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Calculus of Variations

A Penrose inequality for graphs over Kottler space

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Abstract In this work, we prove an optimal Penrose inequality for asymptotically locally hyperbolic manifolds which can be realized as graphs over Kottler space. Such inequality relies heavily on an optimal weighted Alexandrov–Fenchel inequality for the mean convex star-shaped hypersurfaces in Kottler space.

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1 Introduction

The famous Penrose inequality (conjecture) in general relativity, as a refinement of the positive mass theorem [40,43], states that the total mass of a spacetime is no less than the mass of

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its black holes which are measured by the area of its event horizons. When the cosmological constant $\Lambda = 0$, its Riemannian version reads that an asymptotically flat manifold $(\mathcal{M}^n, g)$ with an outermost minimal boundary Σ (a horizon) has the ADM mass

$$m_{ADM} \geq \frac{1}{2}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}, \tag{1.1}$$

provided that the dominant condition $R_g \geq 0$ holds. Here R_g is the scalar curvature of $(\mathcal{M}^n, g)$, $|\Sigma|$ is the area of Σ and ω_{n-1} is the area of the unit $(n-1)$-sphere. Moreover, equality holds if and only if $(\mathcal{M}, g)$ is isometric to the exterior Schwarzschild solution. For the case $n = 3$, (1.1) was proved by Huisken and Ilmanen [28] using the inverse mean curvature flow and by Bray [3] using a conformal flow. Later, Bray's proof was generalized by Bray and Lee [7] to the case $n \leq 7$. For related results and further development, see the excellent surveys [4,35] and also [5,6,19,25,26]. Recently Lam [30] gave an elegant proof of (1.1) for asymptotically flat graphs over $\mathbb{R}^n$ for all dimensions by using Alexandrov- Fenchel inequalities (see [39]). His proof was later extended in [14,29,36]. Very recently, a general Penrose inequality for a higher order mass was conjectured in [21], which is true for the graph cases [21,32] and conformally flat cases [22].

In recent years, there has been great interest to extend the previous results to a spacetime with a negative cosmological constant $\Lambda < 0$. In the time symmetric case, $(\mathcal{M}^n, g)$ is now an asymptotically hyperbolic manifold with an outermost minimal boundary Σ. For the asymptotically hyperbolic manifolds, a mass-like invariant, which generalizes the ADM mass, was introduced by Chruściel et al. [10,11,27]. See also an earlier contribution by Wang [41] for the special case of conformally compact manifolds. For this mass $m^{\mathbb{H}}$ the corresponding Penrose conjecture is

$$m^{\mathbb{H}} \geq \frac{1}{2}\left\{\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n}{n-1}}\right\}, \tag{1.2}$$

provided that the dominant energy condition $R_g \geq -n(n-1)$ holds. This is a very difficult problem. Neves [38] showed that the powerful inverse mean curvature flow of Huisken and Ilmanen [28] alone could not work for proving (1.2). For the special case that the asymptotically hyperbolic manifold can be represented by a graph over the hyperbolic space $\mathbb{H}^n$, Dl-Gicquaud and Sakovich [13] and de Lima and Girão [16] proved this conjecture with a help of a sharp Alexandrov–Fenchel inequality for a weighted mean curvature integral in $\mathbb{H}^n$. More precisely, in [13], several suboptimal inequalities similar to the Alexandrov–Fenchel inequality in the hyperbolic space are given, the sharp inequality (the one that implies the Penrose inequality for hyperbolic graphs) is settled in [16]. Recently there have been many contributions in establishing Alexandrov–Fenchel inequalities in $\mathbb{H}^n$, see [9,23,24,31,42]. Penrose inequalities for the Gauss–Bonnet–Chern mass have been studied in [21,24].

In this paper we are interested in studying asymptotically locally hyperbolic (ALH) manifolds. Let us first introduce the locally hyperbolic metrics. Fix $\kappa = \pm 1, 0$ and suppose $(N^{n-1}, \hat{g})$ is a closed space form of sectional curvature κ. Consider the product manifold $P_\kappa = I_\kappa \times N$, where $I_{-1} = (1, +\infty)$ and $I_0 = I_1 = (0, \infty)$ endowed with the warped product metric

$$b_\kappa = \frac{d\rho^2}{V_\kappa^2(\rho)} + \rho^2 \hat{g}, \quad \rho \in I_\kappa, \quad \text{and} \quad V_\kappa(\rho) = \sqrt{\rho^2 + \kappa}. \tag{1.3}$$

One can easily check that the sectional curvature of (P_κ, b_κ) equals to -1 and thus it is called *locally hyperbolic*. Note that in the case $\kappa = 1$ and $(N, \hat{g})$ is a round sphere, (P_κ, b_κ) is exactly the hyperbolic space. Since there are a lot of work on the case that $\kappa = 1$ and $(N, \hat{g})$ is a round sphere, see the work mentioned above, we will in principle focus on the remaining case, the locally hyperbolic case. Namely, $\kappa = -1, 0$ or $\kappa = 1$ and N is a space form other than the standard sphere. In this case, the mass defined by (1.6) below is a geometric invariant. (See Section 3 in [10]). In order to define this mass, we recall from [10] the following definition of ALH manifolds.

Definition 1.1 A Riemannian manifold $(\mathcal{M}^n, g)$ is called asymptotically locally hyperbolic (ALH) if there exists a compact subset K and a diffeomorphism at infinity $\Phi : \mathcal{M} \setminus K \to N \times (\rho_0, +\infty)$, where $\rho_0 > 1$ such that

$$\|(\Phi^{-1})^* g - b_\kappa\|_{b_\kappa} + \|\nabla^{b_\kappa}\left((\Phi^{-1})^* g\right)\|_{b_\kappa} = O(\rho^{-\tau}), \quad \tau > \frac{n}{2}, \tag{1.4}$$

and

$$\int_{\mathcal{M}} V_\kappa \; |R_g + n(n-1)| dV_g < \infty. \tag{1.5}$$

Then a mass type invariant of $(\mathcal{M}^n, g)$ with respect to Φ, which we call ALH mass, can be defined by

$$m_{(\mathcal{M},g)} = c_n \lim_{\rho\to\infty} \int_{N_\rho} \Big(V_\kappa (\mathrm{div}^{b_\kappa} e - d\,\mathrm{tr}^{b_\kappa} e) + (\mathrm{tr}^{b_\kappa} e) dV_\kappa - e(\nabla^{b_\kappa} V_\kappa, \cdot)\Big)\, \nu d\mu, \tag{1.6}$$

where $e := (\Phi^{-1})^* g - b_\kappa$, $N_\rho = \{\rho\} \times N$, ν is the outer normal of N_ρ induced by b_κ and $d\mu$ is the area element with respect to the induced metric on N_ρ, ϑ_{n-1} is the area of N

$$\vartheta_{n-1} = |N| \quad \text{and} \quad c_n = \frac{1}{2(n-1)\vartheta_{n-1}}.$$

For this mass, there is a corresponding Penrose conjecture.

Conjecture 1 *Let $(\mathcal{M}, g)$ be an ALH manifold with an outermost minimal horizon Σ. Then the mass*

$$m_{(\mathcal{M},g)} \geq \frac{1}{2}\left(\left(\frac{|\Sigma|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}} + \kappa \left(\frac{|\Sigma|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}}\right),$$

provided that $\mathcal{M}$ satisfies the dominant condition

$$R_g + n(n-1) \geq 0. \tag{1.7}$$

Moreover, equality holds if and only if $(\mathcal{M}, g)$ is a Kottler space.

The Kottler space, or Kottler–Schwarzschild space, is an analogue of the Schwarzschild space in the context of asymptotically locally hyperbolic manifolds which is introduced as follows. We consider the metric

$$g_{\kappa,m} = \frac{d\rho^2}{V_{\kappa,m}^2(\rho)} + \rho^2 \hat{g}, \quad V_{\kappa,m} = \sqrt{\rho^2 + \kappa - \frac{2m}{\rho^{n-2}}}. \tag{1.8}$$

Let $\rho_{\kappa,m}$ be the largest positive root of $V_{\kappa,m}$. Then the triple

$$(P_{\kappa,m} = [\rho_{\kappa,m}, +\infty) \times N,\; g_{\kappa,m},\; V_{\kappa,m})$$

is a complete vacuum static data set with the negative cosmological constant $-n$ which satisfies

$$\bar{\Delta} V_{\kappa,m} g_{\kappa,m} - \bar{\nabla}^2 V_{\kappa,m} + V_{\kappa,m} Ric_{g_{\kappa,m}} = 0 \quad \text{and} \quad R_{g_{\kappa,m}} = -n(n-1). \tag{1.9}$$

We remark here that throughout the all paper, $\bar{\Delta}$ and $\bar{\nabla}$ denote the Laplacian and covariant derivative with respect to the metric $g_{\kappa,m}$.

Remark that in (1.8) if $\kappa \geq 0$, the parameter m is always positive; if $\kappa = -1$, the parameter m can be negative. In fact, m belongs to the following interval

$$m \in [m_c, +\infty) \quad \text{and} \quad m_c = -\frac{(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}}. \tag{1.10}$$

Comparing with the case of the asymptotically hyperbolic, this is a new and interesting situation. The corresponding positive mass theorem looks now like

Conjecture 2 *Let $(\mathcal{M}, g)$ be an ALH manifold ($\kappa = -1$ case without boundary). Then the mass*

$$m_{(\mathcal{M},g)} \geq m_c = -\frac{(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}},$$

provided that $\mathcal{M}$ satisfies the dominant condition (1.7).

These problems were first studied by Chruściel and Simon [12]. Recently, Lee and Neves [33,34] used the powerful inverse mean curvature flow to obtain a Penrose inequality for 3 dimensional conformally compact ALH manifolds if the mass $m \leq 0$. Roughly speaking, they managed to show that the inverse mean curvature flow of Huisken and Ilmanen does work for ALH with $\kappa = 0, -1$, though Neves [38] has previously showed that it alone does not work for the asymptotically hyperbolic manifolds, i.e., $\kappa = 1$. Very recently, de Lima and Girão [17] proved Conjecture 1 for a class of graphical ALH for all dimensions $n \geq 3$, in the range $m \in [0, \infty)$.

Motivated by these work and our previous wok on the Gauss–Bonnet–Chern mass, in this paper we want to show Conjecture 1 for a class of graphical ALH for all dimensions $n \geq 3$, in the full range

$$m \in [m_c, \infty) = \left[-\frac{(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}}, \infty \right).$$

In order to state our results, let us introduce the corresponding Kottler–Schwarzschild spacetime in general relativity

$$-V_{\kappa,m}^2 dt^2 + g_{\kappa,m}.$$

We consider its Riemannian version, namely $Q_{\kappa,m} = \mathbb{R} \times P_{\kappa,m}$ with the metric

$$\tilde{g}_{\kappa,m} = V_{\kappa,m}^2 dt^2 + g_{\kappa,m}. \tag{1.11}$$

It is well-known that $\tilde{g}_{\kappa,m}$ is an Einstein metric, i.e.

$$Ric_{\tilde{g}_{\kappa,m}} + n\tilde{g}_{\kappa,m} = 0,$$

which actually follows from (1.9). Now let m be any fixed number

$$m \in [m_c, \infty).$$

We identify $P_{\kappa,m}$ with the slice $\{0\} \times P_{\kappa,m} \subset Q_{\kappa,m}$ and consider a graph over $P_{\kappa,m}$ or over a subset $P_{\kappa,m}\backslash\Omega$ in $Q_{\kappa,m}$, where Ω is a compact smooth subset containing $\{0\} \times \partial P_{\kappa,m}$. A graph associated to a smooth function $f : P_{\kappa,m}\backslash\Omega \to \mathbb{R}$ is a manifold $\mathcal{M}^n$ with the induced metric from $(Q_{\kappa,m}, \tilde{g}_{\kappa,m})$, i.e.

$$g = V_{\kappa,m}^2(\rho)\bar{\nabla} f \otimes \bar{\nabla} f + g_{\kappa,m}. \tag{1.12}$$

Definition 1.2 We say $\mathcal{M}^n \subset Q_{\kappa,m}$ is an ALH graph over $P_{\kappa,m}\backslash\Omega$ (associated to a smooth function $f : P_{\kappa,m}\backslash\Omega \to \mathbb{R}$) if there exists a compact subset K and a diffeomorphism at infinity $\Phi : \mathcal{M} \setminus K \to N \times (\rho_0, +\infty) \subset P_{\kappa,m}\backslash\Omega$, where $\rho_0 > 1$ such that

$$\|(\Phi^{-1})^*g - g_{\kappa,m}\|_{g_{\kappa,m}} + \|\bar{\nabla}\left((\Phi^{-1})^*g\right)\|_{g_{\kappa,m}} = O(\rho^{-\tau}), \quad \tau > \frac{n}{2}, \tag{1.13}$$

or equivalently,

$$|V\bar{\nabla} f|_{g_{\kappa,m}} + |V\bar{\nabla}^2 f + \bar{\nabla} V\bar{\nabla} f|_{g_{\kappa,m}} = O(\rho^{-\frac{\tau}{2}}), \quad \tau > \frac{n}{2}, \tag{1.14}$$

and

$$\int_{\mathcal{M}} V_{\kappa,m}\, |R_g + n(n-1)| dV_g < \infty. \tag{1.15}$$

An ALH graph over $P_{\kappa,m}\backslash\Omega$ in $Q_{\kappa,m}$ must be an ALH manifold in the sense of Definition 1.1. Conversely, if a graph over $P_{\kappa,m}\backslash\Omega$ in $Q_{\kappa,m}$ is an ALH manifold, then it is also an ALH graph in the sense of Definition 1.2. In other words, for a graph over $P_{\kappa,m}\backslash\Omega$ in $Q_{\kappa,m}$, Definition 1.1 and Definition 1.2 are equivalent. For the proof see Appendix B.

We now state the main results of this paper.

Theorem 1.3 *Suppose $\mathcal{M} \subset Q_{\kappa,m}$ is an ALH graph over $P_{\kappa,m}$ with inner boundary Σ, associated to a function $f : P_{\kappa,m}\backslash\Omega \to \mathbb{R}$. Assume that Σ is in a level set of f and $|\bar{\nabla} f(x)| \to \infty$ as $x \to \Sigma$. Then we have*

$$m_{(\mathcal{M},g)} = m + c_n \int_{\mathcal{M}} \left\langle \frac{\partial}{\partial t}, \xi \right\rangle (R_g + n(n-1)) dV_g + c_n \int_{\Sigma} V_{\kappa,m} H d\mu, \tag{1.16}$$

where H is the mean curvature of Σ in $(P_{\kappa,m}, g_{\kappa,m})$ and ξ is the unit outer normal of $(\mathcal{M}, g)$ in $(Q_{\kappa,m}, \tilde{g}_{\kappa,m})$. Moreover, if in addition the dominant energy condition

$$R_g + n(n-1) \geq 0 \tag{1.17}$$

holds, we have

$$m_{(\mathcal{M},g)} \geq m + c_n \int_{\Sigma} V_{\kappa,m} H d\mu. \tag{1.18}$$

Remark 1.4 For any ALH graph over the whole $P_{\kappa,m}$, we have

$$m_{(\mathcal{M},g)} \geq m \geq m_c, \tag{1.19}$$

provided that the dominant energy condition $R_g + n(n-1) \geq 0$ holds, since in this case

$$m_{(\mathcal{M},g)} = m + c_n \int_{\mathcal{M}} \left\langle \frac{\partial}{\partial t}, \xi \right\rangle (R_g + n(n-1)) dV_g \geq m \geq m_c.$$

This can be viewed as a version of the positive mass theorem in this setting. See Conjecture 2.

Comparing with the work of [17], which considers graphs over the local hyperbolic space P_κ, our setting enables us to consider the negative mass range. In order to obtain a Penrose type inequality, we need to establish a Minkowski type inequality in the Kottler space. This motivates us to study geometric inequalities in the Kottler space. The corresponding Minkowski type inequality is proved in the following Theorem.

Theorem 1.5 *Let Σ be a compact embedded hypersurface which is star-shaped with positive mean curvature in $P_{\kappa,m}$, then we have*

$$\int_\Sigma V_{\kappa,m} H d\mu \geq (n-1)\vartheta_{n-1}\left(\left(\frac{|\Sigma|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}} - \left(\frac{|\partial P_{\kappa,m}|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}}\right) + (n-1)\kappa\vartheta_{n-1}\left(\left(\frac{|\Sigma|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}} - \left(\frac{|\partial P_{\kappa,m}|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}}\right), \tag{1.20}$$

where $\partial P_{\kappa,m} = \{\rho_{\kappa,m}\} \times N$. Equality holds if and only if Σ is a slice.

In this paper by star-shaped we mean that Σ can be represented as a graph over $\{\rho_{\kappa,m}\} \times N^{n-1}$ in $P_{\kappa,m}$.

When $m = 0$, i.e, $P_{\kappa,m} = P_\kappa$, which is a locally hyperbolic space, Theorem 1.5 was proved in [17]. When $m \neq 0$, $P_{\kappa,m}$ has no constant curvature. A similar inequality was first proved by Brendle–Hung–Wang in their work on anti-de Sitter Schwarzschild space [9]. Our proof of Theorem 1.5 uses crucially their work.

One can check easily that for the Kottler space $P_{\kappa,m}$ the area of its horizon $\partial P_{\kappa,m}$ satisfies

$$m = \frac{1}{2}\left(\left(\frac{|\partial P_{\kappa,m}|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}} + \kappa\left(\frac{|\partial P_{\kappa,m}|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}}\right). \tag{1.21}$$

Combining (1.18), (1.20) and (1.21), we immediately obtain the Penrose inequality for ALH graphs.

Theorem 1.6 *If $\mathcal{M} \subset Q_{\kappa,m}$ is an ALH graph as in Theorem 1.3, so that its horizon $\Sigma \subset (P_{\kappa,m}, g_{\kappa,m})$ is star-shaped with positive mean curvature, then*

$$m_{(\mathcal{M},g)} \geq \frac{1}{2}\left(\left(\frac{|\Sigma|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}} + \kappa\left(\frac{|\Sigma|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}}\right). \tag{1.22}$$

Equality is achieved by the Kottler space.

When $n = 3$, as mentioned above, this inequality was proved by Lee and Neves [33,34], even without the graphical condition. When $m = 0$ it was proved by de Lima and Girão [17]. However, if one restricts himself only to the case $m = 0$, by (1.16) and the dominant energy condition (1.17) one has $m_{(\mathcal{M},g)} \geq 0$, which means that (1.22) is interesting only if the volume $|\Sigma|$ of Σ is not so small, in the case $\kappa = -1$. This remark was also pointed out in [17]. Our result, Theorem 1.6, remedies this problem.

It is easy to show that the Kottler–Schwarzschild space $P_{\kappa,m}$ can be represented as an ALH graph in $(Q_{\kappa,m'}, \tilde{g}_{\kappa,m'})$ over $P_{\kappa,m'}$, if $m' \leq m$. In general we believe that the class of ALH graphs over $P_{\kappa,m}$ with smaller m is larger than the class of ALH graphs over $P_{\kappa,m}$ with bigger m. That is, we believe the class of ALH graphs with $m = 0$ considered in the paper of de Lima-Girão contains the class of ALH graphs with $m > 0$ and the class with $m_c < 0$ is the biggest. In Appendix A, we show that it is true at least for rotationally symmetric graphs.

By the above results and the results in [16], it is clear that the class of ALH graphs with negative mass we consider here can not be represented as ALH graphs over $P_{\kappa,0}$ in $Q_{\kappa,0}$, since, otherwise the ALH mass is positive. Moreover, in Appendix A we give examples of ALH manifolds with positive ALH mass, which can be represented as an ALH graph over $P_{-1,m'}$ with $m' < 0$, but can not be represented as an ALH graph over $P_{-1,0}$.

The rigidity in Theorem 1.6 should follow from the argument of Huang and Wu [29]. We will return to this problem later.

2 Kottler–Schwarzschild space

As stated in the introduction, the Kottler space, or Kottler–Schwarzschild space, is an analogue of the Schwarzschild space in the setting of asymptotically locally hyperbolic manifolds. Let $(N^{n-1}, \hat{g})$ be a closed space form of constant sectional curvature κ. Then the n-dimensional Kottler–Schwarzschild space $P_{\kappa,m} = [\rho_{\kappa,m}, \infty) \times N$ is equipped with the metric

$$g_{\kappa,m} = \frac{d\rho^2}{V_{\kappa,m}^2(\rho)} + \rho^2\hat{g}, \quad V_{\kappa,m} = \sqrt{\rho^2 + \kappa - \frac{2m}{\rho^{n-2}}}. \tag{2.1}$$

Remark that in (2.1), in order to have a positive root $\rho_{\kappa,m}$ of $\phi(\rho) := \rho^2 + \kappa - \frac{2m}{\rho^{n-2}}$, if $\kappa \geq 0$, the parameter m should be always positive; if $\kappa = -1$, the parameter m can be negative. In fact, in this case, m belongs to the following interval

$$m \in [m_c, +\infty) \quad \text{and} \quad m_c = -\frac{(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}}. \tag{2.2}$$

Here the certain critical value m_c comes from the following. If $m \leq 0$, one can solve the equation

$$\phi'(\rho) = 2\rho + (n-2)\frac{2m}{\rho^{n-1}} = 0,$$

to get the root $\rho_h = (-(n-2)m)^{\frac{1}{n}}$. Note the fact that $\phi(\rho_h) \leq 0$, which yields

$$m \geq -\frac{(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}}.$$

By a change of variable $r = r(\rho)$ with

$$r'(\rho) = \frac{1}{V_{\kappa,m}(\rho)}, \quad r(\rho_{\kappa,m}) = 0,$$

we can rewrite $P_{\kappa,m}$ as $P_{\kappa,m} = [0, \infty) \times N$ equipped with the metric

$$g_{\kappa,m} := \bar{g} := dr^2 + \lambda_\kappa(r)^2\hat{g}, \tag{2.3}$$

where $\lambda_\kappa : [0, \infty) \to [\rho_{\kappa,m}, \infty)$ is the inverse of $r(\rho)$, i.e., $\lambda_\kappa(r(\rho)) = \rho$. It is easy to check

$$\lambda_\kappa'(r) = V_{\kappa,m}(\rho) = \sqrt{\kappa + \lambda_\kappa(r)^2 - 2m\lambda_\kappa(r)^{2-n}}, \tag{2.4}$$

$$\lambda_\kappa''(r) = \lambda_\kappa(r) + (n-2)m\lambda_\kappa(r)^{1-n}. \tag{2.5}$$

By the definition of $\rho_{\kappa,m}$, we know that

$$\lambda_\kappa''(r) \geq 0 \text{ for } r \in [0, \infty).$$

One can also verify

$$\lambda_\kappa(r) = O(e^r) \text{ as } r \to \infty. \tag{2.6}$$

We take $\kappa = -1$ as example to verify (2.6).

$$\begin{aligned}
r(\rho) &= \int_{\rho_{-1,m}}^{\rho} \frac{1}{\sqrt{-1+s^2-2ms^{2-n}}} ds \\
&= \int_1^{\rho} \frac{1}{\sqrt{-1+s^2}} ds + \int_{\rho_{-1,m}}^{1} \frac{1}{\sqrt{-1+s^2-2ms^{2-n}}} ds \\
&\quad + \int_1^{\rho} \left(\frac{1}{\sqrt{-1+s^2-2ms^{2-n}}} - \frac{1}{\sqrt{-1+s^2}} \right) ds \\
&= \ln(2\sqrt{\rho^2-1}+2\rho) - c - \frac{m}{n}\rho^{-n} + O(\rho^{-n-2}) \text{ as } \rho \to \infty.
\end{aligned}$$

Here $c = \ln 2 + \int_{\rho_{-1,m}}^{1} \frac{1}{\sqrt{-1+s^2-2ms^{2-n}}} ds$. By Taylor expansion, we have

$$\frac{e^{r(\rho)+c}}{4} + e^{-(r(\rho)+c)} = (1+o(1))\rho + o(1),$$

which implies $\lambda_\kappa(r) = \rho = O(e^r)$ as $r \to \infty$.

Let $R_{\alpha\beta\gamma\delta}$ denote the Riemannian curvature tensor in $P_{\kappa,m}$. Let $\bar{\nabla}$ and $\bar{\Delta}$ denote the covariant derivative and the Laplacian on $P_{\kappa,m}$, respectively. The Riemannian and Ricci curvature of $(P_{\kappa,m}, \bar{g})$ are given by

$$\begin{aligned}
R_{ijkl} &= \lambda_\kappa(r)^2(\kappa - \lambda'_\kappa(r)^2)(\hat{g}_{ik}\hat{g}_{jl} - \hat{g}_{il}\hat{g}_{jk}) = (2m\lambda_\kappa^{-n} - 1)(\bar{g}_{ik}\bar{g}_{jl} - \bar{g}_{il}\bar{g}_{jk}), \\
R_{ijkr} &= 0, \\
R_{irjr} &= -\lambda_\kappa(r)\lambda''_\kappa(r)\hat{g}_{ij} = -(1+(n-2)m\lambda_\kappa^{-n})\bar{g}_{ij}. \\
Ric(\bar{g}) &= -\left(\frac{\lambda''_\kappa(r)}{\lambda_\kappa(r)} - (n-2)\frac{\kappa - \lambda'_\kappa(r)^2}{\lambda_\kappa(r)^2} \right)\bar{g} - (n-2)\left(\frac{\lambda''_\kappa(r)}{\lambda_\kappa(r)} + \frac{\kappa - \lambda'_\kappa(r)^2}{\lambda_\kappa(r)^2} \right) dr^2 \\
&= (-(n-1) + (n-2)m\lambda_\kappa(r)^{-n})\bar{g} - n(n-2)m\lambda_\kappa(r)^{-n}dr^2.
\end{aligned}$$

It follows from (2.6) that

$$|R_{\alpha\beta\gamma\delta} + \bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}|_{\bar{g}} = O(e^{-nr}), \quad |\bar{\nabla}_\mu R_{\alpha\beta\gamma\delta}|_{\hat{g}} = O(e^{-nr}); \tag{2.7}$$

$$|Ric(\bar{g}) + (n-1)\bar{g}|_{\bar{g}} = O(e^{-nr}). \tag{2.8}$$

3 The ALH mass of graphs in the kottler spaces

First, one can check directly

Lemma 3.1 *The Kottler space $(P_{\kappa,m}, g_{\kappa,m})$ is an ALH manifold with the ALH mass*

$$m_{(P_{\kappa,m}, g_{\kappa,m})} = m.$$

Second, instead of computing the ALH mass with V_κ in (1.5) one can compute it with $V_{\kappa,m}$ by using the following Lemma

Lemma 3.2 *We have*

$$m_{(\mathcal{M},g)} = m+c_n \lim_{\rho\to\infty}\int\limits_{N_\rho} \Big(V_{\kappa,m}(\mathrm{div}^{g_{\kappa,m}}\tilde{e}-d\mathrm{tr}^{g_{\kappa,m}}\tilde{e})+(\mathrm{tr}^{g_{\kappa,m}}\tilde{e})dV_{\kappa,m}-\tilde{e}(\nabla^{g_{\kappa,m}}V_{\kappa,m},\cdot)\Big)\bar{\nu}d\mu, \tag{3.1}$$

where $\tilde{e} := (\Phi^{-1})^*g - g_{\kappa,m}$ *and* $\bar{\nu}$ *denotes the outer normal of* N_ρ *induced by* $g_{\kappa,m}$.

Proof First note that

$$e = (\Phi^{-1})^*g - b_\kappa = \tilde{e} + (g_{\kappa,m} - b_\kappa),$$

thus we have

$$\begin{aligned} m_{(\mathcal{M},g)} &= m_{(P_{\kappa,m},g_{\kappa,m})} + c_n \lim_{\rho\to\infty}\int\limits_{N_\rho}\Big(V_\kappa(\mathrm{div}^{b_\kappa}\tilde{e} - d\mathrm{tr}^{b_\kappa}\tilde{e})+(\mathrm{tr}^{b_\kappa}\tilde{e})dV_\kappa - \tilde{e}(\nabla^{b_\kappa}V_\kappa,\cdot)\Big)\nu d\mu \\ &= m + c_n \lim_{\rho\to\infty}\int\limits_{N_\rho}\Big(V_\kappa(\mathrm{div}^{b_\kappa}\tilde{e} - d\mathrm{tr}^{b_\kappa}\tilde{e}) + (\mathrm{tr}^{b_\kappa}\tilde{e})dV_\kappa - \tilde{e}(\nabla^{b_\kappa}V_\kappa,\cdot)\Big)\nu d\mu. \end{aligned}$$

Then using the fact that $g_{\kappa,m}$ is ALH, one can replace V_κ by $V_{\kappa,m}$, b_κ by $g_{\kappa,m}$ and ν by $\bar{\nu}$ in (1.6) without changing mass, that is,

$$\begin{aligned} &\lim_{\rho\to\infty}\int\limits_{N_\rho}\Big(V_\kappa(\mathrm{div}^{b_\kappa}\tilde{e} - d\mathrm{tr}^{b_\kappa}\tilde{e}) + (\mathrm{tr}^{b_\kappa}\tilde{e})dV_\kappa - \tilde{e}(\nabla^{b_\kappa}V_\kappa,\cdot)\Big)\nu d\mu \\ &= \lim_{\rho\to\infty}\int\limits_{N_\rho}\Big(V_{\kappa,m}(\mathrm{div}^{g_{\kappa,m}}\tilde{e} - d\mathrm{tr}^{g_{\kappa,m}}\tilde{e}) + (\mathrm{tr}^{g_{\kappa,m}}\tilde{e})dV_{\kappa,m} - \tilde{e}(\nabla^{g_{\kappa,m}}V_{\kappa,m},\cdot)\Big)\bar{\nu}d\mu. \end{aligned}$$

This implies the desired result. □

According to [37], the second term in (3.1) is also an integral invariant when the reference metric is taken as the Kottler–Schwarzschild metric $g_{\kappa,m}$ rather than b_κ. In the spirit of [14,15], one can estimate the second term since $(P_{\kappa,m}, g_{\kappa,m}, V_{\kappa,m})$ satisfies the static equation (1.9). Therefore we can prove Theorem 1.3 for the graphs over a Kottler–Schwarzschild space which extends the previous works of graphs over the Euclidean space, hyperbolic space as well as the locally hyperbolic spaces.

Proof of Theorem 1.3 The proof of this theorem follows in the spirit of the one in [14,15]. For the convenience of readers, we sketch it. Denote $(\mathcal{M}, g) \subset (Q_{\kappa,m}, \tilde{g}_{\kappa,m})$ with the unit outer normal ξ and the shape operator $B = -\nabla^{\tilde{g}_{\kappa,m}}\xi$. Define the Newton tensor inductively by

$$T_r = S_r I - BT_{r-1}, \quad T_0 = I,$$

where S_r denotes the r-th mean curvature of $(\mathcal{M}, g)$ with respect to ξ. Let $\{\epsilon_i\}_{i=1}^n$ be a local orthonormal frame on $\mathcal{M}$, then a direct computation gives (or see (3.3) in [1] for the proof)

$$\mathrm{div}_g T_r := \sum_{i=1}^n(\nabla_{\epsilon_i}T_r)(\epsilon_i) = -B(\mathrm{div}_g T_{r-1}) - \sum_{i=1}^n(\tilde{R}(\xi, T_{r-1}(\epsilon_i))\epsilon_i)^T, \tag{3.2}$$

where $\tilde{R}$ denotes the curvature tensor of $(Q_{\kappa,m}, \tilde{g}_{\kappa,m})$ and $(\tilde{R}(\xi, T_{r-1}(\epsilon_i))\epsilon_i)^T$ denotes the tangential component of $\tilde{R}(\xi, T_{r-1}(\epsilon_i))\epsilon_i$.

Using the fact that $\frac{\partial}{\partial t}$ is a Killing vector field, one can check directly (or refer to (8.4) in [1] for the proof)

$$\mathrm{div}_g\left(T_r\left(\frac{\partial}{\partial t}\right)^T\right) = \left\langle \mathrm{div}_g T_r, \left(\frac{\partial}{\partial t}\right)^T\right\rangle + (r+1)S_{r+1}\left\langle \frac{\partial}{\partial t}, \xi\right\rangle, \tag{3.3}$$

where $(\frac{\partial}{\partial t})^T$ is the tangential component of $\frac{\partial}{\partial t}$ along $\mathcal{M}$.

Combining (3.2) and (3.3) together, we get the following flux-type formula (for $r = 1$)

$$\mathrm{div}_g\left(T_1\left(\frac{\partial}{\partial t}\right)^T\right) = 2S_2\left\langle \frac{\partial}{\partial t}, \xi\right\rangle + Ric_{\tilde{g}_{\kappa,m}}\left(\xi, \left(\frac{\partial}{\partial t}\right)^T\right). \tag{3.4}$$

Denote by

$$e_0 = (V_{\kappa,m})^{-1}\frac{\partial}{\partial t}.$$

In the local coordinates $x = (x_1, \cdots, x_n)$ of $(P_{\kappa,m}, g_{\kappa,m})$, the tangent space $T\mathcal{M}^n$ is spanned by

$$Z_i = (V_{\kappa,m}\bar{\nabla}_i f)e_0 + \frac{\partial}{\partial x_i},$$

and thus

$$\xi = \frac{1}{\sqrt{1 + V_{\kappa,m}^2|\bar{\nabla} f|^2}}(e_0 - V_{\kappa,m}\bar{\nabla} f),$$

which implies

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)^T &= V_{\kappa,m}e_0 - \frac{V_{\kappa,m}}{\sqrt{1 + V_{\kappa,m}^2|\bar{\nabla} f|^2}}\xi \\ &= \frac{V_{\kappa,m}^3|\bar{\nabla} f|^2}{1 + V_{\kappa,m}^2|\bar{\nabla} f|^2}e_0 + \frac{V_{\kappa,m}^2}{1 + V_{\kappa,m}^2|\bar{\nabla} f|^2}\bar{\nabla} f.\end{aligned}$$

On the other hand $(\frac{\partial}{\partial t})^T := ((\frac{\partial}{\partial t})^T)^i Z_i$ which yields

$$\left(\left(\frac{\partial}{\partial t}\right)^T\right)^i = \frac{V_{\kappa,m}^2\bar{\nabla}^i f}{1 + V_{\kappa,m}^2|\bar{\nabla} f|^2}. \tag{3.5}$$

Note that the shape operator of $\mathcal{M}^n$ is given by (cf. (4.5) in [24] for instance)

$$\begin{aligned}B^i_j = \frac{V_{\kappa,m}}{\sqrt{1 + V_{\kappa,m}^2|\bar{\nabla} f|^2}}&\left(\bar{\nabla}^i\bar{\nabla}_j f + \frac{\bar{\nabla}^i f\bar{\nabla}_j V_{\kappa,m}}{V_{\kappa,m}(1 + V_{\kappa,m}^2|\bar{\nabla} f|^2)} + \frac{\bar{\nabla}^i V_{\kappa,m}\bar{\nabla}_j f}{V_{\kappa,m}}\right. \\ &\left. - \frac{V^2\bar{\nabla}^i f\bar{\nabla}^s f\bar{\nabla}_s\bar{\nabla}_j f}{1 + V_{\kappa,m}^2|\bar{\nabla} f|^2}\right).\end{aligned} \tag{3.6}$$

By the decay property of metric (1.12) together with (3.5), one can check that

$$g_{ij}\left(T_1\left(\frac{\partial}{\partial t}\right)^T\right)^i \bar{\nu}^j \approx (g_{\kappa,m})_{ij}\left(T_1\left(\frac{\partial}{\partial t}\right)^T\right)^i \bar{\nu}^j$$
$$= (T_1)^i_p \frac{V_{\kappa,m}^2 \bar{\nabla}^p f}{1+V_{\kappa,m}^2|\bar{\nabla} f|^2}\bar{\nu}_i \approx (T_1)^i_p \frac{V_{\kappa,m}^2 \bar{\nabla}^p f}{\sqrt{1+V_{\kappa,m}^2|\bar{\nabla} f|^2}}\bar{\nu}_i, \quad (3.7)$$

where $\approx$ means that the two terms differ only by the terms that vanish at infinity after integration.

With expression (3.6) and applying the similar argument in the proof of (4.11) in [24], one can check that

$$V_{\kappa,m}(\bar{\nabla}^j \tilde{e}^i{}_j - \bar{\nabla}^i \tilde{e}_j{}^j) - (\tilde{e}_{ij}\bar{\nabla}^j V_{\kappa,m} - \tilde{e}_j{}^j \bar{\nabla}^i V_{\kappa,m}) = (T_1)^i_p \frac{V_{\kappa,m}^2 \bar{\nabla}^p f}{\sqrt{1+V_{\kappa,m}^2|\bar{\nabla} f|^2}}.$$

As in the proof of Theorem 1.4 in [24], integrating by parts gives an extra boundary term that

$$\lim_{\rho\to\infty}\int_{N_\rho}\Big(V_{\kappa,m}(div^{g_{\kappa,m}}\tilde{e} - d\,\mathrm{tr}^{g_{\kappa,m}}\tilde{e}) + (\mathrm{tr}^{g_{\kappa,m}}\tilde{e})dV_{\kappa,m} - \tilde{e}(\nabla^{g_{\kappa,m}}V_{\kappa,m},\cdot)\Big)\bar{\nu}\,d\mu$$
$$+\int_\Sigma V_{\kappa,m}H\left(\frac{V_{\kappa,m}^2|\bar{\nabla} f|^2}{1+V_{\kappa,m}^2|\bar{\nabla} f|^2}\right)d\mu$$
$$= \lim_{\rho\to\infty}\int_{N_\rho}(T_1)^i_p \frac{V_{\kappa,m}^2 \bar{\nabla}^p f}{\sqrt{1+V_{\kappa,m}^2|\bar{\nabla} f|^2}}\bar{\nu}_i\,d\mu + \int_\Sigma V_{\kappa,m}H\left(\frac{V_{\kappa,m}^2|\bar{\nabla} f|^2}{1+V_{\kappa,m}^2|\bar{\nabla} f|^2}\right)d\mu.$$

Next using (3.7) and the assumption that $|\bar{\nabla} f(x)| \to \infty$ as $x \to \Sigma$, we have

$$\lim_{\rho\to\infty}\int_{N_\rho}(T_1)^i_p \frac{V_{\kappa,m}^2 \bar{\nabla}^p f}{\sqrt{1+V_{\kappa,m}^2|\bar{\nabla} f|^2}}\bar{\nu}_i\,d\mu + \int_\Sigma V_{\kappa,m}H\left(\frac{V_{\kappa,m}^2|\bar{\nabla} f|^2}{1+V_{\kappa,m}^2|\bar{\nabla} f|^2}\right)d\mu$$
$$= \lim_{\rho\to\infty}\int_{N_\rho} g_{ij}\left(T_1\left(\frac{\partial}{\partial t}\right)^T\right)^i \bar{\nu}^j d\mu + \int_\Sigma V_{\kappa,m}H\,d\mu.$$

Finally integrating (3.4) and revoking Lemma 3.2, we finally obtain

$$m_{(\mathcal{M},g)} = m + c_n\int_{\mathcal{M}}\left(2S_2\left\langle\frac{\partial}{\partial t},\xi\right\rangle + Ric_{\tilde{g}_{\kappa,m}}\left(\xi,\left(\frac{\partial}{\partial t}\right)^T\right)\right)dV_g + c_n\int_\Sigma V_{\kappa,m}H\,d\mu. \quad (3.8)$$

From the Gauss equation we obtain

$$R_g = R_{\tilde{g}_{\kappa,m}} - 2Ric_{\tilde{g}_{\kappa,m}}(\xi,\xi) + 2S_2.$$

Since $\tilde{g}_{\kappa,m}$ is an Einstein metric, we have

$$R_g = -n(n-1) + 2S_2 \quad \text{and} \quad Ric_{\tilde{g}_{\kappa,m}}\left(\xi,\left(\frac{\partial}{\partial t}\right)^T\right) = 0.$$

Combining all the things together, we complete the proof of the theorem. □

4 Inverse mean curvature flow

Let Σ_0 be a star-shaped, strictly mean convex closed hypersurface in $P_{\kappa,m}$ parametrized by $X_0 : N \to P_{\kappa,m}$. Since the case $\kappa = 1$ has been considered in [9], we focus on the case $\kappa = 0$ or -1. Consider a family of hypersurfaces $X(\cdot, t) : N \to P_{\kappa,m}$ evolving by the inverse mean curvature flow:

$$\frac{\partial X}{\partial t}(x,t) = \frac{1}{H(x,t)}\nu(x,t), \quad X(x,0) = X_0(x), \tag{4.1}$$

where $\nu(\cdot, t)$ is the outward normal of $\Sigma_t = X(N, t)$.

Let us first fix the notations. Let g_{ij}, h_{ij} and $d\mu$ denote the induced metric, the second fundamental form and the volume element of Σ_t, respectively. Let ∇ and Δ denote the covariant derivative and the Laplacian on Σ_t, respectively. We always use the Einstein summation convention. Let $|A|^2 = g^{ij}g^{kl}h_{ik}h_{jl}$.

We collect some evolution equations in the following lemma. For the proof see for instance [20].

Lemma 4.1 *Along flow (4.1), we have the following evolution equations.*

(1) *The volume element of Σ_t evolves under*

$$\frac{\partial}{\partial t}d\mu = d\mu.$$

Consequently,

$$\frac{\partial}{\partial t}|\Sigma_t| = |\Sigma_t|.$$

(2) *h_i^j evolves under*

$$\begin{aligned}\frac{\partial h_i^j}{\partial t} = &\frac{\Delta h_i^j}{H^2} + \frac{|A|^2}{H^2}h_i^j - \frac{2h_i^k h_k^j}{H} - \frac{2\nabla_i H \nabla^j H}{H^3}\\ &+\frac{1}{H^2}g^{kl}(2g^{pj}R_{qikp}h_l^q - g^{pj}R_{qkpl}h_i^q - R_{qkil}h^{qj} + R_{\nu k\nu l}h_i^j)\\ &+\frac{1}{H^2}g^{kl}g^{qj}(\overline{\nabla}_q R_{\nu kli} + \overline{\nabla}_l R_{\nu ikq}) - \frac{2}{H}g^{kj}R_{\nu i\nu k}.\end{aligned}$$

(3) *The mean curvature evolves under*

$$\frac{\partial H}{\partial t} = \frac{\Delta H}{H^2} - 2\frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} - \frac{Ric(\nu,\nu)}{H}.$$

(4) *The function $V_{\kappa,m}$ evolves under*

$$\frac{\partial}{\partial t}V_{\kappa,m} = \frac{p}{H},$$

where $p := \langle \overline{\nabla} V_{\kappa,m}, \nu\rangle$ is the support function of Σ.

(5) *The function $\chi = \frac{1}{\langle \lambda_\kappa \partial_r, \nu\rangle}$ evolves under*

$$\frac{\partial \chi}{\partial t} = \frac{\Delta\chi}{H^2} - \frac{2|\nabla\chi|^2}{\chi H^2} - \frac{|A|^2}{H^2}\chi + \frac{-\chi Ric(\nu,\nu) + \chi^2\lambda_\kappa Ric(\nu,\partial_r)}{H^2}. \tag{4.2}$$

(6) *The function p, defined above, evolves under*

$$\frac{\partial p}{\partial t}=\frac{\overline{\nabla}^2 V_{\kappa,m}(\nu,\nu)}{H}+\frac{1}{H^2}\langle \nabla V_{\kappa,m},\nabla H\rangle,$$

and thus

$$\frac{d}{dt}\int_{\Sigma_t} p d\mu = n\int_{\Sigma_t}\frac{V_{\kappa,m}}{H}d\mu.$$

□

Since Σ_0 is star-shaped, we can write Σ_0 as a graph of a function over N:

$$\Sigma_0=\{(u_0(x),x): x\in N\}.$$

It is well known that there exists a maximal time interval $[0,T^*)$, $0<T^*\leq\infty$, such that the flow exists and any $X(\cdot,t)$, $t\in[0,T^*)$ are also graphs of functions u over N:

$$\Sigma_t=\{(u(x,t),x): x\in N\}.$$

Define a function $\varphi(\cdot,t): N\to\mathbb{R}$ by

$$\varphi(x,t)=\int_0^{u(x,t)}\frac{1}{\lambda_\kappa(r)}dr,$$

where $\lambda_\kappa(r)$ is defined in (2.3).

Let

$$v=\sqrt{1+|\nabla_{\hat{g}}\varphi|^2_{\hat{g}}}.$$

In term of the local coordinates x^i on N, the induced metric and the second fundamental form of Σ_t are given, respectively, by

$$g_{ij}=\lambda_\kappa^2(\hat{g}_{ij}+\varphi_i\varphi_j),\quad h_{ij}=\frac{\lambda_\kappa}{v}(\lambda_\kappa'(\hat{g}_{ij}+\varphi_i\varphi_j)-\varphi_{ij}). \tag{4.3}$$

Here $\varphi_i=\nabla_i^{\hat{g}}\varphi$ and $\varphi_{ij}=\nabla_i^{\hat{g}}\nabla_j^{\hat{g}}\varphi$. Thus the mean curvature is given by

$$H=g^{ij}h_{ij}=(n-1)\frac{\lambda_\kappa'}{\lambda_\kappa v}-\frac{\tilde{g}^{ij}\varphi_{ij}}{\lambda_\kappa v}, \tag{4.4}$$

where $\tilde{g}^{ij}=\hat{g}^{ij}-\frac{\varphi^i\varphi^j}{v^2}$.

Along flow (4.1), the graph function u evolves under

$$\frac{\partial u}{\partial t}=\frac{v}{H}. \tag{4.5}$$

Hence

$$\frac{\partial\varphi}{\partial t}=\frac{v}{\lambda H}=\frac{v^2}{(n-1)\lambda_\kappa'-\tilde{g}^{ij}\varphi_{ij}}:=\frac{1}{F(u,\nabla_{\hat{g}}\varphi,\nabla_{\hat{g}}{}^2\varphi)}. \tag{4.6}$$

By the parabolic maximum principle, we can derive the C^0 and C^1 estimates.

Proposition 4.2 *Let* $\underline{u}(t) = \inf_N u(\cdot, t)$ *and* $\bar{u}(t) = \sup_N u(\cdot, t)$. *Then*

$$\lambda_\kappa(\underline{u}(t)) \geq e^{\frac{1}{n-1}t}\lambda_\kappa(\underline{u}(0)), \quad \lambda_\kappa(\bar{u}(t)) \leq e^{\frac{1}{n-1}t}\lambda_\kappa(\bar{u}(0)). \tag{4.7}$$

Proof At the point where $u(\cdot, t)$ attains its minimum, we have $v = 1$ and $\varphi_{ij} \geq 0$, and hence

$$H \leq \frac{(n-1)\lambda_\kappa'(u)}{\lambda_\kappa(u)}.$$

Thus from (4.5) we infer that

$$\frac{d}{dt}\inf_N \lambda_\kappa(\underline{u}(t)) \geq (n-1)\lambda_\kappa(\underline{u}(t)), \tag{4.8}$$

from which the first assertion follows. The second one is proved in a similar way by considering the maximum point of $u(\cdot, t)$. □

To derive the C^1 estimate, we need to estimate the upper and lower bounds for H.

Proposition 4.3 *We have* $H \leq n - 1 + O(e^{-\frac{1}{n-1}t})$ *and* $H \geq Ce^{-\frac{1}{n-1}t}$ *for some positive constant* C *depending only on* n, m *and* Σ_0.

Proof By Lemma 4.1 and (2.8), we have

$$\frac{\partial}{\partial t}H^2 = \frac{\Delta H^2}{H^2} - \frac{3}{2}\frac{|\nabla H^2|^2}{H^4} - 2|A|^2 + 2(n-1) + O(e^{-nr}). \tag{4.9}$$

In view of the inequality $|A|^2 \geq \frac{1}{n-1}H^2$, by using Proposition 4.2 and the maximum principle, we deduce

$$\frac{d}{dt}\sup_N H(\cdot, t)^2 \leq -\frac{2}{n-1}\sup_N H(\cdot, t)^2 + 2(n-1) + O(e^{-\frac{n}{n-1}t}).$$

The first assertion follows.

For the second assertion, we take derivative s of (4.6) with respect to t and get

$$\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial t}\right) = -\frac{1}{F^2}\frac{\partial F}{\partial\varphi_i}\left(\frac{\partial\varphi}{\partial t}\right)_i - \frac{1}{F^2}\frac{\partial F}{\partial\varphi_{ij}}\left(\frac{\partial\varphi}{\partial t}\right)_{ij} - \frac{2(n-1)\lambda_\kappa\lambda_\kappa''}{v^2F^2}\frac{\partial\varphi}{\partial t}.$$

Since $\lambda_\kappa''(r) \geq 0$, by using the maximum principle, we have

$$\frac{d}{dt}\sup_N \frac{\partial\varphi}{\partial t}(\cdot, t) \leq 0. \tag{4.10}$$

Taking into account of (4.6) and Proposition 4.2, we conclude that

$$H \geq C\frac{v}{\lambda_\kappa} \geq Ce^{-\frac{1}{n-1}t}.$$

□

Proposition 4.4 *We have* $|\nabla_{\hat{g}}\varphi|_{\hat{g}} = O(e^{-\frac{1}{(n-1)^2}t})$ *and* $v = 1 + O(e^{-\frac{1}{(n-1)^2}t})$.

Proof Let $\omega = \frac{1}{2}|\nabla_{\hat{g}}\varphi|^2_{\hat{g}}$. Since

$$\frac{\partial\varphi}{\partial t} = \frac{v}{\lambda_\kappa H} := \frac{1}{F(u, \nabla_{\hat{g}}\varphi, \nabla_{\hat{g}}{}^2\varphi)}, \tag{4.11}$$

one can verify that the evolution equation of ω is

$$\frac{\partial \omega}{\partial t} = \frac{\tilde{g}^{ij}}{v^2 F^2}\omega_{ij} - \frac{1}{F^2}\frac{\partial F}{\partial \varphi_i}\omega_i - \frac{2(n-2)\kappa}{v^2 F^2}\omega - \frac{\tilde{g}^{ij}}{v^2 F^2}\hat{g}^{kl}\varphi_{ik}\varphi_{jl} - \frac{2(n-1)\lambda_\kappa \lambda_\kappa''}{v^2 F^2}\omega. \tag{4.12}$$

Notice that $vF = \lambda H$ and $-\kappa \leq \lambda_\kappa^2 - 2m\lambda_\kappa^{2-n}$. Using (2.4), Proposition 4.2 and 4.3, we have

$$\begin{aligned} -\frac{2(n-2)\kappa}{v^2F^2} - \frac{2(n-1)\lambda_\kappa\lambda_\kappa''}{v^2F^2} &\leq \frac{2(n-2)(\lambda_\kappa^2 - 2m\lambda_\kappa^{2-n})}{\lambda_\kappa^2 H^2} - \frac{2(n-1)(1+(n-2)m\lambda_\kappa^{-n})}{H^2} \\ &= -\frac{2}{H^2} - \frac{2(n-2)(n+1)m}{\lambda_\kappa^n H^2} \\ &\leq -\frac{2}{(n-1)^2} + Ce^{-\frac{2}{n-1}t} + Ce^{-\frac{n-2}{n-1}t}. \end{aligned} \tag{4.13}$$

Thus by using the maximum principle on (4.12) we have

$$\frac{\partial}{\partial t}\sup_N \omega(\cdot, t) \leq \left(-\frac{2}{(n-1)^2} + Ce^{-\frac{2}{n-1}t}\right)\sup_N \omega(\cdot, t), \tag{4.14}$$

which implies $\omega = O(e^{-\frac{2}{(n-1)^2}t})$. The assertion follows. □

Remark 4.5 Proposition 4.4 implies that the star-shapedness of Σ_t is preserved. Thus as long as the flow exists, we have $\langle \partial_r, \nu \rangle > 0$ and a graph representation of Σ_t.

Proposition 4.6 *There exists a positive constant C depending only on n, m and Σ_0, such that $H \geq C$.*

Proof Recall the function $\chi = \frac{1}{\langle \lambda(r)\partial_r, \nu\rangle}$. Proposition 4.2 and 4.4 ensure that χ is well defined and there exists $C > 0$ such that $C^{-1}e^{-\frac{1}{n-1}t} \leq \chi \leq Ce^{-\frac{1}{n-1}t}$.

By Lemma 4.1 and (2.8), we have

$$\frac{\partial}{\partial t}\log H = \frac{\Delta \log H}{H^2} - \frac{|\nabla \log H|^2}{H^2} - \frac{|A|^2}{H^2} + \frac{n-1}{H^2} + \frac{1}{H^2}O(e^{-nr}) \tag{4.15}$$

and

$$\frac{\partial}{\partial t}\log \chi = \frac{\Delta \log \chi}{H^2} - \frac{|\nabla \log \chi|^2}{H^2} - \frac{|A|^2}{H^2} + \frac{1}{H^2}O(e^{-nr}). \tag{4.16}$$

Combining (4.9) and (4.15) and using Proposition 4.2, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}(\log \chi - \log H) &= \frac{\Delta(\log \chi - \log H)}{H^2} \\ &\quad + \frac{\langle \nabla(\log H + \log \chi), \nabla(\log H - \log \chi)\rangle}{H^2} \\ &\quad - \frac{n-1}{H^2} + \frac{Ce^{-\frac{n}{n-1}t}}{H^2}. \end{aligned}$$

Using Proposition 4.3 and the maximum principle, we have

$$\frac{d}{dt}\sup_N(\log \chi - \log H)(\cdot, t) \leq -\frac{1}{n-1} + Ce^{-\frac{2}{n-1}t} + Ce^{-\frac{n-2}{n-1}t}. \tag{4.17}$$

Hence $e^{\log \chi - \log H} \leq Ce^{-\frac{1}{n-1}t}$. Note that $\chi = \frac{v}{\lambda}$. Consequently, $H \geq C$. □

With the help of Proposition 4.6, we are able to improve Proposition 4.4.

Proposition 4.7 *We have* $|\nabla_{\hat{g}}\varphi|_{\hat{g}} = O(e^{-\frac{1}{n-1}t})$ *and* $v = 1 + O(e^{-\frac{1}{n-1}t})$.

Proof We need the following refinement of (4.13), by taking Proposition 4.6 into account:

$$-\frac{2(n-2)\kappa}{v^2F^2} = -\frac{2(n-2)\kappa}{\lambda_\kappa^2 H^2} \le Ce^{-\frac{2}{n-1}t};$$

$$\begin{aligned}-\frac{2(n-1)\lambda_\kappa\lambda_\kappa''}{v^2F^2} &= -\frac{2(n-1)(1+\frac{n-2}{2}m\lambda_\kappa^{-n})}{H^2}\\ &\le -\frac{2}{(n-1)} + Ce^{-\frac{2}{n-1}t} + Ce^{-\frac{n-2}{n-1}t}.\end{aligned}$$

Then the proof follows the same way as Proposition 4.4. □

We now derive the C^2 estimates.

Proposition 4.8 *The second fundamental form* h_{ij} *is uniformly bounded. Consequently,* $|\nabla^2_{\hat{g}}\varphi|_{\hat{g}} \le C$.

Proof Let $M_i^j = Hh_i^j$. By Lemma 4.1, we have that M_i^j evolves under

$$\begin{aligned}\frac{\partial M_i^j}{\partial t} &= \frac{\Delta M_i^j}{H^2} - 2\frac{\nabla^k H\nabla_k M_i^j}{H^3} - 2\frac{\nabla_i H\nabla^j H}{H^2}\\ &\quad -2\frac{M_i^k M_k^j}{H^2} + \frac{2(n-1)M_i^j}{H^2} + \left(\frac{|M|}{H^2}+1\right)O(e^{-\frac{n}{n-1}t}).\end{aligned}$$

Hence the maximal eigenvalue μ of M_i^j satisfies

$$\frac{\partial\mu}{\partial t} = -2\frac{\mu^2}{H^2} + \frac{2(n-1)\mu}{H^2} + \left(\frac{\mu}{H}+1\right)O(e^{-\frac{n}{n-1}t}). \tag{4.18}$$

In view of Proposition 4.3 and 4.6, by using the maximum principle we know that μ is uniformly bounded from above. Combining the fact $C_1 \le H \le C_2$, we conclude that h_i^j is uniformly bounded both from above and below. □

Proposition 4.2–4.6 ensure the uniform parabolicity of Eq. (4.6). With the C^2 estimates, we can derive the higher order estimates via standard parabolic Krylov and Schauder theory, which allows us to obtain the long time existence for the flow.

Proposition 4.9 *The flow* (4.1) *exists for* $t \in [0,\infty)$.

□

With Proposition 4.2–4.7 at hand, we can follow the same argument of Proposition 15 and 16 in [9] to obtain improved estimates for H and h_i^j.

Proposition 4.10 $H = n-1+O(te^{-\frac{2}{n-1}t})$ *and* $|h_i^j - \delta_i^j| \le O(t^2e^{-\frac{2}{n-1}t})$.

□

Consequently, we have

Proposition 4.11 $|\nabla^2_{\hat{g}}\varphi|_{\hat{g}} \le O(t^2e^{-\frac{1}{n-1}t})$.

Proof Using Proposition 4.2 and 4.7, we get

$$\lambda'_\kappa = \lambda_k + O(e^{-\frac{t}{n-1}}), \quad \frac{1}{v} = O(e^{-\frac{2t}{n-1}}). \tag{4.19}$$

It follows from Proposition 4.11 that

$$\left| h_{ij} - \frac{\lambda'_\kappa}{\lambda_\kappa v} g_{ij} \right|_g \le |h_{ij} - g_{ij}|_g + (n-1)\left| \frac{\lambda'_\kappa}{\lambda_\kappa v} - 1 \right| \le O(t^2 e^{-\frac{2}{n-1}t}). \tag{4.20}$$

On the other hand,

$$g_{ij} = \lambda_\kappa^2 \hat{g}_{ij} + \varphi_i \varphi_j = O(e^{\frac{2}{n-1}t})\hat{g}_{ij}.$$

Thus from (4.3) we see

$$|\varphi_{ij}|_{\hat{g}} = \frac{\lambda_\kappa}{v} |h_{ij} - \frac{\lambda'_\kappa}{\lambda_\kappa v} g_{ij}|_{\hat{g}} \le O(t^2 e^{-\frac{1}{n-1}t}). \tag{4.21}$$

□

If we do more delicate analysis, we may improve the estimates given in Proposition 4.11 to $o(e^{-\frac{1}{n-1}t})$ as in the work of Gerhardt for the inverse mean curvature flow in $\mathbb{H}^n$. (see also [18]). Here we avoid to do so, as in the work of Brendle et al. [9]. We remark that on a general asymptotically hyperbolic manifolds such estimates may be difficult to obtain, cf. the work of Neves [38].

5 Minkowski type inequalities

We start this section with

Theorem 5.1 ([9]) *Let Σ be a compact embedded hypersurface which is star-shaped with positive mean curvature in $(\rho_{\kappa,m}, \infty) \times N^{n-1}$. Let Ω be the region bounded by Σ and the horizon $\partial M = \{\rho_{\kappa,m}\} \times N$. Then*

$$\int_\Sigma V_{\kappa,m} H d\mu \ge n(n-1)\int_\Omega V_{\kappa,m} dvol + (n-1)\kappa\vartheta_{n-1}\left(\left(\frac{|\Sigma|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}} - \left(\frac{|\partial M|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}}\right). \tag{5.1}$$

Equality holds if and only if $\Sigma = \{\rho\} \times N$ for some $\rho \in [\rho_{\kappa,m}, \infty)$.

When $\kappa = 1$, Theorem 5.1 was proved in [9]; when $\kappa = 0, -1$, the proof follows from a similar argument, which is even simpler. For the convenience of the reader, we include it here. To prove this theorem, we need the following two lemmas.

Lemma 5.2 *The functional*

$$Q_1(t) := \frac{\int_{\Sigma_t} V_{\kappa,m} H d\mu - n(n-1)\int_{\Omega_t} V_{\kappa,m} dvol + (n-1)\kappa\rho_{\kappa,m}^{n-2}\vartheta_{n-1}}{|\Sigma_t|^{\frac{n-2}{n-1}}} \tag{5.2}$$

is monotone non-increasing along flow (4.1).

Proof The proof of this lemma can be found in [9]. For completeness, we include the calculations here. To simplify the notation, we denote $\rho_0 = \rho_{\kappa,m}$. In view of Lemma 4.1 and integrating by parts, we calculate

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Sigma_t} V_{\kappa,m} H d\mu \\
&= -\int_{\Sigma_t} \frac{1}{H}\Delta V_{\kappa,m} d\mu - \int_{\Sigma_t} \frac{V_{\kappa,m}}{H}(|A|^2 + Ric(\nu,\nu))d\mu + \int_{\Sigma_t}(p + V_{\kappa,m}H)d\mu \\
&= -\int_{\Sigma_t} \frac{V_{\kappa,m}}{H}|A|^2 + \int_{\Sigma_t}(2p + V_{\kappa,m}H)d\mu \\
&\le \int_{\Sigma_t}\left(2p + \frac{n-2}{n-1}V_{\kappa,m}H\right)d\mu, \qquad (5.3)
\end{aligned}
$$

where in the third line we used the simple fact $\Delta V_{\kappa,m} = \bar{\Delta} V_{\kappa,m} - \bar{\nabla}^2 V_{\kappa,m}(\nu,\nu) - Hp$ and (1.9).

Then we use the divergence theorem to deal with the first term that

$$
\begin{aligned}
\int_{\Sigma_t} p d\mu &= \int_{\Sigma_t} \langle \bar{\nabla} V_{\kappa,m}, \nu\rangle d\mu \\
&= \int_{\Omega_t} \bar{\Delta} V_{\kappa,m} dvol + ((n-2)m + \rho_0^n)\vartheta_{n-1} \\
&= n\int_{\Omega_t} V_{\kappa,m} dvol + \left(\frac{n}{2}\rho_0^n + \frac{n-2}{2}\kappa\rho_0^{n-2}\right)\vartheta_{n-1}, \qquad (5.4)
\end{aligned}
$$

where in the last equality we used the relation $2m = \rho_0^n + \kappa\rho_0^{n-2}$ and the fact $\bar{\Delta} V_{\kappa,m} = nV_{\kappa,m}$ which follows from (1.9).

Similarly, by Lemma 4.1 and (5.4), we have

$$
\frac{d}{dt}\int_{\Omega_t} nV_{\kappa,m} dvol = n\int_{\Sigma_t} \frac{V_{\kappa,m}}{H} d\mu. \qquad (5.5)
$$

Also a Heintze–Karcher type inequality proved by Brendle [8] is needed to estimate the third term, that is,

$$
(n-1)\int_{\Sigma_t} \frac{V_{\kappa,m}}{H} d\mu \ge n\int_{\Omega_t} V_{\kappa,m} dvol + \rho_0^n \vartheta_{n-1}. \qquad (5.6)
$$

Hence substituting (5.4), (5.6) into (5.3) together with (5.5), we infer

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{\Sigma_t} V_{\kappa,m} H d\mu - n(n-1)\int_{\Omega_t} V_{\kappa,m} dvol\right) \\
&\le \int_{\Omega_t} 2nV_{\kappa,m} dvol + (n\rho_0^n + (n-2)\kappa\rho_0^{n-2})\vartheta_{n-1}
\end{aligned}
$$

$$
+\int_{\Sigma_t} \frac{n-2}{n-1} V_{\kappa,m} H d\mu - \left(n^2 \int_{\Omega_t} V_{\kappa,m} dvol + n\rho_0^n \vartheta_{n-1} \right)
$$

$$
= \frac{n-2}{n-1} \left(\int_{\Sigma_t} V_{\kappa,m} H d\mu - n(n-1) \int_{\Omega_t} V_{\kappa,m} dvol + (n-1)\kappa\rho_0^{n-2}\vartheta_{n-1} \right).
$$

Taking into account of Lemma 4.1 (1), we get the assertion. □

Lemma 5.3

$$
\liminf_{t\to\infty} Q_1(t) \geq (n-1)\kappa \vartheta_{n-1}^{\frac{1}{n-1}}.
$$

Proof In view of (5.4), it suffices to prove

$$
\liminf_{t\to\infty} \frac{\int_{\Sigma_t} V_{\kappa,m} H d\mu - (n-1)\int_{\Sigma_t} p d\mu}{|\Sigma_t|^{\frac{n-2}{n-1}}} \geq (n-1)\kappa \vartheta_{n-1}^{\frac{1}{n-1}}. \tag{5.7}
$$

From (4.4), Proposition 4.7 and 4.11, we have

$$
H = \frac{1}{v}\left((n-1)\frac{\lambda'_\kappa}{\lambda_\kappa} - \frac{1}{\lambda_\kappa}\Delta_{\hat{g}}\varphi \right) + O(t^2 e^{-\frac{3t}{n-1}}). \tag{5.8}
$$

Using Proposition 4.7 and the expressions of λ_κ, λ'_κ, and v, we get

$$
V_{\kappa,m} = \lambda'_\kappa = \lambda_k \left(1 + \frac{\kappa}{2}(\lambda_\kappa)^{-2} \right) + O(e^{-\frac{4t}{n-1}}), \quad \frac{1}{v} = 1 - \frac{1}{2}|\nabla_{\hat{g}}\varphi|^2_{\hat{g}} + O(e^{-\frac{4t}{n-1}}) \tag{5.9}
$$

and

$$
\sqrt{\det g} = \left(\lambda_\kappa^{n-1} + \frac{1}{2}|\nabla_{\hat{g}}\varphi|^2_{\hat{g}}\lambda_\kappa^{n-1} + O(e^{\frac{n-5}{n-1}t}) \right)\sqrt{\det \hat{g}}. \tag{5.10}
$$

Hence we have

$$
\begin{aligned}
\int_{\Sigma_t} V_{\kappa,m} H d\mu &= (n-1)\int_N (\lambda_\kappa^n + \kappa\lambda_\kappa^{n-2}) d\mu_{\hat{g}} - \int_N \lambda_\kappa^{n-1}\Delta_{\hat{g}}\varphi d\mu_{\hat{g}} + O(e^{\frac{n-3}{n-1}t}) \\
&= (n-1)\int_N (\lambda_\kappa^n + \kappa\lambda_\kappa^{n-2}) d\mu_{\hat{g}} + \int_N (n-1)\lambda_k^{n-4}|\nabla_{\hat{g}}\lambda_\kappa|^2 d\mu_{\hat{g}} + O(e^{\frac{n-3}{n-1}t}),
\end{aligned} \tag{5.11}
$$

where in the second line, we have integrated by parts and used the fact

$$
|\nabla_{\hat{g}}\lambda_\kappa - \lambda_\kappa^2 \nabla_{\hat{g}}\varphi|_{\hat{g}} = |\lambda_k - \lambda'_k||\nabla_{\hat{g}} u|_{\hat{g}} = O(e^{-\frac{t}{n-1}}). \tag{5.12}
$$

Meanwhile, we infer from (2.4), (5.9), (5.10) and (5.12) that

$$
\begin{aligned}
-\int_{\Sigma_t} p d\mu &= \int_{\Sigma_t} (V_{\kappa,m} - \langle \bar{\nabla} V_{\kappa,m}, \nu\rangle) d\mu - \int_{\Sigma_t} V_{\kappa,m} d\mu \\
&\geq \int_{\Sigma_t} (V_{\kappa,m} - |\bar{\nabla} V_{\kappa,m}|) d\mu - \int_{\Sigma_t} V_{\kappa,m} d\mu \\
&= \frac{\kappa}{2} \int_N \lambda_\kappa^{n-2} d\mu_{\hat{g}} - \int_N \lambda_\kappa^n \left(1 + \frac{1}{2}\kappa\lambda_\kappa^{-2} + \frac{1}{2}\lambda_\kappa^{-4}|\nabla\lambda_\kappa|^2\right) d\mu_{\hat{g}} + O(e^{\frac{n-3}{n-1}t}) \\
&= -\int_N \lambda_\kappa^n \left(1 + \frac{1}{2}\lambda_\kappa^{-4}|\nabla\lambda_\kappa|^2\right) d\mu_{\hat{g}} + O(e^{\frac{n-3}{n-1}t}) \qquad (5.13)
\end{aligned}
$$

(5.11) and (5.13) imply that (5.7) is reduced to prove

$$
(n-1)\kappa \int_N \lambda_\kappa^{n-2} + \frac{n-1}{2} \int_N \lambda_\kappa^{n-4} |\nabla\lambda_\kappa|^2 \geq (n-1)\kappa \vartheta_{n-1}{}^{\frac{1}{n-1}} \left(\int_N \lambda_\kappa^{n-1}\right)^{\frac{n-2}{n-1}}. \qquad (5.14)
$$

When $\kappa = 1$, it was already observed in [9] that (5.14) is a non-sharp version of Beckner's Sobolev type inequality, Lemma 5.4. When $\kappa = -1$, by the Hölder inequality, we have

$$
\int_N \lambda_\kappa^{n-2} \leq \vartheta_{n-1}{}^{\frac{1}{n-1}} \left(\int_N \lambda_\kappa^{n-1}\right)^{\frac{n-2}{n-1}},
$$

which implies (5.14). When $\kappa = 0$, (5.14) is trivial. Hence we show (5.7) and complete the proof. □

Lemma 5.4 ([2]) *For every positive function f on $\mathbb{S}^{n-1}$, we have*

$$
\begin{aligned}
&(n-1) \int_{\mathbb{S}^{n-1}} f^{n-2} dvol_{\mathbb{S}^{n-1}} + \frac{n-2}{2} \int_{\mathbb{S}^{n-1}} f^{n-4} |\nabla f|^2_{g_{\mathbb{S}^{n-1}}} dvol_{\mathbb{S}^{n-1}} \\
&\geq (n-1)\omega_{n-1}^{\frac{1}{n-1}} \left(\int_{\mathbb{S}^{n-1}} f^{n-1} dvol_{\mathbb{S}^{n-1}}\right)^{\frac{n-2}{n-1}}.
\end{aligned}
$$

Proof Theorem 4 in [2] gives that

$$
\begin{aligned}
&(n-1) \int_{\mathbb{S}^{n-1}} w^2 dvol_{\mathbb{S}^{n-1}} + \frac{2}{n-2} \int_{\mathbb{S}^{n-1}} |\nabla w|^2_{g_{\mathbb{S}^{n-1}}} dvol_{\mathbb{S}^{n-1}} \\
&\geq (n-1)\omega_{n-1}^{\frac{1}{n-1}} \left(\int_{\mathbb{S}^{n-1}} w^{\frac{2(n-1)}{n-2}} dvol_{\mathbb{S}^{n-1}}\right)^{\frac{n-2}{n-1}}.
\end{aligned}
$$

for every positive smooth function w. Set $w = f^{\frac{n-2}{2}}$, one gets the desired result. □

Remark 5.5 It is easy to see that the above inequality holds also on the quotients of spherical space form.

Proof of Theorem 5.1 Note that $|\partial M| = \rho_0^{n-1}\vartheta_{n-1}$. The inequality (5.1) follows directly from Lemma 5.2 and Lemma 5.3. When the equality holds, we have the equality in (5.3), which forces $|A|^2 = \frac{1}{n-1}H^2$ and hence Σ is umbilic. When $m \neq 0$, an umbilic hypersurface must be a slice $\{\rho\} \times N$. When $m = 0$, it follows from the equality case in (5.14) that λ_κ is constant, which implies again Σ is a slice $\{\rho\} \times N$. □

We now prove another version of Alexandrov–Fenchel inequalities, which is applicable to prove Penrose inequalities.

Theorem 5.6 *Let Σ be a compact embedded hypersurface which is star-shaped with positive mean curvature in $(\rho_0 = \rho_{\kappa,m}, \infty) \times N^{n-1}$. Let Ω be the region bounded by Σ and the horizon $\partial M = \{\rho_0\} \times N$. Then*

$$\int V_{\kappa,m} H d\mu \geq (n-1)\kappa\vartheta_{n-1}\left(\left(\frac{|\Sigma|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}} - \left(\frac{|\partial M|}{\vartheta_{n-1}}\right)^{\frac{n-2}{n-1}}\right)$$
$$+(n-1)\vartheta_{n-1}\left(\left(\frac{|\Sigma|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}} - \left(\frac{|\partial M|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}}\right).$$

Equality holds if and only if $\Sigma = \{\rho\} \times N$ for some $\rho \in [\rho_{\kappa,m}, \infty)$.

Proof To simplify the notation, we define

$$\mathbb{J}(\Sigma_t) := n\int_{\Omega_t} V_{\kappa,m} dvol \quad \text{and} \quad \mathbb{K}(\Sigma_t) := \vartheta_{n-1}\left(\left(\frac{|\Sigma_t|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}} - \left(\frac{|\partial M|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}}\right).$$

By (5.5) and (5.6), we have

$$\frac{d}{dt}\int_{\Omega_t} nV_{\kappa,m} dvol = \int_{\Sigma_t} n\frac{V_{\kappa,m}}{H} d\mu$$
$$\geq \frac{n^2}{n-1}\int_{\Omega_t} V_{\kappa,m} dvol + \frac{n}{n-1}\rho_0^n\vartheta_{n-1}.$$

Hence

$$\frac{d}{dt}\left(n\int_{\Omega_t} V_{\kappa,m} dvol + \rho_0^n\vartheta_{n-1}\right) \geq \frac{n}{n-1}\left(n\int_{\Omega_t} V_{\kappa,m} dvol + \rho_0^n\vartheta_{n-1}\right).$$

Taking into account of Lemma 4.1 (1), we find that

$$\frac{d}{dt}\frac{\mathbb{J}(\Sigma_t) - \mathbb{K}(\Sigma_t)}{\left(\frac{|\Sigma_t|}{\vartheta_{n-1}}\right)^{\frac{n}{n-1}}} \geq 0. \tag{5.15}$$

It suffices to show when the initial surface Σ satisfies

$$\mathbb{J}(\Sigma) \leq \mathbb{K}(\Sigma), \tag{5.16}$$

otherwise the assertion follows directly from Theorem 5.1. By the monotonicity (5.15), we divide the proof into two cases.

Case 1 There exists some $t_1 \in (0, \infty)$ such that

$$\mathbb{J}(\Sigma_{t_1}) - \mathbb{K}(\Sigma_{t_1}) = n \int\limits_{\Omega_t} V_{\kappa,m} dvol + \rho_0^n \vartheta_{n-1} - \vartheta_{n-1} \left(\frac{|\Sigma_{t_1}|}{\vartheta_{n-1}} \right)^{\frac{n}{n-1}} = 0.$$

and

$$\mathbb{J}(\Sigma_t) - \mathbb{K}(\Sigma_t) = n \int\limits_{\Omega_t} V_{\kappa,m} dvol + \rho_0^n \vartheta_{n-1} - \vartheta_{n-1} \left(\frac{|\Sigma_t|}{\vartheta_{n-1}} \right)^{\frac{n}{n-1}} \le 0 \text{ for } t \in [0, t_1].$$

From (5.4), we know that

$$\int\limits_{\Sigma_t} p d\mu - (n-2) m \vartheta_{n-1} - \omega_{n-1} \left(\frac{|\Sigma_t|}{\vartheta_{n-1}} \right)^{\frac{n}{n-1}} \le 0 \text{ for } t \in [0, t_1].$$

For $t \in [0, t_1]$, by (5.3), we check that

$$\begin{aligned}
&\frac{d}{dt} \left(\int\limits_{\Sigma_t} V_{\kappa,m} H d\mu + 2(n-1) m \vartheta_{n-1} - (n-1) \vartheta_{n-1} \left(\frac{|\Sigma_t|}{\vartheta_{n-1}} \right)^{\frac{n}{n-1}} \right) \\
&\le \frac{n-2}{n-1} \int\limits_{\Sigma_t} V_{\kappa,m} H d\mu + 2 \int\limits_{\Sigma_t} p d\mu - n \vartheta_{n-1} \left(\frac{|\Sigma_t|}{\vartheta_{n-1}} \right)^{\frac{n}{n-1}} \\
&= \frac{n-2}{n-1} \left(\int\limits_{\Sigma_t} V_{\kappa,m} H d\mu - (n-1) \vartheta_{n-1} \left(\frac{|\Sigma_t|}{\vartheta_{n-1}} \right)^{\frac{n}{n-1}} \right) \\
&\quad + 2 \int\limits_{\Sigma_t} p d\mu - 2 \vartheta_{n-1} \left(\frac{|\Sigma_t|}{\vartheta_{n-1}} \right)^{\frac{n}{n-1}} \\
&\le \frac{n-2}{n-1} \left(\int\limits_{\Sigma_t} V_{\kappa,m} H d\mu - (n-1) \vartheta_{n-1} \left(\frac{|\Sigma_t|}{\vartheta_{n-1}} \right)^{\frac{n}{n-1}} + 2(n-1) m \vartheta_{n-1} \right).
\end{aligned}$$

Hence the quantity

$$Q_2(t) := \frac{\int_{\Sigma_t} V_{\kappa,m} H d\mu + 2(n-1) m \vartheta_{n-1} - (n-1) \vartheta_{n-1} \left(\frac{|\Sigma_t|}{\vartheta_{n-1}} \right)^{\frac{n}{n-1}}}{\left(\frac{|\Sigma_t|}{\vartheta_{n-1}} \right)^{\frac{n-2}{n-1}}}$$

is nonincreasing for $t \in [0, t_1]$. Using (1.21) and Theorem 5.1, we obtain

$$Q_2(0) \ge Q_2(t_1) = Q_1(t_1) \ge (n-1) \kappa \vartheta_{n-1}.$$

Case 2 For all $t \in [0, \infty)$, we have

$$\mathbb{J}(\Sigma_t) - \mathbb{K}(\Sigma_t) \le 0.$$

From above, we know that $Q_2(t)$ is monotone non-increasing in $t \in [0, \infty)$. Thus it suffices to show that

$$\liminf_{t \to \infty} Q_2(t) \ge (n-1) \kappa \vartheta_{n-1}^{\frac{1}{n-1}}. \tag{5.17}$$

By the Hölder inequality and (5.10) we have

$$\vartheta_{n-1}\left(\frac{|\Sigma(t)|}{\vartheta_{n-1}}\right)^{n/(n-1)} \leq \int_N (\sqrt{\det(g)})^{n/(n-1)}$$
$$= \int_N \lambda_\kappa^n \left(1 + \frac{n}{2(n-1)}\lambda_\kappa^{-4}|\nabla\lambda_\kappa|^2 + O(e^{-\frac{4t}{n-1}})\right). \tag{5.18}$$

Combining (5.9) and (5.18), we note that (5.17) is reduced to prove

$$(n-1)\kappa\int_N \lambda_\kappa^{n-2} + \frac{n-2}{2}\int_N \lambda_\kappa^{n-4}|\nabla\lambda_\kappa|^2 \geq (n-1)\kappa\,\vartheta_{n-1}{}^{\frac{1}{n-1}}\left(\int_N \lambda_\kappa^{n-1}\right)^{\frac{n-2}{n-1}}. \tag{5.19}$$

When $\kappa = 1$, (5.19) follows from the sharp version of Beckner's Sobolev type inequality on $\mathbb{S}^{n-1}$. See also Remark 5.5. When $\kappa = -1$, by the Hölder inequality, we have

$$\int_N \lambda_\kappa^{n-2} \leq \vartheta_{n-1}{}^{\frac{1}{n-1}}\left(\int_N \lambda_\kappa^{n-1}\right)^{\frac{n-2}{n-1}},$$

which implies (5.19). When $\kappa = 0$, (5.19) is trivial. Hence we show (5.17). It is easy to show that equality implies that Σ is geodesic. We complete the proof. □

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Appendix A: Examples of ALH graphs

We begin this appendix by showing that any Kottler space $P_{\kappa,m}(m > m_c)$ with metric (2.1), i.e.

$$g_{\kappa,m} = \frac{d\rho^2}{V_{\kappa,m}^2(\rho)} + \rho^2\hat{g}, \quad V_{\kappa,m} = \sqrt{\rho^2 + \kappa - \frac{2m}{\rho^{n-2}}}. \tag{6.1}$$

can be represented as an ALH graph over another Kottler space $P_{\kappa,m'}$ $(m_c \leq m' < m)$ in the ambient space $Q_{\kappa,m'} = \mathbb{R} \times P_{\kappa,m'}$, which is equipped with the Riemannian metric

$$V_{\kappa,m'}(\rho)^2 dt^2 + g_{\kappa,m'}.$$

Obviously one only needs to find a rotational symmetric function $f = f(\rho)$ satisfying

$$\left(\rho^2 + \kappa - \frac{2m'}{\rho^{n-2}}\right)\left(\frac{\partial f}{\partial\rho}\right)^2 = \frac{1}{\rho^2 + \kappa - \frac{2m}{\rho^{n-2}}} - \frac{1}{\rho^2 + \kappa - \frac{2m'}{\rho^{n-2}}}.$$

$m' < m$ implies that the right hand side is positive for $\rho > 0$. Let $\rho_0 := \rho_{\kappa,m}$ be the largest positive root of

$$\phi(\rho) := \rho^2 + \kappa - \frac{2m}{\rho^{n-2}} = 0.$$

When ρ approaches ρ_0, we have $\frac{\partial f}{\partial \rho} = O((\rho - \rho_0)^{-\frac{1}{2}})$, so that one can solve that

$$f(\rho) = \int_{\rho_0}^{\rho} \frac{1}{\sqrt{s^2 + \kappa - \frac{2m'}{s^{n-2}}}} \sqrt{\frac{1}{s^2 + \kappa - \frac{2m}{s^{n-2}}} - \frac{1}{s^2 + \kappa - \frac{2m'}{s^{n-2}}}} \; ds.$$

Its horizon is $\{\{\rho_0\} \times N : \rho_0^n + \kappa \rho_0^{n-2} = 2m\}$ which implies (1.21). Also one can check directly that the ALH mass (1.6) of the Kottler space (6.1) is exactly m.

With the same method, one can represent all rotationally symmetric graphs (with horizon) over $P_{\kappa,m}$ in $Q_{\kappa,m}$ as rotationally symmetric graphs over $P_{\kappa,m'}$ in $Q_{\kappa,m'}$ for $m' < m$. We believe that this statement is also true for non-rotationally symmetric graphs, i.e., all graphs over $P_{\kappa,m}$ in $Q_{\kappa,m}$ can be represented as graphs over $P_{\kappa,m'}$ in $Q_{\kappa,m'}$ for $m' < m$.

In the next example, we show that for any $m > m_c$ there are ALH graphs over $P_{\kappa,m'}$ in $Q_{\kappa,m'}$ ($m_c \le m' < m$) with a horizon and the dominant condition $R + n(n-1) \ge 0$, which can not be represented as ALH graphs in $Q_{\kappa,m}$, and can also not be represented as ALH graphs in $Q_{\kappa,m''}$ for $m'' > m$.

We consider a class of metrics which are perturbation of the Kottler–Schwarzschild spaces. For this purpose, let $(N^{n-1}, \hat{g})$ be a closed space form of constant sectional curvature $\kappa = -1$. Fixing $t \in (-\infty, 1)$. From now we consider a family of metrics

$$g_{m,a} = \frac{d\rho^2}{V_{m,a}^2(\rho)} + \rho^2 \hat{g}, \quad V_{m,a} = \sqrt{\rho^2 - 1 - \frac{2m}{\rho^{n-2}} - \frac{a}{\rho^{n-t}}}.$$

Here the parameter m belongs to the following interval

$$m \in [m_c, +\infty) \quad \text{and} \quad m_c = -\frac{(n-2)^{\frac{n-2}{2}}}{n^{\frac{n}{2}}}. \tag{6.2}$$

and the parameter $a \le 0$. When $a = 0$, they are just the Kottler–Schwarzschild spaces. Let $\rho_{m,a}$ be the largest positive root of

$$\rho^2 - 1 - \frac{2m}{\rho^{n-2}} - \frac{a}{\rho^{n-t}} = 0.$$

It is clear that $\rho_{m,a}$ is increasing in m and in a, provided it is well defined.
As in Sect. 2, by a change of variable $r = r(\rho)$ with

$$r'(\rho) = \frac{1}{V_{m,a}(\rho)}, \quad r(\rho_{m,a}) = 0$$

we write the above metric in the warped product form on $[0, \infty) \times N$ as follows

$$g_{m,a} := \bar{g} := dr^2 + \lambda_{m,a}(r)^2 \hat{g}, \tag{6.3}$$

where $\lambda_{m,a} : [0, \infty) \to [\rho_{m,a}, \infty)$ is the inverse of $r(\rho)$, i.e., $\lambda_{m,a}(r(\rho)) = \rho$. For simplicity, we omit sometimes the subscripts m, a if there is no confusion. It is easy to check that

$$\begin{aligned} Ric(\bar{g}) = &-\left((n-1) - (n-2)m\lambda^{-n} + \frac{-n-t+4}{2} a\lambda^{t-2-n}\right)\bar{g} \\ &-(n-2)\left(nm\lambda^{-n} + \frac{n-t+2}{2} a\lambda^{t-2-n}\right) dr^2 \\ R(\bar{g}) = &-n(n-1) + (n-1)(t-2)a\lambda^{t-2-n}. \end{aligned}$$

As a consequence, we get
Fact 1 For all $m > m_c$ and $a < 0$, we have

$$R(\bar{g}) + n(n-1) > 0$$

When $a = 0$, then

$$R(\bar{g}) + n(n-1) \equiv 0.$$

Moreover, for all $m > m_c$ and $a \le 0$ close to 0 in order to well define $\rho_{m,a}$, we have $V_{m,a}|R(\bar{g}) + n(n-1)|$ is integrable.

By the definition of $\rho_{m,a}$, we know that

$$\lambda''_{m,a}(r) \ge 0 \quad \text{for} \quad r \in [0, \infty)$$

and

$$\lambda_{m,a}(r) = O(e^r) \quad \text{as} \quad r \to +\infty.$$

An immediate result is the following.

Fact 2 For all $m > m_c$, $a \le 0$ and $|a|$ is sufficiently small, $g_{m,a}$ is an ALH metric and has a horizon $\{\rho_{m,a}\} \times N$. Moreover, its ALH mass is exactly m.

Now we consider the metric $g_{m,a}$ as a graph over some Kottler–Schwarzschild space $g_{m_1,0}$ with $m_1 < m$. More precisely, we have

Fact 3 For all $m > m_c$, there exist $b < 0$ and $m_1 \in (m_c, m)$ such that for any $a \in [b, 0)$ and for any $\rho > \rho_{m,a}$ there holds

$$V_{m_1,0}(\rho) > V_{m,a}(\rho).$$

To show this fact, we first observe that for all $\rho > 1$,

$$V_{m+a/2,0}(\rho) > V_{m,a}(\rho).$$

We fix $\epsilon_1 \in (0, (m - m_c)/2)$ and set $m_1 = m - \epsilon_1$. It is clear for all $a \in (-2\epsilon_1, 0)$ and for all $\rho > 1$

$$V_{m-\epsilon_1,0}(\rho) > V_{m,a}(\rho).$$

On the other hand, for all $a \in (-2\epsilon_1, 0)$, there holds $\rho_{m,a} > \rho_{m,-2\epsilon_1} > 0$ provided they are well defined. If $\rho_{m,-2\epsilon_1} > 1$ we are done with $b = -\epsilon_1$. If $\rho_{m,-2\epsilon_1} \le 1$, we could choose $b \in (-2\epsilon_1, 0)$ with the small absolute value such that for all $\rho \in (\rho_{m,-2\epsilon_1}, 1]$ we have

$$V_{m-\epsilon_1,0}(\rho) > V_{m,b}(\rho).$$

Now we take $m_1 = m - \epsilon_1$ and Fact 3 follows.

By Fact 3, as the beginning of this appendix, we see that the metric $g_{m,a}$ could be written as a rotationally symmetric ALH graph over P_{-1,m_1} in Q_{-1,m_1} (recall P_{-1,m_1} and Q_{-1,m_1} are defined in Section 1).

Fact 4 For all $m > m_c$, $a < 0$ and $|a|$ is sufficiently small, the metric $g_{m,a}$ on $[\rho_{m,a}, \infty) \times N$ can not be realized as a graph over $P_{-1,m}$ in $Q_{-1,m}$ with a horizon.

Suppose that the fact were not true, ie. $g_{m,a}$ would be represented as an ALH graph over $P_{-1,m}$ in $Q_{-1,m}$. It follows that the horizon $(\{\rho_{m,a}\} \times N, g_{m,a|\{\rho_{m,a}\}\times N})$ has volume large than or equal to the volume $\left|\{\rho_{m,0}\} \times N\right|$. This contradicts the fact $\rho_{m,a} < \rho_{m,0}$. It is clear that it can also not be realized as a graph in $Q_{-1,m''}$ with $m'' > m$.

Fact 5 For all $m > m_c$, there exist $m_2 > m$, $m_1 < m$ and $a < 0$ such that the metric $g_{m_2,a}$ on $(\rho_{m_2,a}, \infty) \times N$ can not be realized as a graph over $P_{-1,m}$ in $Q_{-1,m}$ with a horizon, but it can be realized as a graph over P_{-1,m_1} in Q_{-1,m_1} with a horizon. Recall that the metric $g_{m_2,a}$ has ALH mass m_2.

In view of Facts 3 and 4, there exists $a < 0$ and $m_1 < m$, such that $\rho_{m_1,0} < \rho_{m,a} < \rho_{m,0}$ and for $\rho > \rho_{m,a}$, $V_{m_1,0}(\rho) > V_{m,a}(\rho)$ holds. Fixing such a, we can choose $m_2 > m$ such that $\rho_{m_1,0} < \rho_{m_2,a} < \rho_{m,0}$ and for $\rho > \rho_{m_2,a}$, $V_{m_1,0}(\rho) > V_{m_2,a}(\rho)$ holds. Hence Fact 5 yields. In particular, when $m = 0$, we can find some metric with positive ALH mass, which can not be realized as a graph over $P_{-1,0}$ in $Q_{-1,0}$ with a horizon, but it can be realized as a graph over P_{-1,m_1} in Q_{-1,m_1} with a horizon. Here $m_1 < 0$.

In particular, fact 5 provides examples of ALH metrics with positive ALH mass, which can be represented an ALH graph over $P_{-1,m'}$ with $m' < 0$, but can be not represented as an ALH graph over $P_{-1,0}$.

Since the above metrics have $R + n(n-1) > 0$, one can perturb these metrics to obtain non-rotationally symmetric ALH graphs with similar properties.

Appendix B: Definitions of ALH graphs

In this appendix we show for a graph over $P_{\kappa,m}\backslash\Omega$ in $Q_{\kappa,m}$, Definition 1.1 and Definition 1.2 are equivalent.

Proposition 7.1 *A graph over $P_{\kappa,m}\backslash\Omega$ in $Q_{\kappa,m}$ is an ALH graph in the sense of Definition 1.2 if and only if it is an ALH manifold in the sense of Definition 1.1.*

Proof We prove the "only if" part. Since the Kottler–Schwarzschild space $(P_{\kappa,m}, g_{\kappa,m})$ is ALH in the sense of Definition 1.1, there exists a compact subset $K_0 \subset P_{\kappa,m}$ and a diffeomorphism at infinity $\Phi_0 : P_{\kappa,m} \setminus K_0 \to N \times (\rho_0, +\infty) \subset P_\kappa$, where $\rho_0 > 1$ such that

$$\|(\Phi_0^{-1})^* g_{\kappa,m} - b_\kappa\|_{b_\kappa} + \|\nabla^{b_\kappa}\left((\Phi_0^{-1})^* g_{\kappa,m}\right)\|_{b_\kappa} = O(\rho^{-\tau}), \quad \tau > \frac{n}{2}. \tag{7.1}$$

Since $(\mathcal{M}^n, g)$ is an ALH graph over $P_{\kappa,m}\backslash\Omega$ in the sense of Definition 1.2, there exists a compact subset K and a diffeomorphism at infinity $\Phi_1 : \mathcal{M} \setminus K \to N \times (\tilde{\rho}_0, +\infty) \subset P_{\kappa,m}\backslash(K_0 \cup \Omega)$ such that

$$\|(\Phi_1^{-1})^* g - g_{\kappa,m}\|_{g_{\kappa,m}} + \|\bar{\nabla}\left((\Phi_1^{-1})^* g\right)\|_{g_{\kappa,m}} = O(\tilde{\rho}^{-\tau}), \quad \tau > \frac{n}{2}, \tag{7.2}$$

where $\tilde{\rho}$ is such that $(y, \tilde{\rho}) = \Phi_0^{-1}(x, \rho) \in N \times (\tilde{\rho}_0, \infty)$. Define $\Phi : \mathcal{M} \setminus K \to N \times (\rho_0, +\infty) \subset P_\kappa$ by $\Phi = \Phi_0 \circ \Phi_1$, then it is easy to see from (7.1) and (7.2) that

$$\|(\Phi^{-1})^* g - b_\kappa\|_{b_\kappa} + \|\nabla^{b_\kappa}\left((\Phi^{-1})^* g\right)\|_{b_\kappa} = O(\rho^{-\tau}), \quad \tau > \frac{n}{2}.$$

The integrability condition (1.5) follows directly from (1.15), since at infinity, $V_{\kappa,m}$ and V_κ are comparable. The "if" part can be proved in a similar way. □

References

1. Alías, L.J., de Lira, J.H.S., Malacarne, J.M.: Constant higher-order mean curvature hypersurfaces in Riemannian spaces. J. Inst. Math. Jussieu **5**(4), 527–562 (2006)
2. Beckner, W.: Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality. Ann. Math. **138**, 213–242 (1993)
3. Bray, H.L.: Proof of the Riemannian Penrose inequality using the positive mass theorem. J. Differ. Geom. **59**(2), 177–267 (2001)
4. Bray, H.L.: On the positive mass, Penrose, an ZAS inequalities in general dimension. In: Surveys in Geometric Analysis and Relativity, Adv. Lect. Math. (ALM), vol. 20. Int. Press, Somerville (2011)
5. Bray, H.L., Chruściel, P.T.: The Penrose Inequality, The Einstein Equations and the Large Scale Behavior of Gravitational Fields. Birkhäuser, Basel, pp. 39–70 (2004)
6. Bray, H.L., Khuri, M.A.: P.D.E'.s which imply the Penrose conjecture. Asian J. Math. **15**(4), 557–610 (2011)
7. Bray, H.L., Lee, D.A.: On the Riemannian Penrose inequality in dimensions less than eight. Duke Math. J. **148**(1), 81–106 (2009)
8. Brendle, S.: Hypersurfaces of constant mean curvature in deSitter–Schwarzschild space. Pub. Math. de l'IHES. **117**(1), 247–269 (2013)
9. Brendle, S., Hung, P.-K., Wang, M.-T.: A Minkowski-type inequality for hypersurfaces in the Anti-deSitter–Schwarzschild manifold. arXiv:1209.0669
10. Chruściel, P., Herzlich, M.: The mass of asymptotically hyperbolic Riemannian manifolds. Pac. J. Math. **212**(2), 231–264 (2003)
11. Chruściel, P., Nagy, G.: The mass of spacelike hypersurface in asymptotically anti-de Sitter space–times. Adv. Theor. Math. Phys. **5**, 697–754 (2002). gr-qc/0110014
12. Chruściel, P., Simon, W.: Towards the classification of static vacuum spacetimes with negative cosmological constant. J. Math. Phys. **42**(4), 1779–1817 (2001)
13. Dahl, M., Gicquaud, R., Sakovich, A.: Penrose type inequalities for asymptotically hyperbolic graphs. Annales Henri Poincaré **14**(5), 1135–1168 (2013). doi:10.1007/s00023-012-0218-4
14. de Lima, L.L., Girão, F.: The ADM mass of asymptotically flat hypersurfaces. Trans. Amer. Math. Soc. (to appear). arXiv:1108.5474
15. de Lima, L.L., Girão, F.: Positive mass and Penrose type inequalities for asymptotically hyperbolic hypersurfaces. arXiv:1201.4991
16. de Lima, L.L., Girão, F.: An Alexandrov–Fenchel-type inequality in hyperbolic space with an application to a Penrose inequality. arXiv:12090669
17. de Lima, L.L., Girão, F.: A penrose inequalities for asymptotically locally hyperbolic graphs. arXiv:1304.7882
18. Ding, Q.: The inverse mean curvature flow in rotationally symmetric spaces. Chin. Ann. Math. Ser. B **32**(1), 27–44 (2011)
19. Freire, A., Scwartz, F.: Mass-capacity inequalities for conformally flat manifolds with boundary. Comm. Partial. Differ. Equ, **39**(1), 98–119 (2014)
20. Gerhardt, C.: Inverse curvature flows in hyperbolic space. J. Differ. Geom. **89**(3), 487–527 (2011)
21. Ge, Y., Wang, G., Wu, J.: A new mass for asymptotically flat manifolds. arXiv:1211.3645
22. Ge, Y., Wang, G., Wu, J.: Hyperbolic Alexandrov–Fenchel quermassintegral inequalities I. arXiv:1303.1714
23. Ge, Y., Wang, G., Wu, J.: Hyperbolic Alexandrov–Fenchel quermassintegral inequalities II. J. Differ. Geom. (to appear). arXiv:1304.1417
24. Ge, Y., Wang, G., Wu, J.: The GBC mass for asymptotically hyperbolic manifolds. arXiv:1306.4233
25. Gibbons, G.W.: Some comments on gravitational entropy and the inverse mean curvature flow. Class. Quantum Gravity **16**(6), 1677–1687 (1999)
26. Herzlich, M.: A Penrose-like inequality for the mass of Riemannian asymptotically flat manifolds. Commun. Math. Phys. **188**, 121–133 (1997)
27. Herzlich, M.: Mass formulae for asymptotically hyperbolic manifolds. In: AdS/CFT Correspondence: Einstein Metrics and Their Conformal Boundaries, pp. 103–121. Eur. Math. Soc, Zürich (2005)
28. Huisken, G., Ilmanen, T.: The inverse mean curvature flow and the Riemannian Penrose inequality. J. Differ. Geom. **59**, 353–437 (2001)
29. Huang, L.-H., Wu, D.: The equality case of the Penrose inequality for asymptotically flat graphs. Trans. Amer. Math. Soc. (to appear). arXiv:1205.2061
30. Lam, M.-K.G.: The graph cases of the Riemannian positive mass and Penrose inequality in all dimensions. arXiv.org/1010.4256

31. Li, H., Wei, Y., Xiong, C.: A geometric ineqality on hypersurface in hyperbolic space. Adv. Math. **253**, 152–162 (2014)
32. Li, H., Wei, Y., Xiong, C.: The Gauss–Bonnet–Chern mass for graphic manifolds. Ann. Global. Anal. Geom. **45**(4) 251–266 (2014)
33. Lee, D., Neves, A.: A static uniqueness theorem for higher genus Kottler metrics, slides of a talk delivered at Tsinghua Sanya International Mathematical Forum
34. Lee, D., Neves, A.: The Penrose inequality for asymptotically locally hyperbolic spaces with nonpositive mass. arXiv:1310.3002
35. Mars, M.: Topical review: present status of the Penrose inequality. Class. Quantum Gravity **26**(19), 193001 (2009)
36. Mirandola, H., Vitório, F.: The positive mass Theorem and Penrose inequality for graphical manifolds. arXiv:1304.3504
37. Michel, B.: Geometric invariance of mass-like asymptotic invariants. J. Math. Phys. **52**(5), 052504 (2011)
38. Neves, A.: Insufficient convergence of inverse mean curvature flow on asymptotically hyperbolic manifolds. J. Differ. Geom. **84**(1), 711–748 (2010)
39. Schneider, R.: Convex Bodies: the Brunn–Minkowski Theory. Cambridge University, Cambridge, MR1216521 (1993)
40. Schoen, R., Yau, S.-T.: On the proof of the positive mass conjecture in general relativity. Commun. Math. Phys. **65**, 45–76 (1979)
41. Wang, X.: Mass for asymptotically hyperbolic manifolds. J. Differ. Geom. **57**, 273–299 (2001)
42. Wang, G., Xia, C.: Isoperimetric type problems and Alexandrov–Fenchel type inequalities in the hyperbolic space. Adv. Math. **259**, 532–556 (2014)
43. Witten, E.: A new proof of the positive energy theorem. Commun. Math. Phys. **80**, 381–402 (1981)