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Calculus of Variations

Spherical symmetrization and the first eigenvalue of geodesic disks on manifolds

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Abstract Given a manifold M, we build two spherically symmetric model manifolds based on the maximum and the minimum of its curvatures. We then show that the first Dirichlet eigenvalue of the Laplace–Beltrami operator on a geodesic disk of the original manifold can be bounded from above and below by the first eigenvalue on geodesic disks with the same radius on the model manifolds. These results may be seen as extensions of Cheng's eigenvalue comparison theorems, where the model constant curvature manifolds have been replaced by more general spherically symmetric manifolds. To prove this, we extend Rauch's and Bishop's comparison theorems to this setting.

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1 Introduction

Optimal domains in isoperimetric inequalities relating eigenvalues to geometrical quantities such as volume and surface area quite often display some degree of symmetry. In many instances, this symmetry is actually the maximal possible, such as in the Rayleigh–Faber–Krahn and the Szegö–Weinberger inequalities, corresponding to Dirichlet and Neumann boundary conditions for Euclidean domains, respectively. It is thus quite natural that symmetrization plays a fundamental role in this aspect of spectral theory and is at the heart of many isoperimetric inequalities of this type. The Rayleigh–Faber–Krahn inequality, for instance, is a consequence of the fact that Schwarz symmetrization does not increase the Dirichlet integral while leaving the L^2 norm unchanged. Even in some cases where the minimiser is not one but two balls, this symmetrization plays a role, as happens not only in the case of the second Dirichlet eigenvalue, but also when other restrictions are enforced—see, for instance, [6, 12].

However, Schwarz and other similar symmetrization procedures are mostly Euclidean techniques, and do not extend to manifolds in general. This does not mean that symmetry does not play a similarly fundamental role in isoperimetrical eigenvalue inequalities on manifolds. One such example is Hersch's result for two-dimensional spheres [15], which states that among all surfaces with the same area which are homeomorphic to $\mathbb{S}^2$, the round sphere (canonical metric) maximises the first nontrivial eigenvalue.

The purpose of the present paper is to develop the usage of symmetrization techniques in the case of manifolds, allowing us to derive comparison isoperimetric inequalities for eigenvalues in this context. To this end, we shall consider a symmetrization procedure based on curvature. More precisely, given a complete n-manifold M and a point p in M such that we have lower and upper bounds for the radial Ricci and sectional curvatures within a geodesic disk of radius r_0, which depend only on the distance t to the point p, we build two spherically symmetric manifolds centred at a point p^* and whose curvatures are determined by the respective bounds. In this way, we are then able to obtain that the first eigenvalue of this geodesic disk with Dirichlet boundary conditions is bounded from above and below by the first Dirichlet eigenvalue on geodesic disks centred at p^* on these two manifolds—see Theorems 3.6 and 4.4 for the precise statements of these results.

The above results may be seen as extensions of Cheng's bounds for the first eigenvalue, where the comparison is made between a geodesic disk on M and those on spaces of constant curvature which are obtained by taking lower and upper bounds of the curvature [10, 11]. The starting point behind Theorems 3.6 and 4.4 is twofold. On the one hand, it should be possible to replace the constant curvature spaces in Cheng's results by spherically symmetric spaces, in such a way that these still yield curvature bounds which imply the desired eigenvalue bounds. On the other hand, spherically symmetric manifolds posses a relatively simple characterization and the first Dirichlet eigenvalue on a geodesic disk is given by the zero of a solution to a second order ordinary differential equation. Thus, not only do there exist many bounds for these eigenvalues, some of which providing quite accurate estimates—see [2, 3, 5, 14], for instance—but also this reduction allows us to estimate the first eigenvalue of a disk on a general n-dimensional manifold by solving a one-dimensional spectral problem which, by construction, will be more accurate than Cheng's methods.

The structure of the paper is as follows. In the next section, we lay out the background to the problem and the necessary basic definitions, including the characterizations of the relevant quantities in the case of spherically symmetric manifolds. The bound for eigenvalues in the case where the radial Ricci curvature is bounded from below is derived in Sect. 3, together with some of its consequences. The case where the radial sectional curvature is bounded from above is dealt with in Sect. 4. Both situations require the extension of other comparison results to the spherically symmetric setting (as opposed to the constant curvature setting), which we believe to be interesting in their own right, such as Rauch's and Bishop's comparison theorems—see Theorems 4.1, 3.3 and 4.2, respectively. In Sect. 5 we briefly discuss some properties of the model manifolds, such as their maximum domain of existence. Finally, in the last section we present some examples illustrating our results.

2 Preliminaries

In this section we recall the notion of spherically symmetric Riemannian manifold with respect to a point and show some of their geometrical and spectral properties.

Given a complete n-dimensional Riemannian manifold M with $n \geq 2$, metric g and Levi-Civita connection ∇, for any fixed point $p \in M$, the exponential map $\exp_p : \mathscr{D}_p \to M \backslash C(p)$ is a diffeomorphism from a star-shaped open set $\mathscr{D}_p$ of T_pM with

$$\mathscr{D}_p = \left\{ t\xi | 0 \leq t < d_\xi, \xi \in S_p^{n-1} \right\}$$

onto the open set $M \backslash C(p)$, where $C(p)$ is the cut locus of p, a closed set of zero n-Hausdorff measure, S_p^{n-1} is the unit sphere of T_pM, and d_ξ is defined by

$$d_\xi = d_\xi(p) := \sup\{t > 0 | \gamma_\xi(s) = \gamma_{(p,\xi)}(s) := \exp_p(s\xi) \text{ is the unique minimal geodesic joining } p \text{ and } \gamma_\xi(t)\}.$$

Clearly, this exponential map provides a maximal normal geodesic coordinate chart at p. As in [2,3,7], we introduce two important maps. For a fixed vector $\xi \in T_pM$, $|\xi| = 1$, let $\xi^\perp$ be the orthogonal complement of $\{\mathbb{R}\xi\}$ in T_pM and $\tau_t : T_pM \to T_{\exp_p(t\xi)}M$ the parallel translation along γ_ξ. The path of linear transformations $\mathbb{A}(t,\xi) : \xi^\perp \to \xi^\perp$ is given by

$$\mathbb{A}(t,\xi)\eta = (\tau_t)^{-1} Y_\eta(t),$$

where $Y_\eta(t) = d(\exp_p)_{(t\xi)}(t\eta)$ is the Jacobi field along γ_ξ satisfying $Y_\eta(0) = 0$, and $(\nabla_t Y_\eta)(0) = \eta$. This operator satisfies the Jacobi equation $\mathbb{A}'' + \mathscr{R}\mathbb{A} = 0$ with initial conditions $\mathbb{A}(0,\xi) = 0$, $\mathbb{A}'(0,\xi) = I$, where $\mathscr{R}(t)$ is the self-adjoint operator on $\xi^\perp$, $\mathscr{R}(t)\eta = (\tau_t)^{-1} R(\gamma_\xi'(t), \tau_t\eta)\gamma_\xi'(t)$. The trace of the later operator is just the radial Ricci tensor along unit speed geodesics starting from p,

$$\mathrm{Ricci}_{(\gamma_\xi(t))}(\gamma_\xi'(t), \gamma_\xi'(t)).$$

By Gauss's lemma, the first fundamental form of the Riemannian metric g on $M \backslash C(p)$ in the spherical geodesic coordinate chart can be expressed by

$$ds^2(\exp_p(t\xi)) = dt^2 + |\mathbb{A}(t,\xi) d\xi|^2, \quad \forall t\xi \in \mathscr{D}_p.$$

Fixing an o.n. basis $\{\eta_i, i \geq 2\}$ of $\xi^\perp = T_\xi S_p^{n-1}$, and extending it to a local frame ξ_i of S_p^{n-1}, we consider the metric components $g_{ij}(t,\xi)$, $i, j \geq 1$, in this coordinate system $\{t, \xi_i, i \geq 2\}$, and define on $\mathscr{D}_p$ a function $J > 0$ by

$$J^{n-1} = \sqrt{|g|} := \sqrt{\det[g_{ij}]}, \tag{2.1}$$

that is, $\sqrt{|g|} = \det \mathbb{A}(t,\xi)$, and $dV_M = J^{n-1} dt d\sigma$ is the volume element of $M \backslash C(p)$, where $d\sigma$ denotes the $(n-1)$-dimensional volume element on $\mathbb{S}^{n-1} \equiv S_p^{n-1} \subseteq T_pM$. If

$r(x) = d(x, p)$ denotes the intrinsic distance to the point p, then for $x \in M \backslash (C(p) \cup \{p\})$, the unit vector field

$$v_x = \nabla r_{(x)}$$

is the radial unit tangent vector at x, according to the definition given in [18]. To see this we only have to recall that for any $\xi \in S_p^{n-1}$ and $t > 0$, $\nabla r_{(\gamma_\xi(t))} = \gamma_\xi'(t)$ is valid away from the cut locus of p (cf. [13]). Applying (2.1), the volume of a geodesic ball of radius r and centered at p, is given by

$$V(B(p,r)) = V(B(p,r)\backslash C(p)) = \int_{S_p^{n-1}} \left(\int_0^{\min\{r, d_\xi\}} \det(\mathbb{A}(t,\xi)) dt \right) d\sigma.$$

Thus, for r smaller than the injectivity radius at p, i.e. $r < inj(p) = d(p, C(p)) = \min_\xi d_\xi$, we have

$$V(B(p,r)) = \int_0^r \int_{S_p^{n-1}} \det(\mathbb{A}(t,\xi)) d\sigma dt. \tag{2.2}$$

For each $\xi \in S_p^{n-1}$, the cut point $\exp_p(d_\xi \xi)$ is either a conjugate point to p, which implies $\det(\mathbb{A}(d_\xi, \xi)) = 0$, or $\exp_p$ is not injective at $d_\xi \xi$. We also recall the following inequality about $r(x)$ (cf. [21], Prop. 39, and pp. 266–267), with $\partial_r = \nabla r$ as a vector of differentiation (see Prop. 7, on p. 47 of the same reference),

$$\partial_r \Delta r \leq \partial_r \Delta r + \|\text{Hess}\, r\|^2 = -\text{Ricci}(\partial_r, \partial_r), \quad \text{with } \Delta r = \partial_r \ln(\sqrt{|g|}),$$

which implies that

$$J'' + \frac{1}{(n-1)} \text{Ricci}(\gamma_\xi'(t), \gamma_\xi'(t))\, J \leq 0, \tag{2.3}$$

$$J(0,\xi) = 0, \quad J'(0,\xi) = 1. \tag{2.4}$$

We are interested in comparing our manifolds with model manifolds which are spherically symmetric with respect to a base point and whose Ricci and sectional curvatures bound those of the original manifolds. The following definitions are helpful to clarify these concepts. Let

$$l(p) := \sup_{x \in M} r(x) = \max_\xi d_\xi.$$

To see the last equality holds we take a sequence $x_n \in M$ such that $d(p, x_n) \to l(p)$ when $n \to +\infty$, and let γ_{ξ_n} be a minimizing unit speed geodesic connecting p to x_n. Then $d(p, x_n) \leq d_{\xi_n}$, proving that $l(p) \leq \max_\xi d_\xi$. If M is complete not compact, this reasoning also shows that both $l(p)$ and $\max_\xi d_\xi$ are infinite. In the later case, we conclude there exists ξ with $d_\xi = +\infty$. Naturally, M is compact iff $l(p)$ is finite. In this case $M = \bar{B}(p, l(p))$, where $\partial B(p, l(p)) = \{q \in M : d(p,q) = l(p)\}$ constitutes a nonempty closed subset $C^+(p)$ of $C(p)$, since a minimizing geodesic connecting p with any of its elements cannot minimize distance after reaching it, at maximum distance $l(p)$. In particular they are of the form $q = exp_p(d_\xi \xi)$, with $d_\xi = l(p) \geq inj(p)$.

Definition 2.1 A domain $\Omega = \exp_p([0, l) \times S_p^{n-1}) \subset M \backslash C(p)$, with $l < inj(p)$, is said to be spherically symmetric with respect to a point $p \in \Omega$, if and only if the matrix $\mathbb{A}(t, \xi)$ satisfies $\mathbb{A}(t, \xi) = f(t)I$, for a function $f \in C^2([0, l))$, with $f(0) = 0$, $f'(0) = 1$, and $f|(0, l) > 0$.

In this case the Riemannian metric of M can be expressed on Ω by

$$ds^2 = dt^2 + f(t)^2|d\xi|^2, \quad \forall \xi \in S_p^{n-1}, 0 < t < l. \tag{2.5}$$

Thus, $J(t, \xi) = f(t)$. If $\bar{\Omega} = M$ and M is a compact Riemannian manifold, then $l < +\infty$ and we are assuming $d_\xi = l = l(p) = inj(p)$ for all ξ. In this case by continuity of $\mathbb{A}(t, \xi)$, $f(l)$ is defined and is equal to $J(l, \xi)$, being zero only if $exp_p(l\xi)$ is a conjugate point $\forall \xi$.

Spherically symmetric manifolds are also sometimes called generalized space forms as in the work of Katz and Kondo [18], and a standard model for such manifolds is given by the quotient manifold of the warped product $M^* = [0, l) \times_f \mathbb{S}^{n-1}$, with the metric (2.5) (see [21], p. 13 and [20], pp. 204–211, and Chapter 7, for notation and properties). Here f satisfies the conditions of Definition 2.1, with all pairs $(0, \xi)$ identified with a single point p (see [2]). This metric is of class C^k, $k \geq 0$, if $f \in C^k((0, l))$ and of class C^{k+3} at $t = 0$ with vanishing $2d$-derivatives at $t = 0$, for all $2d \leq k + 3$ (see [21], p. 13). For $r < l$,

$$V(B(p, r)) = w_n \int_0^r f^{n-1}(t)dt, \tag{2.6}$$

where w_n denotes the $(n-1)$-volume of the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$, and by the co-area formula

$$A(\partial B(p, r)) = \frac{d}{dr} V(B(p, r)) = w_n f^{n-1}(r).$$

If $l = +\infty$ and the metric is of class C^2 then M^* is complete, since geodesics starting at p are defined for all $t \in \mathbb{R}$. If l is finite and $f(l) = 0$ then M^* "closes", and defines a one-point compactification metric space $\bar{M}^* = M^* \cup \{q^*\}$ by identifying all pairs (l, ξ) with a single point q^*, and extending the distance function as $d(q^*, (t, \xi)) = l - t$. This space will be a Riemannian metric space if at the closing point the metric (2.5) can be extended continuously, that is, at $t = l$, f is C^3 with $f'(l) = -1$ and $f''(l) = 0$. In this case, the metric is of class C^k at the closing point, if the even derivatives of f satisfy at $t = l$ conditions analogous to those satisfied at $t = 0$.

For the particular case of surfaces $(n = 2)$, if $|f'(t)| \leq 1$, then

$$\phi(t, \theta) = (f(t) \cos\theta, f(t) \sin\theta, h(t)),$$

with $h(t) = \int_0^t \sqrt{1 - (f'(s))^2}$, defines an isometric embedding of M^* into a surface of revolution of $\mathbb{R}^3$. If M^* has negative sectional curvature at p no such local embedding exists near p, since $f'(t)$ is nondecreasing (see (2.11)).

We will use the following concept.

Definition 2.2 Given a continuous function $k : [0, l) \to \mathbb{R}$, we say that M has a radial Ricci curvature lower bound $(n-1)k$ at the point p if

$$\text{Ricci}(v_x, v_x) \geq (n-1)k(r(x)), \quad \forall x \in M \backslash C(p) \cup \{p\}, \tag{2.7}$$

where Ricci is the Ricci curvature of M.

Similarly, we define

Definition 2.3 Given a continuous function $k : [0, l) \to \mathbb{R}$, we say that M has a radial sectional curvature upper bound k at the point p if

$$K(v_x, V) \leq k(r(x)), \quad \forall x \in M \backslash C(p) \cup \{p\}, \tag{2.8}$$

where $V \perp v_x$, $V \in S_x^{n-1} \subseteq T_x M$, and $K(v_x, V)$ denotes the sectional curvature of the plane spanned by v_x and V.

Remark 2.4 Since the radial distance is given by $r(x) = d(p, x)$, for $x = \gamma_\xi(t)$, the parameter t may be seen as the argument of the continuous function $k : [0, l) \to \mathbb{R}$ in Definition 2.2. Additionally, $\frac{d}{dt}|_x = \nabla r_{(x)} = v_x$, which implies our conditions (2.7) and (2.8) become $\text{Ricci}(\frac{d}{dt}, \frac{d}{dt}) \geq (n-1)k(t)$ and $K(\frac{d}{dt}, V) \leq k(t)$, respectively. We also consider the same definitions holding only on a geodesic ball $B(p, r_0)$ of M.

We shall now construct naturally defined optimal continuous functions $k_\mp(p, t)$ satisfying Definitions 2.2 and 2.3, respectively, with respect to the base point p. We first recall that $\beta(t, x, w) = (\gamma_{(x,w)}(t), \gamma'_{(x,w)}(t))$ can be seen as an integral curve of a vector field on TM, depending smoothly on the variables (t, x, w), that we restrict to $w = \xi \in T_x M$ with $\|\xi\| = 1$, for each $x \in M$, that is (x, ξ) lies in the unit sphere bundle $SM \subset TM$, and define the normalized radial Ricci tensor, smoothly defined for all $(x, \xi) \in SM$, $t \in \mathbb{R}$, as

$$\text{Ricci}_{rad}(x, \xi, t) = \frac{1}{(n-1)} \text{Ricci}(\gamma'_{(x,\xi)}(t), \gamma'_{(x,\xi)}(t)).$$

Since the map $(x, \xi) \to d_\xi(x)$ is continuous, the set $\mathscr{D} = \{(x, \xi, t) \in SM \times [0, +\infty) : 0 \leq t < d_\xi(x)\}$ is an open set of $SM \times [0, +\infty)$, with closure $\bar{\mathscr{D}} = \{(x, \xi, t) \in SM \times [0, +\infty) : 0 \leq t \leq d_\xi(x)\}$. Then we define,

$$k_-(x, t) := \min_{\{\xi : (t, x, \xi) \in \bar{\mathscr{D}}\}} \text{Ricci}_{rad}(x, \xi, t), \quad x \in M, \quad 0 \leq t < l(x), \tag{2.9}$$

$$k_+(x, t) := \max_{\left\{\substack{\xi, V \,:\, (x, \xi), (\gamma_{(x,\xi)}(t), V) \in SM \\ V \perp \gamma'_{(x,\xi)}(t)}\right\}} K(\gamma'_{(x,\xi)}(t), V), \quad x \in M, \quad 0 \leq t < inj(x). \tag{2.10}$$

If $l(x) < +\infty$ (resp. $inj(x) < +\infty$), then $k_-(x, t)$ (resp. $k_+(x, t)$) can be extended continuously to $t = l(x)$ (resp. $t = inj(x)$). Furthermore, if M is closed then $inj(M) = \min_{x \in M} inj(x)$ is a positive constant. The proof that the functions $k_\pm(x, t)$ are continuous is an application of the uniform continuity of continuous functions on compact sets. We have $k_-(p, t) \leq \text{Ricci}_{rad}(p, \xi, t)$ for any ξ s.t. $0 \leq t < d_\xi(p)$, and $k_+(p, t) \geq K(V, \gamma'_{(p,\xi)}(t))$ for any $0 \leq t < inj(p)$, ξ and $V \perp \gamma'_{(p,\xi)}(t) \cap S^{n-1}_{\gamma_{(p,\xi)(t)}}$, and so (2.7) and (2.8) hold, respectively. They are optimal in the sense that, if $k(t)$ is in the conditions of Definition 2.2 (resp. 2.3), then $k(t) \leq k_-(p, t)$ (resp. $k(t) \geq k_+(p, t)$). Further remarks will be described in Sect. 5, clarifying the chosen domains for $k_\pm(p, t)$.

The radial sectional curvature and the radial component of the Ricci tensor of a model space $M^* = [0, l) \times_f \mathbb{S}^{n-1}$, with f of class C^2, are respectively given by (cf. Proposition 42 and Corollary 43 of chapter 7 in [20] or subsection 2.3 of chapter 3 in [21])

$$\begin{aligned} &K\left(\tfrac{d}{dt}, V\right) = R\left(\tfrac{d}{dt}, V, \tfrac{d}{dt}, V\right) = -\tfrac{f''(t)}{f(t)} \quad \text{for} \;\; V \in T_\xi \mathbb{S}^{n-1}, |V|_g = 1, \\ &\text{Ricci}\left(\tfrac{d}{dt}, \tfrac{d}{dt}\right) = -(n-1)\tfrac{f''(t)}{f(t)}. \end{aligned} \tag{2.11}$$

Thus, Definitions 2.2 and 2.3 are satisfied with equality in (2.7) and (2.8) respectively, and $k(t) = -f''(t)/f(t)$. We need to require $f \in C^2((0, l))$ to define the curvature tensor away from p. Furthermore, if $f''(0) = 0$, and f is C^3 at $t = 0$, then $\exists \lim_{t\to 0} k(t) = -f'''(0)$. Though ∇r is not defined at $x = p$, $k(t)$ is usually required to be continuous at $t = 0$, as in the above definitions, which amounts to require f to be C^3 at $t = 0$. This is a natural condition when n equals two, since then the radial sectional curvature and the sectional curvature coincide and the extension is defined. A space form with constant sectional curvature k is also a spherically symmetric manifold and in this particular situation we have

$$f(t) = \begin{cases} \dfrac{\sin\sqrt{k}t}{\sqrt{k}}, & l = \dfrac{\pi}{\sqrt{k}} \quad k > 0, \\ t, & l = +\infty \quad k = 0, \\ \dfrac{\sinh\sqrt{-k}t}{\sqrt{-k}}, & l = +\infty \quad k < 0. \end{cases} \tag{2.12}$$

Given a Riemannian manifold M of class C^1 with metric g of class C^0, the fundamental tone of a domain Ω is given by

$$\lambda^*(\Omega) = \inf_{\phi\in H_0^1(\Omega)} \frac{\int_\Omega \|\nabla\phi\|^2}{\int_\Omega \phi^2},$$

where $H_0^1(\Omega)$ is the completion of the class of functions $\phi \in C^1(\Omega)$ with compact support in the interior of Ω, for the H^1-norm $\|\phi\|_{H^1}^2 = \int_\Omega \phi^2 + \int_\Omega \|\nabla\phi\|^2$. If M is C^2 and g is C^1, and $\partial\Omega$ is piecewise C^2, by Rayleigh's theorem $\lambda^*(\Omega)$ corresponds to the first eigenvalue λ_1 of the Dirichlet problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega \text{ and } \quad u_{|\partial\Omega} = 0.$$

Moreover, $H_0^1(\Omega) \cap C^2(\Omega)$ is just the space of functions in $C^2(\Omega) \cap C^0(\bar{\Omega})$ that vanish at the boundary. When M is closed and $\Omega = M$, the Dirichlet problem turns out to be the closed problem, and taking the constant function $\phi = 1$ we see that for all $r > l$, $\lambda^*(B(p,r)) = \lambda^*(M) = 0$. Furthermore, in this case $M = \bar{B}(p,l)$ with $l = l(p)$, and we may ask if $\lim_{r\to l^-} \lambda^*(B(p,r)) = 0$ or, equivalently, if there exists an increasing sequence $R_m \to l$ such that the decreasing sequence $\lambda^*(B(p, R_m))$ converges to zero. This corresponds to $\int \|\nabla\phi_m\|^2 \to 0$ when $m \to +\infty$, where ϕ_m is a $\lambda_1(B(p, R_m))$-eigenfunction normalized such that $\int \phi_m^2 = |M| = \int_M 1$. If M is smooth and $\partial B(p,l)$ is a smooth submanifold of codimension at least two in M (recall that $\partial B(p,l)$ is a subset of $C(p)$), then Chavel and Feldman proved in [8] that $\lambda^*(M\backslash\Omega(\tau)) \to 0$, when $\tau \to 0$, where $\Omega(\tau) = \{x : d(x, \partial B(p,l)) < \tau\}$. Since $M\backslash\Omega(\tau)$ is a compact subset of $B(p,l)$, it is contained in $B(p,l')$ for some $l' < l$, and Chavel and Feldman's result leads to the conclusion that $\lambda_1(B(p,l')) \to 0$, when $l' \to l^-$. Alternative stronger conditions are for example to find a sequence $\phi_m \in H_0^1(B(p, R_m))$ that converges to the constant function 1 for the H^1-norm, or for $n \geq 3$, when $(V(B(p, R_{m+1})) - V(B(p, R_m)))/(R_{m+1} - R_m)^2 \to 0$ when $m \to \infty$, as we can see by taking $\phi_m(x) = y_m(r(x))$ with y_m defined in the proof of next lemma. The following Lemma 2.5 shows that this is a general property of all closing model spaces, even if the Riemannian metric is not defined at the closing point, since we do not require $f'(l) = -1$, and $f''(l) = 0$. In the later case, $\partial B(p,l)$ is a point that can have a conic singularity in which case the tangent space at this point is not well defined, generalizing Chavel and Feldman's result to the non smooth case. A proof for the case of n-spheres can also be found in [7, p. 50].

Lemma 2.5 *Assume M^* is a generalized space form $[0,l)\times_f \mathbb{S}^{n-1}$ with $f\in C^2([0,l))$ and C^3 at $t=0$, $f(0)=f''(0)=0$, $f'(0)=1$, closing at $t=l$, i.e. $f(l)=0$. If for some $\epsilon>0$, $f\in C^1([0,l+\epsilon))$ in case $n=2$, or $f\in C^2([0,l+\epsilon))$ in case $n\geq 3$, then* $\lim_{r\to l^-}\lambda_1(B(p,r))=0$.

Proof For $r<l$, we denote by $B_r:=B(p,r)$, that has a C^2 boundary, and by $B_l=M^*$. Let $V(r):=|B_r|=\int_{B_r}1$. For any increasing sequence $R_m\nearrow l$, $R_m<R_{m+1}<l$ we define a continuous function $y_m:[0,l)\to[0,1]$, that for $n\geq 3$ is given by

$$y_m(r)=\begin{cases}1 & 0\leq r\leq R_m\\ \dfrac{(R_{m+1}-r)}{(R_{m+1}-R_m)} & R_m\leq r\leq R_{m+1}\\ 0 & R_{m+1}\leq r<l,\end{cases}$$

and for $n=2$,

$$y_m(r)=\begin{cases}1 & 0\leq r\leq R_m\\ \dfrac{\ln\left(\dfrac{l-r}{l-R_{m+1}}\right)}{\ln\left(\dfrac{l-R_m}{l-R_{m+1}}\right)} & R_m\leq r\leq R_{m+1}\\ 0 & R_{m+1}\leq r<l.\end{cases}$$

Then $\phi_m(x)=y_m(r(x))\in H_0^1(B_{R_{m+1}})$, where r is the distance function to p in M^*. Recall that $r(x)$ is Lipschitz continuous on all M^* with $\|\nabla r\|\leq 1$ a.e.. Thus, for $n\geq 3$,

$$\int_M|\phi_m-1|^2\leq|M^*\backslash B_{R_m}|=|M^*|-V(R_m)\to 0\quad\text{when}\quad m\to+\infty,$$
$$\int_M\|\nabla(\phi_m-1)\|^2\leq\frac{V(R_{m+1})-V(R_m)}{(R_{m+1}-R_m)^2}.$$

We will prove that for some suitable sequence R_m, we have $\frac{V(R_{m+1})-V(R_m)}{(R_{m+1}-R_m)^2}\to 0$ as well, what proves that $\phi_m\to 1$ for the H^1-norm. Set $F(s)=(f(s))^{n-1}$. Then for $n\geq 3$, $F(l)=F'(l)=0$. We apply a Taylor's formula for s close to l, $F(s)=F(l)+F'(l)(s-l)+\psi(s-l)(s-l)^2$, where

$$\psi(s-l)=\int_0^1(1-t)F''(l+t(s-l))dt.$$

Considering a constant $C>0$ such that $|\psi(s-l)|\leq C$, for $|l-s|<\epsilon$, and a sufficiently small constant $0<\delta<1$, then setting $R_m=l-\delta^m$ we have by (2.6)

$$\frac{V(R_{m+1})-V(R_m)}{(R_{m+1}-R_m)^2}=\frac{w_n}{(R_{m+1}-R_m)^2}\int_{R_m}^{R_{m+1}}\psi(s-l)(s-l)^2ds$$
$$\leq\frac{Cw_n}{(R_{m+1}-R_m)^2}\int_{R_m}^{R_{m+1}}(s-l)^2ds=\frac{Cw_n}{(\delta^m-\delta^{m+1})^2}\int_{\delta^{m+1}}^{\delta^m}s^2ds$$

$$= \frac{C w_n}{(\delta^m - \delta^{m+1})^2} \frac{1}{3} ((\delta^m)^3 - (\delta^{m+1})^3)) = \frac{C w_n}{3} \frac{(1-\delta^3)}{(1-\delta)^2} \delta^m.$$

The last expression converges to zero when $m \to +\infty$, what proves that $\phi_m \to 1$ in H^1.

If we consider the case $n = 2$, by the assumptions, $f(s) = \xi(s)(s-l)$ where $\xi(s) = \int_0^1 f'(l + t(s-l))dt$ is a bounded function for s close to l. We chose $\delta_m = \frac{1}{m!}$ and set $R_m = l - \delta_m$. Therefore,

$$\int_{M^*} |\phi_m - 1|^2 \le \int_{M^* \setminus B_{R_m}} 1 = |M^*| - V(R_m) \to 0, \quad \text{when } m \to +\infty,$$

and for a constant $C > 0$, for all m

$$\begin{aligned}
\int_{M^*} \|\nabla(\phi_m - 1)\|^2 &\le C \frac{1}{\left(\ln\left(\frac{l-R_m}{l-R_{m+1}}\right)\right)^2} \int_{R_m}^{R_{m+1}} \frac{1}{(l-s)^2}(l-s)ds \\
&= C \frac{1}{\left(\ln\left(\frac{l-R_m}{l-R_{m+1}}\right)\right)^2} (\ln(l-R_m) - \ln(l-R_{m+1})) \\
&= C \frac{1}{\ln\left(\frac{l-R_m}{l-R_{m+1}}\right)} = \frac{C}{\ln(m+1)} \to 0 \quad \text{when} \quad m \to +\infty.
\end{aligned}$$

Consequently, $\phi_m \to 1$ for the H^1-norm. Thus, in both cases

$$\lambda_1(B_{R_m}) \le \frac{\int_{M^*} \|\nabla \phi_m\|^2}{\int_{M^*} \phi_m^2} \to 0,$$

when m goes to $+\infty$. □

We would like to point out that for any Riemannian manifold of dimension $n \ge 3$, the double limit

$$\lim_{s>t, s,t \to l^-} \frac{V(B(p,s)) - V(B(p,t))}{(s-t)^2}$$

does not exist in general, but if $A(\partial B(p,t))$ is a function on t that can be extended at $t = l$ and differentiable at that extension, then by using the co-area formula, we have the iterated limit

$$\lim_{t \to l^-} \lim_{s \to t^+} \frac{V(B(p,s)) - V(B(p,t))}{(s-t)^2} = \frac{1}{2} \frac{d}{dt}_{|t=l} A(\partial B(p,t)).$$

On the other hand, if for suitable increasing sequences R_m the first limit exits, and taking $s = R_{m+1}$ and $t = R_m$, as we did in the above proof, then it agrees with the second limit.

3 Generalized comparison theorems for manifolds with radial Ricci curvature bounded from below

We start this section by showing an analog of Proposition 3 of Chapter 2 in [7], for generalized space forms:

Lemma 3.1 *If $T(t)$ is any solution of*

$$[f(t)^{n-1}T']' + \lambda f(t)^{n-1}T = 0, \tag{3.1}$$

where $f(t) > 0$ on the interval $(0, \beta)$, then for $\Re = T'$ we have

$$[f(t)^{n-1}\Re']' + \left\{\lambda + (n-1)\left[\frac{f'(t)}{f(t)}\right]'\right\} f(t)^{n-1}\Re = 0. \tag{3.2}$$

Furthermore, $\Re|(0, \beta) < 0$ whenever $T|(0, \beta) > 0$ and $\lambda > 0$.

Proof By straightforward computation we can get (3.2) from (3.1) directly. Since $f(t) > 0$ on the interval $(0, \beta)$, and

$$f(t)^{n-1}T'(t) = -\lambda \int_0^t f(s)^{n-1}T ds,$$

the second claim of the proposition follows. □

In the remaining part of this section, M is assumed to be a complete n-dimensional Riemannian manifold M with a radial Ricci curvature lower bound $-(n-1)f''(t)/f(t)$ with respect to a given point p, and r denotes the distance function to p. Denote by $B(p, r_0)$ the open geodesic ball with center p and radius r_0 of M, and $V_n(p^-, r_0)$ the geodesic ball with center p^- and radius r_0 of an n-dimensional spherically symmetric manifold $M^- = [0, l) \times_f \mathbb{S}^{n-1}$ with respect to p^-. Let $a_\xi := \min\{d_\xi, r_0\}$ on M. We always assume $r_0 < \min\{l(p), l\}$.

Lemma 3.2 *The eigenfunction corresponding to the first Dirichlet eigenvalue of $V_n(p^-, r_0)$ may be chosen to be non-negative and is a radial function $\phi(t)$ satisfying $\phi'(t) < 0$ for $0 < t < r_0$.*

Proof The Laplacian on a spherically symmetric manifold in geodesic spherical coordinates at p^- is given by

$$\Delta = \frac{d^2}{dt^2} + (n-1)\frac{f'(t)}{f(t)}\frac{d}{dt} + \frac{1}{f^2(t)}\Delta_{\mathbb{S}^{n-1}}.$$

Then the first eigenfunction is radial satisfying

$$\frac{d^2\phi}{dt^2} + (n-1)\frac{f'(t)}{f(t)}\frac{d\phi}{dt} + \lambda_1(V_n(p^-, r_0))\phi = 0. \tag{3.3}$$

The last statement now follows from the previous lemma. □

We define a quantity on $M\backslash C(p)$ by

$$\theta(t, \xi) = \left[\frac{J(t, \xi)}{f(t)}\right]^{n-1}.$$

Theorem 3.3 *(Generalized Bishop's comparison theorem I) Given $\xi \in S_p^{n-1}$, and a model space $M^- = [0, l) \times_f \mathbb{S}^{n-1}$, under the curvature assumption on the radial Ricci tensor,* $\mathrm{Ricci}(v_x, v_x) \geq -(n-1)f''(t)/f(t)$, *for $x = \gamma_{(p,\xi)}(t)$ with $t < \min\{d_\xi, l\}$ (resp. with $t < \min\{a_\xi, l\}$) the function $\theta(t, \xi)$ is nonincreasing in t. In particular, for all $t < \min\{d_\xi, l\}$ (resp. $t < \min\{a(\xi), l\}$) we have $J(t, \xi) \leq f(t)$. Furthermore, this inequality is strict for all $t \in (t_0, t_1]$, with $0 \leq t_0 < t_1 < \min\{d_\xi, l\}$, if the above curvature assumption holds with a strict inequality for t in the same interval.*

Proof From the assumption on the radial Ricci curvature tensor, and (2.3), with initial conditions (2.4), the function $J(t, \xi)$ satisfies the following differential inequality

$$\begin{cases} J'' + k(t)J \leq 0, \quad 0 \leq t < l, \\ J(0, \xi) = 0, J'(0, \xi) = 1, \end{cases} \tag{3.4}$$

where $k(t) = -f''(t)/f(t)$. On the other hand, $y(t) = f(t)$ is the unique solution of the equation

$$\begin{cases} y'' + k(t)y = 0, \\ y(0) = 0, y'(0) = 1, \\ y > 0 \quad \text{on} \quad (0, l). \end{cases} \tag{3.5}$$

Consequently, on an interval $(0, l)$ on which $y(t) = f(t) > 0$, we have $J''f - f''J \leq 0$, that is $(J'f - f'J)' \leq 0$. The initial conditions for $J(t)$ and $f(t)$ then yield $J'f - f'J \leq 0$. Hence, $(J/y)' = (J/f)' \leq 0$, whenever $y(t) = f(t) > 0$ on $(0, l)$. Thus J/f is a nonincreasing function. Furthermore, by applying L'Hôpital's rule, we have

$$\lim_{t\to 0} \frac{J(t, \xi)}{f(t)} = \lim_{t\to 0} \frac{J'(t, \xi)}{f'(t)} = 1.$$

Consequently, for $t < d_\xi$, $J(t, \xi) \leq f(t)$ holds. Under the assumption on strict inequality for the radial Ricci curvature holding for $t \in (t_0, t_1]$, then $(J/f)' < 0$, i.e. J/f is decreasing in the same interval, and the last assertion holds. □

Remark 3.4 The proof of the first part of the above theorem may be found in [17] (with the wrong sign for $k(t)$).

As another consequence we have the following volume comparison result.

Corollary 3.5 *Under the curvature assumption of Theorem 3.3, we have*

$$V(B(p, r_0)) \leq V(V_n(p^-, r_0)),$$

with equality if and only if $B(p, r_0)$ *is isometric to* $V_n(p^-, r_0)$.

Proof The volume inequality follows immediately from (2.2) and Theorem 3.3. Now let us suppose that the equality of the volumes holds. Then $J(t, \xi) = f(t)$, for all t smaller than a_ξ. As in the proof of Bishops's comparison theorem II in p. 72–73 of [7], this implies at each point t with $J(t, \xi) = f(t)$ that $tr\, U^2 = (tr\, U)^2$, where $U = \mathbb{A}'\mathbb{A}^{-1}$, thus U is a scalar matrix and so is $\mathbb{A}$ with $\mathbb{A}(\xi, t) = f(t)I$. Hence, the metric of $B(p, r_0)$ is of the form (2.5), that is $B(p, r_0)$ is isometric to $V_n(p^-, r_0)$. □

The next theorem is proved using a similar argument to that of Cheng's in [10] and the previous corollary:

Theorem 3.6 *Let* M *be a complete* n*-dimensional Riemannian manifold with a radial Ricci curvature lower bound* $(n-1)k(t) = -(n-1)f''(t)/f(t)$ *with respect to the point* p. *We then have*

$$\lambda_1(B(p, r_0)) \leq \lambda_1(V_n(p^-, r_0)), \tag{3.6}$$

where $\lambda_1(\cdot)$ *denotes the first eigenvalue of the corresponding geodesic ball. Moreover, the equality in (3.6) holds if and only if* $B(p, r_0)$ *is isometric to* $V_n(p^-, r_0)$.

Proof Let ϕ be the first nonnegative eigenfunction of $V_n(p^-, r_0)$ which, by Lemma 3.2, depends only on the radial variable. Since $\phi \circ r$ vanishes on $\partial B(p, r_0)$, from the Rayleigh characterization we obtain

$$\lambda_1(B(p, r_0)) \leq \frac{\int (d\phi \circ r, d\phi \circ r)}{\int (\phi \circ r)^2}.$$

As in [10], we shall use spherical geodesic coordinates centred at p under the integral. Therefore,

$$\int\limits_{B(p,r_0)} (d\phi \circ r, d\phi \circ r) = \int\limits_{\xi \in S^{n-1}} \left[\int\limits_0^{a(\xi)} \left(\frac{d\phi}{dt} \right)^2 f(t)^{n-1} \theta(t, \xi) dt \right] d\sigma,$$

$$\int\limits_{B(p,r_0)} (\phi \circ r)^2 = \int\limits_{\xi \in S^{n-1}} \left[\int\limits_0^{a(\xi)} \phi(t)^2 f(t)^{n-1} \theta(t, \xi) dt \right] d\sigma,$$

where $d\sigma$ is the canonical measure of $\mathbb{S}^{n-1} \equiv S_p^{n-1}$.

On the other hand, we have

$$\begin{aligned} \int\limits_0^{a(\xi)} \left(\frac{d\phi}{dt} \right)^2 f(t)^{n-1} \theta(t, \xi) dt &= \phi \left(\frac{d\phi}{dt} \right) f(t)^{n-1} \theta(t, \xi) \Big|_0^{a(\xi)} \\ &- \int\limits_0^{a(\xi)} \frac{\phi(t)}{f(t)^{n-1}\theta(t, \xi)} \frac{d}{dt} \left[f(t)^{n-1} \theta(t, \xi) \cdot \frac{d\phi}{dt} \right] f(t)^{n-1} \theta(t, \xi) dt, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &\frac{1}{f(t)^{n-1}\theta(t, \xi)} \frac{d}{dt} \left[f(t)^{n-1} \theta(t, \xi) \frac{d\phi}{dt} \right] \\ &= \frac{d^2\phi}{dt^2} + \left[\frac{(n-1) f'(t)}{f(t)} + \frac{1}{\theta(t, \xi)} \frac{d\theta(t, \xi)}{dt} \right] \frac{d\phi}{dt} \\ &= \frac{d^2\phi}{dt^2} + \left\{ \frac{(n-1) f'(t)}{f(t)} + (n-1) \frac{f(t)}{J(t)} \left[\frac{J(t)}{f(t)} \right]' \right\} \frac{d\phi}{dt}. \end{aligned} \tag{3.8}$$

Since, by Lemma 3.2, $\frac{d\phi}{dt} < 0$ for $0 < t < r_0$, and (3.3) holds on $(0, l)$, we have

$$\begin{aligned} &- \int\limits_0^{a(\xi)} \frac{\phi(t)}{f(t)^{n-1}\theta(t, \xi)} \frac{d}{dt} \left[f(t)^{n-1} \theta(t, \xi) \cdot \frac{d\phi}{dt} \right] f(t)^{n-1} \theta(t, \xi) dt \\ &\leq \int\limits_0^{a(\xi)} \phi^2 \lambda_1(V_n(p^-, r_0)) \cdot f(t)^{n-1} \theta(t, \xi) dt. \end{aligned} \tag{3.9}$$

Thus, substituting (3.9) into (3.7), and using that $\phi(a_\xi) \frac{d\phi}{dt}(a_\xi) \leq 0$ gives

$$\int\limits_0^{a(\xi)} \left(\frac{d\phi}{dt} \right)^2 f(t)^{n-1} \theta(t, \xi) dt \leq \phi(a(\xi)) \left(\frac{d\phi}{dt} \right) (a(\xi)) f(a(\xi))^{n-1} \theta(a(\xi), \xi)$$

$$+\int_0^{a(\xi)} \phi^2\lambda_1(V_n(p^-,r_0))f(t)^{n-1}\theta(t,\xi)dt$$
$$\leq \int_0^{a(\xi)} \phi^2\lambda_1(V_n(p^-,r_0))f(t)^{n-1}\theta(t,\xi)dt.$$

Consequently, we have proved that

$$\int_{\xi\in S^{n-1}}\left[\int_0^{a(\xi)}\left(\frac{d\phi}{dt}\right)^2 f(t)^{n-1}\theta(t,\xi)dt\right]d\sigma$$
$$\leq \int_{\xi\in S^{n-1}}\left[\int_0^{a(\xi)}\lambda_1(V_n(p^-,r_0))\phi^2 f(t)^{n-1}\theta(t,\xi)dt\right]d\sigma.$$

Hence, $\lambda_1(B(p,r_0)) \leq \lambda_1(V_n(p^-,r_0))$. When equality holds, we have that $a(\xi)=r_0$ for almost all $\xi \in S^{n-1}$. Thus, $a(\xi)\equiv r_0$ for all ξ. We can then conclude that $J(t,\xi)=f(t)$, and so $V(B(p,r_0))=V(V_n(p^-,r_0))$ which implies, by Corollary 3.5, that $B(p,r_0)$ is isometric to $V_n(p^-,r_0)$. □

Corollary 3.7 *Under the curvature conditions of the previous theorem, holding for all $t < l(p) = l$ where $M^- = [0,l)\times_f \mathbb{S}^{n-1}$, if M is closed and M^- also closes i.e. $f(l)=0$ and satisfies the conditions on Lemma 2.5, then for all ξ, $\exp_p(l\xi)$ is a conjugate point of p, and* $\lim_{r\to l^-}\lambda_1(B(p,r))=0$.

Proof This follows from Theorem 3.6 and Lemma 2.5. Furthermore, by Theorem 3.3 we must have $J(l,\xi)=0$ for all ξ, that is $\exp_p(l\xi)$ is a conjugate point. □

Remark 3.8 Theorem 3.6 is a generalization of Cheng's Theorem 1.1 in [10], since space forms are spherically symmetric manifolds with constant $k(t)$. On the other hand, the choice of a suitable spherically symmetric model space adapted to each base point gives us a finer estimate of the first eigenvalue.

4 Generalized comparison theorems for manifolds with radial sectional curvature bounded from above

In order to prove our second result we shall first present some generalizations of Rauch's and Bishop's comparison theorems. We essentially keep the same notation, but we will now denote by M^+ the model space $[0,l)\times_f \mathbb{S}^{n-1}$ spherically symmetric with respect to a point p^+, with metric (2.5) and $f(t)$ satisfying the conditions in Definition 2.1, and by $V_n(p^+,r_0)$ its geodesic ball of radius r_0 and center p^+.

Theorem 4.1 (Generalized Rauch's comparison theorem) *Suppose M has a radial sectional curvature upper bound $k(t) = -\frac{f''(t)}{f(t)}$ along a given unit speed geodesic $\gamma(t)=\gamma_{(p,\xi)}(t)$, for $t \leq \min\{c_\xi, l\}$, where $c_\xi \geq d_\xi$ is a first conjugate point $\gamma_{(p,\xi)}(c_\xi\xi)$ along γ. Let $\beta \leq$* $\min\{c_\xi,l\}$. *For any normal Jacobi field Y along $\gamma_{|[0,\beta]}$ satisfying $Y(0)=0$, set*

$$\psi_{k(t)} = |Y|'(0)f(t).$$

Then on $(0, \beta)$ *we have*

$$\frac{|Y|'}{|Y|} \geq \frac{\psi'_{k(t)}}{\psi_{k(t)}}, \quad \left[\frac{|Y|}{\psi_{k(t)}}\right]' \geq 0, \quad |Y| \geq \psi_{k(t)}. \tag{4.1}$$

Equality occurs in any of the first two inequalities in (4.1) *at* $t_0 \in (0, \beta)$ *if and only if there exists a unit vector field* E*, parallel along* γ *and pointwise orthogonal to* γ *such that* $Y = \psi_{k(t)} E$ *on* $[0, t_0]$.

Proof Here we use a method similar to that in the proof of Rauch's comparison theorem in pages 67–68 of [7]. Away from conjugate points we have $|Y|' = g(Y, \nabla_t Y)|Y|^{-1}$, which implies, by our assumption and the Cauchy-Schwarz inequality, that

$$\begin{aligned}|Y|'' &= g(Y, \nabla_t Y)'|Y|^{-1} + g(Y, \nabla_t Y)(|Y|^{-1})' \\ &= |Y|^{-3}\{|\nabla_t Y|^2|Y|^2 - g(Y, \nabla_t Y)^2 - g(Y, \mathscr{R}Y)|Y|^2\} \\ &\geq \frac{f''(t)}{f(t)}|Y|.\end{aligned} \tag{4.2}$$

Therefore, one obtains $\{\psi_{k(t)}|Y|' - \psi'_{k(t)}|Y|\}' \geq 0$, with $f(0) = Y(0) = 0$, and the first two inequalities in (4.1) follow directly. Furthermore, applying L'Hôpital's rule we have

$$\lim_{t\to 0} \frac{|Y|(t)}{\psi_{k(t)}} = \lim_{t\to 0} \frac{|Y|'(t)}{|Y|'(0)f'(t)} = 1,$$

which yields the last inequality in (4.1). When equality holds on one of the first two equivalent inequalities in (4.1), at a given point t_0, the same holds in (4.2) for all $t \in [0, t_0]$ as well, as a consequence of the elementary fact that a nonnegative and nondecreasing C^1 function on $[0, \beta)$, with two zeros at $t = 0$ and $t = t_0$, must be constant on $[0, t_0]$. Then, on $[0, t_0]$, $g(Y, \nabla_t Y)^2 = |\nabla_t Y|^2|Y|^2$ and $g(Y, \mathscr{R}Y) = -(f''(t)/f(t))|Y|^2$. In particular, $\nabla_t Y$ is a multiple of $Y(t)$. Hence, on $[0, t_0]$, $Y = \psi_{k(t)}E$, with $E(t) = \nabla_t Y/|\nabla_t Y|$ a parallel unit vector field along $\gamma(t)$. □

Using the above result and now following a similar method to that in the proof of Bishop's comparison Theorem I on page 69 of [7], we obtain

Theorem 4.2 (Generalized Bishop's comparison theorem II) *Suppose* M *has a radial sectional curvature upper bound given by* $k(t) = -\frac{f''(t)}{f(t)}$ *for* $t < \beta \leq \min\{inj_c(p), l\}$*, where* $inj_c(p) = \inf_\xi c_\xi$*, with* c_ξ *defined as in previous Theorem* 4.1. *Then on* $(0, \beta)$

$$\left[\frac{\sqrt{|g|}}{f^{n-1}}\right]' \geq 0, \quad \sqrt{|g|}(t) \geq f^{n-1}(t), \tag{4.3}$$

and equality occurs in the first inequality at $t_0 \in (0, \beta)$ *if and only if*

$$\mathscr{R} = -\frac{f''(t)}{f(t)}, \quad \mathbb{A} = f(t)I,$$

on all of $[0, t_0]$.

Proof As in [7], we consider the self-adjoint positive definite matrix on $(0, \beta)$ defined by $\mathscr{B} = \mathbb{A}^*\mathbb{A}$. Then $\det \mathscr{B} = (\det \mathbb{A})^2$. For any given $\alpha \in (0, \beta)$, we can choose an orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ of $\{\gamma'(0)\}^\perp$, composed by eigenvectors of $\mathscr{B}(\alpha)$, and define $\eta_j(t) =$

$\mathbb{A}(t)e_j$, $j = 1, 2, \ldots, n-1$. Obviously, $\eta_j(0) = \mathbb{A}(0)e_j = 0$. Then as in [7], we have $(\ln \det \mathbb{A})'(\alpha) \geq (n-1) f'(\alpha)/f(\alpha)$, by using the above Theorem 4.1, which implies

$$\frac{(\det \mathbb{A})'}{\det \mathbb{A}} \geq \frac{[f^{n-1}(t)]'}{f^{n-1}(t)}.$$

Now first inequality in (4.3) follows directly. Furthermore, by applying L'Hôpital's rule $(n-1)$ times, we have $\lim_{t\to 0} \sqrt{g}(t)/f(t)^{n-1} = \lim_{t\to 0} \mathbb{A}'(t)/f'(t) = 1$, which implies second inequality of (4.3). The proof in the equality case follows as in the proof of Theorem 4.1. □

For convenience, we also state Barta's Lemma [4,7], which plays an important role in the proof of Theorem 4.4.

Lemma 4.3 (Barta [4,7]) *Let Ω be a normal domain in a Riemannian manifold, and $g \in C^2(\Omega) \cap C^0(\bar{\Omega})$, with $g|\Omega > 0$ and $g|\partial\Omega = 0$. Then*

$$\inf_{\Omega} \left(\frac{\Delta g}{g} \right) \leq -\lambda(\Omega) \leq \sup_{\Omega} \left(\frac{\Delta g}{g} \right),$$

where $\lambda(\Omega)$ denotes the lowest Dirichlet eigenvalue of the domain Ω.

We are now in a position to prove the following generalization of Cheng's result ([11]).

Theorem 4.4 *Suppose M is a complete n-dimensional Riemannian manifold with a radial sectional curvature upper bound $k(t) = -f''(t)/f(t)$ with respect to the point p. Then, for $r_0 < \min\{inj(p), l\}$, we have*

$$\lambda_1(B(p, r_0)) \geq \lambda_1(V_n(p^+, r_0)), \tag{4.4}$$

where $\lambda_1(\cdot)$ denotes the lowest Dirichlet eigenvalue of the corresponding geodesic ball. Furthermore, equality in (4.4) holds if and only if the two geodesic balls are isometric.

Proof As in the proof of Theorem 5 on p. 71 in [7], let $\phi : [0, r_0] \to [0, \infty)$ be a nonnegative radial eigenfunction of $\lambda_1(V_n(p^+, r_0))$ of M^+. Then, as noted in the proof of Lemma 3.2, this eigenfunction satisfies (3.3), with $\phi'(0) = \phi(0) = 0$, $\phi \geq 0$ on $[0, r_0)$, and $\frac{d\phi}{dt} < 0$ on $(0, r_0)$. Define a function $F : \overline{B(p, r_0)} \to [0, \infty)$, by $F(\exp_p t\xi) = \phi(t)$, for $(t, \xi) \in [0, r_0] \times \mathbb{S}_p^{n-1}$. Then by a straightforward calculation as in [7], and using Theorem 4.2, we have

$$\frac{\Delta F}{F}(\exp_p t\xi) \leq \frac{1}{\phi}\left[\phi'' + (n-1)\frac{f'(t)}{f(t)}\phi'\right] = -\lambda_1(V_n(p^+, r_0)).$$

Hence Barta's Lemma yields

$$-\lambda_1(B(p, r_0)) \leq \sup \frac{\Delta F}{F}(\exp_p t\xi) \leq -\lambda_1(V_n(p^+, r_0)),$$

implying the inequality (4.4). The last claim is a direct application of Theorem 4.2. □

Remark 4.5 If $f(l_+) = 0$ for some l_+ smaller than $inj(p)$, and the model space M^+ closes at l_+ in a sufficiently regular way—see Sect. 2—, the above result still holds for r_0 up to l_+^- in the trivial sense that the lower bound is given by the (vanishing) first eigenvalue of the resulting closed manifold.

5 Existence of the model spaces and their applicability

Given a manifold M and a point p in M, in order to apply the generalisations of Bishop's comparison theorems for the volume given in Sects. 3 and 4 and Theorems 3.6 and 4.4 to estimate the first eigenvalue of a disk centred at p and radius t, it is necessary that the corresponding model manifolds are defined for the value of t in question. One situation where these manifolds will cease to exist is when the function f which is used to define them stops being positive at some value of t. However, and as may be seen from Theorem 4.4, other situations may occur. We shall now illustrate some of these possibilities and then focus mainly on the non-compact case.

A solution $f = y(t)$ of (3.5) defined on a maximal interval $[0, l)$, where $t = l$ is the first nonzero zero of f, defines a model space $M^* = [0, l) \times_f \mathbb{S}^{n-1}$, that is complete if $l = +\infty$. For finite l, M^* closes as explained in Sect. 2.

A straightforward consequence of standard comparison results for solutions of second order ordinary differential equations such as (3.5) is the following comparison result for model spaces, that can also be seen as a consequence of Theorem 3.3 using (2.11):

Proposition 5.1 *Given two model spaces $M_i = [0, l_i) \times_{f_i} \mathbb{S}^{n-1}$, $i = 1, 2$, if the radial curvatures $k_i(t) = -f_i''(t)/f_i(t)$ satisfy $k_2(t) \geq k_1(t)$ for all $t \in (0, l)$, where $l \leq \min\{l_1, l_2\}$, then $f_1(t) \geq f_2(t)$ on $(0, l)$. In particularly if $f_1(l_1) = 0$, then $l_2 \leq l_1$, i.e M_2 closes before or at the same time as M_1. Furthermore, if strict inequality holds between the k_i's, then we also have strict inequality for the functions f_i.*

The above result immediately implies that, given a manifold M and a point p on M, the model manifold M^- is always defined while M^+ is. However, the fact that a model manifold M^+ is defined is not enough to ensure the corresponding comparison results may be applied, as the following simple example illustrates.

Let C be an infinite cylinder of unit radius and take any point p on C. In this case, since the curvature vanishes, both model manifolds coincide with Euclidean space and we immediately obtain that, as long as both theorems may be applied, the first eigenvalue of the disk centred at p on the cylinder is given by $j_{0,1}^2/t^2$, where $j_{0,1}$ denotes the first positive zero of the Bessel function J_0. However, for t larger than π, the disk will contain points from the cut locus of p, and the conditions in Theorem 4.4 no longer hold. In fact, since the boundary of the disk has now become disconnected, Theorem 3.6 ensures that there is no equality and thus the first eigenvalue of the disk of radius t must be strictly smaller than that of its Euclidean counterpart whenever t is larger than π. A similar reasoning applies to the volume comparison Theorem 4.2. This shows that it is possible for the function f corresponding to the model surface M^+ to remain strictly positive, but the volume and eigenvalues bounds are no longer valid.

We shall now discuss the domain of definition of our model spaces $M^{\pm}$ and see that while M^- can actually be defined up to the maximum distance $l(p)$ in the original manifold M, M^+ is only defined at most up to $inj(p)$. Set $k_-(t) := k_-(p, t)$ and $k_+(t) := k_+(p, t)$ as defined by (2.9) and (2.10), respectively, for t in $[0, l(p))$ and $[0, inj(p))$, respectively. Let $f_\pm$ be the corresponding solutions of (3.5) with respect to $k_\pm(t)$, respectively. If we fix $t < l(p)$, then there exists some $\xi \in S_p^{n-1}$ s.t. $t < d_\xi$. Applying Theorem 3.3, we have $J(t, \xi) \leq f_-(t)$. Since $J(t, \xi)$ cannot vanish, the same holds for $f_-(t)$, that is, the model M^- is defined at least up to $l_- = l(p)$. The choice of domain of definition $[0, inj(p))$ for $k_+(p, t)$ is a consequence of the fact that the proof of Theorem 4.4 is only valid in this range, as explained above, that is, M^+ only represents a model space for M for $t < l_+$ with l_+ at most $inj(p)$.

For the general case, we have thus proved the following

Proposition 5.2 *Given M and $p \in M$, the model space $M^- = [0, l_-) \times_{f_-} \mathbb{S}^{n-1}$ is defined for $l_- = l(p)$, and f_- satisfying $f_-''(t)/f_-(t) = -k_-(t)$, for all $t < l_-$. If $l_- < +\infty$ and $f_-(l_-) = 0$, then any $q = exp_p(d_\xi \xi) \in C^+(p)$ is a conjugate point of p. The model space $M^+ = [0, l_+) \times_{f_+} \mathbb{S}^{n-1}$ satisfying $f_+''(t)/f_+(t) = -k_+(t)$ is defined for $t \in [0, l_+)$ for some $l_+ \le l = inj(p)$, and $f_+(t) \le f_-(t)$, $\forall t < l_+$.*

Remark 5.3 By a result due to Klingenberg (see [9], Theorem 5.9 (1)) when M is a closed even dimensional Riemannian manifold of positive sectional curvature and $inj(p) = inj(M)$, then $C(p)$ has a conjugate point. Complete Riemannian manifolds of nonpositive sectional curvature have no conjugate points, and the exponential map from any point p is a covering map. It is also known that there are closed surfaces with no conjugate points as shown by examples constructed in [1].

We observe that if M is closed and we have the same curvature bound $k(t)$ for all base points, then our comparison theorems reduce to Cheng's case of $k(t)$ constant. Intuitively we see that the parameter t of the functions $k_\mp(x, t)$ defined in (2.9) and (2.10), respectively, has meaning only when it depends on the point x, unless these functions are constant. To be more precise, we derive the following:

Lemma 5.4 *If M is closed, then $k_-(t) := \min_{x \in M} k_-(x, t)$, defined for $0 \le t \le inj(M)$, is a constant function on t.*

Proof We fix $inj(M) > t > 0$ and $s \in \mathbb{R}$ s.t. $inj(M) > t+s > 0$. There exist $(x_0, \xi_0) \in SM$ s.t. $k_-(t+s) = k_-(x_0, t+s) = \text{Ricci}_{rad}(x_0, \xi_0, t+s)$. On the other hand $\text{Ricci}_{rad}(x_0, \xi_0, t+s) = \text{Ricci}_{rad}(x_s, \xi_s, t)$, where $(x_s, \xi_s) = \beta(s, x_0, \xi_0)$. But $\text{Ricci}_{rad}(x_s, \xi_s, t) \ge k_-(x_s, t) \ge k_-(t)$. We have shown that $k_-(t+s) \ge k_-(t)$. For the same reason $k_-(t) = k_-((t+s)-s) \ge k_-(t+s)$, what shows that $k_-(t)$ is constant. □

For the non compact case we have

Lemma 5.5 *If $k_-(x, t)$ does not depend on the variable x then it is constant as a function on (x, t).*

The proof is similar to that of Lemma 5.4 and similar conclusions may be drawn for an upper bound for the radial sectional curvature.

We shall now turn our attention to the noncompact case, for which $l(p) = +\infty$. This implies that the model manifold M^- based on the minimum of the curvature is also noncompact and a disk with arbitrary large radius centred at the base point exists.

It remains to consider the situation of M^+. More precisely, given a complete non compact Riemannian manifold M, we want to know under which conditions on the function $k_+(p, t)$, defined by (2.10), the model space can be defined for all time. From what was stated above, if the sectional curvature is nonpositive it follows that the disk centred at the base point of the model manifold may also be extended for all t. We are thus mainly interested in the situation where $k(t)$ takes on positive values, and to find out when M^+ will be defined for all t under these conditions. This is equivalent to finding out under which conditions the solution $f(t)$ of (3.5) remains positive for all positive t, in which case the corresponding model space will have a pole at the base point and is of the form $M^+ = [0, +\infty) \times_f \mathbb{S}^{n-1}$. The non-existence of zeros on the interval $(0, \infty)$ is related to the oscillation theory of ordinary differential equations developed by Hille in [16] and we shall now apply it to our problem.

We say that a solution of an ordinary differential equation is oscillatory on the interval (a, ∞) for some positive a, if there are infinitely many zeros on this interval. If the solution has at most one zero point on (a, ∞), then we say it is non-oscillatory. By the Sturm separation theorem, we know if a solution of an ordinary differential equation is oscillatory, then all remaining solutions of this ordinary differential equation are oscillatory. An ordinary differential equation is said to be oscillatory if its solutions are oscillatory, otherwise, it is non-oscillatory.

Define now

$$\bar{k}(t) := t \int_t^{\infty} k(\tau) d\tau,$$

and write

$$\lim_{t\to\infty} \sup \bar{k}(t) = k^*, \quad \lim_{t\to\infty} \inf \bar{k}(t) = k_*. \tag{5.1}$$

A result by Hille in [16] states that

Theorem 5.6 *Given the ordinary differential equation $f''(t) + k(t) f(t) = 0$, where $k(t)$ is a nonnegative function defined for positive t and belongs to $L^1((\varepsilon, 1/\varepsilon))$ for each $\epsilon > 0$, if this equation is non-oscillatory for large t, then $k_* \leq \frac{1}{4}$ and $k^* \leq 1$. Both estimates are the best possible of their kind.*

We define the multi-parameter family of functions $\Phi(t)$ $(t > 0)$ by

$$\Phi(t) = \frac{b_2 b_3^2 e^{-b_3 t}}{b_1 t - b_2 e^{-b_3 t} + b_2}, \tag{5.2}$$

for constants b_1, b_2 and b_3 satisfying

$$0 < b_1 < 1, \quad b_3 > 0, \quad b_1 + b_2 b_3 = 1.$$

The following result is proved in [19]

Theorem 5.7 *For the initial value problem* (3.5) *we have*

- (I) *if $k(t) \leq \Phi(t)$ or $k(t) \leq \frac{1}{4(t+1)^2}$ for $t > 0$, where $\Phi(t)$ is defined by* (5.2)*, then it has a positive solution on* $(0, \infty)$*;*
- (II) *if $k(t) \geq \alpha$ for some positive constant $\alpha > 0$, then it cannot have a positive solution on* $(0, \infty)$*;*
- (III) *if $k_* > \frac{1}{4}$ or $k^* > 1$, where k_* and k^* are defined by* (5.1)*, then it cannot have a positive solution on* $(0, \infty)$.

Applying this to our problem we obtain the following

Corollary 5.8 *Given an n-dimensional ($n \geq 2$) complete manifold M, and a point $p \in M$, if its radial sectional curvature is bounded from above by $k(t)$ with respect to the point p, where $k(t)$ is a continuous function with respect to the distance parameter $t = d(p, \cdot)$, then under the assumptions in (I) of Theorem* 5.7*, the spherically symmetric manifold $M^+ := (0, \infty) \times_f \mathbb{S}^{n-1}$, where the function f is the unique solution of* (3.5)*, can be used as the model space of M, while under the assumptions in (II) or (III) of Theorem* 5.7*, it is not possible to find a spherically symmetric manifold with a pole as the model space of M.*

6 Examples

We shall now illustrate our results by obtaining eigenvalue bounds for disks on surfaces. In this case ($n = 2$), when $|f'_{\pm}(t)| \leq 1$ we may take the corresponding isometric surfaces of revolutions into $\mathbb{R}^3$, centred at the origin as defined in Sect. 2. In this instance, and placing both model surfaces at the origin, we see that M^- is below M^+ and this will be the only intersection point in case of strict inequality of the curvatures.

We shall examine three different situations, namely, both positive and negative curvatures (a torus), positive curvature (an elliptic paraboloid) and negative curvature (a saddle). In the case of the torus, we are able to compute explicitly K and $K_{\pm}$, while in the other two situations although it is possible to compute K explicitly, $K_{\pm}$ have to be computed numerically. In all three examples the function f is computed numerically by solving equation (3.5) and it is not to be expected that this equation may be solved in closed form in general—there are some situations for the torus, for instance, where it will be possible to find f explicitly (see [19]), but we shall not pursue that here. Our perspective is that, just like in Cheng's case, we approximate a spectral problem for a partial differential operator by the much simpler situation of the determination of the spectrum of an ordinary differential operator.

Example 6.1 **Torus:** Consider the torus $\mathscr{T}_\epsilon$ obtained by rotating the circle $(x-1)^2 + z^2 = \epsilon^2 (0 < \epsilon < 1)$ in the xz-plane around the z-axis. This may be parameterized by

$$\begin{cases} x = (1 + \epsilon \cos v) \cos u \\ y = (1 + \epsilon \cos v) \sin u \\ z = \epsilon \sin v \end{cases} \tag{6.1}$$

with $u, v \in [0, 2\pi)$, while the corresponding Gaussian curvature depends only on v and is given by

$$K = \frac{\cos v}{\epsilon(1 + \epsilon \cos v)}.$$

Given a point p on $\mathscr{T}_\epsilon$, we are interested in determining bounds for K on the boundary of the disk centred at p and with radius t, say $B(p, t)$, so that we can then apply Theorems 3.6 and 4.4 above.

We first note that for any circle on the torus obtained by fixing u in (6.1) the curvature will be decreasing for v on $[0, \pi)$ and increasing on $[\pi, 2\pi)$. On the other hand, the points with the largest and smallest values of v in $\partial B(p, t)$ will have the same value of u as the point p. This yields that, for a point p with coordinates $(u, v) = (u_0, v_0)$, the minimum and maximum values of the curvature in $\partial B(p, t)$ are given by

$$K_-(t) = \begin{cases} \dfrac{\cos(v_0 + t\epsilon^{-1})}{\epsilon[1 + \epsilon \cos(v_0 + t\epsilon^{-1})]}, & 0 < t < (\pi - v_0)\epsilon \\ \dfrac{-1}{\epsilon(1-\epsilon)}, & (\pi - v_0)\epsilon \leq t \leq \epsilon\pi \end{cases}$$

and

$$K_+(t) = \begin{cases} \dfrac{\cos(v_0 - t\epsilon^{-1})}{\epsilon[1 + \epsilon \cos(v_0 - t\epsilon^{-1})]}, & 0 < t < \epsilon v_0 \\ \dfrac{1}{\epsilon(1+\epsilon)}, & \epsilon v_0 \leq t \leq \epsilon\pi \end{cases},$$

respectively.

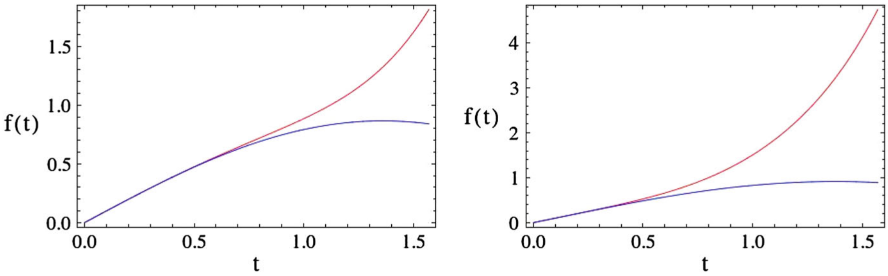

Fig. 1 Functions f for the torus with v_0 equal to 0 (*left*) and $\pi/2$ (*right*); the *upper* and *lower curves* correspond to K_- and K_+, respectively

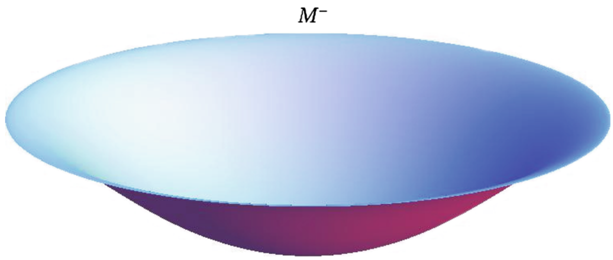

Fig. 2 Surface M^- corresponding to a point p on the torus having maximum positive curvature ($v_0 = 0$)

In order to obtain the model surfaces $M^{\pm}$, we now need to solve equation (3.5) with $k(t)$ replaced by each of the functions $K_{\pm}$ given by the expressions above. Solving the corresponding equations numerically for particular points p and ϵ equal to $1/2$ yields the functions f shown in Fig. 1.

In the case where v_0 is equal to zero, the corresponding model surface M^+ is a round sphere, as the maximum of the curvature remains constant, and our results coincide with Cheng's. The surface M^- corresponding to the same point p is shown on Fig. 2. We note that in this case M^- stops being isometrically embeddable in $\mathbb{R}^3$ for $t \approx 1.097$, due to the fact that f' becomes larger than one at this point. However, the comparison theorems remain valid after that and as long as f remains positive. To obtain the corresponding bounds, we then need to compute the first eigenvalue for a geodesic disk on these surfaces, which is obtained from equation (3.3). The values obtained for the two examples above are shown in Figs. 3 and 4. In all examples we have opted for showing the graphs of $t^2\lambda$ in order to keep the graphs bounded as t approaches zero.

Example 6.2 **Elliptic paraboloid:** As a second example we consider the surface of the elliptic paraboloid given by

$$z = x^2 + 4y^2.$$

Note that it is still possible to compute the curvature at each point on this surface, which is given by

$$K(x, y) = \frac{16}{(1 + 4x^2 + 64y^2)^2}.$$

The difficulty lies in the determination of the maximum and minimum values of K on a geodesic ball centred at a point p, and thus all the following computations were done

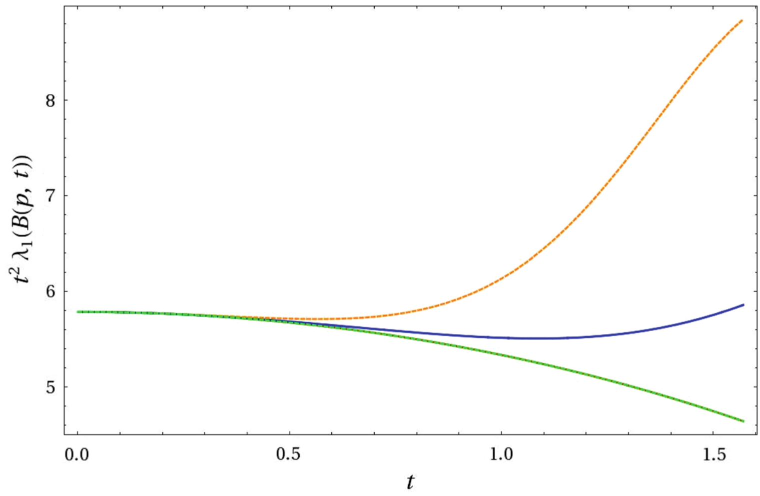

Fig. 3 Upper and lower bounds corresponding to $t^2\lambda_1(B(p,t))$ of a disk centred at a point p on the torus having $v_0 = 0$; the *top curve* corresponds to Cheng's upper bound shown here for comparison (Cheng's lower bound coincides with ours when $v_0 = 0$)

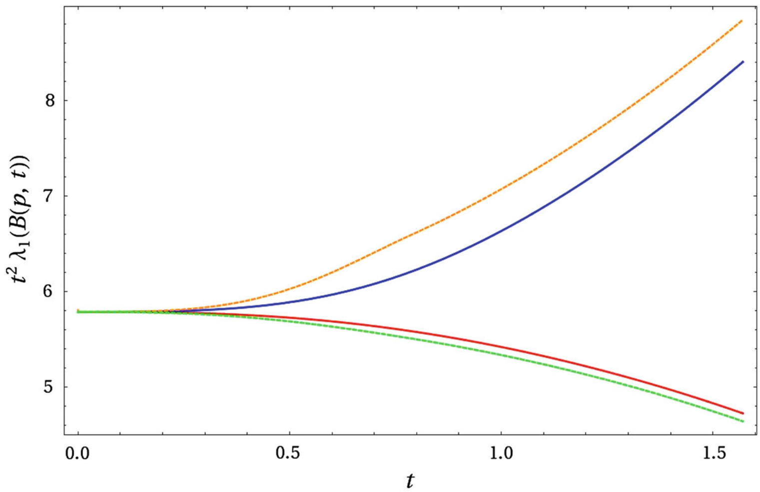

Fig. 4 Upper and lower bounds corresponding to $t^2\lambda_1(B(p,t))$ of a disk centred at a point p on the torus having $v_0 = \pi/2$; the *outer curves* correspond to Cheng's bounds shown here for comparison

numerically. For the case where the point p is the vertex of the parabola, we have determined the corresponding model surfaces which are shown in Fig. 5. We note that while M^- is defined for all positive t, M^+ is only defined up to $t \approx 1.7314$.

The graph of $t^2\lambda$ on geodesic disks of radius t centred at the pole of these surfaces are shown in Fig. 6. In agreement with Lemma 2.5, we see that λ converges to zero as t approaches the value where M^+ closes.

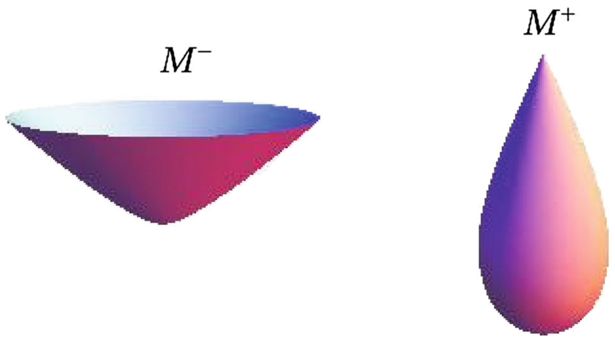

Fig. 5 Surfaces $M^{\pm}$ corresponding to p being the vertex of the elliptic paraboloid $z = x^2 + 4y^2$

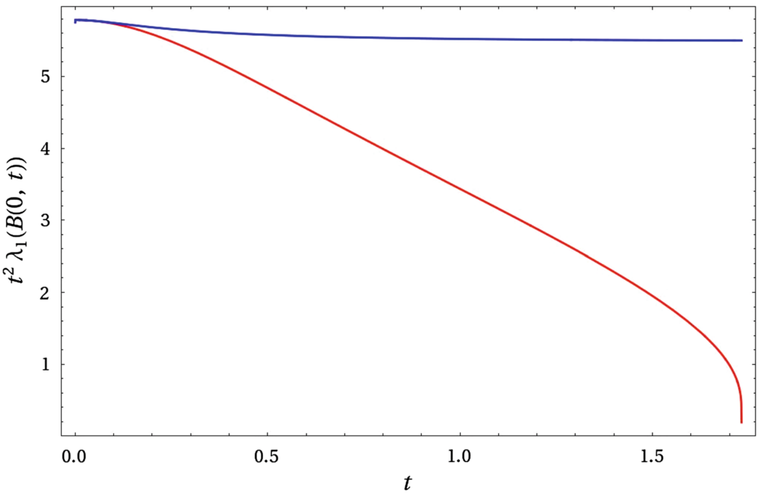

Fig. 6 Upper and lower bounds obtained for the first eigenvalue of the disk centred at the origin for the elliptic paraboloid defined by $z = x^2 + 4y^2$

Example 6.3 **Saddle:** Finally, we consider an example with negative curvature, the saddle surface given by

$$z = x^2 - y^2.$$

Again we may compute the curvature at each point which is found to be

$$K(x, y) = -\frac{4}{(1 + 4x^2 + 4y^2)^2},$$

while the values of $K_{\pm}$ have to be found numerically, together with the corresponding functions f. Due to the fact that both $K_{\pm}$ will be negative, it is not possible to isometrically embed the resulting surfaces $M^{\pm}$ in R^3.

In this case both model surfaces are defined for all values of the radius, since the curvature is always negative. In Fig. 7 we plot the corresponding values of $t^2\lambda$ for t in [0, 100], while Fig. 8 shows an upper bound for the percentage of the error made.

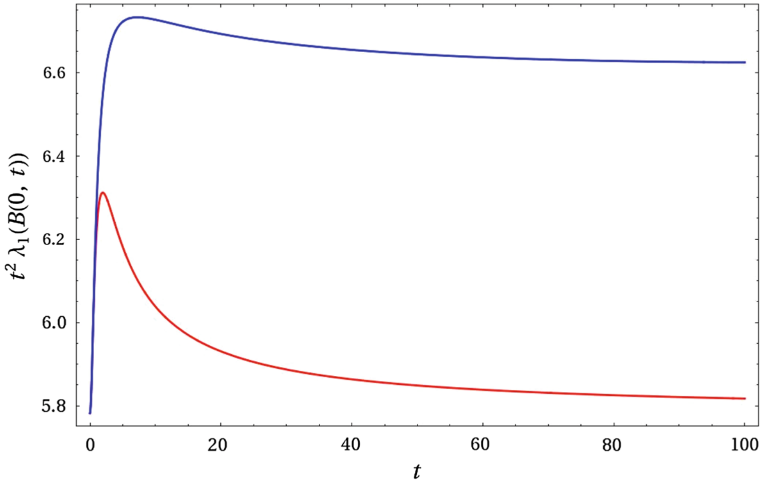

Fig. 7 Upper and lower bounds obtained for the first eigenvalue of the disk centred at the origin for the saddle surface given by $z = x^2 - y^2$

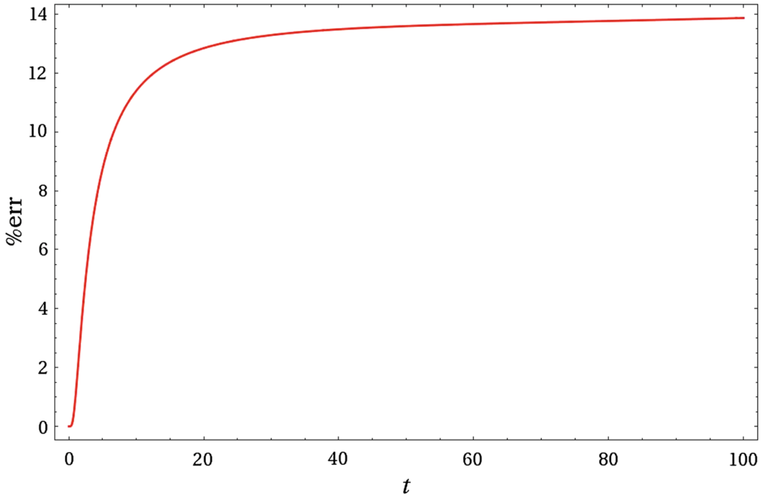

Fig. 8 Upper bound for the error (percentwise) for the first eigenvalue of the disk centred at the origin for the saddle surface given by $z = x^2 - y^2$

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