Calculus of Variations

Spinning Q-balls in Abelian Gauge Theories with positive potentials: existence and non existence

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Abstract We study the existence of cylindrically symmetric electro-magneto-static solitary waves for a system of a nonlinear Klein–Gordon equation coupled with Maxwell's equations in presence of a positive mass and of a nonnegative nonlinear potential. Nonexistence results are provided as well.

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1 Introduction, motivations and results

In recent years great attention has being paid to some classes of systems of partial differential equations that provide a model for the interaction of matter with electromagnetic field. Such theories are known in literature as Abelian Gauge Theories, and in this framework a crucial rôle is played by systems whose field equation is the Klein–Gordon's one. In particular, we recall the papers [2–7,9,12,13,16,21,26–29,36,39], where existence or non existence results are proved in the whole physical space for systems of Klein–Gordon–Maxwell type.

Here we are interested in a particular class of solutions, consisting in the so called *solitary waves*, i.e. solutions of a field equation whose energy travels as a localized packet. This kind of solutions plays an important rôle in these theories because of their relationship with solitons. "Soliton" is the name by which solitary waves are known when they exhibit some strong form of stability; they appear in many situations of mathematical physics, such as

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classical and quantum field theory, nonlinear optics, fluid mechanics and plasma physics (for example, see [14, 18, 33]). Therefore, the first step to prove the existence of solitons is to prove the existence of solitary waves, as we will do.

Our starting point is the following system, obtained by the interaction of a Klein–Gordon field with Maxwell's equations, which is, therefore, a model for electrodynamics:

$$\begin{cases} (\partial_t + iq\phi)^2\psi - (\nabla - iq\mathbf{A})^2\psi + W'(\psi) = 0, \\ \operatorname{div}(\partial_t\mathbf{A} + \nabla\phi) = q\left(\operatorname{Im}\frac{\partial_t\psi}{\psi} + q\phi\right)|\psi|^2, \\ \nabla \times (\nabla \times \mathbf{A}) + \partial_t(\partial_t\mathbf{A} + \nabla\phi) = q\left(\operatorname{Im}\frac{\partial_t\psi}{\psi} - q\mathbf{A}\right)|\psi|^2. \end{cases} \tag{1.1}$$

Here $\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$, $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{A} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$, see [4] for the derivation of the general system and for a detailed description of the physical meaning of the unknowns.

We are interested in standing waves solutions of system (1.1), under the assumption that W possesses some good invariants (necessary to be considered in Abelian Gauge Theories), typically some conditions of the form

$$W(e^{i\alpha}u) = W(u) \quad \text{and} \quad (W')(e^{i\alpha}u) = e^{i\alpha}W'(u)$$

for any function u and any $\alpha \in \mathbb{R}$. Thus we look for solutions having the special form

$$\psi(x, t) = u(x)e^{iS(x,t)}, u : \mathbb{R}^3 \to \mathbb{R}, \ S(x, t) = S_0(x) - \omega t \in \mathbb{R}, \ \omega \in \mathbb{R}, \tag{1.2}$$

$$\partial_t\mathbf{A} = 0, \ \partial_t\phi = 0. \tag{1.3}$$

In this way the previous system reads as

$$\begin{cases} -\Delta u + |\nabla S - q\mathbf{A}|^2 u - \left(\frac{\partial S}{\partial t} + q\phi\right)^2 u + W'(u) = 0, \\ \frac{\partial}{\partial t}\left[\left(\frac{\partial S}{\partial t} + q\phi\right)u^2\right] - \operatorname{div}[(\nabla S - q\mathbf{A})u^2] = 0, \\ \operatorname{div}\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla\phi\right) = q\left(\frac{\partial S}{\partial t} + q\phi\right)u^2, \\ \nabla \times (\nabla \times \mathbf{A}) + \frac{\partial}{\partial t}\left(\frac{\partial\mathbf{A}}{\partial t} + \nabla\phi\right) = q(\nabla S - q\mathbf{A})u^2, \end{cases} \tag{1.4}$$

where the equations are the matter equation, the charge continuity equation, the Gauss equation and the Ampère equation, respectively.

Three different types of finite energy, stationary nontrivial solutions can be considered:

- electrostatic solutions: $\mathbf{A} = 0, \phi \neq 0$;
- magnetostatic solutions: $\mathbf{A} \neq 0, \phi = 0$;
- electro-magneto-static solutions: $\mathbf{A} \neq 0, \phi \neq 0$.

Under suitable assumptions, all these types of solutions may exist.

Existence and nonexistence of electrostatic solutions for system (1.4) have been proved under different assumptions on W: in [12] and [13] the following potential (or more general ones) has been considered:

$$W(s) = \frac{1}{2}s^2 - \frac{s^p}{p}, \ s \geq 0.$$

In [4] the case $4 < p < 6$, in [13] the case $2 < p < 6$ and in [12] the remaining cases are studied.

In [3] and [29] the existence of electrostatic solutions has been studied for the first time when the potential W is nonnegative. In particular the existence of radially symmetric, electrostatic solutions has been analyzed in both papers, and it turns out that all these solutions have zero angular momentum.

Here we are interested in electro-magneto-static solutions when $W \geq 0$; in particular, we shall study the existence of vortices, which are solutions with non vanishing angular momentum, namely solutions with $S_0(x) = l\theta(x)$–θ is the polar function in cylindrical coordinates–i.e. of the form

$$\psi(t, x) = u(x)e^{i(l\theta(x) - \omega t)}, \ l \in \mathbb{Z} \setminus \{0\}, \tag{1.5}$$

and we will see that the angular momentum $\mathbf{M}_m$ of the matter field of a vortex does not vanish (see Remark 2.3); this fact justifies the name "vortex". These kinds of solutions are also known as spinning Q-balls; in this regard we recall the pioneering paper of Rosen [35] and of Coleman [11]. Coleman was the first to use the name Q-ball, referring to spherically symmetric solutions. Vortices in the nonlinear Klein–Gordon–Maxwell equations with a *non-negative* nonlinear term $W(s)$ with $W(0) = 0$ are also considered in Physics literature with the name of *gauged spinning Q-balls*, the name balls being used even if they do not exhibit a spherical symmetry, as in the case treated in this paper. More precisely, spinning axially symmetric Q-balls have been constructed by Volkov and Wohnert [39], and have already been analysed also in [1,9,19,20]. For a review of the problem of constructing classical field theory solutions describing stationary vortex rings we refer to [32], where applications in relativistic field theories and non-linear optics is presented.

However, in most of the previous considerations the existence of such solutions is discussed only qualitatively, so that almost no solutions of this type are explicitly known. Indeed, the mathematical existence of spinning Q-balls was given for the first time in [5], though some numerical results are known since [22]. Therefore, this paper is a contribution to an existence theory which is still at the very beginning.

By (1.5), system (1.4) becomes

$$-\Delta u + \left[|l\nabla\theta - q\mathbf{A}|^2 - (\omega - q\phi)^2\right]u + W'(u) = 0, \tag{1.6}$$

$$-\Delta\phi = q(\omega - q\phi)u^2, \tag{1.7}$$

$$\nabla \times (\nabla \times \mathbf{A}) = q(l\nabla\theta - q\mathbf{A})u^2, \tag{1.8}$$

which is the Klein–Gordon–Maxwell system we have investigated. Moreover, though system (1.6)–(1.8) was obtained by means of considerations on gauge invariance of W, from a mathematical point of view we can also replace (1.6) with

$$-\Delta u + \left[|l\nabla\theta - q\mathbf{A}|^2 - (\omega - q\phi)^2\right]u + W_u(x, u) = 0,$$

i.e. we could let W depend on the x–variable. More precisely, in order to use our functional approach, we let W depend on $(\sqrt{x_1^2 + x_2^2}, x_3)$, but we do not require any positivity far from 0, in contrast to the usual Ambrosetti–Rabinowitz condition. We think that this fact is quite interesting, both from a mathematical and a physical point of view: for example, it may happen that the potential is inactive in some cylinder, or, even more interestingly, out of a cylinder, as it happens where strong magnetic potential are present in linear accelerators.

According to what just said, in the second section we will show a new existence result for system (1.6)–(1.8) under general assumptions on the nonnegative potential W. We were inspired by the approach of [5], and for this reason, the functional structure is the same one of that article. However, our hypotheses on W imply, in particular, that the potential $W(s)$ might be 0 for values of s different from 0, in contrast to all previous results, where the potential W was assumed to lye above a parabola. This corresponds to the situation in which, for values of the unknown different from 0, there is no interaction among particles (see [13,29]).

Moreover, even more interestingly, we show the existence of solutions for *all* possible values of the charge q. We believe this is a very nice result, since for the first time in literature

from the seminal paper by Coleman [11], in which the charge was supposed small, as in all the subsequent papers in our bibliography, we give existence results *for all* values of the charge.

In conclusion, though our assumptions are weaker, our results are stronger than those found so far.

Remark 1.1 If we consider the electrostatic case, i.e. $-\Delta u + W'(u) = 0$, calling "rest mass" of the particle u the quantity

$$\int_{\mathbb{R}^3} W(u)\,dx,$$

see [7], our assumptions on W imply that we are dealing a priori with systems for particles having positive mass, which is, of course, the physical interesting case.

Entering into details, we shall study system (1.6)–(1.8) under the following hypothesis on the potential W:

(W1) $W(s) \geq 0$ for all $s \geq 0$;
(W2) W is of class C^2 with $W(0) = W'(0) = 0$, $W''(0) = m^2 > 0$;
(W3) setting

$$W(s) = \frac{m^2}{2}s^2 + N(s), \tag{1.9}$$

we assume that there exist positive constants c_1, c_2, p, ℓ, with $2 < \ell \leq p < 6$, such that for all $s \geq 0$ there holds

$$|N'(s)| \leq c_1 s^{\ell-1} + c_2 s^{p-1}.$$

Moreover, though we are interested in positive solutions, it is convenient to extend W to all of $\mathbb{R}$ setting

$$W(s) = W(-s) \quad \text{for every} \quad s < 0.$$

System (1.6)–(1.8) was introduced in [5] assuming (W1), (W2), (W3) and the fundamental requirement

$$\inf_{s>0}\left(\frac{W(s)}{\frac{m^2}{2}s^2}\right) < 1. \tag{1.10}$$

We immediately see that assumption (W3) plus (1.10) is equivalent to require that there exists $s_0 > 0$ such that $N(s_0) < 0$, the first step in the classical "Berestycki–Lions" approach. In this paper we will use an hypothesis different from (1.10), which will let us prove our main result without any restriction on the charge q, in contrast to all previous results.

Indeed, we will assume
(W4) there exist $\tau > 2$,

$$D \geq \begin{cases} 3(1+l^2)^{\frac{\tau-2}{2}}2^{3\tau/2-5}m^{4-\tau} & \text{if } q \leq 1, \\ 3(1+l^2)^{\frac{\tau-2}{2}}2^{3\tau/2-5}m^{4-\tau}q^{3(\tau-2)} & \text{if } q > 1 \end{cases}$$

and $\varepsilon_0 > 0$ with $\varepsilon_0 = \varepsilon_0(q)$ if $q > 1$, such that

$$N(s) \leq -D|s|^\tau \quad \text{for all } s \in [0, \varepsilon_0].$$

It is clear that functions of the type $N(s) = |s|^p - |s|^q$, $2 < q < p$, satisfy (W4). Of course, (W4) implies that there exists $s_0 > 0$ such that $N(s_0) < 0$, but (W4) permits to prove existence results for any $q > 0$ and suitable potentials W, see Theorem 1.3.

Remark 1.2 We emphasize the fact that (W1) and (W4) together imply that D cannot be as large as desired, since the condition $W \geq 0$ forces D to depend on ε_0 and q. However, we remark that the parameter ε_0 is allowed to depend on q only when $q > 1$, hence D does *not* depend on the charge q when $q \leq 1$, but it depends only on m and l. As a consequence, the class of admissible potentials does *not* depend on the value of the charge q, whenever $q \leq 1$.

As usual, for physical reasons, we look for solutions having finite energy, i.e. $(u, \phi, \mathbf{A}) \in H^1 \times \mathcal{D}^1 \times (\mathcal{D}^1)^3$, where $H^1 = H^1(\mathbb{R}^3)$ is the usual Sobolev space, and $\mathcal{D}^1 = \mathcal{D}^1(\mathbb{R}^3)$ is the completion of $\mathscr{D} = C_C^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\mathcal{D}^1}^2 := \int_{\mathbb{R}^3} |\nabla u|^2 \, dx$ (see Sect. 2.2 for the precise functional setting).

Before giving our main result, we remark that, as in [5], the parameter ω is an unknown of the problem.

Theorem 1.3 *Assume (W1), (W2), (W3), (W4), let $l \in \mathbb{Z}$ and $q \geq 0$. Then, system (1.6)–(1.8) admits a finite energy solution in the sense of distributions $(u, \omega, \phi, \mathbf{A})$, $u \neq 0$, $\omega > 0$ such that*

- *the maps u, ϕ depend only on the variables $r = \sqrt{x_1^2 + x_2^2}$ and x_3;*
-

$$\int_{\mathbb{R}^3} \frac{u^2}{r^2} \, dx \in \mathbb{R};$$

- *the magnetic potential $\mathbf{A}$ has the following form:*

$$\mathbf{A} = a(r, x_3)\nabla\theta = a(r, x_3) \left(\frac{x_2}{r^2}\mathbf{e_1} - \frac{x_1}{r^2}\mathbf{e_2} \right). \tag{1.11}$$

If $q = 0$, then $\phi = 0$ and $\mathbf{A} = \mathbf{0}$. If $q > 0$, then $\phi \neq 0$. Moreover, $\mathbf{A} \neq \mathbf{0}$ if and only if $l \neq 0$.

Remark 1.4 By definition, the angular momentum is the quantity which is preserved by virtue of the invariance under space rotations of the Lagrangian with respect to the origin. Using the gauge invariant variables, we get:

$$\mathbf{M} = \mathbf{M}_m + \mathbf{M}_f,$$

where

$$\mathbf{M}_m = \int_{\mathbb{R}^3} \left[-x \times (\nabla u \partial_t u) + x \times \frac{\rho \mathbf{j}}{q^2 u^2} \right] dx$$

and

$$\mathbf{M}_f = \int_{\mathbb{R}^3} x \times (\mathbf{E} \times \mathbf{H}) \, dx.$$

Here $\mathbf{M}_m$ refers to the "matter field" and $\mathbf{M}_f$ to the "electromagnetic field", while ρ and $\mathbf{j}$ denote the electric charge and the current density, respectively.

We will see below that the solution found in Theorem 1.3 has nontrivial angular momentum, see Remark 2.3.

Remark 1.5 When $l = 0$ and $q > 0$ the last part of Theorem 1.3 states the existence of electrostatic solutions, namely finite energy solutions with $u \neq 0$, $\phi \neq 0$ and $\mathbf{A} = \mathbf{0}$. This result is a variant of a recent ones (see [3,29]).

Moreover, let us observe that under general assumptions on W, magnetostatic solutions (i.e. with $\omega = \phi = 0$) do not exist. In fact also the following proposition is proved in [5]:

Remark 1.6 [5, Proposition 8] Assume that W satisfies the assumptions $W(0) = 0$ and $W'(s)s \geq 0$. Then (1.6)–(1.8) has no solutions with $\omega = \phi = 0$ (see [31, Proposition 1.2] for a related result).

In our setting, we are able to prove the following nonexistence results:

Theorem 1.7 *If u is a finite energy solution of* (1.6) *with*

$$\int_{\mathbb{R}^3} N(u) \, dx \in \mathbb{R},$$

and

- $\omega^2 < m^2$ and either $N \geq 0$ or $N'(s)s \leq 6N(s)$ for all $s \in \mathbb{R}$, *or*
- $N'(s)s \geq 2N(s)$ for all $s \in \mathbb{R}$,

then $u \equiv 0$.

A natural consequence is the following

Corollary 1.8 *If $u \in L^p(\mathbb{R}^3)$ is a finite energy solution of* (1.6)–(1.8), *and*

- $\omega^2 < m^2$ and

$$N(u) = \begin{cases} \dfrac{|u|^p}{p}, & p \leq 6, \\ -\dfrac{|u|^p}{p}, & p \geq 6, \end{cases}$$

  or

-

$$N(u) = \begin{cases} \dfrac{|u|^p}{p}, & p \geq 2, \\ -\dfrac{|u|^p}{p}, & p \leq 2, \end{cases}$$

then $u \equiv 0$.

Remark 1.9 Theorem 1.7 implies that, in general, in order to have vortices with $N \geq 0$ it is necessary to have a "large" frequency. We are not aware of similar results in the theory of vortices, and we believe that such a result can shed a new light on this subject.

In Sect. 4 we shall prove another existence result concerning a different kind of solutions, namely solutions having fixed L^2 norm. In general these solutions cannot be obtained from the solutions found in Theorem 1.3, for example via a rescaling argument, and we shall focus on the case $\int_{\mathbb{R}^3} u^2 dx = 1$, which corresponds to solutions having a density of probability equal to 1. An analogous result could be obtained for $\int_{\mathbb{R}^3} u^2 dx = c \in \mathbb{R}^+$, but the physical

meaning of this kind of solutions is not clear to us. Indeed, in different situations it may happen that if $\int_{\mathbb{R}^3} u^2 = c$ is fixed a priori, then solutions appear only for certain values of c: a typical example is in the context of boson stars, when solutions with fixed energy do exist if and only if $c < M_C$, the Chandrasekhar limit mass (see [23,30]).

Our result is the following

Proposition 1.10 *Under the hypotheses of Theorem 1.3, there exists $\mu \in \mathbb{R}$ and a solution in the sense of distributions for the system*

$$-\Delta u + \left[|l\nabla\theta - q\mathbf{A}|^2 - (\omega - q\phi)^2\right]u + W'(u) = \mu u,$$
$$-\Delta\phi = q(\omega - q\phi)u^2,$$
$$\nabla \times (\nabla \times \mathbf{A}) = q(l\nabla\theta - q\mathbf{A})u^2,$$

such that $\int_{\mathbb{R}^3} u^2 dx = 1$. Moreover, if $\omega^2 \leq m^2$ and $N'(s)s \geq 0$ for all $s \in \mathbb{R}$, then $\mu > 0$.

Due to the presence of the multiplier μ, we give the following

Definition 1.11 We call *effective mass* of the system the quantity $\tilde{m} = m^2 - \mu$.

2 Preliminary setting

2.1 Standing wave solutions and vortices

Substituting (1.2) and (1.3) in (1.4), we get the following equations in $\mathbb{R}^3$:

$$-\Delta u + \left[|\nabla S_0 - q\mathbf{A}|^2 - (\omega - q\phi)^2\right]u + W'(u) = 0, \tag{2.1}$$

$$-\text{div}\left[(\nabla S_0 - q\mathbf{A})u^2\right] = 0, \tag{2.2}$$

$$-\Delta\phi = q(\omega - q\phi)u^2, \tag{2.3}$$

$$\nabla \times (\nabla \times \mathbf{A}) = q(\nabla S_0 - q\mathbf{A})u^2. \tag{2.4}$$

We can easily observe that (2.2) follows from (2.4): as a matter of fact, applying the divergence operator to both sides of (2.4), we immediately get (2.2). Then we are reduced to study system (2.1), (2.3), (2.4).

We are interested in finite-energy solutions—the most relevant physical case—i.e. solutions of system (2.1)–(2.4) for which the following energy is finite:

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + |\nabla\phi|^2 + |\nabla \times \mathbf{A}|^2 + (|\nabla S_0 - q\mathbf{A}|^2 + (\omega - q\phi)^2)u^2\right)dx$$
$$+ \int_{\mathbb{R}^3} W(u)dx \tag{2.5}$$

Furthermore, in order to study the behavior of some particular functional which will be introduced later on, it is useful to give the electric charge Q a specific representation in terms of the solution u, as (see e.g. [5], p.644):

$$Q = q\sigma, \tag{2.6}$$

where

$$\sigma = \int_{\mathbb{R}^3} (\omega - q\phi)u^2 \, dx. \tag{2.7}$$

However, our strategy will consist in fixing a real number σ and then find a solution u which turns out to verify (2.7).

Remark 2.1 When $u = 0$, the only finite energy gauge potentials which solve (2.3), (2.4) are the trivial ones $\mathbf{A} = \mathbf{0}, \phi = 0$.

In particular, following [5], we shall look for solutions of the above system which are known in literature as vortices. In order to do that, we need some preliminaries. First, set

$$\Sigma = \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0\Big\},$$

and define the map

$$\theta : \mathbb{R}^3 \setminus \Sigma \to \frac{\mathbb{R}}{2\pi\mathbb{Z}},$$
$$\theta(x_1, x_2, x_3) = \operatorname{Im} \log(x_1 + ix_2).$$

The following definition is crucial:

Definition 2.2 A finite energy solution $(u, S_0, \phi, \mathbf{A})$ of (2.1)–(2.4) is called *vortex* if $S_0 = l\theta$ for some $l \in \mathbb{Z} \setminus \{0\}$.

Of course, in this case, ψ has the form

$$\psi(t, x) = u(x)e^{i(l\theta(x) - \omega t)}, \; l \in \mathbb{Z} \setminus \{0\}. \tag{2.8}$$

Remark 2.3 In [5, Proposition 7] it was proved that if $(u, \omega, \phi, \mathbf{A})$ is a non trivial, finite energy solution of (2.1)–(2.4), then the angular momentum $\mathbf{M}_m$ has the expression

$$\mathbf{M}_m = - \left[\int_{\mathbb{R}^3} (l - qa)(\omega - q\phi)u^2 dx \right] \mathbf{e_3}, \tag{2.9}$$

and, if $l \neq 0$, it does not vanish. Hence, in this case, the name "vortex" is justified and by Theorem 1.3 the existence of a spinning Q-ball is guaranteed.

Now, observe that $\theta \in C^\infty \left(\mathbb{R}^3 \setminus \Sigma, \frac{\mathbb{R}}{2\pi\mathbb{Z}} \right)$, and, with an abuse of notation, we set

$$\nabla\theta(x) = \frac{x_2}{x_1^2 + x_2^2}\mathbf{e_1} - \frac{x_1}{x_1^2 + x_2^2}\mathbf{e_2},$$

where $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ is the standard frame in $\mathbb{R}^3$.

Using the *Ansatz* (2.8), Eqs. (2.1), (2.3), (2.4) give rise to Eqs. (1.6), (1.7), (1.8), which is the Klein–Gordon–Maxwell system we shall study from now on.

Remark 2.4 If $\mathbf{A} = \left(\frac{x_2}{x_1^2 + x_2^2}, -\frac{x_1}{x_1^2 + x_2^2}, 0 \right)$, we obviously get $\nabla \times \mathbf{A} = 0$. Viceversa, if $\mathbf{A}$ is irrotational and it solves (1.8), then $\mathbf{A} = \frac{l}{q}\nabla\theta$. In such a case, system (1.6)–(1.8) reduces to the one considered in [29], where, by Theorem 1.7, we can now say that the nontrivial solution found therein is such that $\omega^2 \geq m^2$.

2.2 Functional approach

We shall follow the functional approach of [5], with minor changes in some parts. Anyway, our main Theorem 1.3 has been proved thanks to completely new results (see Lemma 3.4 and Proposition 3.5), which let us avoid any bound on q, differently from [5].

First, we denote by $L^p \equiv L^p(\mathbb{R}^3)$ $(1 \leq p < +\infty)$ the usual Lebesgue space endowed with the norm

$$\|u\|_p^p := \int_{\mathbb{R}^3} |u|^p \, dx.$$

We also recall the continuous embeddings

$$H^1(\mathbb{R}^3) \hookrightarrow \mathcal{D}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \quad \text{and} \quad H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3) \quad \forall \, p \in [2, 6], \tag{2.10}$$

being 6 the critical exponent for the Sobolev embedding $\mathcal{D}^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$. Here $H^1 \equiv H^1(\mathbb{R}^3)$ denotes the usual Sobolev space with norm

$$\|u\|_{H^1}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx$$

and $\mathcal{D}^1 = \mathcal{D}^1(\mathbb{R}^3)$ is the completion of $\mathcal{D} = C_C^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^1}^2 := \int_{\mathbb{R}^3} |\nabla u|^2 \, dx,$$

induced by the scalar product $(u, v)_{\mathcal{D}^1} := \int_{\mathbb{R}^3} \nabla u \cdot \nabla v \, dx$.

Moreover, we need the weighted Sobolev space $\hat{H}^1 \equiv \hat{H}_l^1(\mathbb{R}^3)$, depending on a fixed integer l, whose norm is given by

$$\|u\|_{\hat{H}^1}^2 = \int_{\mathbb{R}^3} \left[|\nabla u|^2 + \left(1 + \frac{l^2}{r^2} \right) u^2 \right] dx, \, l \in \mathbb{Z},$$

where $r = \sqrt{x_1^2 + x_2^2}$. Clearly $\hat{H}^1 = H^1$ if and only if $l = 0$. Moreover, it is not hard to see that

$$C_C^\infty(\mathbb{R}^3) \cap \hat{H}^1(\mathbb{R}^3) \text{ is dense in } \hat{H}^1(\mathbb{R}^3). \tag{2.11}$$

We set

$$H = \hat{H}^1 \times \mathcal{D}^1 \times \left(\mathcal{D}^1 \right)^3,$$

$$\|(u, \phi, \mathbf{A})\|_H^2 = \int_{\mathbb{R}^3} \left[|\nabla u|^2 + \left(1 + \frac{l^2}{r^2} \right) u^2 + |\nabla \phi|^2 + |\nabla \mathbf{A}|^2 \right] dx.$$

We shall denote by $u = u(r, x_3)$ any real function in $\mathbb{R}^3$ which depends only on the cylindrical coordinates (r, x_3), and we set

$$\mathscr{D}_\sharp = \left\{ u \in \mathscr{D} : u = u(r, x_3) \right\}.$$

Finally, we shall denote by $\mathcal{D}_\sharp^1$ the closure of $\mathscr{D}_\sharp$ in the $\mathcal{D}^1$ norm and by $\hat{H}_\sharp^1$ the closed subspace of $\hat{H}^1$ whose functions are of the form $u = u(r, x_3)$.

Now, we consider the functional

$$J(u, \phi, \mathbf{A}) = \frac{1}{2} \int_{\mathbb{R}^3} \left[|\nabla u|^2 - |\nabla \phi|^2 + |\nabla \times \mathbf{A}|^2 \right] dx$$

$$+ \frac{1}{2} \int_{\mathbb{R}^3} \left[|l\nabla\theta - q\mathbf{A}|^2 - (\omega - q\phi)^2 \right] u^2 dx + \int_{\mathbb{R}^3} W(u) dx, \quad (2.12)$$

where $(u, \phi, \mathbf{A}) \in H$. Formally, Eqs. (1.6)–(1.8) are the Euler–Lagrange equations of the functional J, and, indeed, standard computations show that the following lemma holds:

Lemma 2.5 *Assume that W satisfies (W3). Then the functional J is of class C^1 on H and Eqs. (1.6)–(1.8) are its Euler–Lagrange equations.*

By the above lemma it follows that any critical point $(u, \phi, \mathbf{A}) \in H$ of J is a weak solutions of system (1.6)–(1.8), namely

$$\int_{\mathbb{R}^3} \left[\nabla u \cdot \nabla v + \left[|l\nabla\theta - q\mathbf{A}|^2 - (\omega - q\phi)^2 \right] uv + W'(u)v \right] dx = 0 \, \forall \, v \in \hat{H}^1, \quad (2.13)$$

$$\int_{\mathbb{R}^3} \left[\nabla \phi \cdot \nabla w - qu^2(\omega - q\phi)w \right] dx = 0 \, \forall \, w \in \mathcal{D}^1, \quad (2.14)$$

$$\int_{\mathbb{R}^3} \left[(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{V}) - qu^2(l\nabla\theta - q\mathbf{A}) \cdot \mathbf{V} \right] dx = 0 \, \forall \, \mathbf{V} \in (\mathcal{D}^1)^3. \quad (2.15)$$

2.3 Solutions in the sense of distributions

Since $\mathcal{D}$ is *not* contained in $\hat{H}^1$, a solution $(u, \phi, \mathbf{A}) \in H$ of (2.13)–(2.15) need not be a solution of (1.6)–(1.8) in the sense of distributions on $\mathbb{R}^3$. However, we will show that the singularity of $\nabla\theta(x)$ on Σ is removable in the following sense:

Theorem 2.6 *Let $(u_0, \phi_0, \mathbf{A}_0) \in H$, $u_0 \geq 0$ be a solution of (2.13)–(2.15) (i.e. a critical point of J). Then $(u_0, \phi_0, \mathbf{A}_0)$ is a solution of system (1.6)–(1.8) in the sense of distributions, namely*

$$\int_{\mathbb{R}^3} \left[\nabla u_0 \cdot \nabla v + \left[|l\nabla\theta - q\mathbf{A}_0|^2 - (\omega - q\phi_0)^2 \right] u_0 v + W'(u_0)v \right] dx = 0 \, \forall \, v \in \mathcal{D}, \quad (2.16)$$

$$\int_{\mathbb{R}^3} \left[\nabla \phi_0 \cdot \nabla w - qu_0^2(\omega - q\phi_0)w \right] dx = 0 \, \forall \, w \in \mathcal{D}, \quad (2.17)$$

$$\int_{\mathbb{R}^3} \left[(\nabla \times \mathbf{A}_0) \cdot (\nabla \times \mathbf{V}) - qu_0^2(l\nabla\theta - q\mathbf{A}_0) \cdot \mathbf{V} \right] dx = 0 \, \forall \, \mathbf{V} \in (\mathcal{D})^3. \quad (2.18)$$

A proof of Theorem 2.6 was given in [5].

Let us now remark that the presence of the term $-\int_{\mathbb{R}^3} |\nabla\phi|^2 dx$ gives the functional J a strong indefiniteness, namely any nontrivial critical point of J has infinite Morse index. It turns out that a direct approach to finding critical points for J is very hard. For this reason, as usual in this setting, it is convenient to introduce a *reduced functional*.

2.4 The reduced functional

Writing Eq. (1.7) as

$$-\Delta\phi + q^2 u^2 \phi = q\omega u^2, \tag{2.19}$$

then we can verify that the following holds:

Proposition 2.7 [13, Proposition 2.2] *For every $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in \mathcal{D}^1$ which solves (2.19) and there exists $S > 0$ such that*

$$\|\phi_u\| \leq q S \|u\|_{12/5}^2 \quad \text{for every} \quad u \in H^1(\mathbb{R}^3). \tag{2.20}$$

Lemma 2.8 *If $u \in \hat{H}_\sharp^1(\mathbb{R}^3)$, then the solution $\phi = \phi_u$ of (2.19) belongs to $\mathcal{D}_\sharp^1(\mathbb{R}^3)$.*

The proof is an adaptation of the analogue in [13] and is thus omitted.

By the lemma above, we can define the map

$$u \in \hat{H}_\sharp^1(\mathbb{R}^3) \mapsto Z_\omega(u) = \phi_u \in \mathcal{D}_\sharp^1 \text{ solves (2.19)}. \tag{2.21}$$

Since ϕ_u solves (2.19), clearly we have

$$d_\phi J(u, Z_\omega(u), \mathbf{A}) = 0, \tag{2.22}$$

where J is defined in (2.12) and $d_\phi J$ denotes the partial differential of J with respect to ϕ.

Following the lines of the proof of [12, Proposition 2.1], using Lemma 2.8, we can easily prove the following result:

Proposition 2.9 *The map Z_ω defined in (2.21) is of class C^1 and*

$$(Z_\omega'[u])[v] = 2q \left(\Delta - q^2 u^2\right)^{-1} [(q\phi_u - \omega)uv] \quad \forall u, v \in \mathcal{D}_\sharp^1. \tag{2.23}$$

For $u \in H^1(\mathbb{R}^3)$, let $\Phi = \Phi_u$ be the solution of (2.19) with $\omega = 1$; then Φ solves the equation

$$-\Delta\Phi_u + q^2 u^2 \Phi_u = qu^2, \tag{2.24}$$

and clearly

$$\phi_u = \omega\Phi_u. \tag{2.25}$$

Now let $q > 0$; then, by maximum principle arguments, one can show that for any $u \in H^1(\mathbb{R}^3)$ the solution Φ_u of (2.24) satisfies the following estimate, first proved in [28]:

$$0 \leq \Phi_u \leq \frac{1}{q}. \tag{2.26}$$

Now, if $(u, \mathbf{A}) \in \hat{H}^1 \times \left(\mathcal{D}^1\right)^3$, we introduce the *reduced action functional*

$$\tilde{J}(u, \mathbf{A}) = J(u, Z_\omega(u), \mathbf{A}).$$

Recalling that J and the map $u \to Z_\omega(u) = \phi_u$ are of class C^1 by Lemma 2.5 and Proposition 2.9) respectively, functional $\tilde{J}$ is of class C^1. as well. Now, by using the chain rule and (2.22), it is standard to show that the following Lemma holds:

Lemma 2.10 *If $(u, \mathbf{A})$ is a critical point of $\tilde{J}$, then $(u, Z_\omega(u), \mathbf{A})$ is a critical point of J (and viceversa).*

From (2.24) we have

$$\int_{\mathbb{R}^3} qu^2 \Phi_u dx = \int_{\mathbb{R}^3} |\nabla \Phi_u|^2 dx + q^2 \int_{\mathbb{R}^3} u^2 \Phi_u^2 dx, \tag{2.27}$$

which is another way of writing (2.22).

Now, by (2.25) and (2.27), we have:

$$\tilde{J}(u, \mathbf{A}) = J(u, Z_\omega(u), \mathbf{A}) = \frac{1}{2} \int_{\mathbb{R}^3} \left[|\nabla u|^2 - |\nabla \phi_u|^2 + |\nabla \times \mathbf{A}|^2 \right] dx$$

$$+ \frac{1}{2} \int_{\mathbb{R}^3} \left[|l\nabla\theta - q\mathbf{A}|^2 - (\omega - q\phi_u)^2 \right] u^2 dx + \int_{\mathbb{R}^3} W(u) dx$$

$$= \frac{1}{2} \int_{\mathbb{R}^3} \left[|\nabla u|^2 + |\nabla \times \mathbf{A}|^2 + |l\nabla\theta - q\mathbf{A}|^2 u^2 \right] dx + \int_{\mathbb{R}^3} W(u) dx$$

$$- \frac{\omega^2}{2} \int_{\mathbb{R}^3} (1 - q\Phi_u) u^2 dx.$$

Then

$$\tilde{J}(u, \mathbf{A}) = I(u, \mathbf{A}) - \frac{\omega^2}{2} K_q(u), \tag{2.28}$$

where $I : \hat{H}^1 \times \left(\mathcal{D}^1\right)^3 \to \mathbb{R}$ and $K_q : \hat{H}^1 \to \mathbb{R}$ are defined as

$$I(u, \mathbf{A}) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + |\nabla \times \mathbf{A}|^2 + |l\nabla\theta - q\mathbf{A}|^2 u^2 \right) dx + \int_{\mathbb{R}^3} W(u) dx \tag{2.29}$$

and

$$K_q(u) = \int_{\mathbb{R}^3} (1 - q\Phi_u) u^2 dx. \tag{2.30}$$

Now, let us introduce the *reduced energy functional*, defined as

$$\hat{\mathcal{E}}(u, \mathbf{A}) = \mathcal{E}(u, Z_\omega(u), \mathbf{A}),$$

where, as in (2.5),

$$\mathcal{E}(u, \phi, \mathbf{A}) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + |\nabla \phi|^2 + |\nabla \times \mathbf{A}|^2 + (|l\nabla\theta - q\mathbf{A}|^2 + (\omega - q\phi)^2) u^2 \right) dx$$

$$+ \int_{\mathbb{R}^3} W(u) dx. \tag{2.31}$$

By using (2.27) and (2.25), we easily find that

$$\hat{\mathcal{E}}(u, \mathbf{A}) = I(u, \mathbf{A}) + \frac{\omega^2}{2} K_q(u). \tag{2.32}$$

Recalling (2.6) and (2.7), we note that

$$Q = q\sigma = q\omega K_q(u)$$

represents the (electric) charge, so that, if $u \neq 0$, we can write

$$\hat{\mathcal{E}}(u, \mathbf{A}) = I(u, \mathbf{A}) + \frac{\omega^2}{2} K_q(u) = I(u, \mathbf{A}) + \frac{\sigma^2}{2K_q(u)}.$$

Then for any $\sigma \neq 0$, the functional $E_{\sigma,q} : (\hat{H}^1 \setminus \{0\}) \times (\mathcal{D}^1)^3 \to \mathbb{R}$, defined by

$$E_{\sigma,q}(u, \mathbf{A}) = I(u, \mathbf{A}) + \frac{\omega^2}{2} K_q(u) = I(u, \mathbf{A}) + \frac{\sigma^2}{2K_q(u)} \tag{2.33}$$

represents the energy on the configuration $(u, \omega\Phi_u, \mathbf{A})$ having charge $Q = q\sigma$ or, equivalently, frequency $\omega = \frac{\sigma}{K_q(u)}$.

The following lemma holds (see [5, Lemma 13]):

Lemma 2.11 *The functional*

$$\hat{H}^1 \ni u \mapsto K(u) = \int_{\mathbb{R}^3} (1 - q\Phi_u)u^2 dx$$

is differentiable and for any $u, v \in \hat{H}^1$ we have

$$K'(u)[v] = 2 \int_{\mathbb{R}^3} (1 - q\Phi_u)^2 uv \, dx. \tag{2.34}$$

Introducing $E_{\sigma,q}$ turns out to be a useful choice, as the following easy consequence shows (see [5, Proposition 14]):

Proposition 2.12 *Let $\sigma \neq 0$ and let $(u, \mathbf{A}) \in \hat{H}^1 \times (\mathcal{D}^1)^3$, $u \neq 0$ be a critical point of $E_{\sigma,q}$. Then, if we set $\omega = \frac{\sigma}{K_q(u)}$, $(u, Z_\omega(u), \mathbf{A})$ is a critical point of J.*

Therefore, by Proposition 2.12 and Theorem 2.6 we are reduced to study the critical points of $E_{\sigma,q}$, which is a functional bounded from below, since all its components are nonnegative.

However $E_{\sigma,q}$ contains the term $\int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2$, which is not a Sobolev norm in $(\mathcal{D}^1)^3$. In order to avoid consequent difficulties, we introduce a suitable manifold $V \subset \hat{H}^1 \times (\mathcal{D}^1)^3$ in the following way: first, we set

$$\mathcal{A}_0 := \left\{ \mathbf{X} \in C_C^\infty(\mathbb{R}^3 \setminus \Sigma, \mathbb{R}^3) : \mathbf{X} = b(r, z)\nabla\theta; \ b \in C_C^\infty(\mathbb{R}^3 \setminus \Sigma, \mathbb{R}) \right\},$$

and we denote by $\mathcal{A}$ the closure of $\mathcal{A}_0$ with respect to the norm of $(\mathcal{D}^1)^3$. We now consider the space

$$V := \hat{H}_\sharp^1 \times \mathcal{A}, \tag{2.35}$$

and we set $U = (u, \mathbf{A}) \in V$ with

$$\|U\|_V = \|(u, \mathbf{A})\|_V = \|u\|_{\hat{H}_\sharp^1} + \|\mathbf{A}\|_{(\mathcal{D}^1)^3}.$$

We need the following result, for whose proof see [5, Lemma 15]:

Lemma 2.13 *If* $\mathbf{A} \in \mathcal{A}$, *then*

$$\int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2 dx = \int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2 dx.$$

Working in V has two advantages: first, the components $\mathbf{A}$ of the elements in V are divergence free, so that the term $\int_{\mathbb{R}^3} |\nabla \times \mathbf{A}|^2$ can be replaced by $\|\mathbf{A}\|^2_{(\mathcal{D}^1)^3} = \int_{\mathbb{R}^3} |\nabla \mathbf{A}|^2$. Second, the critical points of J constrained on V satisfy system (1.6)–(1.8); namely V is a "natural constraint" for J.

3 Proof of Theorem 1.3

In this section we shall always assume that W satisfies (W1)–(W4) and we will show that $E_{\sigma,q}$ constrained on V as in (2.35) has a minimum which is a nontrivial solution of system (1.6)–(1.8).

We start with the following a priori estimate on minimizing sequences, whose proof is similar to the proof of [5, Lemma 18]:

Lemma 3.1 *For any* $\sigma, q > 0$, *any minimizing sequence* $(u_n, \mathbf{A}_n) \subset V$ *for* $E_{\sigma,q}|_V$ *is bounded in* $\hat{H}^1 \times (\mathcal{D}^1)^3$.

Proposition 3.2 *For any* $\sigma, q > 0$ *there exists a minimizing sequence* $U_n = (u_n, \mathbf{A}_n)$ *of* $E_{\sigma,q}|_V$, *with* $u_n \geq 0$ *and which is also a Palais–Smale sequence for* $E_{\sigma,q}$, *i.e.*

$$E'_{\sigma,q}(u_n, \mathbf{A}_n) \to 0.$$

Proof Let $(u_n, \mathbf{A}_n) \subset V$ be a minimizing sequence for $E_{\sigma,q}|_V$. It is not restrictive to assume that $u_n \geq 0$. Otherwise, we can replace u_n with $|u_n|$ and we still have a minimizing sequence (see (2.31)). By Ekeland's Variational Principle (see [15]) we can also assume that $(u_n, \mathbf{A}_n)$ is a Palais–Smale sequence for $E_{\sigma,q}|_V$, namely we can assume that

$$E'_{\sigma,q}|_V(u_n, \mathbf{A}_n) \to 0.$$

By using the same technique used to prove Theorem 16 in [6], it follows that $(u_n, \mathbf{A}_n)$ is a Palais–Smale sequence also for $E_{\sigma,q}$, that is

$$E'_{\sigma,q}(u_n, \mathbf{A}_n) \to 0.$$

$\square$

A fundamental tool in proving the existence result, is given by the following

Lemma 3.3 *For any* $\sigma, q > 0$ *and for any minimizing sequence* $(u_n, \mathbf{A}_n) \subset V$ *for* $E_{\sigma,q}|_V$, *there exist positive numbers* $a_1 < a_2$ *such that*

$$a_1 \leq \int_{\mathbb{R}^3} (1 - q\Phi_{u_n})u_n^2 dx \leq a_2 \quad \text{for every} \quad n \in \mathbb{N}$$

and

$$a_1 \leq \int_{\mathbb{R}^3} u_n^2 dx \leq a_2 \quad \text{for every} \quad n \in \mathbb{N}.$$

Proof The upper bounds are an obvious consequence of Lemma 3.1 and of (2.26), so that we only prove the lower bounds.

Since $E_{\sigma,q}(u_n, \mathbf{A}_n) \to \inf_V E_{\sigma,q}$, from (2.33) we immediately get that there exists $a_1 > 0$ such that

$$\frac{1}{\int_{\mathbb{R}^3}(1 - q\Phi_{u_n})u_n^2 dx} \leq \frac{1}{a_1} \quad \text{for every} \quad n \in \mathbb{N},$$

and thus all the claims follow. □

As a corollary of the previous lemma, we have the following result, whose proof is now very easy, but whose consequences are crucial:

Lemma 3.4 *For any $\sigma, q > 0$*

$$\inf_V E_{\sigma,q} > 0.$$

Proof Assume by contradiction that $\inf_V E_{\sigma,q} = 0$. Hence, there would exist a sequence $(u_n, \mathbf{A}_n)_n \subset V$ such that $E_{\sigma,q}(u_n, \mathbf{A}_n) \to 0$ as $n \to \infty$. Since both I and K_q are nonnegative, from (2.33) we get

$$I(u_n, \mathbf{A}_n) \to 0 \quad \text{and} \quad \frac{1}{K_q(u_n)} \to 0 \quad \text{as} \quad n \to \infty.$$

In particular,

$$\int_{\mathbb{R}^3}(1 - q\Phi_{u_n})u_n^2 dx \to \infty \quad \text{as} \quad n \to \infty,$$

and thus, by (2.26),

$$\int_{\mathbb{R}^3} u_n^2 dx \to \infty \quad \text{as} \quad n \to \infty,$$

a contradiction to Lemma 3.3. □

The following result, which turns out to be a crucial one, is the only point where assumption (W4) is used.

Lemma 3.5 *There exists $\sigma_0 > 0$ such that there exists $u_0 \in \hat{H}^1$ with*

$$E_{\sigma_0,q}(u_0, 0) < m\sigma_0.$$

Moreover, if $q \leq 1$, then σ_0 depends only on D and m, while, if $q > 1$, then σ_0 depends on D, m and q.

Proof Let us define

$$v(x) := \begin{cases} 1 - \sqrt{(r-2)^2 + x_3^2}, & (r-2)^2 + x_3^2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

We define the set $A_\lambda := \{(r, x_3) \in \mathbb{R}^3 \text{ s.t. } (r - 2\lambda)^2 + x_3^2 \leq \lambda^2\}$ and we compute

$$|A_\lambda| = \int_{A_\lambda} dx_1 dx_2 dx_3 = 4\pi^2\lambda^3 = \lambda^3|A_1|. \tag{3.1}$$

Of course, $v \in \hat{H}_r^1$ and, for a future need, we also compute

$$\int_{\mathbb{R}^3} v^2 dx = \int_{A_1} \left(1 - \sqrt{(r-2)^2 + x_3^2}\right)^2 dx_1 dx_2 dx_3 = \frac{2}{3}\pi^2,$$

$$\int_{\mathbb{R}^3} v \, dx = \int_{A_1} \left(1 - \sqrt{(r-2)^2 + x_3^2}\right) dx_1 dx_2 dx_3 = \frac{4}{3}\pi^2,$$

$$\int_{\mathbb{R}^3} |\nabla v|^2 dx = \int_{A_1} dx_1 dx_2 dx_3 = 4\pi^2. \tag{3.2}$$

Moreover, for $\varepsilon \in (0, \varepsilon_0)$ and $\lambda \geq 1$ we define

$$u_{\varepsilon,\lambda}(x) = \varepsilon^2 \lambda v\left(\frac{x}{\lambda}\right).$$

We also choose ε and λ such that

$$\varepsilon\lambda \leq 1, \tag{3.3}$$

so that $0 \leq u_{\varepsilon,\lambda} \leq \varepsilon < \varepsilon_0$ in $\mathbb{R}^3$.

Then we have

$$E_{\sigma_\lambda, q}(u_{\varepsilon,\lambda}, 0) = \int_{\mathbb{R}^3} \left[\frac{1}{2}|\nabla u_{\varepsilon,\lambda}|^2 + \frac{l^2}{r^2}\frac{u_{\varepsilon,\lambda}^2}{2} + W(u_{\varepsilon,\lambda})\right] dx + \frac{\sigma^2}{2K_q(u_{\varepsilon,\lambda})}$$

$$= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_{\varepsilon,\lambda}|^2 + \frac{l^2}{2}\int_{\mathbb{R}^3} \frac{u_{\varepsilon,\lambda}^2}{r^2} + \frac{m^2}{2}\int_{\mathbb{R}^3} u_{\varepsilon,\lambda}^2$$

$$+ \int_{\mathbb{R}^3} N(u_{\varepsilon,\lambda}) \, dx + \frac{\sigma^2}{2K_q(u_{\varepsilon,\lambda})}. \tag{3.4}$$

Now, observe that in A_λ we have

$$r \geq 2\lambda - \sqrt{\lambda^2 - x_3^2} \geq \lambda,$$

so that, thanks to (3.1), we can estimate

$$\int_{\mathbb{R}^3} \frac{u_{\varepsilon,\lambda}^2}{r^2} dx_1 dx_2 dx_3 = \varepsilon^4 \int_{A_\lambda} \frac{\left(\lambda - \lambda\sqrt{\left(\frac{r}{\lambda} - 2\right)^2 + \frac{x_3^2}{\lambda^2}}\right)^2}{r^2} dr dx_3$$

$$\leq \varepsilon^4 \int_{A_\lambda} \frac{\left(\lambda - \sqrt{(r - 2\lambda)^2 + x_3^2}\right)^2}{\lambda^2} dr dx_3$$

$$\leq \varepsilon^4 \int_{A_\lambda} \left(\frac{\lambda - \sqrt{(r - 2\lambda)^2 + x_3^2}}{\lambda}\right)^2 dr dx_3 \leq \varepsilon^4 |A_\lambda| = 4\pi^2 \varepsilon^4 \lambda^3.$$

$$\tag{3.5}$$

By the change of variables $y = x/\lambda$ we immediately get

$$\int\limits_{A_\lambda} |\nabla u_{\varepsilon,\lambda}|^2 dx = \varepsilon^4 \lambda^3 \int\limits_{A_1} |\nabla v|^2 dx,$$

$$\int\limits_{A_\lambda} (u_{\varepsilon,\lambda})^\vartheta dx = \varepsilon^{2\vartheta} \lambda^{\vartheta+3} \int\limits_{A_1} v^\vartheta dx \quad \forall \theta > 0.$$

Therefore, (3.2), (3.4), (3.5) and (W4) imply

$$E_{\sigma,q}(u_{\varepsilon,\lambda}, 0) \le 2\pi^2 \varepsilon^4 \lambda^3 + \frac{m^2 \pi^2}{3} \varepsilon^4 \lambda^5 + 2\pi^2 l^2 \varepsilon^4 \lambda^3$$

$$- D\varepsilon^{2\tau} \lambda^{\tau+3} \int\limits_{A_1} v^\tau dx + \frac{\sigma^2}{2K_q(u_{\varepsilon,\lambda})}. \qquad (3.6)$$

Now, let us note that

$$-\Delta \Phi_{u_{\varepsilon,\lambda}} = q u_{\varepsilon,\lambda}^2 (1 - q\Phi_{u_{\varepsilon,\lambda}}) \le q u_{\varepsilon,\lambda}^2,$$

so that, by the Comparison Principle, for every $x \in \mathbb{R}^3$ we have

$$\Phi_{u_{\varepsilon,\lambda}}(x) \le \frac{q}{4\pi} \int\limits_{\mathbb{R}^3} \frac{u_{\varepsilon,\lambda}^2(x-y)}{|y|} dy = \frac{q\varepsilon^4 \lambda^5}{4\pi} \int\limits_{\mathbb{R}^3} \frac{v^2(y)}{|x - \lambda y|} dy \le \frac{q}{2} \varepsilon^4 \lambda^4. \qquad (3.7)$$

Indeed:

$$\int\limits_{\mathbb{R}^3} \frac{v^2(y)}{|x - \lambda y|} dy \le \int\limits_{A_1} \frac{1}{|x - \lambda y|} dy = \frac{1}{\lambda^3} \int\limits_{A_{1/\lambda}} \frac{1}{|x - z|} dz$$

$$= \frac{1}{\lambda^3} \int\limits_{A_{1/\lambda} - x} \frac{1}{|z|} dz \le \frac{1}{\lambda^3} \int\limits_{B(0,1/\lambda)} \frac{1}{|z|} dz = \frac{2\pi}{\lambda},$$

and (3.7) follows.

As a consequence,

$$K_q(u_{\varepsilon,\lambda}) = \int\limits_{\mathbb{R}^3} u_{\varepsilon,\lambda}^2 \left(1 - q\Phi_{u_{\varepsilon,\lambda}}\right) dx \ge \int\limits_{\mathbb{R}^3} u_{\varepsilon,\lambda}^2 \left(1 - \frac{q^2}{2} \varepsilon^4 \lambda^4\right) dx$$

$$= \frac{2}{3}\pi^2 (1 - \frac{q^2}{2} \varepsilon^4 \lambda^4) \varepsilon^4 \lambda^5.$$

Hence, choosing

$$\varepsilon^4 \lambda^4 \le 1/q^2, \qquad (3.8)$$

(3.6) becomes

$$E_{\sigma,q}(u_{\varepsilon,\lambda}, 0) \le 2\pi^2 \varepsilon^4 \lambda^3 + \frac{m^2 \pi^2}{3} \varepsilon^4 \lambda^5 + 2\pi^2 l^2 \varepsilon^4 \lambda^3$$

$$- D\varepsilon^{2\tau} \lambda^{\tau+3} \int\limits_{A_1} v^\tau dx + \frac{3\sigma^2}{\pi^2 \varepsilon^4 \lambda^5}.$$

Now, take

$$\varepsilon^4 \lambda^5 = \frac{6\sigma}{m\pi^2}, \tag{3.9}$$

so that (3.8) implies

$$\lambda \geq \frac{6\sigma}{m\pi^2} q^2. \tag{3.10}$$

With this choice we find

$$E_{\sigma,q}(u_{\varepsilon,\lambda}, 0) \leq 12\frac{\sigma}{m}(1+l^2)\lambda^{-2} + 2m\sigma - E\lambda^{3-3\tau/2} + \frac{m\sigma}{2},$$

where we have set $E = D(6\sigma/m\pi^2)^{\tau/2} \int v^\tau$.

Let us show that we can find $\lambda \geq \max\{1, 3q^2\sigma/m\pi^2\}$ (and thus $\varepsilon \leq 1$) satisfying (3.3) and (3.8) such that

$$12\frac{\sigma}{m}(1+l^2)\lambda^{-2} + \frac{5}{2}m\sigma - E\lambda^{3-3\tau/2} \leq m\sigma,$$

that is

$$\frac{12}{m}(1+l^2) + \frac{3}{2}m\lambda^2 - F\lambda^{5-3\tau/2} \leq 0, \tag{3.11}$$

where $F = D(6/m\pi^2)^{\tau/2}\sigma^{\tau/2-1} \int v^\tau$. Also note that $5 - 3\tau/2 < 2$, since $\tau > 2$.

Indeed, we choose

$$\lambda \geq \frac{\sqrt{8(1+l^2)}}{m}, \tag{3.12}$$

so that we can estimate the left hand side of (3.11) with

$$\frac{12}{m}(1+l^2) + \frac{3}{2}m\lambda^2 - F\lambda^{5-3\tau/2} \leq 3m\lambda^2 - F\lambda^{5-3\tau/2},$$

and the last quantity is non positive as soon as

$$\lambda \leq \left(\frac{F}{3m}\right)^{2/3(\tau-2)}. \tag{3.13}$$

Summing up, from (3.3), (3.8), (3.9), (3.10), (3.12) and (3.13), we are led to solve the following set of conditions:

$$\frac{6\sigma}{m\pi^2} \leq \lambda \tag{3.14}$$

$$\frac{6\sigma}{m\pi^2} q^2 \leq \lambda \tag{3.15}$$

$$\frac{\sqrt{8(1+l^2)}}{m} \leq \lambda \tag{3.16}$$

$$\lambda \leq \left(\frac{F}{3m}\right)^{2/3(\tau-2)}. \tag{3.17}$$

Now, if $q \leq 1$, (3.14) implies (3.15). Then, choose σ such that

$$\frac{6\sigma}{m\pi^2} \geq \frac{\sqrt{8(1+l^2)}}{m},$$

i.e.

$$\sigma \geq \frac{\pi^2\sqrt{8(1+l^2)}}{6}. \tag{3.18}$$

Hence, from (3.14) and (3.17), we must solve

$$\frac{6\sigma}{m\pi^2} \le \lambda \le \left(\frac{G}{3m}\right)^{2/3(\tau-2)} \sigma^{1/3},$$

where $F = G\sigma^{\frac{\tau-2}{2}}$, so that G is independent of σ.

Of course, such a choice of λ is possible provided that

$$\sigma \le \left(\frac{m^{\tau-4}\pi^{2\tau-6}}{6^{\tau-3}3} D \int_{\mathbb{R}^3} v^\tau dx\right)^{1/(\tau-2)}$$

$$\le \left(\frac{m^{\tau-4}\pi^{2\tau-6}}{6^{\tau-3}3} D \int_{\mathbb{R}^3} v^2 dx\right)^{1/(\tau-2)} \tag{3.19}$$

In conclusion, (3.18), (3.19) and (3.2) imply

$$\frac{\pi^2\sqrt{8(1+l^2)}}{6} \le cD^{1/(\tau-2)} m^{(\tau-4)/(\tau-2)},$$

which is true by (W4).

On the other hand, if $q > 1$, proceeding as above, we find a suitable λ provided that

$$\frac{\pi^2\sqrt{8(1+l^2)}}{6} \le cD^{1/(\tau-2)} m^{(\tau-4)/(\tau-2)} \frac{1}{q^3}.$$

In any case, the lemma holds. $\qquad\square$

As a consequence, we can prove the following

Lemma 3.6 *There exists $c > 0$ and a minimizing sequence $U_n = (u_n, \mathbf{A}_n) \subset V$ of $E_{\sigma_0, q}|_V$ such that*

$$\int_{\mathbb{R}^3} (|u_n|^\ell + |u_n|^p) dx \ge c > 0 \quad \text{for every} \quad n \in \mathbb{N}.$$

Proof By Lemma 3.5 we know that there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$E_{\sigma_0, q}(u_n, \mathbf{A}_n) \le m\sigma_0 - \delta,$$

which implies in particular that

$$\frac{m^2}{2} \int_{\mathbb{R}^3} u_n^2 dx + \int_{\mathbb{R}^3} N(u_n)\, dx + \frac{\sigma_0^2}{2\int_{\mathbb{R}^3} u_n^2 dx} \le m\sigma_0 - \delta.$$

Thus

$$\int_{\mathbb{R}^3} N(u_n)\, dx \le m\sigma_0 - \delta - \left(\frac{m^2}{2}\int_{\mathbb{R}^3} u_n^2 dx + \frac{\sigma^2}{2\int_{\mathbb{R}^3} u_n^2 dx}\right) \le -\delta,$$

since $a/(2b) + b/(2a) \ge 1$ for any $a, b > 0$. Then

$$\left|\int_{\mathbb{R}^3} N(u_n)\, dx\right| \ge \delta \quad \text{for all } n \ge n_0,$$

and (W2) imply the claim, up to a relabelling of the sequence. $\qquad\square$

By Lemma 3.1 we know that any minimizing sequence $U_n := (u_n, \mathbf{A}_n) \subset V$ of $E_{\sigma_0,q}|_V$ weakly converges (up to a subsequence). Observe that $E_{\sigma_0,q}$ is invariant by translations along the z-axis, namely for every $U \in V$ and $L \in \mathbb{R}$ we have

$$E_{\sigma_0,q}(T_L U) = E_{\sigma_0,q}(U),$$

where

$$T_L(U)(x, y, z) = U(x, y, z + L). \tag{3.20}$$

As a consequence of this invariance, we have that $(u_n, \mathbf{A}_n)$ does not contain in general a strongly convergent subsequence. To overcome this difficulty, we will show that there exists a minimizing sequence $(u_n, \mathbf{A}_n)$ of $E_{\sigma_0,q}|_V$ which, up to translations along the z-direction, weakly converges to a non–trivial limit $(u_0, \mathbf{A}_0)$. Eventually, we will show that $(u_0, \mathbf{A}_0)$ is a critical point of $E_{\sigma,q}$ for a suitable $\sigma > 0$.

In order to proceed with this strategy, we start proving the following weak compactness result, whose proof is an adaptation of [5, Proposition 22], but whose statement is more general:

Proposition 3.7 *There exists a Palais–Smale sequence $U_n = (u_n, \mathbf{A}_n)$ of $E_{\sigma_0,q}$ which weakly converges to $(u_0, \mathbf{A}_0)$, $u_0 \geq 0$ and $u_0 \neq 0$.*

Proof By Proposition 3.2, we know that there exists a minimizing sequence $U_n = (u_n, \mathbf{A}_n)$ of $E_{\sigma_0,q}|_V$, with $u_n \geq 0$ and which is also a Palais–Smale sequence for $E_{\sigma_0,q}$. Moreover, by Lemma 3.6, we know that there exists $c > 0$ such that

$$\|u_n\|_{L^\ell}^\ell + \|u_n\|_{L^p}^p \geq c > 0 \quad \text{for } n \text{ large.} \tag{3.21}$$

By Lemma 3.1 the sequence $\{U_n\}$ is bounded in $\hat{H}^1 \times \left(\mathcal{D}^1\right)^3$, so we can assume that it weakly converges. However the weak limit could be trivial. We will show that there is a sequence of integers j_n such that $V_n := T_{j_n} U_n \rightharpoonup U_0 = (u_0, \mathbf{A}_0)$ in $H^1 \times \left(\mathcal{D}^1\right)^3$, with $u_0 \neq 0$, see (3.20).

For any integer j we set

$$\Omega_j = \{(x_1, x_2, x_3) : j \leq x_3 < j + 1\}.$$

In the following we denote by c various positive absolute constants which may vary also from line to line. We have for all n,

$$
\begin{aligned}
\|u_n\|_{L^\ell}^\ell &= \sum_j \left(\int_{\Omega_j} |u_n|^\ell dx\right)^{1/\ell} \left(\int_{\Omega_j} |u_n|^\ell dx\right)^{\frac{\ell-1}{\ell}} \\
&\leq \sup_j \|u_n\|_{L^\ell(\Omega_j)} \sum_j \left(\int_{\Omega_j} |u_n|^\ell dx\right)^{\frac{\ell-1}{\ell}} \\
&\leq c \sup_j \|u_n\|_{L^\ell(\Omega_j)} \sum_j \|u_n\|_{H^1(\Omega_j)}^{\ell-1} \\
&= c \sup_j \|u_n\|_{L^\ell(\Omega_j)} \|u_n\|_{H^1(\mathbb{R}^3)}^{\ell-1} \leq \text{(since } \|u_n\|_{H^1(\mathbb{R}^3)} \text{ is bounded)} \\
&\leq c \sup_j \|u_n\|_{L^\ell(\Omega_j)} \quad \text{for all } n \geq 1.
\end{aligned}
\tag{3.22}
$$

In the same way we get

$$\|u_n\|_{L^p}^p \leq c \sup_j \|u_n\|_{L^p(\Omega_j)} \quad \text{for all } n \geq 1. \tag{3.23}$$

Then, by (3.21), (3.22) and (3.23) it immediately follows that, for n large, we can choose an integer j_n such that

$$\|u_n\|_{L^\ell(\Omega_{j_n})} + \|u_n\|_{L^p(\Omega_{j_n})} \geq c > 0. \tag{3.24}$$

Now set

$$(u_n', \mathbf{A}_n') = U_n'(x_1, x_2, x_3) = T_{j_n}(U_n) = U_n(x_1, x_2, x_3 + j_n).$$

Since $(U_n')_n$ is again a minimizing sequence for $E_{\sigma_0, q}|_V$, by Lemma 3.1 the sequence $\{u_n'\}$ is bounded in $\hat{H}^1(\mathbb{R}^3)$; then (up to a subsequence) it weakly converges to $u_0 \in \hat{H}^1(\mathbb{R}^3)$. Clearly $u_0 \geq 0$, since $u_n' \geq 0$. We want to show that $u_0 \neq 0$. Now, let $\varphi = \varphi(x_3)$ be a nonnegative, C^∞-function whose value is 1 for $0 < x_3 < 1$ and 0 for $|x_3| > 2$. Then, the sequence $\varphi u_n'$ is bounded in $H_0^1(\mathbb{R}^2 \times (-2, 2))$, and moreover $\varphi u_n'$ has cylindrical symmetry. Then, using the compactness result of Esteban–Lions [17], we have that, up to a subsequence,

$$\varphi u_n' \to \varphi u_0 \text{ in } L^\ell(\mathbb{R}^2 \times (-2, 2)), \text{ in } L^p(\mathbb{R}^2 \times (-2, 2)) \text{ and a.e. in } \mathbb{R}^2 \times (-2, 2). \tag{3.25}$$

Moreover for $r = p, \ell$ we clearly have

$$\|\varphi u_n'\|_{L^r(\mathbb{R}^2 \times (-2,2))} \geq \|u_n'\|_{L^r(\Omega_0)} = \|u_n\|_{L^r(\Omega_{j_n})}. \tag{3.26}$$

Then by (3.25), (3.26) and (3.24) we have

$$\|\varphi u_0\|_{L^\ell(\mathbb{R}^2 \times (-2,2))} + \|\varphi u_0\|_{L^p(\mathbb{R}^2 \times (-2,2))} \geq c > 0.$$

Thus we have that $u_0 \neq 0$, as claimed. $\qquad \square$

In order to approach the conclusion, we need

Proposition 3.8 *For every $q > 0$ there exists $\sigma > 0$ such that $E_{\sigma,q}$ has a critical point $(u_0, \mathbf{A}_0)$, $u_0 \neq 0$, $u_0 \geq 0$.*

Proof By Proposition 3.7, there exists a sequence $U_n = (u_n, \mathbf{A}_n)$ in V, with $u_n \geq 0$ and such that

$$E_{\sigma_0, q}'(u_n, \mathbf{A}_n) \to 0 \tag{3.27}$$

and

$$(u_n, \mathbf{A}_n) \rightharpoonup (u_0, \mathbf{A}_0), \, u_0 \geq 0, \, u_0 \neq 0.$$

We now show that there exists $\sigma > 0$ such that $U_0 = (u_0, \mathbf{A}_0)$ is a critical point of $E_{\sigma,q}$. By (3.27), in particular we get that

$$dE_{\sigma_0, q}(U_n)[w, 0] \to 0 \text{ and } dE_{\sigma_0, q}(U_n)[0, \mathbf{w}] \to 0, \text{ for any } (w, \mathbf{w}) \in \hat{H}^1 \times \left(C_C^\infty\right)^3.$$

Then for any $w \in \hat{H}^1$ and $\mathbf{w} \in \left(C_C^\infty\right)^3$ we have

$$\partial_u I(U_n)[w] + \partial_u \left(\frac{\sigma_0^2}{2K_q(u_n)}\right)[w] \to 0 \tag{3.28}$$

and

$$\partial_{\mathbf{A}} I(U_n)[\mathbf{w}] \to 0, \tag{3.29}$$

where ∂_u and $\partial_{\mathbf{A}}$ denote the partial derivatives of I with respect to u and $\mathbf{A}$, respectively. So from (3.28) we get for any $w \in \hat{H}^1$,

$$\partial_u I(U_n)[w] - \frac{\sigma_0^2 K_q'(u_n)}{2\left(K_q(u_n)\right)^2}[w] \to 0,$$

which can be written as follows:

$$\partial_u I(U_n)[w] - \frac{\omega_n^2 K_q'(u_n)}{2}[w] \to 0, \tag{3.30}$$

where we have set

$$\omega_n = \frac{\sigma_0}{K_q(u_n)}.$$

By Lemma 3.3 we have that (up to a subsequence)

$$\omega_n \to \omega_0 > 0.$$

Then by (3.30) we get for any $w \in \hat{H}^1$

$$\partial_u I(U_n)[w] - \frac{\omega_0^2 K_q'(u_n)}{2}[w] \to 0. \tag{3.31}$$

Now, let Φ_n be the solution in $\mathcal{D}^1$ of the equation

$$-\Delta\Phi_n + q^2 u_n^2 \Phi_n = q u_n^2. \tag{3.32}$$

Since $\{u_n\}$ is bounded in H^1 and since Φ_n solves (3.32), by (2.20) we have that $\{\Phi_n\}$ is bounded in $\mathcal{D}^1$ and, checking with test functions in $C_C^\infty(\mathbb{R}^3)$, it is easy to see that (up to a subsequence) its weak limit Φ_0 is a weak solution of

$$-\Delta\Phi_0 + q^2 u_0^2 \Phi_0 = q u_0^2. \tag{3.33}$$

Moreover, by Lemma 2.11, we have

$$K_q'(u_n)[w] = 2\int_{\mathbb{R}^3} u_n w(1 - q\Phi_n)^2 dx \quad \text{and} \quad K_q'(u_0)[w] = 2\int_{\mathbb{R}^3} u_0 w(1 - q\Phi_0)^2 dx \tag{3.34}$$

for every $w \in \hat{H}^1$.

We claim that

$$K_q'(u_n)[w] \to K_q'(u_0)[w] \quad \text{for any} \quad w \in \hat{H}^1. \tag{3.35}$$

Indeed, by (2.11), for any $w \in \hat{H}^1$ and every $\varepsilon > 0$, there exists $w_\varepsilon \in C_C^\infty \cap \hat{H}^1$ such that $\|w - w_\varepsilon\|_{\hat{H}^1} < \varepsilon$. Then,

$$K_q'(u_n)[w] - K_q'(u_0)[w] = K_q'(u_n)[w - w_\varepsilon]$$
$$+ [K_q'(u_n) - K_q'(u_0)][w_\varepsilon] - K_q'(u_0)[w_\varepsilon - w].$$

But the sequence of operators $(K'(u_n))_n$ is bounded in $(\hat{H}^1)'$, while $[K_q'(u_n) - K_q'(u_0)][w_\varepsilon] \to 0$ by the Rellich Theorem. The claim follows.

Similar estimates show that for any $w \in \hat{H}^1$

$$\partial_u I(U_n)[w] \to \partial_u I(U_0)[w]. \tag{3.36}$$

Then, passing to the limit in (3.31), by (3.35) and (3.36), we get

$$\partial_u I(U_0)[w] - \frac{\omega_0^2 K_q'(u_0)}{2}[w] = 0 \quad \text{for any} \quad w \in \hat{H}^1. \tag{3.37}$$

On the other hand, similar arguments show that we can pass to the limit also in $\partial_{\mathbf{A}} I(U_n)[\mathbf{w}]$ and have

$$\partial_{\mathbf{A}} I(U_n)[\mathbf{w}] \to \partial_{\mathbf{A}} I(U_0)[\mathbf{w}] \text{ for all } \mathbf{w} \in \left(C_C^\infty\right)^3. \tag{3.38}$$

From (3.29) and (3.38) we get

$$\partial_{\mathbf{A}} I(U_0)[\mathbf{w}] = 0 \text{ for all } \mathbf{w} \in \left(C_C^\infty\right)^3,$$

and, by density, for any $\mathbf{w} \in \left(\mathcal{D}^1\right)^3$. From (3.37) we thus deduce that $U_0 = (u_0, \mathbf{A}_0)$ is a critical point of $E_{\sigma,q}$ with $\sigma = \omega_0 K_q(u_0) > 0$. □

Now we are ready to prove the main existence Theorem 1.3.

Proof of Theorem 1.3 The first part of Theorem 1.3 immediately follows from Propositions 2.12, 3.8 and Theorem 2.6. In fact, if the couple $(u_0, \mathbf{A}_0)$ is like in Proposition 3.8, by Proposition 2.12 and Theorem 2.6 we deduce that $(u_0, \omega_0, \phi_0, \mathbf{A}_0)$ with $\omega_0 = \frac{\sigma}{K_q(u_0)}$, $\phi_0 = Z_{\omega_0}(u_0)$, solves (1.6)–(1.8).

Now assume $q = 0$, then, by (1.7) and (1.8), we easily deduce that $\phi_0 = 0$ and $\mathbf{A}_0 = 0$. Finally assume that $q > 0$. Then, since $\omega_0 > 0$, by (1.7) we deduce that $\phi_0 \neq 0$. Moreover by (1.8) we deduce that $\mathbf{A}_0 \neq 0$ if and only if $l \neq 0$. □

4 Solutions with full probability

Throughout this section we are concerned with a different approach to system (1.6)–(1.8): namely, we look for solutions having full probability and we prove Proposition 1.10. From a physical point of view such solutions are the most relevant ones, and in general they cannot be obtained from the solutions found in Theorem 1.3 by a rescaling argument, unless some homogeneity in the potential is given. However, this is not the case if $N \neq 0$.

Therefore, we will work in the new manifold $\tilde{V} := V \cap \mathcal{S}$, where

$$\mathcal{S} = \left\{(u, \mathbf{A}) \in V : \int_{\mathbb{R}^3} u^2 dx = 1\right\}.$$

We follow the lines of the previous part of the paper, and for this reason we will be sketchy, though some differences will appear. For example, we begin with the following

Proposition 4.1 *For any $\sigma, q \geq 0$ there exists a minimizing sequence $U_n = (u_n, \mathbf{A}_n)$ of $E_{\sigma,q}|_{\tilde{V}}$, with $u_n \geq 0$, and a sequence $(\mu_n)_n \in \mathbb{R}$, such that*

$$E'_{\sigma,q}(u_n, \mathbf{A}_n)(v, \mathbf{B}) - \mu_n \int_{\mathbb{R}^3} u_n v \, dx \to 0 \quad \forall (v, \mathbf{B}) \in \tilde{V}.$$

Moreover, $(\mu_n)_n$ converges to some $\mu \in \mathbb{R}$ as $n \to \infty$.

Proof Let $(u_n, \mathbf{A}_n) \subset V$ be a minimizing sequence for $E_{\sigma,q}|_{\tilde{V}}$. Working with $u_n \geq 0$, or replacing u_n with $|u_n|$ if necessary, we still have a minimizing sequence (see (2.31)). By

Ekeland's Variational Principle we can also assume that $(u_n, \mathbf{A}_n)$ is a Palais–Smale sequence for $E_{\sigma,q}|_{\tilde{V}}$, namely we can assume that

$$E'_{\sigma,q}|_{\tilde{V}}(u_n, \mathbf{A}_n) \to 0,$$

i.e. there exists a sequence $(\mu_n)_n$ in $\mathbb{R}$ with

$$E'_{\sigma,q}(u_n, \mathbf{A}_n)(v, \mathbf{B}) - \mu_n \int_{\mathbb{R}^3} u_n v dx \to 0, \ \forall \, v \in \hat{H}^1_\sharp, \ \forall \, \mathbf{B} \in \mathcal{A}. \tag{4.1}$$

Taking $(u_n, \mathbf{A}_n)$ as a test function and using $\int_{\mathbb{R}^3} u_n^2 dx = 1$ for all $n \in \mathbb{N}$, we get

$$E'_{\sigma,q}(u_n, \mathbf{A}_n)(u_n, \mathbf{A}_n) - \mu_n \int_{\mathbb{R}^3} u_n^2 dx = E'_{\sigma,q}(u_n, \mathbf{A}_n)(u_n, \mathbf{A}_n) - \mu_n \to 0. \tag{4.2}$$

From (4.2) we get

$$\begin{aligned}
\mu_n &= E'_{\sigma,q}(u_n, \mathbf{A}_n)(u_n, \mathbf{A}_n) + o(1) \\
&= \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} |l\nabla\theta - q\mathbf{A}_n|^2 u_n^2 dx + \int_{\mathbb{R}^3} W'(u_n)u_n dx \\
&\quad + \int_{\mathbb{R}^3} |\nabla \times \mathbf{A}_n|^2 dx + q \int_{\mathbb{R}^3} u_n^2 |\mathbf{A}_n|^2 dx + \sigma K'_q(u_n)u_n dx + o(1),
\end{aligned} \tag{4.3}$$

where $o(1) \to 0$ as $n \to \infty$. Thus, since all the terms in the right-hand side of (4.3) are bounded, as already shown for Lemma 3.1, we get that also $(\mu_n)_n$ is bounded; hence, there exists $\mu \in \mathbb{R}$ such that, up to a subsequence, $\mu_n \to \mu$ as $n \to \infty$. □

Now we restate Proposition 3.7 which still holds in this case thanks to Proposition 4.1, hence we get

Proposition 4.2 *There exists a Palais–Smale sequence $U_n = (u_n, \mathbf{A}_n)$ of $E_{\sigma_0,q}$ which weakly converges to $(u_0, \mathbf{A}_0)$, $u_0 \geq 0$ and $u_0 \neq 0$.*

In order to prove Proposition 1.10 we should just notice that the analogue of Proposition 3.8 still holds using Proposition 4.1 and Proposition 4.2. Hence, we just restate the result of Proposition 3.8 as follows:

Proposition 4.3 *For every $q > 0$ there exists $\sigma > 0$ such that $E_{\sigma,q}$ has a critical point $(u_0, \mathbf{A}_0)$, $u_0 \neq 0$, $u_0 \geq 0$.*

Finally, we conclude with the

Proof of Proposition 1.10 It is a natural consequence of what already proved, exactly as done for the proof of Theorem 1.3 in the previous section. Namely, since Proposition 4.3 holds by Proposition 4.1 and Proposition 4.2, we can conclude that our claim is true thanks to Propositions 4.3, 2.12 and Theorem 2.6.

Now, suppose that $\omega^2 \leq m^2$ and $N'(s)s \geq 0$. Passing to the limit as $n \to \infty$ in (4.1) with $v = u_0$ and $\mathbf{B} = \mathbf{0}$, as in the proof of Proposition 3.8, we get

$$\int_{\mathbb{R}^3} [|\nabla u_0|^2 + |l\nabla\theta - q\mathbf{A}_0|^2 u_0^2 - (\omega - q\phi_{u_0})^2 u_0^2 + W'(u_0)u_0] dx = \mu \int_{\mathbb{R}^3} u_0^2 dx,$$

which can be written as

$$\int_{\mathbb{R}^3} [|\nabla u_0|^2 + |l\nabla\theta - q\mathbf{A}|^2 u_0^2 + (m^2 - \omega^2)u_0^2 - (q\phi - 2\omega)u_0^2 q\phi_{u_0}$$

$$+ N'(u_0)u_0]dx = \mu \int_{\mathbb{R}^3} u_0^2 dx.$$

Thanks to (2.25), (2.26) and to the hypotheses under consideration, we get $\mu > 0$, so that the effective mass (see Definition 1.11) is strictly less than the original mass. $\qquad\square$

5 Non-existence of standing solutions

In this section we shall prove Theorem 1.7. To this purpose, we re-write the usual system using (1.9), so that we deal with

$$-\Delta u + \left[|l\nabla\theta - q\mathbf{A}|^2 + m^2 - (\omega - q\phi)^2\right]u + N'(u) = 0, \tag{5.1}$$

$$-\Delta\phi = q(\omega - q\phi)u^2, \tag{5.2}$$

$$\nabla \times (\nabla \times \mathbf{A}) = q(l\nabla\theta - q\mathbf{A})u^2. \tag{5.3}$$

Proof of Theorem 1.7 If $\mathbf{A} = \mathbf{0}$, in [12] a variational identity for solutions of (5.1) was given. However, the same identity holds when $\mathbf{A} \neq \mathbf{0}$, and it reads as follows:

$$0 = -\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\nabla\phi|^2 dx - 3\Omega \int_{\mathbb{R}^3} u^2 dx$$

$$- 3q \int_{\mathbb{R}^3} (2\omega - q\phi)\phi u^2 dx + 6 \int_{\mathbb{R}^3} F(u)dx, \tag{5.4}$$

where we have set $\Omega = m^2 - \omega^2$, $F(s) = \int_0^s f(t)\,dt$ and

$$f(u) = -|l\nabla\theta - q\mathbf{A}|^2 u - N'(u).$$

Since ϕ solves (5.2), we have

$$\int_{\mathbb{R}^3} |\nabla\phi|^2 dx = q \int_{\mathbb{R}^3} (\omega - q\phi)u^2\phi dx; \tag{5.5}$$

substituting (5.5) into (5.4) and computing $F(u)$ we get

$$0 = -\int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} \left[3\Omega + 5q\omega\phi - 2q^2\phi^2 + 3|l\nabla\theta - q\mathbf{A}|^2\right]u^2 dx$$

$$- 6 \int_{\mathbb{R}^3} N(u)\,dx. \tag{5.6}$$

By (2.25) and (2.26), if $N \geq 0$ and $\omega^2 < m^2$, we get $u \equiv 0$.

Moreover, since u solves (5.1), we have

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |l\nabla\theta - q\mathbf{A}|^2 u^2 dx + m^2 \int_{\mathbb{R}^3} u^2 dx - \int_{\mathbb{R}^3} (\omega - q\phi)^2 u^2 dx + \int_{\mathbb{R}^3} N'(u)u dx = 0;$$

(5.7)

substituting the expression $\int_{\mathbb{R}^3} |\nabla u|^2 dx$ taken from (5.7) into (5.6), we obtain

$$0 = q \int_{\mathbb{R}^3} (q\phi - 3\omega)u^2\phi \, dx - 2 \int_{\mathbb{R}^3} |l\nabla\theta - q\mathbf{A}|^2 u^2 dx$$

$$+ 2(\omega^2 - m^2) \int_{\mathbb{R}^3} u^2 dx + \int_{\mathbb{R}^3} [N'(u)u - 6N(u)]dx.$$

(5.8)

Thanks to (2.25) and (2.26), all the terms in (5.8) are non-positive if $\omega^2 < m^2$, $N'(s)s - 6N(s) \leq 0$; hence $u \equiv 0$.

Finally, when $N'(s)s \geq 2N(s)$, we proceed as follows: from (5.7) we get

$$\Omega \int_{\mathbb{R}^3} u^2 dx = - \int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} |l\theta - q\mathbf{A}|^2 u^2 dx$$

$$- 2q\omega \int_{\mathbb{R}^3} u^2\phi dx + q^2 \int_{\mathbb{R}^3} u^2\phi^2 dx - \int_{\mathbb{R}^3} N'(u)u \, dx.$$

(5.9)

Substituting (5.9) into (5.6) we get

$$0 = 2 \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} qu^2\phi(\omega - q\phi)dx + \int_{\mathbb{R}^3} [3N'(u)u - 6N(u)] \, dx.$$

(5.10)

Analogously, now all the coefficients are non-negative, and thus $u \equiv 0$. ☐

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