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Calculus of Variations

Form-type equations on Kähler manifolds of nonnegative orthogonal bisectional curvature

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Abstract In this paper we prove the existence and uniqueness of the form-type equation on Kähler manifolds of nonnegative orthogonal bisectional curvature.

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1 Introduction

In the previous paper [2], we introduced the form-type Calabi–Yau equation on a compact complex n-dimensional manifold with a balanced metric and with a non-vanishing holomorphic n-form Ω. A balanced metric ω on X is a hermitian metric such that $d\omega^{n-1} = 0$. Given a balanced metric ω_0 on X, let us denote by $\mathcal{P}(\omega_0)$ the set of all smooth real $(n-2, n-2)$-forms ψ such that $\omega_0^{n-1} + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\psi > 0$ on X. Then, for each $\varphi \in \mathcal{P}(\omega_0)$, there exists a balanced metric, which we denote by ω_φ, such that $\omega_\varphi^{n-1} = \omega_0^{n-1} + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\varphi$. We say that such a metric ω_φ is in the balanced class of ω_0. Our aim is to find a balanced metric ω_φ in the balanced class of ω_0 such that

$$\|\Omega\|_{\omega_\varphi} = \text{a constant } C_0 > 0. \tag{1.1}$$

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The geometric meaning of such a metric is that its Ricci curvatures of the hermitian connection and the spin connection are zero. On the other hand, the direct non-Kähler analogue of the Calabi conjecture has recently been solved by Tosatti–Weinkove [9] (see also [5], and the references in [5,9]). In general their solutions provide hermitian Ricci-flat metrics which are not balanced.

As in the Kähler case, Eq. (1.1) can be reformulated in the following form

$$\frac{\omega_\varphi^n}{\omega_0^n} = e^f \frac{\int_X \omega_\varphi^n}{\int_X \omega_0^n}, \tag{1.2}$$

where $f \in C^\infty(X)$ is given and satisfies the compatibility condition:

$$\int\limits_X e^f \omega_0^n = \int\limits_X \omega_0^n. \tag{1.3}$$

We would like to find a solution $\varphi \in \mathcal{P}(\omega_0)$. The Eq. (1.2) is called a *form-type Calabi–Yau equation*, a reminiscent of the classic function type Calabi–Yau equation. We note here that when $n = 2$, the form type equation is reduced automatically to the classic function type equation and the balanced metric is a Kähler metric. Hence in this case the Eq. (1.2) is the classic Calabi–Yau equation and has been solved by Yau in [10]. Therefore, in the following we assume $n \geq 3$.

We have constructed solutions for (1.1) when X is a complex torus [2]. A natural approach to solve (1.2) is to use the continuity method. The openness and uniqueness were discussed in the previous work [2]. We do not know whether there is a geometric obstruction for solving (1.2) in general.

Equation (1.2) is still meaningful on a compact complex manifold with a balanced metric, whose canonical bundle is not holomorphically trivial. Geometrically, solving (1.2) allows us to solve the problem of prescribed volume form on X, in the balanced class of each balanced metric on X. Namely, given any positive (n, n)-form W on X and a balanced metric ω_0, we let

$$e^f = \left(\frac{W}{\omega_0^n}\right) \frac{\int_X \omega_0^n}{\int_X W};$$

then by solving (1.2) we are able to find a metric ω_φ in the balanced class of ω_0 such that ω_φ^n is equal to W, up to a constant rescaling.

It seems to us very hard to understand Eq. (1.2) in general. In this paper, we want to give the mechanics of looking for all solutions within the balanced class of a given balanced metric. The idea is, in some sense, to transfer the form-type Calabi–Yau equation to a function type equation.

So in the following we let (X, η) be an n-dimensional Kähler manifold, $n \geq 3$, and ω_0 be a balanced metric on X. We let on X

$$\mathcal{P}_\eta(\omega_0) = \left\{ v \in C^\infty(X) \mid \omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial} v \wedge \eta^{n-2} > 0 \right\}.$$

For each $u \in \mathcal{P}_\eta(\omega_0)$, we denote by ω_u the unique positive $(1, 1)$-form on X such that

$$\omega_u^{n-1} = \omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial} u \wedge \eta^{n-2} \quad \text{on } X.$$

Then we consider the equation

$$\frac{\omega_u^n}{\omega_0^n} = e^f \frac{\int_X \omega_u^n}{\int_X \omega_0^n}, \tag{1.4}$$

where $f \in C^\infty(X)$ is given and satisfies the compatibility condition (1.3).

In this paper, we are able to solve (1.4), under the assumption that the Kähler metric η has *nonnegative orthogonal bisectional curvature*; that is, for any orthonormal tangent frame $\{e_1, \ldots, e_n\}$ at any $x \in M$, the curvature tensor of η satisfies that

$$R_{i\bar{i}j\bar{j}} \equiv R(e_i, \bar{e}_i, e_j, \bar{e}_j) \geq 0, \quad \text{for all } 1 \leq i, j \leq n \text{ and } i \neq j. \tag{1.5}$$

We remark that nonnegativity of the orthogonal bisectional curvature is weaker than nonnegativity of the bisectional curvature. In fact, the former condition are satisfied by not only complex projective spaces and the Hermitian symmetric spaces, but also some compact Kähler manifolds of dimension ≥ 2 whose holomorphic sectional curvature is strictly negative somewhere. We refer the reader to the recent work Gu–Zhang [4] for the study of nonnegative orthogonal bisectional curvature, which generalizes the earlier work of Mok [7] and Siu–Yau [8].

Our main result is as follows:

Theorem 1 *Let (X, η) be a compact Kähler manifold of nonnegative orthogonal bisectional curvature, and ω_0 be a balanced metric on X. Then, for any smooth function f on X satisfying (1.3), Eq. (1.4) admits a solution $u \in \mathcal{P}_\eta(\omega_0)$, which is unique up to a constant.*

Subsequently, Eq. (1.2) has a solution $\varphi = u\eta^{n-2} \in \mathcal{P}(\omega_0)$. Now we explain how to use Theorem 1 to find all solutions of (1.2) in the balanced class of ω_0 on a compact Kähler manifold (X, η) of nonnegative orthogonal bisectional curvature. Let ω_ψ, for $\psi \in \mathcal{P}(\omega_0)$, be a balanced metric in the balanced class of ω_0. We then let

$$\mathcal{P}_\eta(\omega_\psi) = \left\{v \in C^\infty(X) \mid \omega_\psi^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}v \wedge \eta^{n-2} > 0\right\}.$$

For each $v \in \mathcal{P}_\eta(\omega_\psi)$, we denote by $\omega_{\psi,v}$ the unique positive (1,1)-form on X such that

$$\omega_{\psi,v}^{n-1} = \omega_\psi^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}v \wedge \eta^{n-2}.$$

Such a $(1, 1)$-form $\omega_{\psi,v}$ is still in the balanced class of ω_0. Then we consider the equation

$$\frac{\omega_{\psi,u}^n}{\omega_\psi^n} = e^{f_\psi} \frac{\int_X \omega_{\psi,u}^n}{\int_X \omega_\psi^n}, \tag{1.6}$$

where $f_\psi \in C^\infty(X)$ is given by

$$e^{f_\psi} = e^f \frac{\omega_0^n}{\omega_\psi^n} \frac{\int_X \omega_\psi^n}{\int_X \omega_0^n}$$

and satisfies the compatibility condition.

Replacing ω_0 with ω_ψ and f with f_ψ in Theorem 1, we show that (1.6) admits a solution, denoted by u_ψ, which is unique up to a constant. It then follows that $\varphi = \psi + u_\psi\eta^{n-2} \in \mathcal{P}(\omega_0)$ is a solution to (1.2). Hence, when we vary $\psi \in \mathcal{P}(\omega_0)$, we obtain all solutions $\varphi = \psi + u_\psi\eta^{n-2}$ (which are *infinitely many*) to Eq. (1.2) in the balanced class of ω_0 on X of nonnegative orthogonal bisectional curvature. In particular, the form-type equation on a complex torus is completely settled in this way.

Corollary 2 *Let (X, η) be a compact Kähler manifold of nonnegative orthogonal bisectional curvature, and ω_0 be a balanced metric on X. Let f be a smooth function on X satisfying (1.3). Then for any $\psi \in \mathcal{P}(\omega_0)$, Eq. (1.2) admits a solution $\varphi = \psi + u_\psi\eta^{n-2}$. Here u_ψ is a solution to (1.6) which is unique up to a constant.*

Thus, the idea used in this paper, which is to transfer from the form-type Calabi–Yau equation to a function-type equation, may be useful. Later we will establish the Theorem 1 on any compact Kähler manifold. We need to overcome some difficulties of estimates.

We employ the continuity method to prove Theorem 1. In Sect. 2, we establish an *a priori* C^2 estimate for the solution u. This is the place where we need the curvature condition. The C^2 estimate enables us to obtain a general *a priori* C^0 estimate, by combining the maximum principle and the weak Harnack inequality. This is the content of Sect. 3. We then adapt the Evans–Krylov theory to our form-type equation, and obtain in Sect. 4 the Hölder estimates for second derivatives. The openness is covered by Theorem 3 in our previous paper [2]. For readers' convenience, we briefly indicate the argument in the last section, Sect. 5. The uniqueness is also proved in Sect. 5.

2 C^2 estimates for form-type equations

In this section, we would like to establish the following estimate:

Lemma 3 *Given $F \in C^2(X)$, let $u \in C^4(X)$ satisfy that*

$$\omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2} > 0 \quad \text{on } X,$$

and that

$$\det\left[\omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2}\right] = e^F \det \omega_0^{n-1}. \tag{2.1}$$

Assume that η has nonnegative orthogonal bisectional curvature. Then, we have

$$\Delta_\eta u \le C + C(u - \inf_X u) \qquad \text{on } X, \tag{2.2}$$

and

$$\sup_X |\omega_0^{n-1} + \partial\bar{\partial}u \wedge \eta^{n-2}|_\eta \le C + \left(\sup_X u - \inf_X u\right).$$

Here $\Delta_\eta v = \sum \eta^{i\bar{j}} v_{i\bar{j}}$ denotes the Laplacian of a function v with respect to η, and $C > 0$ is a constant depending only on $\inf_X(\Delta_\eta F)$, $\sup_X F$, η, n, and ω_0.

Here are some conventions: For an $(n-1, n-1)$-form Θ, we denote

$$\Theta = \left(\frac{\sqrt{-1}}{2}\right)^{n-1} (n-1)! \\ \times \sum_{p,q} s(p,q)\Theta_{p\bar{q}} dz^1 \wedge d\bar{z}^1 \cdots \wedge \widehat{dz^p} \wedge d\bar{z}^p \wedge \cdots \wedge d\bar{z}^q \wedge \widehat{d\bar{z}^q} \wedge \cdots \wedge dz^n \wedge d\bar{z}^n,$$

in which

$$s(p,q) = \begin{cases} -1, & \text{if } p > q; \\ 1, & \text{if } p \le q. \end{cases} \tag{2.3}$$

Here we introduce the sign function s so that,

$$dz^p \wedge d\bar{z}^q \wedge s(p,q) dz^1 \wedge d\bar{z}^1 \cdots \wedge \widehat{dz^p} \wedge d\bar{z}^p \wedge \cdots \wedge d\bar{z}^q \wedge \widehat{d\bar{z}^q} \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\ = dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n, \quad \text{for all} \quad 1 \le p, q \le n.$$

We denote

$$\det \Theta = \det(\Theta_{p\bar{q}}).$$

If the matrix $(\Theta_{p\bar{q}})$ is invertible, we denote by $(\Theta^{p\bar{q}})$ the transposed inverse of $(\Theta_{p\bar{q}})$, i.e.,

$$\sum_l \Theta_{i\bar{l}}\Theta^{j\bar{l}} = \delta_{ij}.$$

Note that, for a positive (1, 1)-form ω given by

$$\omega = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

we have

$$\omega^n = \left(\frac{\sqrt{-1}}{2}\right)^n n! \det(g_{i\bar{j}}) dz^1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n,$$

and by our convention,

$$(\omega^{n-1})_{i\bar{j}} = \det(g_{i\bar{j}}) g^{i\bar{j}}.$$

It follows that

$$\det(\omega^{n-1}) = \det(g_{i\bar{j}})^{n-1}, \tag{2.4}$$

and

$$(\omega^{n-1})^{i\bar{j}} = \frac{g_{i\bar{j}}}{\det(g_{i\bar{j}})}.$$

In the following, the subscripts such as ", p" stand for the ordinary local derivatives; for example,

$$\eta_{i\bar{j},k} = \frac{\partial \eta_{i\bar{j}}}{\partial z^k}, \quad \eta_{i\bar{j},l\bar{m}} = \frac{\partial^2 \eta_{i\bar{j}}}{\partial z^l \partial \bar{z}^m}. \tag{2.5}$$

For a function h we can omit the comma: $h_l = h_{,l}$, $h_{l\bar{m}} = h_{,l\bar{m}}$, etc. Unless otherwise indicated, all the summations below range from 1 to n. We remark that, under the convention, Eq. (1.4) can be rewritten as

$$\frac{\det[\omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2}]}{\det \omega_0^{n-1}} = e^{(n-1)f}\left(\frac{\int_X \omega_u^n}{\int_X \omega_0^n}\right)^{n-1},$$

which is convenient for deriving the estimates.

Proof of Lemma 3 Let

$$\Psi_u = \Psi + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2}, \quad \text{where} \quad \Psi = \omega_0^{n-1}.$$

Let

$$\phi = \frac{\sum_{i,j} \eta_{i\bar{j}} (\Psi_u)_{i\bar{j}}}{\det \eta} - Au,$$

where $A > 0$ is a large constant to be determined. Using wedge products, the function ϕ can also be written as

$$\begin{aligned}\phi &= \frac{n\eta \wedge \Psi_u}{\eta^n} - Au \\ &= (h + \Delta_\eta u) - Au, \quad \text{where } h = \frac{n\eta \wedge \omega_0^{n-1}}{\eta^n}.\end{aligned} \tag{2.6}$$

Consider the operator

$$L\phi = (n-1)\sum_{k,l} \Psi_u^{k\bar{l}} \left(\frac{\sqrt{-1}}{2} \partial\bar{\partial}\phi \wedge \eta^{n-2} \right)_{k\bar{l}}.$$

Suppose that ϕ attains its maximum at some point P in X. We choose a normal coordinate system such that at P, $\eta_{i\bar{j}} = \delta_{ij}$ and $d\eta_{i\bar{j}} = 0$. Then, we rotate the axes so that at P we have $(\Psi_u)_{p\bar{q}} = \delta_{pq}(\Psi_u)_{p\bar{p}}$. Thus, for any smooth function v on X, we have at P that

$$(n-1)\left(\frac{\sqrt{-1}}{2} \partial\bar{\partial} v \wedge \eta^{n-2} \right)_{i\bar{j}} = \delta_{ij} \sum_{p \neq i} v_{p\bar{p}} + (1 - \delta_{ij}) v_{j\bar{i}}. \tag{2.7}$$

By (2.7) we obtain that

$$(\Psi_u)_{i\bar{i}} = \Psi_{i\bar{i}} + \frac{1}{n-1} \sum_{q \neq i} u_{q\bar{q}}, \tag{2.8}$$

$$(\Psi_u)_{i\bar{j}} = \Psi_{i\bar{j}} + \frac{u_{j\bar{i}}}{n-1} = 0, \quad \text{for all } i \neq j. \tag{2.9}$$

It follows that

$$\sum_{i=1}^n (\Psi_u)_{i\bar{i}} = \sum_{i=1}^n \Psi_{i\bar{i}} + \sum_{i=1}^n u_{i\bar{i}} = h + \Delta_\eta u. \tag{2.10}$$

Furthermore, we have

$$(\Psi_u)_{i\bar{j},p} = \Psi_{i\bar{j},p} + \frac{\delta_{ij}}{n-1} \sum_{q \neq i} u_{q\bar{q}p} + \frac{1 - \delta_{ij}}{n-1} u_{j\bar{i}p}, \tag{2.11}$$

and

$$\begin{aligned}(\Psi_u)_{i\bar{i},p\bar{p}} = \Psi_{i\bar{i},p\bar{p}} &+ \frac{1}{n-1} \sum_{k \neq i} u_{k\bar{k}p\bar{p}} + \frac{1}{n-1} \sum_{k \neq i} u_{k\bar{k}} \left(\sum_{j \neq k, j \neq i} \eta_{j\bar{j},p\bar{p}} \right) \\ &- \frac{1}{n-1} \sum_{a \neq i, b \neq i, a \neq b} u_{a\bar{b}} \eta_{b\bar{a},p\bar{p}}.\end{aligned}$$

Note that under the normal coordinate system, the curvature $(R_{i\bar{j}k\bar{l}})$ of η reads

$$R_{i\bar{j}k\bar{l}} = -\eta_{i\bar{j},k\bar{l}} + \sum_{a,b} \eta^{a\bar{b}} \eta_{i\bar{b},k} \eta_{a\bar{j},\bar{l}} = -\eta_{i\bar{j},k\bar{l}}, \quad \text{at} \quad P.$$

This together with (2.9) imply that

$$(\Psi_u)_{i\bar{i},p\bar{p}} = \Psi_{i\bar{i},p\bar{p}} + \frac{1}{n-1}\sum_{k\neq i} u_{k\bar{k}p\bar{p}} - \frac{1}{n-1}\sum_{k\neq i} u_{k\bar{k}}\left(\sum_{j\neq k, j\neq i} R_{j\bar{j}p\bar{p}}\right) - \sum_{a\neq i, b\neq i, a\neq b} \Psi_{a\bar{b}} R_{a\bar{b}p\bar{p}}. \tag{2.12}$$

We compute at P that

$$L\phi = (n-1)\sum_l (\Psi_u)^{l\bar{l}} \left(\frac{\sqrt{-1}}{2}\partial\bar{\partial}\phi \wedge \eta^{n-2}\right)_{l\bar{l}} = \sum_l \sum_{p\neq l} (\Psi_u)^{l\bar{l}} \phi_{p\bar{p}}.$$

Note that

$$0 = \phi_p(P) = h_p + (\Delta_\eta u)_p - Au_p. \tag{2.13}$$

Differentiating once more to obtain that

$$0 \geq \phi_{p\bar{p}}(P) = h_{p\bar{p}} + (\Delta_\eta u)_{p\bar{p}} - Au_{p\bar{p}}.$$

It follows that

$$\begin{aligned} 0 \geq L\phi &= \sum_l \sum_{p\neq l} (\Psi_u)^{l\bar{l}} \phi_{p\bar{p}} \\ &= \sum_l \sum_{p\neq l} (\Psi_u)^{l\bar{l}} [h_{p\bar{p}} + (\Delta_\eta u)_{p\bar{p}}] - A \sum_l \sum_{p\neq l} (\Psi_u)^{l\bar{l}} u_{p\bar{p}}. \end{aligned} \tag{2.14}$$

Notice that

$$\begin{aligned} &\sum_l \sum_{p\neq l} (\Psi_u)^{l\bar{l}} [h_{p\bar{p}} + (\Delta_\eta u)_{p\bar{p}}] \\ &= \sum_l \sum_{p\neq l} (\Psi_u)^{l\bar{l}} h_{p\bar{p}} + \sum_{l,a} \sum_{p\neq l} (\Psi_u)^{l\bar{l}} u_{a\bar{a}p\bar{p}} + \sum_l \sum_{p\neq l} (\Psi_u)^{l\bar{l}} \sum_{a,b} \eta^{a\bar{b}}_{,p\bar{p}} u_{a\bar{b}} \\ &= \sum_l \sum_{p\neq l} (\Psi_u)^{l\bar{l}} h_{p\bar{p}} + \sum_{l,a} \sum_{p\neq l} (\Psi_u)^{l\bar{l}} u_{a\bar{a}p\bar{p}} + \sum_{l,a} \sum_{p\neq l} (\Psi_u)^{l\bar{l}} R_{a\bar{a}p\bar{p}} u_{a\bar{a}} \\ &\quad - (n-1) \sum_l \sum_{p\neq l} \sum_{a\neq b} (\Psi_u)^{l\bar{l}} R_{b\bar{a}p\bar{p}} \Psi_{b\bar{a}}, \qquad \text{(by (2.9))}. \end{aligned} \tag{2.15}$$

Here the fourth derivative term can be handled by the Eq. (2.1): We rewrite (2.1) as

$$\log\det\Psi_u = F + \log\det\Psi.$$

Differentiating this in the direction of $\partial/\partial z^a$ yields

$$\sum_{k,l} (\Psi_u)^{k\bar{l}} (\Psi_u)_{k\bar{l},a} = (F + \log\det\Psi)_a.$$

Then,

$$\sum_{k,l} (\Psi_u)^{k\bar{l}} (\Psi_u)_{k\bar{l},a\bar{b}} = (F + \log\det\Psi)_{a\bar{b}} + \sum_{k,l,p,q} (\Psi_u)^{k\bar{q}} (\Psi_u)^{p\bar{l}} (\Psi_u)_{k\bar{l},a} (\Psi_u)_{p\bar{q},\bar{b}}.$$

Contracting this with $(\eta^{a\bar{b}})$ and applying the normal coordinates yield that

$$\sum_{l,a}(\Psi_u)^{l\bar{l}}(\Psi_u)_{l\bar{l},a\bar{a}} = \sum_a (F+\log\det\Psi)_{a\bar{a}} + \sum_{k,l,a}\frac{\left|(\Psi_u)_{k\bar{l},a}\right|^2}{(\Psi_u)_{l\bar{l}}(\Psi_u)_{k\bar{k}}}.$$

This together with (2.12) imply that

$$\begin{aligned}
&\sum_{l,a}(\Psi_u)^{l\bar{l}}\Psi_{l\bar{l},a\bar{a}} + \frac{1}{n-1}\sum_{l,a}\sum_{p\neq l}(\Psi_u)^{l\bar{l}}u_{p\bar{p}a\bar{a}}\\
&\quad= \sum_{k,l,a}\frac{\left|(\Psi_u)_{k\bar{l},a}\right|^2}{(\Psi_u)_{l\bar{l}}(\Psi_u)_{k\bar{k}}} + \frac{1}{n-1}\sum_{l,a}(\Psi_u)^{l\bar{l}}\sum_{p\neq l}u_{p\bar{p}}\left(\sum_{m\neq p,m\neq l}R_{m\bar{m}a\bar{a}}\right)\\
&\qquad+\sum_{l,a}(\Psi_u)^{l\bar{l}}\left(\sum_{p\neq l,q\neq l,p\neq q}\Psi_{p\bar{q}}R_{p\bar{q}a\bar{a}}\right) + \Delta_\eta F + \Delta_\eta(\log\det\Psi).
\end{aligned}$$

Combining this with (2.15) yields

$$\begin{aligned}
&\sum_{l}\sum_{p\neq l}(\Psi_u)^{l\bar{l}}(h_{p\bar{p}} + (\Delta_\eta u)_{p\bar{p}})\\
&\quad= \sum_{l,a}\sum_{p\neq l}(\Psi_u)^{l\bar{l}}R_{a\bar{a}p\bar{p}}u_{a\bar{a}} + \sum_{l,a}\sum_{p\neq l}(\Psi_u)^{l\bar{l}}u_{p\bar{p}}\left(\sum_{m\neq p,m\neq l}R_{a\bar{a}m\bar{m}}\right)\\
&\qquad+(n-1)\sum_{k,l,a}\frac{\left|(\Psi_u)_{k\bar{l},a}\right|^2}{(\Psi_u)_{l\bar{l}}(\Psi_u)_{k\bar{k}}} + (n-1)\Delta_\eta F + (n-1)\Delta_\eta(\log\det\Psi)\\
&\qquad+\sum_{l}\sum_{p\neq l}(\Psi_u)^{l\bar{l}}h_{p\bar{p}} - (n-1)\sum_{l,a}(\Psi_u)^{l\bar{l}}\Psi_{l\bar{l},a\bar{a}}\\
&\qquad+(n-1)\sum_{l,a}(\Psi_u)^{l\bar{l}}\left(\sum_{p\neq l,q\neq l,p\neq q}\Psi_{p\bar{q}}R_{p\bar{q}a\bar{a}}\right)\\
&\qquad-(n-1)\sum_{l}\sum_{p\neq l}\sum_{a\neq b}(\Psi_u)^{l\bar{l}}R_{a\bar{b}p\bar{p}}\Psi_{a\bar{b}}.
\end{aligned}\tag{2.16}$$

The first two terms on the right hand side of the above inequality can be handled as follows.

$$\begin{aligned}
&\sum_{l,a}\sum_{p\neq l}(\Psi_u)^{l\bar{l}}R_{a\bar{a}p\bar{p}}u_{a\bar{a}} + \sum_{l,a}\sum_{p\neq l}(\Psi_u)^{l\bar{l}}u_{p\bar{p}}\left(\sum_{m\neq p,m\neq l}R_{a\bar{a}m\bar{m}}\right)\\
&\quad= \sum_{l,a}(\Psi_u)^{l\bar{l}}R_{l\bar{l}a\bar{a}}u_{l\bar{l}} - \sum_{l,a}(\Psi_u)^{l\bar{l}}R_{a\bar{a}l\bar{l}}u_{a\bar{a}} + \sum_{l,p}\sum_{a\neq l}(\Psi_u)^{l\bar{l}}u_{a\bar{a}}R_{a\bar{a}p\bar{p}}\\
&\qquad+\sum_{l,a}\sum_{p\neq l}(\Psi_u)^{l\bar{l}}u_{p\bar{p}}\left(\sum_{m\neq p,m\neq l}R_{a\bar{a}m\bar{m}}\right)\\
&\quad= \frac{1}{2}\sum_{l,a}(\Psi_u)^{l\bar{l}}R_{l\bar{l}a\bar{a}}(u_{l\bar{l}} - u_{a\bar{a}}) + \frac{1}{2}\sum_{l,a}(\Psi_u)^{a\bar{a}}R_{l\bar{l}a\bar{a}}(u_{a\bar{a}} - u_{l\bar{l}})
\end{aligned}$$

$$
\begin{aligned}
&+ (n-1)\sum_l \left(\sum_{m\neq l} R_{m\bar m}\right)(\Psi_u)^{l\bar l}\left[(\Psi_u)_{l\bar l} - \Psi_{l\bar l}\right] \quad \text{(by (2.8))}\\
&= \frac{1}{2}\sum_{l,a} R_{l\bar l a\bar a}\frac{(u_{l\bar l} - u_{a\bar a})[(\Psi_u)_{a\bar a} - (\Psi_u)_{l\bar l}]}{(\Psi_u)_{l\bar l}(\Psi_u)_{a\bar a}}\\
&+ (n-1)^2\sum_l R_{l\bar l} - (n-1)\sum_l (\Psi_u)^{l\bar l}\Psi_{l\bar l}\left(\sum_{m\neq l} R_{m\bar m}\right). \qquad (2.17)
\end{aligned}
$$

Apply (2.8) to estimate the first term of last equality

$$
\begin{aligned}
&\frac{1}{2}\sum_{l,a} R_{l\bar l a\bar a}\frac{(u_{l\bar l} - u_{a\bar a})[(\Psi_u)_{a\bar a} - (\Psi_u)_{l\bar l}]}{(\Psi_u)_{l\bar l}(\Psi_u)_{a\bar a}}\\
&= \frac{n-1}{2}\sum_{l,a} R_{l\bar l a\bar a}\frac{[(\Psi_u)_{a\bar a} - (\Psi_u)_{l\bar l}]^2}{(\Psi_u)_{l\bar l}(\Psi_u)_{a\bar a}}\\
&\quad + \frac{n-1}{2}\sum_{l,a} R_{l\bar l a\bar a}\frac{(\Psi_{l\bar l} - \Psi_{a\bar a})[(\Psi_u)_{a\bar a} - (\Psi_u)_{l\bar l}]}{(\Psi_u)_{l\bar l}(\Psi_u)_{a\bar a}}\\
&\geq (n-1)\sum_{l,a} R_{l\bar l a\bar a}\frac{\Psi_{l\bar l} - \Psi_{a\bar a}}{(\Psi_u)_{l\bar l}}, \qquad \text{by curvature assumption (1.5).} \qquad (2.18)
\end{aligned}
$$

Combining (2.16) with (2.17) and then with (2.18), we obtain

$$
\begin{aligned}
&\sum_l \sum_{p\neq l} (\Psi_u)^{l\bar l}(h_{p\bar p} + (\Delta_\eta u)_{p\bar p})\\
&\quad \geq -C_1(n-1)\sum_l (\Psi_u)^{l\bar l} - (n-1)^2 C_1 + (n-1)\inf \Delta_\eta F. \qquad (2.19)
\end{aligned}
$$

Here and throughout this section, we denote by $C_1 > 0$ a generic constant depending only on Ψ and the curvature of η.

Substituting (2.19) into (2.14) yields

$$
\begin{aligned}
0 \geq L\phi &\geq -A\sum_l \sum_{p\neq l} (\Psi_u)^{l\bar l} u_{p\bar p} - C_1(n-1)\sum_l (\Psi_u)^{l\bar l}\\
&\quad - (n-1)^2 C_1 + (n-1)\inf \Delta_\eta F\\
&= -n(n-1)A + (n-1)A\sum_l (\Psi_u)^{l\bar l}\Psi_{l\bar l} - C_1(n-1)\sum_l (\Psi_u)^{l\bar l}\\
&\quad - (n-1)^2 C_1 + (n-1)\inf \Delta_\eta F.
\end{aligned}
$$

Now we choose $A > 0$ sufficiently large so that

$$
A \inf_X (\min_l \Psi_{l\bar l}) \geq 2C_1.
$$

It follows that

$$\frac{nA}{C_1} + (n-1) - \frac{\inf \Delta_\eta F}{C_1} \geq \sum_{l=1}^{n} (\Psi_u)^{l\bar{l}}$$
$$\geq \left[\sum_{i=1}^{n} (\Psi_u)_{i\bar{i}}\right]^{\frac{1}{n-1}} \left\{\det\left[(\Psi_u)_{i\bar{j}}\right]\right\}^{\frac{-1}{n-1}}$$
$$= \left[\sum_{i=1}^{n} (\Psi_u)_{i\bar{i}}\right]^{\frac{1}{n-1}} e^{\frac{-F}{n-1}} (\det \Psi)^{\frac{-1}{n-1}}.$$

Hence,

$$h + \Delta_\eta u = \sum_{i=1}^{n} (\Psi_u)_{i\bar{i}} \leq C_2 \quad \text{at } P.$$

Here and throughout this section, we denote by C_2 a generic positive constant depending only on n, Ψ, η, $\sup \Delta_\eta F$, and $\sup F$. Therefore, at any point in X,

$$(h + \Delta_\eta u) \leq (h + \Delta_\eta u)(P) + Au - Au(P) \leq C_2 + C_2 \left(u - \inf_X u\right).$$

Since $[(\Psi_u)_{i\bar{j}}]$ is positive definite everywhere, we have

$$|(\Psi_u)_{i\bar{j}}| \leq C_2 + C_2(u - \inf_X u), \quad \text{for all } 1 \leq i, j \leq n.$$

This completes the proof. □

Lemma 3 enables us to establish the C^2 estimate for Eq. (1.4):

Corollary 4 *For any $f \in C^\infty(X)$, let $u \in C^\infty(X)$ be a solution of*

$$\frac{\det(\omega_u^{n-1})}{\det(\omega_0^{n-1})} = e^{(n-1)f} \left(\frac{\int_X \omega_u^n}{\int_X e^f \omega_0^n}\right)^{n-1}, \tag{2.20}$$

where ω_u is a positive $(1,1)$-form on X such that

$$\omega_u^{n-1} = \omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial} u \wedge \eta^{n-2} > 0.$$

Assume that η has nonnegative orthogonal bisectional curvature. Then, we have

$$\Delta_\eta u \leq C + C(u - \inf_X u) \quad \text{on } X, \tag{2.21}$$

and

$$\sup_X |\omega_u^{n-1}|_\eta \leq C + C(\sup_X u - \inf_X u),$$

where $C > 0$ is a constant depending only on f, η, n, and ω_0.

Proof Let

$$F = (n-1)\left(f + \log \int_X \omega_u^n - \log \int_X e^f \omega_0^n\right).$$

To apply Lemma 3, it suffices to estimate $\inf(\Delta_\eta F)$ and $\sup F$. Note that

$$\Delta_\eta F = (n-1)\Delta_\eta f.$$

Applying the maximum principle to (2.20) at the points where u attain its maximum and minimum, respectively, yields a uniform bound for the constant:

$$-\sup f \leq \log \int_X \omega_u^n - \log \int_X e^f \omega_0^n \leq -\inf f.$$

This implies that $\sup |F| \leq (n-1)(\sup f - \inf f)$. □

3 C^0 estimates

In this section, we will derive the following general C^0 estimate. This together with Corollary 4 will settle the C^0 estimate for manifolds of nonnegative orthogonal bisectional curvature.

Lemma 5 *Let (X, η) be an arbitrary Kähler manifold with complex dimension $n \geq 2$. Suppose that $u \in C^2(X)$ satisfies*

$$\Delta u \leq C_1 + C_1(u - \inf_X u),$$
$$\Delta u > -C_2,$$

where Δ stands for the Laplacian with respect to η, and C_1, C_2 are two positive constants. Then,

$$\sup_X u - \inf_X u \leq C,$$

in which $C > 0$ is a constant depending only on η, n, C_1, and C_2.

The proof is based on the following maximum principle (Proposition 6) and the weak Harnack inequality (Proposition 7). We denote for $p > 0$,

$$\|h\|_p = \left(\int h^p \eta^n\right)^{1/p}, \quad \text{for all} \quad h \in L^p(X, \eta).$$

Proposition 6 *Let $v \in C^2(X)$, $v > 0$ on X, satisfy that*

$$\Delta v + cv \geq d \quad \text{on } X, \tag{3.1}$$

where c and d are constants. Then, for any real number $p > 0$,

$$\sup_X v \leq C^{1/p}(1 + |c|)^{n/p}(\|v\|_p + |d|),$$

where $C > 0$ is a constant depending only on η and n.

Proposition 7 *Let $v \in C^2(X)$, $v > 0$ on X and satisfy*

$$\Delta v - cv \leq 0 \quad \text{on} \quad X, \tag{3.2}$$

where c is a constant. Then, there exists a real number $p_0 > 0$, depending on η, n, and c, such that

$$\inf_X v \geq C^{-1/p_0}(1 + |c|)^{-n/p_0}\|v\|_{p_0},$$

where $C > 0$ depends only on η and n.

Proposition 6 and Proposition 7 can be proved by Moser's iteration. The arguments are standard (see, for example, [6]). We are in a position to prove Lemma 5.

Proof of Lemma 5 Let

$$v = u - \inf_X u + 1.$$

Since X is compact, u attains its infimum. Then,

$$v \geq 1, \quad \text{and} \quad \inf_X v = 1.$$

On the other hand, we have

$$\Delta v - C_1 v \leq 0, \tag{3.3}$$

and

$$\Delta v > -C_2. \tag{3.4}$$

Applying Proposition 7 to (3.3), we obtain that

$$\inf_X v \geq C^{-1/p_0}(1 + |C_1|)^{-n/p_0}\|v\|_{p_0}.$$

Here $p_0 > 0$ is a number depending only on η, n, and C_1; $C > 0$ is a constant depending only on η and n. Applying Proposition 6 to (3.4) with $p = p_0$ yields that

$$\sup_X v \leq (C')^{1/p_0}(\|v\|_{p_0} + C_2),$$

where $C' > 0$ depends only on η and n. Combining these two inequalities we have

$$\begin{aligned}\sup_X v &\leq (C')^{1/p_0}\left[C^{1/p_0}(1 + |C_1|)^{n/p_0}\inf_X v + C_2\right]\\ &= (C')^{1/p_0}\left[C^{1/p_0}(1 + |C_1|)^{n/p_0} + C_2\right].\end{aligned}$$

It follows that

$$\sup_X u - \inf_X u \leq \sup_X v \leq C,$$

where $C > 0$ depends only on η, n, C_1, and C_2. □

Let us now return to Eq. (1.4). We let (X, η) be the complex n-dimensional Kähler manifold of nonnegative orthogonal bisectional curvature, and ω_0 be a Hermitian metric on X.

Corollary 8 *For any $f \in C^\infty(X)$, let $u \in C^\infty(X)$ be a solution of*

$$\frac{\det(\omega_u^{n-1})}{\det(\omega_0^{n-1})} = e^{(n-1)f}\left(\frac{\int_X \omega_u^n}{\int_X e^f \omega_0^n}\right)^{n-1},$$

where ω_u is a positive $(1,1)$-form such that

$$\omega_u^{n-1} = \omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2} > 0 \quad \text{on } X.$$

Then,

$$\sup_X |\omega_u^{n-1}|_\eta \leq C,$$

where $C > 0$ is a constant depending only on f, η, n, and ω_0.

Proof By Corollary 4, it suffices to estimate $(\sup u - \inf u)$. Contracting

$$\omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2} > 0$$

with η yields that

$$\Delta_\eta u > -\frac{n\eta \wedge \omega_0^{n-1}}{\eta^n} > -C_2 \qquad \text{on } X.$$

Here the constant $C_2 > 0$ depends only on η, n, and ω_0. We have (2.21), on the other hand. Therefore, the result is an immediate consequence of Lemma 5. ◻

4 Hölder estimates for second derivatives

Let X be a n-dimensional Kähler manifold, η be a Kähler metric on X, and ω_0 be a balanced metric on X. We will establish the following estimate.

Lemma 9 *For $F \in C^2(X)$, let $u \in C^4(X)$ satisfy that*

$$\omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2} > 0 \qquad \text{on } X,$$

and that

$$\det[\omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2}] = e^F \det \omega_0^{n-1}. \tag{4.1}$$

Suppose that

$$\sup_X |\omega_0^{n-1} + \sqrt{-1}/2\partial\bar{\partial}u \wedge \eta^{n-2}|_\eta \le C_3 \tag{4.2}$$

for some constant $C_3 > 0$. Then,

$$\|u\|_{C^{2,\alpha}(X)} \le C,$$

where $0 < \alpha < 1$ and $C > 0$ are constants depending only on C_3, n, ω_0, and η.

We shall apply the Evans–Krylov theory (see, for example, Gilbarg–Trudinger [3, p. 461, Theorem 17.14]), which is on the real fully nonlinear elliptic equation. Note that Evans–Krylov theory is based on the *weak Harnack estimate* (see, for example, [3, p. 246, Theorem 9.22]), which, in turn, makes uses of Aleksandrov's maximum principle (see, for example, [3, p. 222, Lemma 9.3]).

We first adapt Aleksandrov's maximum principle to the complex setting. To see this, we start from the following result (see, for example, Lemma 9.2 in [3]): Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth boundary.

Lemma (Aleksandrov) *For $v \in C^2(\overline{\Omega})$ with $v \le 0$ on $\partial\Omega$, we have*

$$\sup_\Omega v \le \frac{\operatorname{diam}(\Omega)}{\sigma_{2n}^{1/(2n)}} \left(\int_{\Gamma_v^+} |\det D^2 v| \right)^{\frac{1}{2n}}. \tag{4.3}$$

Here σ_{2n} is the volume of unit ball in $\mathbb{C}^n$, $D^2 v$ denotes the real Hessian matrix of v, and Γ_v^+ is the upper contact set of v, i.e.,

$$\Gamma_v^+ = \{y \in \Omega; v(x) \le v(y) + Dv(y)\cdot(x-y) \quad \text{for all} \quad x \in \Omega\}.$$

Then, it suffices to control the real Hessian D^2v by the complex Hessian $(v_{i\bar{j}})$ of v, over Γ_v^+. Note that $\Gamma_v^+ \subset \{y \in \Omega;\, (D^2v)(y) \le 0\}$. We shall make use of the following inequality (comparing with [1, p. 246], we do not need Hadarmad's inequality for semipositive matrices):

Proposition 10 *Let w be a real C^2 function in Ω. For $P \in \Omega$ such that $D^2w \ge 0$,*

$$\det(D^2w) \le 8^n |\det w_{i\bar{j}}|^2 \quad \text{at } P.$$

Proof Recall that

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right), \quad 1 \le i \le n.$$

We denote

$$w_{x^i} = \frac{\partial w}{\partial x^i}, \quad w_{x^i y^j} = \frac{\partial^2 w}{\partial x^i \partial y^j}, \quad \ldots .$$

Then,

$$w_{i\bar{j}} = \frac{1}{4}\left(w_{x^i x^j} + w_{y^i y^j}\right) + \frac{\sqrt{-1}}{4}\left(w_{x^i y^j} - w_{x^j y^i}\right), \quad 1 \le i, j \le n.$$

Since $D^2w \ge 0$ at P, we can choose a coordinate system $(x^1, y^1, \ldots, x^n, y^n)$ near P such that D^2w is diagonalized at P, and hence,

$$w_{x^i x^i} \ge 0, \quad w_{y^i y^i} \ge 0, \quad \text{for all} \quad 1 \le i \le n.$$

Then, under this coordinate system, the complex Hessian of w is also diagonalized, i.e.,

$$w_{i\bar{j}} = \frac{\delta_{ij}}{4}\left(w_{x^i x^i} + w_{y^i y^i}\right).$$

It follows that, at P,

$$\begin{aligned} 16^n |\det w_{i\bar{j}}|^2 &= \prod_{i=1}^n \left(w_{x^i x^i} + w_{y^i y^i}\right)^2 \\ &\ge 2^n \prod_{i=1}^n w_{x^i x^i} \prod_{i=1}^n w_{y^i y^i} \\ &= 2^n \det(D^2w). \end{aligned}$$

Moreover, for any Hermitian matrix $(a^{i\bar{j}}) > 0$ on Γ_v^+, we have by an elementary inequality that

$$\det(a^{i\bar{j}})\det(-v_{i\bar{j}}) \le \left(\frac{-\sum_{i,j} a^{i\bar{j}} v_{i\bar{j}}}{n}\right)^n. \tag{4.4}$$

Now apply Proposition 10 and (4.4) to (4.3) to obtain the following complex version Aleksandrov's maximum principle (compare with [3, p. 222, Lemma 9.3]):

Lemma 11 *Let $(a^{i\bar{j}})$ be a positive definite Hermitian matrix in Ω. For $v \in C^2(\overline{\Omega})$ with $v \le 0$ on $\partial\Omega$,*

$$\sup_\Omega v \le \frac{2^n \operatorname{diam}(\Omega)}{n\,\sigma_{2n}^{1/(2n)}} \left[\int_{\Gamma_v^+} \left|\frac{-\sum a^{i\bar{j}} v_{i\bar{j}}}{\det(a_{i\bar{j}})^{1/n}}\right|^{2n}\right]^{\frac{1}{2n}}.$$

Then, the weak Harnack inequality below (compare with [3, p. 246, Theorem 9.22]) follows from Lemma 11 and the cube decomposition procedure.

Theorem (Krylov–Safonov) *Let $v \in W^{2,2n}(\Omega)$ satisfy $\sum a^{i\bar{j}} v_{i\bar{j}} \leq g$ in Ω, where $g \in L^{2n}(\Omega)$, and $(a^{i\bar{j}})$ satisfies that*

$$0 < \lambda|\zeta|^2 \leq \sum_{i,j} a^{i\bar{j}}(z)\zeta_i\zeta_j \leq \Lambda|\zeta|^2, \quad \text{for all} \quad z \in \Omega \text{ and } \zeta \in \mathbb{C}^n,$$

in which λ and Λ are two constants. Suppose that $v \geq 0$ in an open ball $B_{2R}(y) \subset \Omega$ centered at y of radius $2R$. Then,

$$\left(\frac{1}{|B_R|}\int_{B_R} v^p\right)^{1/p} \leq C\left[\inf_{B_R} v + \frac{R}{\lambda}\|g\|_{L^{2n}(B_{2R})}\right],$$

where $|B_R|$ denotes the measure of B_R, and $p > 0$ and $C > 0$ are constants depending only on n, λ, and Λ.

Let us denote by

$$E[(u_{i\bar{j}})] = \log\det\left[\omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2}\right].$$

To apply Evans–Krylov theory, it remains to check the following two conditions ([3, p. 456]):

(1) E is uniformly elliptic with respect to $(u_{i\bar{j}})$,
(2) E is concave on the range of $(u_{i\bar{j}})$.

As in Sect. 2, we denote $\Psi = \omega_0^{n-1}$ and

$$\Psi_u = \Psi + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2}. \tag{4.5}$$

We use the index convention (2) for an $(n-1, n-1)$-form. Then,

$$E[(u_{i\bar{j}})] = \log\det[(\Psi_u)_{i\bar{j}}],$$

and thus,

$$\frac{\partial E}{\partial(\Psi_u)_{i\bar{j}}} = (\Psi_u)^{i\bar{j}}, \quad \frac{\partial^2 E}{\partial(\Psi_u)_{i\bar{j}}\partial(\Psi_u)_{k\bar{l}}} = -(\Psi_u)^{i\bar{l}}(\Psi_u)^{k\bar{j}}.$$

Clearly, E is concave on $[(\Psi_u)_{i\bar{j}}]$. By (4.1) and (4.2), we know that the eigenvalues of $[(\Psi_u)_{i\bar{j}}]$ with respect to $(\eta_{i\bar{j}})$, have uniform bounds which depend only on F, ω_0, and C_3. Therefore, E is uniformly elliptic with respect to $[(\Psi_u)_{i\bar{j}}]$. Observe that by (4.5), $[(\Psi_u)_{i\bar{j}}]$ depends linearly on $(u_{p\bar{q}})$. Since $(\eta_{k\bar{l}}) > 0$ on X, the conditions (1) and (2) follows immediately from the chain rule.

Now we can apply the procedure in [3, p. 457–461], and this proves Lemma 9. As a corollary, we obtain the Hölder estimate of C^2 for Eq. (1.4).

Corollary 12 *Let (X, η) an n-dimensional Kähler of nonnegative quadratic bisectional curvature, and ω_0 be a Hermitian metric on X. Given any $f \in C^\infty(X)$, let $u \in C^\infty(X)$ be a solution of*

$$\frac{\det(\omega_u^{n-1})}{\det(\omega_0^{n-1})} = e^{(n-1)f}\left(\frac{\int_X \omega_u^n}{\int_X e^f\omega_0^n}\right)^{n-1},$$

where ω_u is a positive $(1, 1)$*-form such that*

$$\omega_u^{n-1} = \omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2} > 0 \quad \text{on} \quad X.$$

Then,

$$\|u\|_{C^{2,\alpha}(X)} \leq C,$$

where $0 < \alpha < 1$ *and* $C > 0$ *are constants depending only on* f, η, n, *and* ω_0.

5 Openness and uniqueness

Throughout this section, we let ω_0 be a balanced metric, and let η be an arbitrary Kähler metric, unless otherwise indicated. We fix $k \geq n+4$, $0 < \alpha < 1$, and a function $f \in C^{k,\alpha}(X)$ satisfying

$$\int_X e^f \omega_0^n = V \equiv \int_X \omega_0^n.$$

Here $C^{k,\alpha}(X)$ is the usual Hölder space on X. Consider for $0 \leq t \leq 1$,

$$\frac{\det(\omega_{u_t}^{n-1})}{\det(\omega_0^{n-1})} = e^{(n-1)tf}\left(\frac{\int_X \omega_{u_t}^n}{\int_X e^{tf}\omega_0^n}\right)^{n-1}, \tag{5.1}$$

where $u_t \in \mathcal{P}_\eta(\omega_0)$. By abuse of notation, in this section we denote

$$\mathcal{P}_\eta(\omega_0) = \left\{v \in C^{k+2,\alpha}(X);\, \omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}v \wedge \eta^{n-2} > 0\right\}.$$

Let

$$T = \{t \in [0,1]; \text{the equation (5.1) has a solution } \; u_t \in C^{k+2,\alpha}(X) \text{ such that } u_t \in \mathcal{P}_\eta(\omega_0).\}. \tag{5.2}$$

Clearly, we have $0 \in T$.

Lemma 13 *Let* T *be the set given as above. Then* T *is open in* $[0, 1]$.

Proof Notice that (5.1) is the same as

$$\frac{\omega_{u_t}^n}{\omega_0^n} = e^{tf}\frac{\int_X \omega_{u_t}^n}{\int_X e^{tf}\omega_0^n}.$$

As in Section 3 of [2], we define

$$M(w) \equiv \log\frac{\omega_w^n}{\omega_0^n} - \log\left(\frac{1}{V}\int_X \omega_w^n\right),$$

for any $w \in \mathcal{P}_\eta(\omega_0)$. Then, $M(w) \in \mathcal{F}^{k,\alpha}(X)$, where $\mathcal{F}^{k,\alpha}(X)$ is the hypersurface in $C^{k,\alpha}(X)$ given by

$$\mathcal{F}^{k,\alpha}(X) = \left\{g \in C^{k,\alpha}(X);\, \int_X e^g \omega_0^n = V\right\}.$$

Now suppose that $t \in T$. Then, the corresponding u_t defines a positive (1, 1)-form ω_{u_t} such that

$$\omega_{u_t}^{n-1} = \omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}u \wedge \eta^{n-2} > 0 \quad \text{on } X;$$

furthermore, u_t satisfies that

$$M(u_t) = tf + \log V - \log\left(\int_X e^{tf}\omega_0^n\right) \in \mathcal{F}^{k,\alpha}(X).$$

The tangent space of $\mathcal{F}^{k,\alpha}(X)$ at $M(u_t)$ is identically the same as the Banach space $\mathcal{E}_t^{k,\alpha}(X)$, which consists of all $h \in C^{k,\alpha}(X)$ such that

$$\int_X h\,\omega_{u_t}^n = 0.$$

In view of the Implicit Function Theorem, it suffices to show that the linearization operator $L_t \equiv M_{u_t}$, given by

$$L_t(v) = \frac{n(\sqrt{-1}/2)\partial\bar{\partial}v \wedge \eta^{n-2} \wedge \omega_{u_t}}{(n-1)\omega_{u_t}^n} - \frac{n\int_X(\sqrt{-1}/2)\partial\bar{\partial}v \wedge \eta^{n-2} \wedge \omega_{u_t}}{(n-1)\int_X \omega_{u_t}^n},$$

is a linear isomorphism from $\mathcal{E}_t^{k+2,\alpha}(X)$ to $\mathcal{E}_t^{k,\alpha}(X)$. This is guaranteed by Lemma 13 in [2]. The proof is thus finished. □

Remark 14 We thank John Loftin for pointing out that the openness argument in [2] also works for η being a *astheno-Kähler* metric, i.e., η is a hermitian metric such that $\partial\bar{\partial}\eta^{n-2} = 0$.

By the results in the previous section, we know that T is also closed, provided that the orthogonal bisectional curvature of η is nonnegative. Therefore, the existence part in Theorem 1 is proved. The uniqueness follows immediately from the following proposition.

Proposition 15 *Let $v \in \mathcal{P}_\eta(\omega_0)$ satisfying*

$$\det\left[\omega_0^{n-1} + (\sqrt{-1}/2)\partial\bar{\partial}v \wedge \eta^{n-2}\right] = \delta \det\omega_0^{n-1}, \tag{5.3}$$

where $\delta > 0$ is a constant. Then, v must be a constant function and $\delta = 1$.

Proof Applying the maximum principle to Eq. (5.3) at the maximum points of v yields that $\delta \le 1$. Similarly, we get $\delta \ge 1$ by considering (5.3) at the minimum points of v. Thus, $\delta = 1$. Then, we apply the arithmetic–geometric mean inequality to obtain

$$\begin{aligned} 1 = \left[\frac{\det\omega_v^{n-1}}{\det\omega_0^{n-1}}\right]^{1/n} &\le 1 + \frac{1}{n}\sum_{i,j=1}^n (\omega_0^{n-1})^{i\bar{j}}\left((\sqrt{-1}/2)\partial\bar{\partial}v \wedge \eta^{n-2}\right)_{i\bar{j}} \\ &= 1 + \frac{\omega_0 \wedge \eta^{n-2} \wedge (\sqrt{-1}/2)\partial\bar{\partial}v}{\omega_0^n} \equiv 1 + Kv. \end{aligned}$$

Note that the linear operator K so defined is uniformly elliptic, by the metric equivalence of η and ω_0 on the compact manifold X. Applying the strong maximum principle to $Kv \ge 0$ yields that v is a constant function. □

Therefore, the proof of Theorem 1 is completed.

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