D}^{1,2}} + \|\phi_u\|_{\mathcal{D}^{1,2}}\right) \\
&\quad \le \left(\int_{\mathbb{R}^3} |\nabla\phi_{u_n}|^2 dx\right)^{1/4}\left(\int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx\right)^{1/4}\left(\|\phi_{u_n}\|_{\mathcal{D}^{1,2}} + \|\phi_u\|_{\mathcal{D}^{1,2}}\right) \\
&\quad = \|\phi_{u_n}\|_{\mathcal{D}^{1,2}}^{1/2}\|\phi_u\|_{\mathcal{D}^{1,2}}^{1/2}\left(\|\phi_{u_n}\|_{\mathcal{D}^{1,2}} + \|\phi_u\|_{\mathcal{D}^{1,2}}\right) \\
&\quad \le \|\phi_{u_n}\|_{\mathcal{D}^{1,2}}^2 + \|\phi_u\|_{\mathcal{D}^{1,2}}^2 \\
&\quad = \int_{\mathbb{R}^3} K(x)(\phi_{u_n}u_n^2 + \phi_u u^2)dx,
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^3} K(x)(\phi_{u_n}u_n - \phi_u u)(u_n - u)dx \ge 0.$$

Combining this with Lemma 4.1, we obtain

$$\begin{aligned}
\|u_n - u\|^2 &= \langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle - \int_{\mathbb{R}^3} K(x)(\phi_{u_n}u_n - \phi_u u)(u_n - u)dx \\
&\quad + \int_{\mathbb{R}^3} (f(x,u_n) - f(x,u))(u_n - u)dx \\
&\le \|\varphi'(u_n)\|_{E^*}\|u_n - u\| - \langle\varphi'(u), u_n - u\rangle + \left(\int_{\mathbb{R}^3} |f(x,u_n) - f(x,u)|^2 dx\right)^{1/2}\cdot |u_n - u|_2 \\
&\to 0,
\end{aligned}$$

that is, $u_n \to u$ ($n \to \infty$). Hence the (PS) condition holds.

Verification of (ii). For simplicity, we assume that $x_0 = 0$ in (f_6). For $r > 0$, let $D(r)$ denotes the cube

$$D(r) = \{(x_1, x_2, x_3) : 0 \le x_i \le r, i = 1, 2, 3\}.$$

Fix $r > 0$ small enough such that $D(r) \subset B(0, \delta)$, where δ is the constant given in (f_6). For arbitrary $k \in \mathbb{N}$, we shall construct an $A_k \in \Gamma_k$ satisfying $\sup_{u\in A_k} \varphi(u) < 0$.

Let $m \in \mathbb{N}$ be the smallest integer such that $m^3 \geq k$. We divide $D(r)$ equally into m^3 small cubes by planes parallel to each face of $D(r)$ and denote them by D_i with $1 \leq i \leq m^3$. We only use D_i with $1 \leq i \leq k$. Set $a = r/m$. Then the edge of D_i has length a. We consider a cube $E_i \subset D_i$ $(i = 1, 2, \ldots, k)$ such that E_i has the same center as that of D_i, the faces of E_i and D_i are parallel and the edge of E_i has length $a/2$. Define $\zeta \in C_0^\infty(\mathbb{R}, [0, 1])$ such that $\zeta(t) = 1$ for $t \in [a/4, 3a/4]$, $\zeta(t) = 0$ for $t \in (-\infty, 0] \bigcup [a, +\infty)$. Define

$$\xi(x) = \zeta(x_1)\zeta(x_2)\zeta(x_3), \qquad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Then $\mathrm{supp}\xi \subset [0, a]^3$. Now for each $1 \leq i \leq k$, we can choose a suitable $y_i \in \mathbb{R}^3$ and define

$$\xi_i(x) = \xi(x - y_i), \qquad \forall x \in \mathbb{R}^3$$

such that

$$\mathrm{supp}\xi_i \subset D_i, \qquad \mathrm{supp}\xi_i \bigcap \mathrm{supp}\xi_j = \emptyset \ \ (i \neq j), \tag{4.11}$$

and

$$\xi_i(x) = 1 \ \ (x \in E_i), \qquad 0 \leq \xi_i(x) \leq 1 \ \ (x \in \mathbb{R}^3).$$

Set

$$V_k = \left\{(t_1, t_2, \ldots, t_k) \in \mathbb{R}^k : \max_{1 \leq i \leq k} |t_i| = 1\right\} \tag{4.12}$$

and

$$W_k = \left\{\sum_{i=1}^k t_i \xi_i(x) : (t_1, t_2, \ldots, t_k) \in V_k\right\}.$$

Observing V_k is homeomorphic to the unit sphere in $\mathbb{R}^k$ by an odd mapping, we get $\gamma(V_k) = k$. Furthermore, $\gamma(W_k) = \gamma(V_k) = k$ because the mapping $(t_1, \ldots, t_k) \longmapsto \sum_{i=1}^k t_i \xi_i(x)$ is odd and homeomorphic. Since W_k is compact, there exists $C_k > 0$ such that

$$\|u\| \leq C_k, \qquad \forall u \in W_k. \tag{4.13}$$

For $0 < s < \varepsilon$ (ε is the constant given in (f_6)) and $u = \sum_{i=1}^k t_i \xi_i(x) \in W_k$, we obtain

$$\begin{aligned} \varphi(su) &\leq \frac{s^2}{2}\|u\|^2 + \frac{s^4}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}_3} F\left(x, s\sum_{i=1}^k t_i \xi_i(x)\right) dx \\ &\leq \frac{s^2}{2} C_k^2 + \frac{s^4}{4} C_0 C_k^4 - \sum_{i=1}^k \int_{D_i} F(x, s t_i \xi_i(x)) dx \end{aligned} \tag{4.14}$$

by (4.13), (4.11) and Lemma 3.5 (i). Observing (4.12), there exists an integer $i_0 \in [1, k]$ such that $|t_{i_0}| = 1$. Then it follows that

$$\begin{aligned} \sum_{i=1}^k \int_{D_i} F(x, s t_i \xi_i(x)) dx &= \int_{E_{i_0}} F(x, s t_{i_0} \xi_{i_0}(x)) dx + \int_{D_{i_0} \setminus E_{i_0}} F(x, s t_{i_0} \xi_{i_0}(x)) dx \\ &\quad + \sum_{i \neq i_0} \int_{D_i} F(x, s t_i \xi_i(x)) dx. \end{aligned} \tag{4.15}$$

Noting that $|t_{i_0}| = 1$, $\xi_{i_0} \equiv 1$ on E_{i_0} and $F(x, u)$ is even in u, we get

$$\int_{E_{i_0}} F(x, st_{i_0}\xi_{i_0}(x))dx = \int_{E_{i_0}} F(x, s)dx. \tag{4.16}$$

By (f_6),

$$\int_{D_{i_0}\setminus E_{i_0}} F(x, st_{i_0}\xi_{i_0}(x))dx + \sum_{i\neq i_0}\int_{D_i} F(x, st_i\xi_i(x))dx \geq -a_2\text{vol}(D(r))s^2, \tag{4.17}$$

where vol$(D(r))$ denotes the volume of $D(r)$, i.e. r^3. Combining (4.14)–(4.17), one has

$$\varphi(su) \leq \frac{s^2}{2}C_k^2 + \frac{s^4}{4}C_0C_k^4 + a_2r^3s^2 - \int_{E_{i_0}} F(x, s)dx.$$

Substituting $s = \varepsilon_n$ and using (1.2), we obtain

$$\varphi(\varepsilon_n u) \leq \varepsilon_n^2\left[\frac{C_k^2}{2} + \frac{\varepsilon_n^2}{4}C_0C_k^4 + a_2r^3 - \left(\frac{a}{2}\right)^3 M_n\right].$$

Since $\varepsilon_n \to 0^+$ and $M_n \to +\infty$ as $n \to \infty$, we choose n_0 large enough such that the right side of the last inequality is negative. Take

$$A_k = \varepsilon_{n_0} W_k.$$

Then we have

$$\gamma(A_k) = \gamma(W_k) = k \qquad \text{and} \qquad \sup_{u\in A_k} \varphi(u) < 0.$$

Consequently, Theorem 1.3 follows from Proposition 2.6. This completes the proof. □

5 Concentration of solutions

In this section, we deal with problem $(SP)_\lambda$ with $\lambda = \lambda_k \to +\infty$.

Theorem 5.1 *Suppose that* (V_3)–(V_4) *and* (K) *are satisfied,* $V^{-1}(0)$ *has nonempty interior* Ω *and there exist* $a_3 > 0$, $p \in (2, 2^*)$ *such that*

$$|f(x, t)| \leq a_3(|t| + |t|^{p-1}), \qquad \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{5.1}$$

Let $(u_k) \subset E$ *be a solution of* $(SP)_\lambda$ *with* $\lambda = \lambda_k$. *If* $\lambda_k \to +\infty$ *and* $\|u_k\|_{\lambda_k} \leq C$ *for some* $C > 0$ *and all k, then, passing to a subsequence,* $u_k \to \bar{u}$ *in* $L^s(\mathbb{R}^3)$ *for* $s \in (2, 2^*)$, $\bar{u}$ *is a weak solution of*

$$\begin{cases} -\Delta u + \frac{1}{4\pi}\left((K(x)u^2) * \frac{1}{|x|}\right) K(x)u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

and $\bar{u} = 0$ *a.e. in* $\mathbb{R}^3\setminus V^{-1}(0)$. *If moreover* $V \geq 0$ *and* (f_1) *is satisfied, then* $u_k \to \bar{u}$ *in* E.

We note that $\bar{u} \in H_0^1(\Omega)$ if $V^{-1}(0) = \overline{\Omega}$ and $\partial\Omega$ is locally Lipschitz continuous (see [7]). Before proving the above theorem we point out some of its consequences.

Corollary 5.1 *Let $(u_\lambda, \phi_\lambda)$ be the solution obtained in Theorem 1.2 (existence result). Then $u_\lambda \to \bar{u}$ in E, $\phi_\lambda \to \phi_{\bar{u}}$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $\lambda \to +\infty$, and $\bar{u}$ is a nontrivial solution of (5.2).*

Proof For $\lambda_k \to +\infty$, set $u_k := u_{\lambda_k}$ be the critical point of φ_{λ_k} obtained in Theorem 1.2. It follows from (3.30) that

$$C_1 \geq c_{\lambda_k} = \varphi_{\lambda_k}(u_k) - \frac{1}{4}\langle \varphi'_{\lambda_k}(u_k), u_k\rangle = \frac{1}{4}\|u_k\|^2_{\lambda_k} + \int_{\mathbb{R}^3} \mathcal{F}(x, u_k)dx \geq \frac{1}{4}\|u_k\|^2_{\lambda_k}.$$

Hence $\left\{\|u_k\|_{\lambda_k}\right\}$ is bounded. So the conclusion of Theorem 5.1 holds.

We show that $\bar{u} \neq 0$. Since $V \geq 0$ and $\langle \varphi'_{\lambda_k}(u_k), u_k\rangle = 0$, we have

$$\|u_k\|^2_{\lambda_k} + N(u_k) = \int_{\mathbb{R}^3} f(x, u_k)u_k dx \leq \varepsilon |u_k|_2^2 + C_\varepsilon |u_k|_p^p.$$

If $\bar{u} = 0$, then $u_k \to 0$ in $L^p(\mathbb{R}^3)$, and therefore

$$\|u_k\|_{\lambda_k} \to 0, \qquad N(u_k) \to 0 \qquad \text{as } k \to \infty$$

(note $|u_{\lambda_k}|_2$ is bounded and ε is arbitrary). Now it follows easily that $\varphi_{\lambda_k}(u_k) \to 0$, a contradiction with the fact $\varphi_{\lambda_k}(u_k) = c_{\lambda_k} \geq \alpha$. □

Proof of Theorem 5.1 We adapt an argument in [7]. We divide the proof into three steps.

(1) Since $\|u_k\| \leq \|u_k\|_{\lambda_k} \leq C$, one has

$$u_k \rightharpoonup \bar{u} \text{ in } E, \qquad u_k \to \bar{u} \text{ in } L^s_{loc}(\mathbb{R}^3) \ (2 \leq s < 2^*), \qquad u_k(x) \to \bar{u}(x) \text{ a.e. } x \in \mathbb{R}^3.$$

For any $\psi \in C_0^\infty(\mathbb{R}^3)$, it follows from the fact $\langle \varphi'_{\lambda_k}(u_k), \psi\rangle = 0$ that

$$\begin{aligned} &\left|\int_{\mathbb{R}^3} V(x)u_k\psi dx\right| \\ &\quad\leq \frac{1}{\lambda_k}\left(\int_{\mathbb{R}^3} |f(x, u_k)\psi| dx + \int_{\mathbb{R}^3} |K(x)\phi_{u_k}u_k\psi| dx + \int_{\mathbb{R}^3} |\nabla u_k \cdot \nabla\psi| dx\right) \\ &\quad\leq \frac{1}{\lambda_k}\left[a_3(|u_k|_2|\psi|_2 + |u_k|_p^{p-1}|\psi|_p) + |K|_2|\phi_{u_k}|_6|\psi|_\infty|u_k|_3 + |\nabla u_k|_2|\nabla\psi|_2\right] \\ &\quad\leq \frac{c}{\lambda_k} \longrightarrow 0 \qquad \text{as } k \to \infty, \end{aligned}$$

and hence

$$\int_{\mathbb{R}^3} V(x)\bar{u}\psi dx = 0, \qquad \forall \psi \in C_0^\infty(\mathbb{R}^3),$$

which implies that $\bar{u} = 0$ a.e. in $\mathbb{R}^3 \backslash V^{-1}(0)$. Now for each $\psi \in C_0^\infty(\Omega)$, since $\langle \varphi'_{\lambda_k}(u_k), \psi\rangle = 0$, it follows that

$$\int_{\mathbb{R}^3} \nabla\bar{u} \cdot \nabla\psi dx + \int_{\mathbb{R}^3} K(x)\phi_{\bar{u}}\bar{u}\psi dx = \int_{\mathbb{R}^3} f(x, \bar{u})\psi dx,$$

i.e., $\bar{u}$ is a weak solution of (5.2) by the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$.

(2) $u_k \to \bar{u}$ in $L^s(\mathbb{R}^3)$ for $2 < s < 2^*$. Arguing indirectly, by Lion's vanishing lemma, there exist δ, $\rho > 0$ and $(x_k) \subset \mathbb{R}^3$ such that

$$\int\limits_{B_\rho(x_k)} (u_k - \bar{u})^2 dx \geq \delta.$$

It is easy to see that $|x_k| \xrightarrow{k} \infty$. So $\mathrm{meas}\big(B_\rho(x_k) \cap \big\{x \in \mathbb{R}^3 : V(x) < b\big\}\big) \to 0$, and

$$\int\limits_{B_\rho(x_k)\cap\{V<b\}} (u_k - \bar{u})^2 dx \leq |u_k - \bar{u}|_3^2 \left(\mathrm{meas}(B_\rho(x_k) \cap \{V < b\})\right)^{1/3} \xrightarrow{k} 0.$$

Thus,

$$\begin{aligned}\|u_k\|_{\lambda_k}^2 &\geq \lambda_k b \int\limits_{B_\rho(x_k)\cap\{V\geq b\}} u_k^2 dx \\ &= \lambda_k b \int\limits_{B_\rho(x_k)\cap\{V\geq b\}} (u_k - \bar{u})^2 dx \\ &= \lambda_k b \left(\int\limits_{B_\rho(x_k)} (u_k - \bar{u})^2 dx - \int\limits_{B_\rho(x_k)\cap\{V<b\}} (u_k - \bar{u})^2 dx \right) \\ &\to +\infty,\end{aligned}$$

a contradiction with the boundedness of $\left\{\|u_k\|_{\lambda_k}\right\}_k$.

(3) Suppose that $V \geq 0$ and (f_1) holds. We show that $u_k \to \bar{u}$ in E. Since $\langle \varphi'_{\lambda_k}(u_k), u_k\rangle = 0$ and $\langle \varphi'_{\lambda_k}(u_k), \bar{u}\rangle = 0$, we have

$$\|u_k\|_{\lambda_k}^2 = \int\limits_{\mathbb{R}^3} f(x, u_k)u_k dx - \int\limits_{\mathbb{R}^3} K(x)\phi_{u_k} u_k^2 dx \tag{5.3}$$

and

$$(u_k, \bar{u})_{\lambda_k} = \int\limits_{\mathbb{R}^3} f(x, u_k)\bar{u} dx - \int\limits_{\mathbb{R}^3} K(x)\phi_{u_k} u_k \bar{u} dx. \tag{5.4}$$

From (5.1) and (f_1), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x, t)| \leq \varepsilon|t| + C_\varepsilon |t|^{p-1}, \qquad \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}.$$

Hence we obtain

$$\begin{aligned}\left|\int\limits_{\mathbb{R}^3} f(x, u_k)(u_k - \bar{u}) dx\right| &\leq \varepsilon \int\limits_{\mathbb{R}^3} |u_k||u_k - \bar{u}| dx + C_\varepsilon \int\limits_{\mathbb{R}^3} |u_k|^{p-1}|u_k - \bar{u}| dx \\ &\leq \varepsilon |u_k|_2 |u_k - \bar{u}|_2 + C_\varepsilon |u_k|_p^{p-1} |u_k - \bar{u}|_p \\ &= o(1)\end{aligned} \tag{5.5}$$

since $u_k \to \bar{u}$ in $L^p(\mathbb{R}^3)$ $(2 < p < 6)$, $(u_k) \subset E$ is bounded and ε has been chosen arbitrarily. Similar to (2.7), we have

$$\left| \int_{\mathbb{R}^3} K(x)\phi_{u_k} u_k (u_k - \bar{u}) dx \right| \leq |\phi_{u_k}|_6 |u_k|_6 \left(\int_{\mathbb{R}^3} K(x)(u_k - \bar{u})^{3/2} dx \right)^{2/3} \to 0. \quad (5.6)$$

Using (5.3)-(5.6) and recalling $\bar{u}(x) = 0$ if $V(x) > 0$, we obtain

$$\|u_k\|^2 \leq \|u_k\|^2_{\lambda_k} = (u_k, \bar{u})_{\lambda_k} + o(1) = \int_{\mathbb{R}^3} \nabla u_k \cdot \nabla \bar{u} dx + o(1) = \|\bar{u}\|^2 + o(1). \quad (5.7)$$

It follows from the weak lower semicontinuity that

$$\|\bar{u}\|^2 \leq \liminf_{k\to\infty} \|u_k\|^2,$$

which, jointly with (5.7), shows that $u_k \to \bar{u}$ in E. The proof is complete. □

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