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Calculus of Variations

Fractional eigenvalues

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Abstract We study the non-local eigenvalue problem

$$2\int_{\mathbb{R}^n} \frac{|u(y)-u(x)|^{p-2}\big(u(y)-u(x)\big)}{|y-x|^{\alpha p}}\,dy + \lambda|u(x)|^{p-2}u(x) = 0$$

for large values of p and derive the limit equation as $p \to \infty$. Its viscosity solutions have many interesting properties and the eigenvalues exhibit a strange behaviour.

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1 Introduction

The problem of minimizing the fractional Rayleigh quotient

$$\inf_{\phi} \frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\phi(y)-\phi(x)|^p}{|y-x|^{\alpha p}}\,dx\,dy}{\int_{\mathbb{R}^n} |\phi(x)|^p\,dx} \tag{1}$$

among all functions ϕ in the class $C_0^\infty(\Omega)$, $\phi \not\equiv 0$ leads to an interesting eigenvalue problem with the non-local Euler-Lagrange equation

$$2\int_{\mathbb{R}^n} \frac{|u(y)-u(x)|^{p-2}\big(u(y)-u(x)\big)}{|y-x|^{\alpha p}}\,dy + \lambda|u(x)|^{p-2}u(x) = 0$$

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in a bounded domain Ω in the n-dimensional Euclidean space. Here $p \geq 2$ and $n < \alpha p < n + p$. It is an essential feature that the solutions may be multiplied by constant factors. We treat the solutions in the viscosity sense and prove, among other things, that positive viscosity solutions are unique (up to a normalization) and that the first eigenvalue is isolated. For sign changing solutions we detect some strange phenomena, caused by the influence of points far away appearing in the domain of integration for the non-local operator. Indeed, it is as if the nodal domains were interacting with each other. In the linear case $p = 2$ the connection to the more familiar *fractional Laplacian* is the principal value formula

$$(-\Delta)^{(2\alpha-n)/2} u\,(x) = -C(n,\alpha)\,\text{P.V.} \int\limits_{\mathbb{R}^n} \frac{u(y)-u(x)}{|y-x|^{2\alpha}}\,dy$$

valid at least in the range $n < 2\alpha < n + 2$. The linear case has been treated in [8,17,20].

To the best of our knowledge, no advanced regularity theory is yet available for $p \neq 2$. To assure continuity for eigenfunctions we have, occasionally, assumed that αp is larger than what appears to be necessary. This is of little importance here, because our main interest is the asymptotic case $p = \infty$. Formally, one has then to minimize the quotient

$$\frac{\left\| \frac{u(y)-u(x)}{|y-x|^{\alpha}} \right\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}}{\|u\|_{L^\infty(\mathbb{R}^n)}}, \qquad 0 < \alpha \leq 1,$$

among all admissible functions u. However, this minimization problem has too many solutions. Therefore the proper limit equation is called for. The equation takes the form

$$\max\left\{\mathcal{L}_\infty u\,(x),\ \mathcal{L}_\infty^- u\,(x) + \lambda u(x)\right\} = 0 \tag{2}$$

in Ω. In this new equation λ is a real parameter (the eigenvalue) and

$$\begin{aligned} \mathcal{L}_\infty u\,(x) &= \sup_{y\in\mathbb{R}^n} \frac{u(y)-u(x)}{|y-x|^{\alpha}} + \inf_{y\in\mathbb{R}^n} \frac{u(y)-u(x)}{|y-x|^{\alpha}} \\ \mathcal{L}_\infty^- u\,(x) &= \inf_{y\in\mathbb{R}^n} \frac{u(y)-u(x)}{|y-x|^{\alpha}}. \end{aligned}$$

The solutions u, referred to as ∞-eigenfunctions, belonging to $C_0(\overline{\Omega})$, and regarded as zero outside Ω, have to be interpreted in the viscosity sense, because the operator $\mathcal{L}_\infty u\,(x)$ is not sufficiently smooth. It is remarkable that the parameter λ behaves like a genuine eigenvalue. Indeed, a non-negative solution $u > 0$ belonging to $C_0(\overline{\Omega})$, exists if and only if λ has the value:

$$\lambda = \frac{1}{\left(\max\limits_{x\in\Omega} \text{dist}(x, \mathbb{R}^n \setminus \Omega)\right)^{\alpha}} \equiv \Lambda_\infty^\alpha.$$

Thus the radius R of the largest inscribed ball in Ω is decisive: $\Lambda_\infty^\alpha = R^{-\alpha}$. If $\alpha = 1$, the eigenvalue $\Lambda_\infty^1 = \Lambda_\infty$ is, incidentally, the same as the one in the differential equation

$$\max\left\{\Lambda_\infty - \frac{|\nabla u|}{u},\ \sum_{i,j} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}\right\} = 0, \tag{3}$$

treated in [12], but the equations are not equivalent. This differential equation is related to finding

$$\min_u \frac{\|\nabla u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\Omega)}}$$

among all $u \in W_0^{1,\infty}(\Omega)$, $u \not\equiv 0$. It is the limit of the Euler-Lagrange equations coming from the minimization of the Rayleigh quotients

$$\frac{\int_\Omega |\nabla u(x)|^p\,dx}{\int_\Omega |u(x)|^p\,dx} \tag{4}$$

as $p \to \infty$. Therefore a comparison of the two problems is of actual interest.

Let us return to the ∞-eigenvalue Eq. (2). A central part of the domain Ω, called the High Ridge, is important. With the notation $\delta(x) = \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)$ and $R = \|\delta\|_\infty$, the set $\Gamma = \{x \in \Omega |\, \delta(x) = R\}$ is the High Ridge. We have discovered the remarkable representation formula

$$u(x) = \frac{\delta(x)^\alpha}{\delta(x)^\alpha + \rho(x)^\alpha},$$

where $\rho(x) = \operatorname{dist}(x, \Gamma)$. The formula is valid in every domain and gives a first ∞-eigenfunction. If $\Gamma_1 \subset \Gamma$ is an arbitrary non-empty closed subset, the same formula, but with $\rho(x)$ replaced by

$$\rho_1(x) = \operatorname{dist}(x, \Gamma_1),$$

also yields an ∞-eigenfunction. Thus *uniqueness is lost*. We do not know whether all positive solutions of (2) are represented. —No such formula is known for the differential Eq. (3). To derive and verify the representation formula we use the Dirichlet problem for the equation

$$\mathcal{L}_\infty u\,(x) = 0 \quad \text{in } \Omega \setminus \Gamma$$

with boundary values 0 and 1. This equation has been treated in [5].

We have included a brief account on the higher eigenvalues, corresponding to sign changing solutions. In this case the ∞-eigenvalue Eq. (2) has to be amended to include the open set $\{u < 0\}$ and the nodal line $\{u = 0\}$, see Eq. (17) on page 38. Strange phenomena occur. First, the nodal domains, which are the connected components of the open sets $\{u > 0\}$ and $\{u < 0\}$, do not have the same *first* ∞-eigenvalue, yet they all come from the same higher ∞-eigenfunction. Second, the restriction of a higher ∞-eigenfunction to one of its nodal domains (and extended as zero) is not an ∞-eigenfunction for the nodal domain in question. Even one-dimensional examples exhibit this, see Sect. 12.

To this one may add that such a behaviour is totally impossible for equations like

$$\Delta u + \lambda u = 0, \quad \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0,$$

and (3). It is the non-local character of our equation that causes such phenomena.

Needless to say, there are many open problems with our fractional, non-local, non-linear eigenvalue problem, both for finite exponents p and for $p = \infty$. For example, the simplicity of the first ∞-eigenvalue Λ_∞^α is valid only in the special case when the High Ridge contains exactly one point. Nonetheless, this does not yet exclude the possibility that the minimizers of the fractional Rayleigh quotient (4) can converge to a unique function, as $p \to \infty$. It stands to reason that the limit procedure $p \to \infty$ should produce the maximal solution, the one with $\Gamma_1 = \Gamma$. But the presently known situation for the "local" problem (3) is also incomplete; see however [6,10,19] for some progress. The higher eigenvalues are mysterious when $p \neq 2$: for none of the equations mentioned is it known that the eigenvalues are countable! This challenging problem about the spectrum is likely to be the most difficult open question in this connection.

2 Preliminaries and notation

To study the fractional Rayleigh quotient (1) the so-called fractional Sobolev spaces[1] $W^{s,p}(\mathbb{R}^n)$ with $0 < s < 1$ are expedient. If $1 < p < \infty$, as usual, the norm is defined through

$$\|u\|^p_{W^{s,p}(\mathbb{R}^n)} = \iint\limits_{\mathbb{R}^n\mathbb{R}^n} \frac{|u(y)-u(x)|^p}{|y-x|^{sp+n}} dxdy + \int\limits_{\mathbb{R}^n} |u|^p dx.$$

The space $W^{s,p}(D)$ for a bounded and open subset D of $\mathbb{R}^n$ is defined similarly and, as usual $W^{s,p}_0(D)$ is defined as the closure of $C^\infty_0(D)$ with respect to the norm $\|\cdot\|_{W^{s,p}(D)}$. The relation between s and our α is $n + sp = \alpha p$. In "Hitchhiker's Guide to the Fractional Sobolev Spaces" one can find most of the useful properties, cf. [7]. We list some of them below.

Theorem 1 (Sobolev-type inequality) *Let $D \subset \mathbb{R}^n$ be bounded and open, $sp < n$ and $s \in (0,1)$. Then there is a constant C such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \left(\iint\limits_{\mathbb{R}^n\mathbb{R}^n} \frac{|u(y)-u(x)|^p}{|y-x|^{sp+n}} dxdy \right)^{\frac{1}{p}},$$

for all $u \in W^{s,p}_0(D)$ and where $p^* = \frac{np}{n-sp}$.

This is Theorem 6.10 on page 49 in [7]. From this one can extract the following estimate.

Theorem 2 *Let $\alpha p > n$. If Ω is a bounded domain in $\mathbb{R}^n$ there exists a constant $C(n,p,\alpha) > 0$ such that*

$$C(n,p,\alpha)\, |\Omega|^{1-\frac{\alpha p}{n}} \int\limits_\Omega |\phi|^p \, dx \leq \int\limits_\Omega \int\limits_\Omega \frac{|\phi(y)-\phi(x)|^p}{|y-x|^{\alpha p}} \, dx\, dy$$

for all $\phi \in C^\infty_0(\Omega)$.

The right-hand side is the so-called Gagliardo seminorm raised to the p^{th} power.

Theorem 3 (Hölder embedding) *Let $D \subset \mathbb{R}^n$ be bounded and open, $sp > n$ and $s \in (0,1)$. Then there is a constant C such that for all $u \in W^{s,p}_0(D)$*

$$\|u\|_{C^{0,\beta}(\mathbb{R}^n)} \leq C\|u\|_{W^{s,p}(\mathbb{R}^n)},$$

where $\beta = (sp-n)/p$.

This is Theorem 8.2 on page 38 in [7] and here

$$\|u\|_{C^{0,\alpha}(D)} = |[u]|_{\alpha,D} + \|u\|_{L^\infty(D)},$$

where we use the notation

$$|[u]|_{\alpha,D} = \left\| \frac{u(x)-u(y)}{|x-y|^\alpha} \right\|_{L^\infty(D\times D)}, \qquad |[u]|_\alpha = |[u]|_{\alpha,\mathbb{R}^n}.$$

[1] These spaces are also known as Aronszajn, Gagliardo or Slobodeckij spaces

Theorem 4 (Compact embedding) *Assume $D \subset \mathbb{R}^n$ to be bounded and open, $p \in [1, \infty)$ and $s \in (0, 1)$. Let u_i be a sequence of functions in $W_0^{s,p}(D)$ such that*

$$\|u_i\|_{W^{s,p}(\mathbb{R}^n)} \leq M < \infty.$$

Then there is a subsequence of u_i converging in $L^q(\mathbb{R}^n)$ for all $q \in [1, p]$.

This result can be found in Theorem 7.1 on page 33 in [7].

It is worth mentioning that asymptotically, as $s \to 1$, the space $W^{s,p}$ becomes $W^{1,p}$, see [4]. The same also holds for the corresponding Euler-Lagrange equation, see [11].

A function $u \in C_0(\overline{\Omega})$ or $u \in W_0^{s,p}(\Omega)$ is always assumed be defined in the whole $\mathbb{R}^n$ by extending it by zero.

3 The Euler-Lagrange equation

Let Ω be a bounded domain in $\mathbb{R}^n$. We consider the problem of minimizing the fractional Rayleigh quotient among all functions ϕ in the class $C_0^\infty(\Omega)$, $\phi \not\equiv 0$:

$$\inf_{\phi} \frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\phi(y)-\phi(x)|^p}{|y-x|^{\alpha p}}\,dx\,dy}{\int_{\mathbb{R}^n} |\phi(x)|^p\,dx} \quad = \quad \lambda_1. \tag{5}$$

It is desirable that

$$n < \alpha p < n + p,$$

but we will often require the narrower bound

$$n < \alpha p < n + p - 1.$$

Occasionally, we take $\alpha p > 2n$ (instead of $> n$) to guarantee regularity. We aim at studying the asymptotic case $p \to \infty$. For p large enough, any exponent $0 < \alpha \leq 1$ is sooner or later included. The usual fractional Sobolev space $W^{s,p}$ has the exponent $n + sp$ in the place of our αp, i.e.

$$s = \alpha - \frac{n}{p}, \quad 0 < s < 1.$$

For us α is more convenient. It is helpful to keep in mind that in the range $\alpha p > n$ one has

$$\int_\Omega\int_\Omega \frac{dx\,dy}{|y-x|^{\alpha p}} = \infty.$$

The inequality

$$C(n, p, \alpha)\,|\Omega|^{1-\frac{\alpha p}{n}} \int_\Omega |\phi|^p\,dx \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\phi(y)-\phi(x)|^p}{|y-x|^{\alpha p}}\,dx\,dy \tag{6}$$

shows that the infimum $\lambda_1 > 0$. We call λ_1 *the first eigenvalue*[2]. It is worth noting that, although $\phi = 0$ in the whole complement $\mathbb{R}^n \setminus \Omega$, the identity

$$\iint\limits_{\mathbb{R}^n\mathbb{R}^n} \frac{|\phi(y)-\phi(x)|^p}{|y-x|^{\alpha p}}\,dx\,dy = \iint\limits_{\Omega\ \Omega} \frac{|\phi(y)-\phi(x)|^p}{|y-x|^{\alpha p}}\,dx\,dy + 2\int\limits_{\mathbb{R}^n\setminus\Omega} dy \int\limits_{\Omega} \frac{|\phi(x)|^p}{|y-x|^{\alpha p}}\,dx$$

has a term from the complement. However, the inequality (6) is valid also with $\Omega \times \Omega$ as the domain of integration in the double integral, see Theorem 2. But the minimization problem is not quite the same if $\mathbb{R}^n \times \mathbb{R}^n$ is replaced by $\Omega \times \Omega$ in the integral. Our choice has the advantage that the property

$$\lambda_1(\Omega) \le \lambda_1(\Upsilon), \quad \text{if } \Upsilon \subset \Omega$$

is evident for subdomains. A simple change of coordinates yields that

$$\lambda_1(\Omega) = k^{\alpha p - n}\lambda_1(k\Omega), \qquad k > 0.$$

This and (6) indicate that *small domains have large first eigenvalues.*

A minimizer of the fractional Rayleigh quotient (5) cannot change sign, since

$$|\phi(y)-\phi(x)| > \big||\phi(y)| - |\phi(x)|\big| \quad \text{when } \phi(y)\phi(x) < 0.$$

The minimizer in the next theorem is called *the first eigenfunction* in Ω.

Theorem 5 *There exists a non-negative minimizer $u \in W_0^{s,p}(\Omega)$, $u \not\equiv 0$, and $u = 0$ in $\mathbb{R}^n \setminus \Omega$. It satisfies the Euler-Lagrange equation*

$$\iint\limits_{\mathbb{R}^n\mathbb{R}^n} \frac{|u(y)-u(x)|^{p-2}\big(u(y)-u(x)\big)\big(\phi(y)-\phi(x)\big)}{|y-x|^{\alpha p}}\,dx\,dy = \lambda \int\limits_{\mathbb{R}^n} |u|^{p-2}u\phi\,dx \tag{7}$$

with $\lambda = \lambda_1$ whenever $\phi \in C_0^\infty(\Omega)$. If $\alpha p > 2n$, the minimizer is in $C^{0,\beta}(\mathbb{R}^n)$ with $\beta = \alpha - 2n/p$.

Proof The existence of a minimizer is proved via the direct method in the Calculus of Variations. First a minimizing sequence of admissible functions ϕ_j is selected. It can be normalized so that $\|\phi_j\|_{L^p(\mathbb{R}^n)} = \|\phi_j\|_{L^p(\Omega)} = 1$. Then we have

$$\iint\limits_{\mathbb{R}^n\mathbb{R}^n} \frac{|\phi_j(y)-\phi_j(x)|^p}{|y-x|^{\alpha p}}\,dx\,dy + \int\limits_{\mathbb{R}^n} |\phi_j|^p\,dx \le \lambda_1 + 1 + 1$$

for large indices j. According to Theorem 4, there is a subsequence that converges in $L^p(\mathbb{R}^n)$. The limit of the subsequence, say u, is in $W_0^{s,p}(\Omega)$ and vanishes outside Ω. Fatou's lemma yields that u is minimizing. So is *a fortiori* $|u|$. Thus the existence of a non-negative minimizer is proved.

To derive the Euler-Lagrange equation, one uses a device due to Lagrange. If u is minimizing, consider the competing function

$$v(x,t) = u(x) + t\phi(x), \quad \phi \in C_0^\infty(\Omega).$$

[2] The name "principal frequency" is synonymous.

The necessary condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(y,t)-v(x,t)|^p}{|y-x|^{\alpha p}}\,dx\,dy}{\int_{\mathbb{R}^n}|v(x,t)|^p\,dx}\right\}=0 \quad \text{at} \quad t=0$$

for a minimum yields the Eq. (7).

Finally, the β-Hölder continuity is a property of the fractional Sobolev space, cf. Theorem 3. This concludes our proof. □

The Euler-Lagrange equation can be written in the form

$$2\int_{\mathbb{R}^n}\phi(x)\,dx\int_{\mathbb{R}^n}\frac{|u(y)-u(x)|^{p-2}\big(u(y)-u(x)\big)}{|y-x|^{\alpha p}}\,dy+\lambda\int_{\mathbb{R}^n}|u|^{p-2}u\phi\,dx=0$$

provided that the double integral converges. To see this, split the double integral in (7) into two, one with $\phi(x)$ and one with $\phi(y)$. Then use symmetry. This counts for the factor 2. By the variational lemma the equation

$$\begin{aligned}\mathcal{L}_p u\,(x) &:= 2\int_{\mathbb{R}^n}\frac{|u(y)-u(x)|^{p-2}\big(u(y)-u(x)\big)}{|y-x|^{\alpha p}}\,dy\\ &= -\lambda|u(x)|^{p-2}u(x)\end{aligned}$$

holds at a. e. point $x\in\Omega$, if the inner integral is summable.[3] A sufficient condition is that u is Lipschitz continuous and $\alpha p < p+n-1$ (instead of $< p+n$). In this case $\mathcal{L}_p u\,(x)$ is continuous in the variable x. In the complement $\mathbb{R}^n\setminus\Omega$ this equation is not valid, but there we instead have the information that $u\equiv 0$. Symbolically we can write the Euler-Lagrange equation as

$$\mathcal{L}_p u\,(x)+\lambda|u(x)|^{p-2}u(x)\;=\;0.$$

We remark that if $u\in C_0^1(\mathbb{R}^n)$ satisfies this equation in Ω, then

$$\iint_{\mathbb{R}^n\mathbb{R}^n}\frac{|u(y)-u(x)|^{p-2}\big(u(y)-u(x)\big)\big(\phi(y)-\phi(x)\big)}{|y-x|^{\alpha p}}\,dx\,dy=\lambda\int_{\mathbb{R}^n}|u|^{p-2}u\phi\,dx$$

holds whenever $\phi\in C_0^\infty(\Omega)$. (See Lemma 10.)

Finally, to be on the safe side, we define the concept of eigenfunctions. They are weak solutions of the Euler-Lagrange equation. Notice that they are defined in the whole space, since we consider them to be extended by zero outside Ω.

Definition 6 We say that $u\not\equiv 0$, $u\in W_0^{s,p}(\Omega)$, $s=\alpha-n/p$, is an *eigenfunction* of Ω, if the Euler-Lagrange Eq. (7) holds for all test functions $\phi\in C_0^\infty(\Omega)$. The corresponding λ is called an *eigenvalue*.

Due to the global nature of the operator $\mathcal{L}_p$ it is not sufficient to prescribe the boundary values only on the boundary $\partial\Omega$, but one has to declare that $u=0$ in the whole complement $\mathbb{R}^n\setminus\Omega$. Indeed, a change of u done outside Ω can influence the entire operator $\mathcal{L}_p u$.

[3] In the linear case this integral operator has been treated as the principal value of a singular integral.

4 Viscosity solutions

The eigenfunctions were defined as the weak solutions to the Euler-Lagrange equation in the usual way with test functions under the integral sign (Definition 2). As we will see, they are also viscosity solutions of the equation

$$\mathcal{L}_p u + \lambda |u|^{p-2} u = 0,$$

provided that they are continuous. This is another notion. We refer to the book [16] for an introduction. The theory of viscosity solutions is based on *pointwise* testing: the equation is evaluated for test functions at points of contact. The viscosity solutions are assumed to be continuous, but the fractional Sobolev space is absent from their definition.

Definition 7 *(Viscosity solutions)* Suppose that the function u is continuous in $\mathbb{R}^n$ and that $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. We say that u is a *viscosity supersolution* in Ω of the equation

$$\mathcal{L}_p u + \lambda |u|^{p-2} u = 0$$

if the following holds: whenever $x_0 \in \Omega$ and $\varphi \in C_0^1(\mathbb{R}^n)$ are such that

$$\varphi(x_0) = u(x_0) \quad \text{and} \quad \varphi(x) \le u(x) \quad \text{for all} \quad x \in \mathbb{R}^n,$$

then we have

$$\mathcal{L}_p \varphi\,(x_0) + \lambda |\varphi(x_0)|^{p-2} \varphi(x_0) \le 0.$$

The requirement for a *viscosity subsolution* is symmetric: the test function is touching from above and the inequality is reversed. Finally, a *viscosity solution* is defined as being both a viscosity supersolution and a viscosity subsolution.

Remark 8 The pointwise inequality

$$\mathcal{L}_p \varphi\,(x_0) + \lambda |u(x_0)|^{p-2} u(x_0) \le 0$$

is valid also if the function $\varphi(x) + C$ touches u at x_0. To see that the constant has no influence, use the following simple monotonicity property for $\psi, \varphi \in C_0^1(\mathbb{R}^n)$:

$$\text{if } \psi \ge \varphi \quad \text{and} \quad \psi(x_0) = \varphi(x_0), \text{ then } \mathcal{L}_p \psi\,(x_0) \ge \mathcal{L}_p \varphi\,(x_0).$$

In order to prove that continuous weak solutions are viscosity solutions we need a comparison principle.

Lemma 9 (Comparison principle) *Let u and v be two continuous functions belonging to $W_0^{s,p}(\mathbb{R}^n)$. Let $D \subset \mathbb{R}^n$ be a domain. If*

- *$v \ge u$ in $\mathbb{R}^n \setminus D$, and*
- *$\mathcal{L}_p v\,(x) \le \mathcal{L}_p u\,(x)$ when $x \in D$ in the sense that*

$$\iint\limits_{\mathbb{R}^n \mathbb{R}^n} \frac{|v(y)-v(x)|^{p-2}\big(v(y)-v(x)\big)\big(\phi(y)-\phi(x)\big)}{|y-x|^{\alpha p}}\,dx\,dy$$
$$\ge \iint\limits_{\mathbb{R}^n \mathbb{R}^n} \frac{|u(y)-u(x)|^{p-2}\big(u(y)-u(x)\big)\big(\phi(y)-\phi(x)\big)}{|y-x|^{\alpha p}}\,dx\,dy$$

whenever $\phi \in W_0^{s,p}(D)$, $\phi \ge 0$,

then $v \ge u$ also in D. That is, $v \ge u$ in $\mathbb{R}^n$.

Proof The integrals are finite by Hölder's inequality. By the assumption, the integral

$$\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\frac{\big[|v(y)-v(x)|^{p-2}\big(v(y)-v(x)\big)-|u(y)-u(x)|^{p-2}\big(u(y)-u(x)\big)\big]\big(\phi(y)-\phi(x)\big)}{|y-x|^{\alpha p}}dxdy$$

is non-negative if $\phi \geq 0$. We aim at showing that the integrand is non-positive for the choice $\phi = (u-v)^+$. The identity

$$|b|^{p-2}b - |a|^{p-2}a \;=\; (p-1)(b-a)\int\limits_0^1 |a+t(b-a)|^{p-2}\,dt$$

with $a = u(y) - u(x)$ and $b = v(y) - v(x)$ gives the formula

$$\begin{aligned}&|v(y)-v(x)|^{p-2}\big(v(y)-v(x)\big) - |u(y)-u(x)|^{p-2}\big(u(y)-u(x)\big)\\ &\quad = (p-1)\big\{u(x)-v(x)-\big(u(y)-v(y)\big)\big\}Q(x,y),\end{aligned}$$

which is to be used in the integrand above. We have abbreviated[4]

$$Q(x,y) = \int\limits_0^1 \Big|\big(u(y)-u(x)\big)+t\Big(\big(v(y)-v(x)\big)-\big(u(y)-u(x)\big)\Big)\Big|^{p-2}\,dt.$$

We see that $Q(x,y) \geq 0$, and $Q(x,y) = 0$ only if $v(y) = v(x)$ and $u(y) = u(x)$. We choose the test function $\phi = (u-v)^+$ and write

$$\psi = u - v = (u-v)^+ - (u-v)^-, \quad \phi = (u-v)^+ = \psi^+.$$

The integrand becomes the factor $(p-1)Q(x,y)/|y-x|^\alpha$ multiplied with

$$\begin{aligned}&[\psi(x)-\psi(y)][\phi(y)-\phi(x)]\\ &\quad = [\psi^+(x)-\psi^-(x)-\psi^+(y)+\psi^-(y)][\psi^+(y)-\psi^+(x)]\\ &\quad = -\big(\psi^+(y)-\psi^+(x)\big)^2 + \big(\psi^-(y)-\psi^-(x)\big)\big(\psi^+(y)-\psi^+(x)\big)\\ &\quad = -\big(\psi^+(y)-\psi^+(x)\big)^2 - \psi^-(y)\psi^+(x) - \psi^-(x)\psi^+(y),\end{aligned}$$

where the formula $\psi^-(x)\psi^+(x) = 0$ was used. The integrand contains only negative terms and, to avoid a contradiction, it is necessary that

$$\psi^+(y) = \psi^+(x) \quad \text{or} \quad Q(x,y) = 0$$

at a. e. point (x, y). Also the latter alternative implies that $\psi^+(y) = \psi^+(x)$. In other words, the identity

$$\big(u(y)-v(y)\big)^+ = \big(u(x)-v(x)\big)^+$$

must hold. It follows that $u(x) - v(x) = C = \text{Constant} \geq 0$ in the set where $u(x) \geq v(x)$. The boundary condition requires that $C = 0$. The claim $v \geq u$ follows. □

Lemma 10 *Let* $f \in C(\Omega)$ *and* $v \in C_0^1(\mathbb{R}^n)$. *If the inequality*

$$\mathcal{L}_p v\,(x) \leq f(x)$$

[4] The idea is obvious in the case $p = 2$.

is valid at each point x in the subdomain $D \subset \Omega$, then the inequality

$$-\iint_{\mathbb{R}^n\mathbb{R}^n} \frac{|v(y)-v(x)|^{p-2}\big(v(y)-v(x)\big)\big(\phi(y)-\phi(x)\big)}{|y-x|^{\alpha p}}\,dx\,dy \le \int_{\Omega} f(x)\phi(x)\,dx \tag{8}$$

holds for all $\phi \in C_0(D)$, $\phi \ge 0$.

Proof Multiply the inequality $\mathcal{L}_p v\,(x) \le f(x)$ with $\phi(x)$ and integrate over D to obtain

$$2\int_D\int_{\mathbb{R}^n} \frac{|v(y)-v(x)|^{p-2}\big(v(y)-v(x)\big)\phi(x)}{|y-x|^{\alpha p}}\,dy\,dx \le \int_{\Omega} f(x)\phi(x)\,dx.$$

We can replace D by $\mathbb{R}^n$ in the outer integration. Switching x and y, we can write

$$-2\int_D\int_{\mathbb{R}^n} \frac{|v(y)-v(x)|^{p-2}\big(v(y)-v(x)\big)\phi(y)}{|y-x|^{\alpha p}}\,dx\,dy \le \int_{\Omega} f(y)\phi(y)\,dy.$$

Notice the minus sign. Adding the expressions we arrive at (8). □

Proposition 11 *Let $\alpha p < n + p - 1$. An eigenfunction $u \in C_0(\overline{\Omega})$ is a viscosity solution of the equation*

$$\mathcal{L}_p u = -\lambda |u|^{p-2} u.$$

Proof We prove the case of a subsolution, assuming for simplicity that $u \ge 0$. Our proof is indirect. If u is not a viscosity subsolution, the *antithesis* is that there exist a testfunction ϕ and a point x_0 in Ω such that

$$\phi \in C_0^1(\mathbb{R}^n), \quad \phi \ge u, \quad \phi(x_0) = u(x_0),$$
$$\mathcal{L}_p\phi\,(x_0) < -\lambda|\phi(x_0)|^{p-2}\phi(x_0).$$

By continuity

$$\mathcal{L}_p\phi\,(x) < -\lambda|\phi(x_0)|^{p-2}\phi(x_0)$$

holds when $x \in B(x_0, 2r)$, where the radius r is small enough. This means that ϕ is a "strict supersolution" in the ball. We need to modify ϕ. For the purpose we choose a smooth radial function $\eta \in C^\infty(\mathbb{R}^n)$ such that $0 \le \eta(x) \le 1$ and

$$\eta(x_0) = 0,$$
$$\eta(x) > 0, \quad \text{when } x \ne x_0,$$
$$\eta(x) = 1, \quad \text{when } |x - x_0| \ge r.$$

Let $\varepsilon > 0$ be small and consider the function

$$v = v_\varepsilon = \phi + \varepsilon\eta - \varepsilon.$$

Outside $B(x_0, r)$ it coincides with ϕ. By Lebesgue's Dominated Convergence Theorem

$$\lim_{\varepsilon\to 0} \mathcal{L}_p v_\varepsilon\,(x) = \mathcal{L}_p\phi\,(x).$$

A closer inspection reveals that, actually, the limit is uniform on compact sets. Since u is continuous, it follows that for a sufficiently small $\varepsilon > 0$

$$\mathcal{L}_p v_\varepsilon\,(x) < -\lambda|u(x)|^{p-2}u(x) = f(x)$$

when $x \in B(x_0, r)$. By the previous lemma this inequality also holds in the weak sense with test functions under the integral sign. Thus Eq. (8) is available.

Now $\mathcal{L}_p v \leq \mathcal{L}_p u$ in the weak sense in $B(x_0, r)$, as described in Lemma 9. By the construction

$$v = \phi \quad \text{in} \quad \mathbb{R}^n \setminus B(x_0, r).$$

In particular,

$$v \geq u \quad \text{in} \quad \mathbb{R}^n \setminus B(x_0, r).$$

By the comparison principle (Lemma 9)

$$v \geq u \quad \text{in} \quad B(x_0, r).$$

But this contradicts the fact that

$$v(x_0) = \phi(x_0) - \varepsilon = u(x_0) - \varepsilon < u(x_0).$$

Thus the antithesis is false. We have proved that u is a viscosity subsolution. —The case of viscosity supersolutions is similar. □

The next result shows that the first eigenfunctions cannot have zeros in the domain.

Lemma 12 (Positivity) *Assume $u \geq 0$ and $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. If u is a viscosity supersolution of the equation $\mathcal{L}_p u = 0$ in Ω, then either $u > 0$ in Ω or $u \equiv 0$.*

Proof Recall that being a supersolution means that $\mathcal{L}_p \psi \leq 0$ for the test functions below. At a point x_0 in Ω where $u(x_0) = 0$ we have for any test function ψ that touches u from below that

$$0 \geq \mathcal{L}_p\, \psi(x_0) = 2 \int_{\mathbb{R}^n} \frac{|\psi(y)|^{p-2}\psi(y)\, dy}{|y - x_0|^{\alpha p}}$$

since $\psi(x_0) = 0$. If $\psi \geq 0$ this implies that $\psi \equiv 0$. But, if $u \not\equiv 0$, we can certainly, using the continuity of u, select a test function ψ so that $0 \leq \psi \leq u$ which is positive at some point. □

It is noteworthy that the result above does not hold true if u is not non-negative in $\mathbb{R}^n \setminus \Omega$. This is related to the fact that the usual Harnack inequality fails for non-local operators in general. See [13], for an explicit counter example in the case $p = 2$.

5 Uniqueness of positive eigenfunctions

We know that a continuous non-negative eigenfunction cannot have any zeros in the domain Ω (Lemma 12). We shall prove that the only positive eigenfunctions are the first ones and also that *the first eigenvalue is simple*. In other words, if u_1 is a minimizer of the Rayleigh quotient, *all positive eigenfunctions are of the form $u(x) = Cu_1(x)$*. First, we have to prove that the minimizer is unique, except for multiplication by constants. Then it will be established that a positive eigenfunction is a minimizer. We will encounter the difficulty with the lack of an adequate regularity theory for our equation. To avoid such issues here, we deliberately take $\alpha p > 2n$, which guarantees the continuity of the eigenfunctions.

We use an elementary inequality for the auxiliary function

$$ß(s, t) = |s^{1/p} - t^{1/p}|^p, \qquad s > 0,\ t > 0.$$

Lemma 13 *The function* $ß(s, t)$ *is convex in the quadrant* $s > 0, t > 0$. *Thus*

$$ß\left(\frac{s_1 + s_2}{2}, \frac{t_1 + t_2}{2}\right) \le \frac{1}{2}ß(s_1, t_1) + \frac{1}{2}ß(s_2, t_2).$$

Moreover, equality holds only for $s_1 t_2 = s_2 t_1$.

Proof As a matter of fact, ß is a solution to the Monge-Ampère equation

$$ß_{ss}ß_{tt} - ß_{st}^2 = 0.$$

A direct calculation yields the expression

$$ß_{ss}(s,t)X^2 + 2ß_{st}(s,t)XY + ß_{tt}(s,t)Y^2 = \tfrac{p-1}{p}|s^{1/p} - t^{1/p}|^{p-2}(st)^{1/p}\left(\frac{s}{X} - \frac{t}{Y}\right)^2$$

for the quadratic form associated with the Hessian matrix. The quadratic form is strictly positive except when $s = t$ or $\frac{s}{X} = \frac{t}{Y}$. The result follows by inspection. □

Theorem 14 *Take* $\alpha p > 2n$. *The minimizer of the Rayleigh quotient is unique, except that it may be multiplied by a constant.*[5]

Proof Our proof is a modification of the proof given in [3]. If u and v are minimizers, so are $|u|$ and $|v|$. Since $|u| > 0$ and $|v| > 0$ in Ω by Lemma 12, we may by continuity assume that $u > 0$ and $v > 0$ from the beginning. Our claim is that $u(x) = Cv(x)$.

Normalize the functions so that

$$\int_{\mathbb{R}^n} u^p\,dx = \int_{\mathbb{R}^n} v^p\,dx = 1$$

and consider the admissible function

$$w = \left(\frac{u^p + v^p}{2}\right)^{1/p}$$

in the Rayleigh quotient. Also

$$\int_{\mathbb{R}^n} w^p\,dx = 1$$

by construction. In the numerator we have, according to the previous lemma,

$$|w(y) - w(x)|^p \le \frac{1}{2}|u(y) - u(x)|^p + \frac{1}{2}|v(y) - v(x)|^p \tag{9}$$

with equality *only* for

$$u(x)v(y) = u(y)v(x). \tag{10}$$

[5] *Note added in proof* In the recent work "A note on positive eigenfunctions and hidden convexity", Archiv der Mathematik, 2012, Volume 99, Issue 4, pp 367-374, by L. Brasco and G. Franzina, the range for α in Theorem 14 and Theorem 16 has been widened.

Divide by $|y-x|^{\alpha p}$, integrate, and use the normalization to conclude that

$$\begin{aligned}\lambda_1 &\le \frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|w(y)-w(x)|^p}{|y-x|^{\alpha p}}\,dx\,dy}{\int_{\mathbb{R}^n} w^p\,dx}\\ &\le \frac{\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(y)-u(x)|^p}{|y-x|^{\alpha p}}\,dx\,dy}{\int_{\mathbb{R}^n} u^p\,dx} + \frac{\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(y)-v(x)|^p}{|y-x|^{\alpha p}}\,dx\,dy}{\int_{\mathbb{R}^n} v^p\,dx}\\ &= \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_1 = \lambda_1.\end{aligned}$$

Thus the only possibility is that equality holds in (9) for x and y in Ω. Thus (10) holds, which proves that $u(x) = Cv(x)$. □

Lemma 15 (Exhaustion) *Let*

$$\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots \subset \Omega, \qquad \Omega = \bigcup \Omega_j.$$

Then

$$\lim_{j\to\infty} \lambda_1(\Omega_j) = \lambda_1(\Omega).$$

Proof Since $\lambda_1(\Omega_1) \ge \lambda_1(\Omega_2) \ge \cdots \ge \lambda_1(\Omega)$ the limit exists. Given $\varepsilon > 0$, there exists a $\phi \in C_0^\infty(\Omega)$ such that

$$\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\phi(y)-\phi(x)|^p}{|y-x|^{\alpha p}}\,dx\,dy}{\int_{\mathbb{R}^n} |\phi|^p\,dx} < \lambda_1(\Omega) + \varepsilon,$$

because $\lambda_1(\Omega)$ is the infimum. For j large enough, $supp(\phi) \subset \Omega_j$ and thus ϕ will do as test function in the Rayleigh quotient also for the subdomain Ω_j. It follows that

$$\lambda_1(\Omega_j) < \lambda_1(\Omega) + \varepsilon$$

for sufficiently large j. □

Any domain Ω can be exhausted by a sequence of *smooth* domains $\Omega_j \subset\subset \Omega$. See for example [15, p. 317-319].

Theorem 16 *Take $\alpha p > 2n$. Then a non-negative eigenfunction minimizes the Rayleigh quotient.*[6]

Proof The proof is based on a construction in [18]; see also [14].

Antithesis: Assume that $v \ge 0$ is a weak solution in Ω of the Euler-Lagrange Eq. (7) with eigenvalue $\lambda > \lambda_1(\Omega)$.

By Theorem 3 v is continuous. As $v \not\equiv 0$ we have that $v > 0$ by Lemma 12. According to Lemma 15 and the remark following it, we can construct a smooth domain $\Omega^* \subset\subset \Omega$ such that also

$$\lambda_1^* = \lambda_1(\Omega^*) < \lambda.$$

Let v^* denote the first eigenfunction in Ω^*; its eigenvalue is λ_1^*. Since $\alpha p > 2n,\ v^* \in C(\overline{\Omega^*})$ and $v^* = 0$ on $\partial\Omega^*$ and in $\mathbb{R}^n \setminus \Omega^*$. Because $v > 0$ in Ω,

$$\min_{\overline{\Omega^*}} v > 0,$$

[6] *Note added in proof* In the recent work "A note on positive eigenfunctions and hidden convexity", Archiv der Mathematik, 2012, Volume 99, Issue 4, pp 367-374, by L. Brasco and G. Franzina, the range for α in Theorem 14 and Theorem 16 has been widened.

and we can arrange it so that

$$v \geq v^* \quad \text{in} \quad \mathbb{R}^n$$

by multiplying v by a suitable constant, if needed.

Let $\phi \in C_0^\infty(\Omega^*)$, $\phi \geq 0$, be a test function. Then the equations are

$$\iint\limits_{\mathbb{R}^n\mathbb{R}^n} \frac{|v^*(y) - v^*(x)|^{p-2}\big(v^*(y) - v^*(x)\big)\big(\phi(y) - \phi(x)\big)}{|y - x|^{\alpha p}}\, dx\, dy$$
$$= \lambda_1^* \int\limits_{\mathbb{R}^n} v^*(y)^{p-1}\phi(y)\, dy \leq \lambda_1^* \int\limits_{\mathbb{R}^n} v(y)^{p-1}\phi(y)\, dy = \lambda \int\limits_{\mathbb{R}^n} (\varkappa v(y))^{p-1}\phi(y)\, dy$$
$$= \iint\limits_{\mathbb{R}^n\mathbb{R}^n} \frac{|\varkappa v(y) - \varkappa v(x)|^{p-2}\big(\varkappa v(y) - \varkappa v(x)\big)\big(\phi(y) - \phi(x)\big)}{|y - x|^{\alpha p}}\, dx\, dy,$$

where we have denoted

$$\varkappa = \left(\frac{\lambda_1^*}{\lambda}\right)^{1/(p-1)} < 1.$$

Symbolically, $\mathcal{L}_p v^* \geq \mathcal{L}_p(\varkappa v)$ in Ω^* and $\varkappa v \geq v^*$ in $\mathbb{R}^n \setminus \Omega^*$. The Comparison Principle (Lemma 9) yields that

$$\varkappa v \geq v^* \qquad (0 < \varkappa < 1).$$

We can repeate the procedure, now starting with the function $\varkappa v$ in the place of v. This yields $\varkappa(\varkappa v) \geq v^*$. By iteration we arrive at

$$\varkappa^j v \geq v^*, \quad j = 1, 2, \ldots$$

Since $\varkappa^j \to 0$ as $j \to \infty$ we obtain the contradiction that $v^* \equiv 0$. □

6 Higher eigenvalues

For a fixed exponent p the set of all eigenvalues form the *spectrum* $\{\lambda\}$. By compactness arguments *the spectrum is a closed set*. The higher eigenvalues are associated with sign-changing eigenfunctions. It is well-known that, for a differential operator like the ordinary Laplacian for instance, a restriction of a higher eigenfunction to one of its nodal domains is a *first* eigenfunction with respect to that subdomain. Then a higher eigenvalue of a domain is a first eigenvalue for any nodal domain. This property holds for many other equations, too. However, we encounter a new phenomenon for our operator. The non-local nature of the problem causes the higher eigenvalues to be too large for this property to hold.

Let us begin by recalling that, given an eigenfunction, its nodal domains are the connected open components of the sets $\{u > 0\}$ and $\{u < 0\}$. In passing, we mention that also the quantities $\lambda_1(\{u > 0\})$ and $\lambda_1(\{u < 0\})$ can be defined in the natural way, although the open sets involved are not always connected ones.

Theorem 17 *If u is a continuous sign changing eigenfunction with eigenvalue $\lambda(\Omega)$, then the strict inequalities*

$$\lambda(\Omega) > \lambda_1(\Omega^+) \quad \textit{and} \quad \lambda(\Omega) > \lambda_1(\Omega^-),$$

hold for the open sets $\Omega^+ = \{u > 0\}$ *and* $\Omega^- = \{u < 0\}$. *Moreover,*

$$\lambda \geq C(n,p,\alpha)\,|\Omega^+|^{-\frac{\alpha p - n}{n}} \quad \textit{and} \quad \lambda \geq C(n,p,\alpha)\,|\Omega^-|^{-\frac{\alpha p - n}{n}}.$$

Proof Let $u = u^+ - u^-$ be the usual decomposition where $u^+ \geq 0$, $u^- \geq 0$. Choose the test function $\phi = u^+$ in the Euler-Lagrange Eq. (7). We need to have command over the sign of the product

$$\begin{aligned}
&[u(y) - u(x)][\phi(y) - \phi(x)] \\
&= [u^+(y) - u^+(x)]^2 - \big(u^-(y) - u^-(x)\big)\big(u^+(y) - u^+(x)\big) \\
&= [u^+(y) - u^+(x)]^2 + u^+(y)u^-(x) + u^+(x)u^-(y),
\end{aligned}$$

where it was used that $u^+(x)u^-(x) = 0$. The Euler-Lagrange equation becomes

$$\begin{aligned}
\lambda \int_\Omega |u^+|^p\,dx &= \iint_{\mathbb{R}^n\mathbb{R}^n} \frac{|u(y) - u(x)|^{p-2}\big(u^+(y) - u^+(x)\big)^2}{|y - x|^{\alpha p}}\,dx\,dy \\
&\quad + 2 \iint_{\mathbb{R}^n\mathbb{R}^n} \frac{|u(y) - u(x)|^{p-2}u^+(y)u^-(x))}{|y - x|^{\alpha p}}\,dx\,dy.
\end{aligned}$$

The formula

$$\begin{aligned}
|u(y) - u(x)|^2 &= \big(u^+(y) - u^+(x)\big)^2 + \big(u^-(y) - u^-(x)\big)^2 \\
&\quad -2\big(u^+(y) - u^+(x)\big)\big(u^-(y) - u^-(x)\big) \\
&= \big(u^+(y) - u^+(x)\big)^2 + \big(u^-(y) - u^-(x)\big)^2 + 2u^+(y)u^-(x) + 2u^+(x)u^-(y)
\end{aligned}$$

implies the estimate

$$\begin{aligned}
\lambda \int_{\Omega^+} |u^+|^p\,dx &\geq \iint_{\mathbb{R}^n\mathbb{R}^n} \frac{|u^+(y) - u^+(x)|^p}{|y - x|^{\alpha p}}\,dx\,dy \\
&\quad + 2^{p/2} \iint_{\mathbb{R}^n\mathbb{R}^n} \frac{\big(u^+(y)u^-(x)\big)^{\frac{p}{2}}}{|y - x|^{\alpha p}}\,dx\,dy.
\end{aligned}$$

It follows that

$$\lambda \geq \lambda_1(\Omega^+) + 2^{p/2} \frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left(u^+(y)u^-(x)\right)^{\frac{p}{2}}}{|y-x|^{\alpha p}}\,dx\,dy}{\int_{\Omega^+} |u^+|^p\,dx}, \tag{11}$$

because u^+ is admissible in the Rayleigh quotient as test function for Ω^+. This clearly shows that we have the *strict* inequality $\lambda > \lambda_1(\Omega^+)$.

By inequality (6) it follows immediately that

$$\lambda \int_{\Omega^+} |u^+|^p\,dx \geq \iint_{\mathbb{R}^n\mathbb{R}^n} \frac{|u^+(y) - u^+(x)|^p}{|y - x|^{\alpha p}}\,dx\,dy \geq C\,|\Omega^+|^{-\frac{\alpha p - n}{n}} \int_{\Omega^+} |u^+|^p\,dx,$$

and so, upon division, $\lambda \geq C\,|\Omega^+|^{-\frac{\alpha p - n}{n}}$. —The proof for Ω^- is symmetric. □

Remark 18 The excess term in (11) can be improved a little, but it is not evident, whether one can get a bound free of the functions u^+ and u^-.

Due to the fact that higher eigenfunctions are sign-changing, there is a gap in the spectrum just above the first eigenvalue λ_1. Consequently, the second eigenvalue is well defined as the number

$$\lambda_2 = \inf\{\lambda > \lambda_1\}.$$

The minimum is attained. (See [1] for the local case.)

Theorem 19 *Take $\alpha p > 2n$. Then the first eigenvalue is isolated.*

Proof Suppose that there is a sequence of eigenvalues $\lambda_k^{'}$ tending to λ_1, $\lambda_k' \neq \lambda_1$. If u_k denotes the corresponding normalized eigenfunction, we have

$$\int_\Omega |u_k|^p \, dx = 1, \quad \lambda_k^{'} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_k(y) - u_k(x)|^p}{|y-x|^{\alpha p}} \, dx \, dy.$$

By compactness (cf. Theorem 4) we can construct a subsequence and a function $u \in W_0^{s,p}(\Omega)$, $s = \alpha - n/p$, such that

$$u_{k_j} \to u \quad \text{in} \quad L^p(\mathbb{R}^n).$$

Extracting a further subsequence we can assume that $\lim u_{k_j}(x) = u(x)$ a.e.. By Fatou's lemma

$$\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(y)-u(x)|^p}{|y-x|^{\alpha p}} \, dx \, dy}{\int_\Omega |u(x)|^p \, dx} \leq \lim_{j\to\infty} \lambda_{k_j}^{'} = \lambda_1.$$

We read off that u is a minimizer and therefore the first eigenfunction. From Lemma 12, either $u > 0$ in Ω or $u < 0$ in Ω. But if $\lambda_k^{'} > \lambda_1$ then u_k must change signs in Ω in view of Theorem 16. Both sets

$$\Omega_k^+ = \{u_k > 0\} \quad \text{and} \quad \Omega_k^- = \{u_k < 0\}$$

are non-empty and their measures cannot tend to zero, because small subdomains have large eigenvalues. Indeed, by Theorem 17

$$\lambda_k^{'} \geq \lambda_1(\Omega_k^+) \geq C\,|\Omega_k^+|^{1-\alpha p/n},$$
$$\lambda_k^{'} \geq \lambda_1(\Omega_k^-) \geq C\,|\Omega_k^-|^{1-\alpha p/n}.$$

Both sets

$$\Omega^+ = \limsup \Omega_{k_j}^+, \quad \Omega^- = \limsup \Omega_{k_j}^-,$$

have positive measure by a selection procedure. Passing to a suitable subsequence we can show that $u \geq 0$ in Ω^+ and $u \leq 0$ in Ω^-. This is never possible for a first eigenfunction. □

7 Passage to infinity

In order to study the asymptotic case $p \to \infty$ we fix α so that

$$0 < \alpha \leq 1$$

and regard p as sufficiently large, say $\alpha p > 2n$. Taking the p^{th} root of the Rayleigh quotient and sending $p \to \infty$ we formally arrive at the minimization problem

$$\inf_{\phi} \frac{\left\| \frac{\phi(y)-\phi(x)}{|y-x|^{\alpha}} \right\|_{L^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)}}{\|\phi\|_{L^{\infty}(\mathbb{R}^n)}} = \inf_{\phi} \frac{\left\| \frac{\phi(y)-\phi(x)}{|y-x|^{\alpha}} \right\|_{L^{\infty}(\Omega \times \Omega)}}{\|\phi\|_{L^{\infty}(\Omega)}} = \Lambda_{\infty}^{\alpha}, \tag{12}$$

where the infimum is taken over all $\phi \in C_0^{\infty}(\Omega)$. It will turn out that

$$\Lambda_{\infty}^{\alpha} = \left(\frac{1}{\max\limits_{x \in \Omega} \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)} \right)^{\alpha},$$

so that the notation is consistent with $\Lambda_{\infty}^{\alpha} = (\Lambda_{\infty})^{\alpha}$. It is clear that the infimum is the same if all points outside Ω are ignored. *The minimum is always attained*, but in the larger space $W_0^{1,\infty}(\Omega)$. Indeed, let $B(x_0, R)$ be the largest open ball contained in Ω. (There may be several such balls). Then the function

$$\phi(x) = [R - |x - x_0|]^{+}$$

solves the minimization problem and yields $\Lambda_{\infty}^{\alpha} = R^{-\alpha}$. To rigorously prove the lower bound for an arbitrary competing $\phi \in C_0^{\infty}(\Omega)$, we notice that if ξ is the closest boundary point from x then

$$\phi(x) = \phi(x) - \phi(\xi) = |x-\xi|^{\alpha} \left| \frac{\phi(x) - \phi(\xi)}{|x-\xi|^{\alpha}} \right| \leq |x - \xi|^{\alpha} |[\phi]|_{\alpha} = \delta(x)^{\alpha} |[\phi]|_{\alpha}.$$

Recall the notation

$$|[\phi]|_{\alpha} = \left\| \frac{\phi(y) - \phi(x)}{|y-x|^{\alpha}} \right\|_{L^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)}, \quad \delta(x) = \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega).$$

Now $\delta(x) \leq R$ and consequently $\|\phi\|_{\infty} \leq R^{\alpha} |[\phi]|_{\alpha}$. It follows that

$$\frac{1}{R^{\alpha}} \leq \frac{|[\phi]|_{\alpha}}{\|\phi\|_{\infty}},$$

as desired. The calculations showing that this minimum is attained can be found in the proof of the next proposition.

Setting λ_p equal to the *first* eigenvalue, the following limit is easy to establish.

Proposition 20 *We have*

$$\lim_{p \to \infty} \sqrt[p]{\lambda_p} = \frac{1}{R^{\alpha}},$$

where $R = \max\{dist(x, \mathbb{R}^n \setminus \Omega)\}$ *is the radius of the largest inscribed ball in the domain* Ω.

Proof Let ϕ be a test function so that

$$\lambda_p \leq \frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\phi(y)-\phi(x)|^p}{|y-x|^{\alpha p}} \, dx \, dy}{\int_{\mathbb{R}^n} |\phi(x)|^p \, dx}.$$

Taking the p^{th} root and letting $p \to \infty$ we obtain the bound

$$\limsup_{p \to \infty} \lambda_p^{\frac{1}{p}} \leq \frac{\left\| \frac{\phi(y) - \phi(x)}{|y-x|^{\alpha}} \right\|_{L^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)}}{\|\phi(x)\|_{L^{\infty}(\mathbb{R}^n)}}.$$

As ϕ we take the distance function $\delta = \delta(x) = [R - |x - x_0|]^+$ for the inscribed ball, the center of which we may assume to be $x_0 = 0$. Then $\|\phi\|_\infty = R$ and a direct computation gives

$$\left|\frac{\delta(y) - \delta(x)}{|y - x|^\alpha}\right| = \frac{\big|\,|y| - |x|\,\big|}{|y - x|^\alpha},$$

from which the desired upper bound follows by calculus.

To get the lower bound, we select an increasing sequence $p_j \to \infty$ such that $\lim \lambda_{p_j}^{1/p_j} = \liminf \lambda_p^{1/p}$. Let u_{p_j} be the corresponding minimizer of the Rayleigh quotient normalized so that

$$\int_\Omega u_{p_j}^{p_j}\, dx = 1, \quad \lambda_{p_j} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\frac{u_{p_j}(y) - u_{p_j}(x)}{|y - x|^\alpha}\right|^{p_j} dx\, dy$$

By the inclusion in Hölder spaces, Theorem 3, a subsequence converges uniformly in $\mathbb{R}^n$ to a function $u \in C_0(\overline{\Omega})$. In particular the normalization is preserved: $\|u\|_{L^\infty(\Omega)} = 1$. In order to avoid an unbounded domain in Hölder's inequality below, we integrate first only over $\Omega \times \Omega$. For a fixed exponent q Fatou's lemma and Hölder's inequality imply

$$\begin{aligned}
&\int_\Omega\int_\Omega \left|\frac{u(y) - u(x)}{|y - x|^\alpha}\right|^q dx\, dy \\
&\le \liminf_{j\to\infty} \int_\Omega\int_\Omega \left|\frac{u_{p_j}(y) - u_{p_j}(x)}{|y - x|^\alpha}\right|^q dx\, dy \\
&\le \liminf_{j\to\infty} |\Omega|^{2\left(1-\frac{q}{p_j}\right)} \left\{\int_\Omega\int_\Omega \left|\frac{u_{p_j}(y) - u_{p_j}(x)}{|y - x|^\alpha}\right|^{p_j} dx\, dy\right\}^{\frac{q}{p_j}} \\
&\le |\Omega|^2 \liminf_{j\to\infty} \left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\frac{u_{p_j}(y) - u_{p_j}(x)}{|y - x|^\alpha}\right|^{p_j} dx\, dy\right\}^{\frac{q}{p_j}} \\
&= |\Omega|^2 \left(\lim_{j\to\infty} \lambda_{p_j}^{1/p_j}\right)^q.
\end{aligned}$$

Taking the qth root of the estimate, then sending $q \to \infty$ and recalling the normalization, we see that the minimum is less than $\liminf_{j\to\infty} \lambda_p^{1/p}$. □

8 The infinity Euler-Lagrange equation

The minimization problem (12) often has too many solutions, because a minimizer can be rather freely modified outside the largest inscribed ball in the domain. To eliminate the "false solutions" we need the limit equation to which the Euler-Lagrange equations tend as $p \to \infty$. The operator

$$\mathcal{L}_\infty u\,(x) = \underbrace{\sup_{y\in\mathbb{R}^n} \frac{u(y) - u(x)}{|y - x|^\alpha}}_{\mathcal{L}_\infty^+ u\,(x)} + \underbrace{\inf_{y\in\mathbb{R}^n} \frac{u(y) - u(x)}{|y - x|^\alpha}}_{\mathcal{L}_\infty^- u\,(x)}$$

is fundamental. The decomposition

$$\mathcal{L}_\infty u\,(x) = \mathcal{L}_\infty^+ u\,(x) + \mathcal{L}_\infty^- u\,(x)$$

is not the ordinary one into positive and negative parts. For positive solutions we will derive the limit equation

$$\max\left\{\mathcal{L}_\infty u\,(x),\ \mathcal{L}_\infty^- u\,(x) + \Lambda_\infty^\alpha u(x)\right\} = 0, \tag{13}$$

and for lack of a better name we refer to this equation as the ∞-*eigenvalue equation*. This "Euler-Lagrange equation" has to be interpreted in the viscosity sense. The notation above indicates that at each point the largest of two numbers is zero.

Definition 21 We say that a non-negative function $u \in C_0(\mathbb{R}^n)$ is a *viscosity supersolution* of the equation

$$\max\left\{\mathcal{L}_\infty u\,(x),\ \mathcal{L}_\infty^- u\,(x) + \Lambda_\infty^\alpha u(x)\right\} = 0$$

in the domain Ω if the conditions

$$\mathcal{L}_\infty \phi\,(x_0) \le 0 \quad \text{and} \quad \mathcal{L}_\infty^- \phi\,(x_0) + \Lambda_\infty^\alpha \phi(x_0) \le 0$$

hold, whenever the test function $\phi \in C_0^1(\mathbb{R}^n)$ touches u from below at the point $x_0 \in \Omega$.

We say that $u \in C_0(\mathbb{R}^n)$ is a *viscosity subsolution* if one of the conditions

$$\mathcal{L}_\infty \psi\,(x_0) \ge 0 \quad \text{or} \quad \mathcal{L}_\infty^- \psi\,(x_0) + \Lambda_\infty^\alpha \psi(x_0) \ge 0$$

holds, whenever the test function $\psi \in C_0^1(\mathbb{R}^n)$ touches u from above at the point $x_0 \in \Omega$.

Finally, u is a *viscosity solution* if it is both a viscosity supersolution and a viscosity subsolution.

A viscosity solution $u \in C_0(\overline{\Omega})$, $u > 0$, is called a first ∞-eigenfunction.

We consider an arbitrary sequence of first eigenvalues λ_p with $p \to \infty$ and denote the corresponding eigenfunction by u_p. The limit procedure requires the following lemma.

Lemma 22 (Positivity) *Let $v \in C_0(\mathbb{R}^n)$ be a viscosity supersolution of the equation $\mathcal{L}_\infty v = 0$ in Ω. If $v \ge 0$ in $\mathbb{R}^n$, then, either $v > 0$ in Ω or $v \equiv 0$.*

Proof The concept means that $\mathcal{L}_\infty \phi\,(x_0) \le 0$ for all test functions touching v from below at a given point x_0 in Ω. Assume now that $v(x_0) = 0$ at some point. Then there is certainly a test function such that $0 \le \phi \le v$ and $\phi(x_0) = v(x_0) = 0$. Hence

$$0 \ge \mathcal{L}_\infty \phi\,(x_0) = \max_{\mathbb{R}^n} \frac{\phi(y)}{|x_0 - y|^\alpha} + \min_{\mathbb{R}^n} \frac{\phi(y)}{|x_0 - y|^\alpha} \ge \frac{\phi(y)}{|x_0 - y|^\alpha},$$

which implies that $\phi \equiv 0$. As in the proof of Lemma 12 we conclude that $v \equiv 0$. □

As we know $\sqrt[p]{\Lambda_p} \to \Lambda_\infty^\alpha$, by Proposition 20. For the eigenfunctions we have to go to subsequences.

Theorem 23 *There exists a subsequence of functions u_p converging uniformly in Ω to a function $u \in C_0(\overline{\Omega})$ which is a viscosity solution in Ω of the ∞-eigenvalue Eq. (13).*

Proof If we normalize the functions so that $\|u_p\|_{L^p} = 1$, then for $sp = \alpha p - n$

$$\|u_p\|_{W^{s,p}(\mathbb{R}^n)} \le C(1 + \sqrt[p]{\lambda_p}).$$

Since $\sqrt[p]{\lambda_p} \to R^{-\alpha}$ we have a bound independent of p. For an arbitrary $\gamma \in (0, \alpha)$, we have a bound on the Hölder norms $\|u_p\|_{C^\gamma(\mathbb{R}^n)}$ for large p's, according to Theorem 3. By Ascoli's theorem we can extract a subsequence $u_j = u_{p_j}$ that converges uniformly in each $C^{\gamma}(\mathbb{R}^n)$ to a function u. It follows that $u \in C_0(\overline{\Omega})$ and $u = 0$ in $\mathbb{R}^n \setminus \Omega$.

Viscosity Supersolution In order to prove that the limit function is a viscosity supersolution in Ω, we assume that ϕ is a test function touching u from below at a point x_0. We may assume that the touching is strict by considering $\phi(x) - |x|^2\eta(x)$, where $\eta \in C_0^\infty(\mathbb{R}^n)$ is a function such that $\eta = 1$ in a neighbourhood of x_0 and $\eta \geq 0$. We can assure that $u_j - \phi$ assumes its minimum at points $x_j \to x_0$. This is standard reasoning. By adding a suitable constant c_j we can arrange it so that $\phi + c_j$ touches u_j from below at the point x_j. Recall that the constant has no influence in the testing procedure according to Remark 8.

Since an eigenfunction is a viscosity solution, we have the inequality

$$\mathcal{L}_{p_j}\phi\,(x_j) + \lambda_{p_j} u_j^{p_j-1}(x_j) \leq 0$$

and writing

$$A_j^{p_j-1} = 2\int_{\mathbb{R}^n} \frac{|\phi(y)-\phi(x_j)|^{p_j-2}\big(\phi(y)-\phi(x_j)\big)^+}{|y-x_j|^{\alpha p_j}}\,dy,$$

$$B_j^{p_j-1} = 2\int_{\mathbb{R}^n} \frac{|\phi(y)-\phi(x_j)|^{p_j-2}\big(\phi(y)-\phi(x_j)\big)^-}{|y-x_j|^{\alpha p_j}}\,dy,$$

$$C_j^{p_j-1} = \lambda_{p_j} u_j^{p_j-1}(x_j),$$

we get the abbreviated form

$$A_j^{p_j-1} + C_j^{p_j-1} \leq B_j^{p_j-1}. \tag{14}$$

According to [5, Lemma 6.5] and Proposition 20

$$A_j \to \mathcal{L}_\infty^+\phi\,(x_0), \quad B_j \to -\mathcal{L}_\infty^-\phi\,(x_0), \quad C_j \to \Lambda_\infty^\alpha \phi(x_0).$$

By dropping either $A_j^{p_j-1}$ or $C_j^{p_j-1}$ in (14) and sending $j \to \infty$, we see that

1. $\mathcal{L}_\infty^+\phi\,(x_0) \leq -\mathcal{L}_\infty^-\phi\,(x_0)$, which is equivalent to $\mathcal{L}_\infty\phi\,(x_0) \leq 0$,
2. $\Lambda_\infty^\alpha\phi(x_0) \leq -\mathcal{L}_\infty^-\phi\,(x_0)$, which is equivalent to $\Lambda_\infty^\alpha\phi(x_0) + \mathcal{L}_\infty^-\phi\,(x_0) \leq 0$.

This proves that we have a viscosity supersolution.

Viscosity subsolution This time the test function ϕ is touching u strictly from above at the point x_0. Now we get the reversed inequality

$$A_j^{p_j-1} + C_j^{p_j-1} \geq B_j^{p_j-1}.$$

We now know that $\phi(x_0) > 0$ by Lemma 22, since we already have proved that u is a viscosity supersolution ($\mathcal{L}_\infty u \leq 0$). If $\mathcal{L}_\infty^-\phi\,(x_0) + \Lambda_\infty^\alpha\phi(x_0) \geq 0$, then the desired inequality

$$\max\{\mathcal{L}_\infty\phi\,(x_0),\ \mathcal{L}_\infty^-\phi\,(x_0) + \Lambda_\infty^\alpha\phi(x_0)\} \geq 0$$

follows immediately. The possibility that $-\mathcal{L}_\infty^-\phi\,(x_0) > \Lambda_\infty^\alpha\phi(x_0) > 0$ remains. Then $B_j > 0$ for large indices. We divide by B_j to obtain

$$\frac{C_j^{p_j-1}}{B_j^{p_j-1}} + \frac{A_j^{p_j-1}}{B_j^{p_j-1}} \geq 1$$

and it follows that

$$\frac{\mathcal{L}^{+}_{\infty}\phi\,(x_0)}{-\,\mathcal{L}^{-}_{\infty}\phi\,(x_0)} \geq 1, \quad \mathcal{L}^{+}_{\infty}\phi\,(x_0) \geq -\,\mathcal{L}^{-}_{\infty}\phi\,(x_0).$$

Thus $\mathcal{L}_{\infty}\phi\,(x_0) \geq 0$. Again the desired inequality holds. This proves that we have a viscosity subsolution. □

9 Pointwise behaviour

Recall that the ∞-eigenvalue equation was formulated for test functions. As we will see, a part of it, namely

$$\mathcal{L}^{-}_{\infty}u\,(x) + \Lambda^{\alpha}_{\infty}u\,(x) \leq 0$$

holds pointwise in Ω. This simplifies the investigations.

We need the auxiliary function $|x - x_0|^{\alpha}$, which acts as a fundamental solution. However it has to be truncated.

Lemma 24 *Let $\alpha < 1$. The truncated "α-cone function"*

$$C_{x_0,R}(x) = \min\{|x - x_0|^{\alpha},\ R^{\alpha}\}$$

satisfies the strict inequality $\mathcal{L}_{\infty}C_{x_0,R}\,(x) < 0$ at every point $x \in B_R(x_0) \setminus \{x_0\}$.

Proof The following estimate holds for $\mathcal{L}^{-}_{\infty}$:

$$\mathcal{L}^{-}_{\infty}C_{x_0,R}\,(x) \leq \frac{C_{x_0,R}(x_0) - C_{x_0,R}(x)}{|x_0 - x|^{\alpha}} = -1.$$

In order to estimate $\mathcal{L}^{+}_{\infty}$ we first remark that, since $\alpha < 1$,

$$\frac{C_{x_0,R}(y) - C_{x_0,R}(x)}{|y - x|^{\alpha}} \to 0, \quad \text{as } y \to x.$$

For $x \neq y$

$$\frac{C_{x_0,R}(y) - C_{x_0,R}(x)}{|y - x|^{\alpha}} \leq \frac{|y - x_0|^{\alpha} - |x - x_0|^{\alpha}}{|y - x|^{\alpha}} < 1,$$

where we have used the inequality

$$C_{x_0,R}(y) \leq |y - x_0|^{\alpha} = |x - x_0 + y - x|^{\alpha} < |x - x_0|^{\alpha} + |y - x|^{\alpha},$$

which is strict when $\alpha \in (0, 1)$, $x \neq x_0$ and $y \neq x$. Hence

$$\mathcal{L}^{+}_{\infty}C_{x_0,R}\,(x) < 1,$$

and the result follows. □

When $\alpha = 1$ the cone needs to be adjusted in order to become a strict supersolution.

Lemma 25 *Let $\alpha = 1$. The truncated Lipschitz cone*

$$C_{x_0,R}(x) = \min\{|x - x_0| - \varepsilon|x - x_0|^2,\ R - \varepsilon R^2\},$$

with $\varepsilon R < 1$ satisfies $\mathcal{L}^{-}_{\infty}C_{x_0,R}\,(x) < 0$ at every point $x \in B_R(x_0) \setminus \{x_0\}$.

Proof The computation is the same as when $\alpha < 1$. □

Lemma 26 *If $u \in C_0(\mathbb{R}^n)$ is a viscosity supersolution of the equation $\mathcal{L}_\infty u = 0$ in an open set D where $u > 0$, and if $u \leq 0$ in $\mathbb{R}^n \setminus D$, then*

$$\mathcal{L}_\infty^- u\,(x) = \inf_{y \in \mathbb{R}^n \setminus D} \frac{u(y) - u(x)}{|x - y|^\alpha}.$$

In other words, the infimum is attained in the complement of D and thus $\mathcal{L}_\infty^- u$ is continuous in D.

Remark 27 In general, $\mathcal{L}_\infty^+ u$ is not continuous.

Proof Take $x \in D$ and define

$$L_x^- = \inf_{y \in \mathbb{R}^n \setminus D} \frac{u(y) - u(x)}{|y - x|^\alpha}.$$

By the hypothesis, $L_x^- < 0$. Let

$$w(y) = u(x) + L_x^- \, C_{x,R}(y),$$

where $C_{x,R}$ is as in Lemma 24 or Lemma 25 with R chosen so that $\operatorname{supp} u \subset B_R(x)$. We now claim that $u \geq w$ in D, which implies the lemma. In order to use the comparison principle in the open set $D \setminus \{x\}$ we see that

1. $\mathcal{L}_\infty w > 0$ in $D \setminus \{x\}$ from Lemma 24,
2. $\mathcal{L}_\infty u \leq 0$ in D,
3. $u \geq w$ in $\mathbb{R}^n \setminus D$,
4. $u(x) = w(x)$.

By the comparison principle in [5], $u \geq w$. Indeed, if there is $x_0 \in D \setminus \{x\}$ such that $u(x_0) < w(x_0)$ then, for a suitable constant C, $w - C$ touches u from below at x_0, contradicting (1) above. (Remark 8 is valid also for $p = \infty$.) □

As a consequence any viscosity supersolution is locally α-Hölder continuous.

Corollary 28 *Under the hypotheses in Lemma 26 u is locally α-Hölder continuous in D. So is, in particular, a first ∞-eigenfunction.*

Proposition 29 *Suppose that $u \in C_0(\mathbb{R}^n)$ is a viscosity solution of the ∞-eigenvalue equation (13) in an open set D. In addition, assume $u > 0$ in D and $u \leq 0$ in $\mathbb{R}^n \setminus D$. If $\mathcal{L}_\infty^- u\,(x_0) + \Lambda_\infty^\alpha u(x_0) < 0$ at some $x_0 \in D$ in the pointwise sense, then $\mathcal{L}_\infty u\,(x_0) = 0$ in the viscosity sense.*

Proof Since we already know that $\mathcal{L}_\infty u \leq 0$ in the viscosity sense, it remains only to prove that $\mathcal{L}_\infty u\,(x_0) \geq 0$. Assume

$$-\varepsilon_0 = \mathcal{L}_\infty^- u\,(x_0) + \Lambda_\infty^\alpha u(x_0) < 0$$

and pick $y_0 \in \mathbb{R}^n \setminus D$ such that

$$\mathcal{L}_\infty^- u\,(x_0) = \frac{u(y_0) - u(x_0)}{|y_0 - x_0|^\alpha}.$$

This is possible due to Lemma 26.

Let $\varphi \in C_0^1(\mathbb{R}^n)$ be a function touching u from above at x_0 and choose $\varphi_0 \in C_0^1(\mathbb{R}^n)$ so that $\varphi \geq \varphi_0 \geq u$ and

$$\frac{\varphi_0(y_0) - u(y_0)}{|y_0 - x_0|^\alpha} < \varepsilon_0.$$

Then

$$\begin{aligned}\mathcal{L}_\infty^- \varphi_0 (x_0) + \Lambda_\infty^\alpha \varphi_0(x_0) &\le \frac{\varphi_0(y_0) - \varphi_0(x_0)}{|y_0 - x_0|^\alpha} + \Lambda_\infty^\alpha \varphi_0(x_0) \\ &= \frac{u(y_0) - u(x_0) + \varphi_0(y_0) - u(y_0)}{|y_0 - x_0|^\alpha} + \Lambda_\infty^\alpha u(x_0) \qquad (15) \\ &< -\varepsilon_0 + \varepsilon_0 = 0.\end{aligned}$$

But on the other hand, φ_0 touches u from above at x_0. Hence, (15) implies $\mathcal{L}_\infty \varphi_0 (x_0) \ge 0$. Since φ touches φ_0 from above at x_0, the monotonicity of $\mathcal{L}_\infty$ (cf. Remark 8) implies $\mathcal{L}_\infty \varphi (x_0) \ge 0$. □

Proposition 30 *Let $\alpha < 1$. Suppose that $u \in C_0(\mathbb{R}^n)$ is a viscosity supersolution of (13) in D, $u > 0$ in D, and $u \le 0$ in $\mathbb{R}^n \setminus D$. If there is a $\varphi \in C_0^1(\mathbb{R}^n)$ touching u from below at $x_0 \in D$, then $\mathcal{L}_\infty^- u (x_0) + \Lambda_\infty^\alpha u(x_0) \le 0$ in the pointwise sense.*

Proof We would like to take u itself as a test function, but this is not allowed. Instead we construct a test function looking like an α-cone with (negative) opening $\mathcal{L}_\infty^- u (x_0)$. The details are spelled out below.

Since φ is C^1 we can choose δ so small that

$$\varphi(x) - \varphi(x_0) > \mathcal{L}_\infty^- u (x_0)|x - x_0|^\alpha \quad \text{in } B_{2\delta}(x_0).$$

Choose R very large and let ψ_δ be a regularised version of

$$\mathcal{L}_\infty^- u (x_0) C_{x_0,R} + \varphi(x_0)$$

such that

$$\psi_\delta = \mathcal{L}_\infty^- u (x_0) C_{x_0,R} + \varphi(x_0) \quad \text{in } \mathbb{R}^n \setminus B_\delta(x_0), \quad \psi_\delta \le \mathcal{L}_\infty^- u (x_0) C_{x_0,R} + \varphi(x_0),$$

where $C_{x_0,R}$ is the truncated α-cone in Lemma 24. By definition

$$\psi_\delta \le \mathcal{L}_\infty^- u (x_0) C_{x_0,R} + u(x_0) \le u.$$

Let η_δ be a cut-off function:

$$\eta_\delta \ge 0, \quad \eta_\delta = 0 \quad \text{in } \mathbb{R}^n \setminus B_{2\delta}(x_0), \quad \eta_\delta = 1 \quad \text{in } B_\delta(x_0).$$

Finally, define $\Psi = \eta_\delta \varphi + (1 - \eta_\delta)\psi_\delta$. One can verify that

$$u \ge \Psi \ge \mathcal{L}_\infty^- u (x_0) C_{x_0,R} + u(x_0), \quad u(x_0) = \Psi(x_0) = \varphi(x_0).$$

In other words, Ψ touches u from below at x_0, and we can conclude

$$\begin{aligned}0 \ge \mathcal{L}_\infty^- \Psi (x_0) + \Lambda_\infty^\alpha \Psi(x_0) &= \Lambda_\infty^\alpha u(x_0) + \inf_{y \in \mathbb{R}^n} \frac{\Psi(y) - u(x_0)}{|y - x_0|^\alpha} \\ &\ge \Lambda_\infty^\alpha u(x_0) + \inf_{y \in \mathbb{R}^n} \frac{\mathcal{L}_\infty^- u (x_0) C_{x_0,R}(y)}{|y - x_0|^\alpha} \\ &= \Lambda_\infty^\alpha u(x_0) + \mathcal{L}_\infty^- u (x_0),\end{aligned}$$

since $\mathcal{L}_\infty^- u (x_0) < 0$. □

Since a continuous function can be touched from below in a dense subset this implies, in view of Lemma 26 that the inequality is true everywhere.

Corollary 31 *Let $\alpha < 1$. Suppose $u \in C_0(\mathbb{R}^n)$ is a non-negative viscosity solution of the ∞-eigenvalue Eq.* (13) *in D, $u > 0$ in D, and $u \leq 0$ in $\mathbb{R}^n \setminus D$. Then $\mathcal{L}_\infty^- u + \Lambda_\infty^\alpha u \leq 0$ in D in the pointwise sense.*

Proof Part d) of Lemma 1.8 in [2] states that the subdifferential of a continuous function is non-empty in a dense subset. That the subdifferential is non-empty is equivalent to the existence of a C^1-function touching from below. Thus, from Lemma 30, $\mathcal{L}_\infty^- u + \Lambda_\infty^\alpha u \leq 0$ holds in a dense subset of D. By Lemma 26, $\mathcal{L}_\infty^- u + \Lambda_\infty^\alpha u$ is a continuous function and hence the inequality holds in the whole D. □

When $\alpha = 1$ the proof has to be modified slightly.

Proposition 32 *Let $\alpha = 1$. If $u \in C_0(\mathbb{R}^n)$ is a non-negative viscosity solution of* (13)*, $u > 0$ in D, and $u \leq 0$ in $\mathbb{R}^n \setminus D$. Then $\mathcal{L}_\infty^- u + \Lambda_\infty u \leq 0$ in D in the pointwise sense.*

Proof By Corollary 28, u is locally Lipschitz continuous and thus by Rade-macher's theorem, u is a.e. differentiable. Take x_0 where u is differentiable. Then it is well known that one can find a C^1 function φ touching u from below at x_0. Moreover, $\mathcal{L}_\infty^- u\,(x_0) \leq -|\nabla u(x_0)| = -|\nabla \varphi(x_0)|$. Given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\varphi(x) \geq \varphi(x_0) + [\mathcal{L}_\infty^- u\,(x_0) - \varepsilon]\, C_{x_0,R}(x) \quad \text{in } B_{2\delta}(x_0).$$

Repeating the procedure with η_δ, ψ_δ and Ψ as in the proof of Proposition 30, we obtain that

$$\mathcal{L}_\infty^- u\,(x_0) + \Lambda_\infty u(x_0) \leq \varepsilon.$$

Since ε was arbitrary, this yields $\mathcal{L}_\infty^- u\,(x_0) + \Lambda_\infty u(x_0) \leq 0$, and this holds at a.e. point in D. By Lemma 26, $\mathcal{L}_\infty^- u + \Lambda_\infty u$ is a continuous function, so this must hold everywhere. □

10 The ground state

Recall that the first ∞-eigenfunctions were defined in Definition 21 as the non-negative solutions in $C_0(\overline{\Omega})$ of the ∞-eigenvalue Eq. (13). We will give a remarkable representation formula for one first ∞-eigenfunction, valid in any domain. In some cases we can assure uniqueness.

We need some concepts related to the geometry of Ω. We denote by $\delta(x)$ the distance function, $\text{dist}(x, \mathbb{R}^n \setminus \Omega)$. This function is Lipschitz continuous and $|\nabla \delta| = 1$ almost everywhere in Ω. We define the *High Ridge* as the set of points where the distance function attains its maximum, i.e.

$$\Gamma = \{x \in \Omega \mid \delta(x) = R\},$$

where as before, R denotes the radius of the largest ball that can be inscribed inside Ω. The function $\delta(x)$ is not differentiable on Γ. The High Ridge is a closed set and $\Omega \setminus \Gamma$ is open. We denote

$$\rho(x) = \text{dist}(x, \Gamma).$$

The quantity Λ_∞^α behaves as a genuine eigenvalue in the sense that it cannot be replaced by any other number in the ∞-eigenvalue equation:

Theorem 33 *Let $u \in C_0(\overline{\Omega})$, $u \not\equiv 0$, be a non-negative solution of*

$$\max\left\{\mathcal{L}_\infty u\,(x), \mathcal{L}_\infty^- u\,(x) + \lambda u(x)\right\} = 0 \quad \text{in } \Omega.$$

Then $\lambda = \Lambda_\infty^\alpha$.

Proof From Proposition 30, $\mathcal{L}_\infty^- u + \lambda u \le 0$ in the pointwise sense and from Lemma 26

$$\mathcal{L}_\infty^- u\,(x) = \inf_{y\in\mathbb{R}^n\setminus\Omega} \frac{u(y)-u(x)}{|y-x|^\alpha} = -\frac{u(x)}{\delta(x)^\alpha}.$$

Eliminating u from the inequality, we obtain that $\lambda \le \frac{1}{\delta(x)^\alpha}$ for all $x \in \Omega$. Hence, $\lambda \le \Lambda_\infty^\alpha$.

Now, assume that $\lambda < \Lambda_\infty^\alpha$. Then $\lambda < \frac{1}{\delta(x)^\alpha}$ and thus $\mathcal{L}_\infty^- u\,(x) + \lambda u(x) < 0$ for all $x \in \Omega$. By Lemma 29, $\mathcal{L}_\infty u = 0$ in Ω, which by the comparison principle ([5, Prop. 11.2]) implies that u is identically zero in Ω. This case was excluded. Hence $\lambda \ge \Lambda_\infty^\alpha$. The result follows. □

An immediate consequence of the theorem above is that any first ∞-eigenfunction minimizes the Rayleigh quotient (12).

The fundamental role of the High Ridge Γ is revealed in:

Theorem 34 *Let u be a first ∞-eigenfunction. Then $\mathcal{L}_\infty^- u\,(x) + \Lambda_\infty^\alpha u\,(x) = 0$ (pointwise) if and only if $x \in \Gamma$. In the complement $\Omega \setminus \Gamma$ the equation $\mathcal{L}_\infty u = 0$ holds in the viscosity sense.*

Proof By Lemma 26

$$\mathcal{L}_\infty^- u\,(x) = \inf_{y\in\mathbb{R}^n\setminus\Omega} \frac{u(y)-u(x)}{|y-x|^\alpha} = \inf_{y\in\mathbb{R}^n\setminus\Omega} \frac{-u(x)}{|y-x|^\alpha} = -\frac{u(x)}{\delta(x)^\alpha}.$$

Thus

$$\mathcal{L}_\infty^- u\,(x) + \Lambda_\infty^\alpha u\,(x) = u(x)\left(\frac{1}{R^\alpha} - \frac{1}{\delta(x)^\alpha}\right) \le 0$$

with equality if and only if $\delta(x) = R$, i.e., if and only if $x \in \Gamma$. □

This provides us with a method to construct first ∞-eigenfunctions, using an equation that does not explicitly contain Λ_∞^α. Let $\Gamma_1 \subset \Gamma$ be an arbitrary closed non-empty subset. According to [5, Thm 1.5], the Dirichlet boundary value problem

$$\begin{cases} \mathcal{L}_\infty u = 0 & \text{in } \Omega \setminus \Gamma_1, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \\ u = 1 & \text{on } \Gamma_1, \end{cases} \tag{16}$$

has a unique viscosity solution in $\Omega \setminus \Gamma_1$, which takes the boundary values continuously. Moreover, $0 \le u \le 1$ by [5, Prop. 11.2]. Moreover, by Lemma 12 we have $0 < u < 1$ in $\Omega \setminus \Gamma_1$. Therefore different subsets yield different solutions! Hence, *uniqueness fails if Γ consists of several points.*

Theorem 35 *The solution of the Dirichlet problem (16) is a first ∞-eigenfunction in Ω.*

Proof We first prove that u is a viscosity supersolution of the ∞-eigenvalue Eq. (13). Take φ touching u from below at $x_0 \in \Gamma_1$. Then by direct pointwise computations

$$\mathcal{L}_\infty \varphi\,(x_0) \le \mathcal{L}_\infty u\,(x_0) \le 0,$$

since $0 \le u \le 1$.

Hence u is a supersolution of $\mathcal{L}_\infty u \le 0$ in the whole Ω. By Lemma 26

$$\mathcal{L}_\infty^- u\,(x) = \inf_{y\in\mathbb{R}^n\setminus\Omega} \frac{u(y)-u(x)}{|y-x|^\alpha} = -\frac{u(x)}{\delta(x)^\alpha}$$

and we can conclude

$$\mathcal{L}_\infty^- u\,(x) + \Lambda_\infty^\alpha u(x) = u(x)\left(\frac{1}{R^\alpha} - \frac{1}{\delta(x)^\alpha}\right) \le 0,$$

for any $x \in \Omega$, since $R \ge \delta(x)$. Thus u is a viscosity supersolution of the ∞-eigenvalue Eq. (13) in Ω.

To prove that u is also a viscosity subsolution of (13), it is enough to verify that $\mathcal{L}_\infty^- u + \Lambda_\infty^\alpha u \ge 0$ on Γ_1. This follows by the same arguments as in the proof of Theorem 34. □

We can give the solution of (16) explicitly in terms of distances. This is a special case of Theorem 1.5 in [5].

Theorem 36 (Representation Formula) *Let* $\rho_1(x) = dist(x, \Gamma_1)$. *The function*

$$u(x) = \frac{\delta(x)^\alpha}{\delta(x)^\alpha + \rho_1(x)^\alpha}$$

solves the problem (16) *and is therefore a first* ∞*-eigenfunction.*

Proof For notational convenience, we drop the index writing Γ for Γ_1 and ρ for ρ_1 in this proof. We first claim that when $x \in \Omega \setminus \Gamma$, the supremum in $\mathcal{L}_\infty^+ u\,(x)$ is attained on Γ or, in other words, that

$$\begin{aligned}
\frac{u(y) - u(x)}{|y - x|^\alpha} &\le \sup_{y \in \Gamma} \frac{u(y) - u(x)}{|y - x|^\alpha} \\
&= \sup_{y \in \Gamma} \frac{1 - \frac{\delta(x)^\alpha}{\delta(x)^\alpha + \rho(x)^\alpha}}{|y - x|^\alpha} \\
&= \frac{\frac{\rho(x)^\alpha}{\delta(x)^\alpha + \rho(x)^\alpha}}{\underbrace{|y_x - x|^\alpha}_{=\rho(x)}} \\
&= \frac{1}{\delta(x)^\alpha + \rho(x)^\alpha}
\end{aligned}$$

for all $y \in \Omega$. This is equivalent to

$$\frac{\delta(y)^\alpha \rho(x)^\alpha - \delta(x)^\alpha \rho(y)^\alpha}{|x - y|^\alpha (\delta(x)^\alpha + \rho(x)^\alpha)} \le 1,$$

or

$$\delta(y)^\alpha \rho(x)^\alpha \le |x - y|^\alpha \delta(x)^\alpha + |x - y|^\alpha \rho(x)^\alpha + \delta(x)^\alpha \rho(y)^\alpha.$$

Since $\alpha \in (0, 1]$, the triangle inequality yields

$$\delta(y)^\alpha \le |x - y|^\alpha + \delta(x)^\alpha, \quad \rho(x)^\alpha \le \rho(y)^\alpha + |x - y|^\alpha.$$

Hence,

$$\delta(y)^\alpha \rho(x)^\alpha \le |x - y|^\alpha \rho(x)^\alpha + \delta(x)^\alpha \rho(x)^\alpha \le |x - y|^\alpha \rho(x)^\alpha + \delta(x)^\alpha (\rho(y)^\alpha + |x - y|^\alpha),$$

which proves the claim.

It remains to verify that u solves (16). Take $x \in \Omega \setminus \Gamma$. From the claim

$$\mathcal{L}_\infty^+ u\,(x) = \frac{1}{\delta(x)^\alpha + \rho(x)^\alpha}.$$

Moreover,

$$\mathcal{L}_{\infty}^{-}u(x) \leq \inf_{y\in\mathbb{R}^n\setminus\Omega} \frac{u(y)-u(x)}{|y-x|^\alpha} = \frac{-u(x)}{\underbrace{|y_x-x|^\alpha}_{=\delta(x)}} = -\frac{1}{\delta(x)^\alpha+\rho(x)^\alpha}.$$

Thus $\mathcal{L}_\infty u \leq 0$ in $\Omega\setminus\Gamma$. If $x\in\Gamma$ then $\mathcal{L}_{\infty}^{-}u(x) \leq \mathcal{L}_{\infty}^{+}u(x) \leq 0$ since u attains its maximum there. Thus, $\mathcal{L}_\infty u \leq 0$ in Ω. By Lemma 26 the infimum in $\mathcal{L}_{\infty}^{-}u(x)$ is attained in $\mathbb{R}^n\setminus\Omega$ so that

$$\mathcal{L}_{\infty}^{-}u(x) = -\frac{1}{\delta(x)^\alpha+\rho(x)^\alpha}.$$

Hence, $\mathcal{L}_\infty u = 0$ in $\Omega\setminus\Gamma$. The boundary values of u on Γ and $\mathbb{R}^n\setminus\Omega$ are 1 and 0. Thus u is a solution of the Dirichlet problem (16). The final result follows from Theorem 35. □

Corollary 37 *A first ∞-eigenfunction that is constant on Γ is given by the representation formula*

$$u(x) = C\frac{\delta(x)^\alpha}{\delta(x)^\alpha+\rho(x)^\alpha}.$$

Proof Let u be a first ∞-eigenfunction. By Lemma 34, $\mathcal{L}_\infty u = 0$ outside Γ so that, up to a multiplicative constant, u satisfies (16). By [5, Thm 1.5] the solution of Eq. (16) is unique. □

Example This certainly implies uniqueness when the High Ridge consists of only one point, as for a ball or a cube. The first eigenfunction for the ball $B(0,R)$ is

$$\frac{(R-|x|)^\alpha}{(R-|x|)^\alpha+|x|^\alpha}.$$

For $\alpha=1$ it becomes $\delta(x)=R-|x|$, which incidentally also solves the differential Eq. (3).

We have seen that if the High Ridge Γ consists of more than one point, we can construct several linearly independent first ∞-eigenfunctions for the same domain Ω. It stands to reason that the limiting procedure $u_{p_j}\to u$ in Sect. 7 yields the maximal solution, in which case $\Gamma_1=\Gamma$. We have no valid proof, except in some symmetric special cases.

We have a geometric criterion to guarantee that the distance function is a first ∞-eigenfunction.

Corollary 38 *Take $\alpha=1$. If the distance function is differentiable outside Γ, then the distance function is a first ∞-eigenfunction.*

Proof The first step is to control $\mathcal{L}_\infty\delta$. Since δ is differentiable outside Γ, $|\nabla\delta|=1$ there. Moreover, δ is Lipschitz continuous with constant 1. Therefore with $h=\nabla\delta(x)$ for $x\notin\Gamma$

$$1 \geq \sup_{y\in\mathbb{R}^n}\frac{\delta(y)-\delta(x)}{|y-x|} \geq \lim_{t\searrow 0}\frac{\delta(x+th)-\delta(x)}{|h|} = 1$$

and

$$-1 \leq \inf_{y\in\mathbb{R}^n}\frac{\delta(y)-\delta(x)}{|y-x|} \leq \lim_{t\nearrow 0}\frac{\delta(x+th)-\delta(x)}{|h|} = -1.$$

Thus, $\mathcal{L}_{\infty}^{+}\delta(x) = -\mathcal{L}_{\infty}^{-}\delta(x) = 1$ or equivalently $\mathcal{L}_\infty\delta(x)=0$ for $x\notin\Gamma$. The result now follows from Theorem 35. □

11 Higher infinity eigenvalues

Also for the higher eigenfunctions it is possible to deduce a limiting equation as $p \to \infty$. The equation is the one for the first eigenfunction in every nodal domain together with a transition condition:

$$\begin{cases} \max\left\{\mathcal{L}_\infty u\,(x),\ \mathcal{L}_\infty^- u\,(x) + \lambda u(x)\right\} = 0 & \text{when } u(x) > 0 \\ \mathcal{L}_\infty u\,(x) = 0 & \text{when } u(x) = 0 \\ \min\left\{\mathcal{L}_\infty u\,(x),\ \mathcal{L}_\infty^+ u\,(x) + \lambda u(x)\right\} = 0 & \text{when } u(x) < 0 \end{cases} \tag{17}$$

The result below can be obtained by following the proof of Theorem 23.

Theorem 39 *Let u_p be a sign-changing eigenfunction with the finite exponent p. Then, upon normalizing u_p, there is a subsequence u_{p_j} converging uniformly in Ω to a function $u \in C_0(\overline{\Omega})$ which is a viscosity solution of Eq. (17) for some $\lambda \geq \Lambda_\infty^\alpha(\Omega)$.*

This leads to the following definition of higher ∞-eigenfunctions:

Definition 40 We say that $u \in C_0(\overline{\Omega})$ is a higher ∞-eigenfunction with eigenvalue λ if u is a sign-changing viscosity solution of Eq. (17).

We give a list of properties that hold for higher ∞-eigenfunctions, which can be proved in the same manner as those for the first ∞-eigenfunctions:

- The infimum in $\mathcal{L}_\infty^- u$ is attained in the set $\{u \leq 0\}$ and the supremum in $\mathcal{L}_\infty^+ u$ is attained in the set $\{u \geq 0\}$. Follows from Lemma 26.
- $\mathcal{L}_\infty u = 0$ in the viscosity sense wherever

$$\mathcal{L}_\infty^- u + \lambda u < 0 \quad \text{and} \quad u > 0$$

  or

$$\mathcal{L}_\infty^+ u + \lambda u > 0 \quad \text{and} \quad u < 0.$$

  See Proposition 29.
- When $u > 0$ then $\mathcal{L}_\infty^- + \lambda u \leq 0$ in the pointwise sense, and when $u < 0$, then $\mathcal{L}_\infty^+ + \lambda u \geq 0$, also in the pointwise sense. See Proposition 30.

We change the notation now so that R_1 denotes the radius of the largest inscribed ball in Ω. We define $R_2 = R_2(\Omega)$ as the largest radius R such that two disjoint open balls of radius R can be inscribed in Ω.

Proposition 41 *If u is a higher ∞-eigenfunction with eigenvalue λ then*

$$\lambda \geq \frac{1}{R_2^\alpha}.$$

Proof Pick $x_0 \in \{u > 0\}$ such that

$$\lambda u(x_0) + \mathcal{L}_\infty^- u\,(x_0) = 0.$$

Such an x_0 exists since otherwise, by Proposition 29, $\mathcal{L}_\infty u = 0$ in $\{u > 0\}$, which by the comparison principle in [5] would force $u \leq 0$.

Since $\mathcal{L}_\infty^- u\,(x_0)$ is attained in $\{u \le 0\}$ (cf. Property 1 above)

$$\begin{aligned}\lambda u(x_0) &= -\mathcal{L}_\infty^- u\,(x_0)\\ &= -\inf_{y\in\mathbb{R}^n\cap\{u\le 0\}} \frac{u(y)-u(x_0)}{|y-x_0|^\alpha}\\ &\ge -\inf_{y\in\mathbb{R}^n\cap\partial\{u>0\}} \frac{u(y)-u(x_0)}{|y-x_0|^\alpha}\\ &\ge \frac{u(x_0)}{\operatorname{dist}(x_0,\partial\{u\le 0\})^\alpha}.\end{aligned}$$

The same can be obtained for $y_0 \in \{u < 0\}$ so that

$$\lambda \ge \max\left(\sup_{x_0\in\{u>0\}} \frac{1}{\operatorname{dist}(x_0,\partial\{u>0\})^\alpha}, \sup_{y_0\in\{u<0\}} \frac{1}{\operatorname{dist}(y_0,\partial\{u<0\})^\alpha}\right) \ge \frac{1}{R_2^\alpha}.$$

□

The above proposition implies that in the case when $R_2 \ne R_1$ we can define the second eigenvalue as

$$\inf\{\lambda : \lambda \text{ is an eigenvalue of } u,\ u \text{ changes signs}\}.$$

There are simple examples of domains with $R_1 = R_2$. If $\alpha < 1$ and if there is a nodal domain compactly contained in Ω, we are able to obtain a better lower bound for the second eigenvalue. We encounter a strange phenomenon when $\alpha \ne 1$, viz. the restriction of a higher ∞-eigenfunction to a nodal domain (and extended as zero) is not a first ∞-eigenfunction with respect to the nodal domain.

Proposition 42 *Assume u to be a higher ∞-eigenfunction with eigenvalue λ. If N is a nodal domain compactly contained in the interior of Ω, then $\lambda > \Lambda_\infty^\alpha(N)$.*

Proof We can assume that $\{u > 0\}$ in N. As before we can find $x_0 \in N$ such that $\mathcal{L}_\infty^- u\,(x_0) + \lambda u(x_0) = 0$. Since $\mathcal{L}_\infty^- u\,(x_0)$ is attained in $\{u \le 0\}$,

$$\lambda \ge -\frac{\mathcal{L}_\infty^- u\,(x_0)}{u(x_0)} \ge \inf_{y\in\partial N} \frac{1}{|y-x_0|^\alpha} = \frac{1}{\operatorname{dist}(x_0,\partial N)^\alpha} \ge \Lambda_\infty^\alpha(N). \tag{18}$$

Assume now towards a contradiction that $\lambda = \Lambda_\infty^\alpha(N)$. Then equality holds all the way in (18), so that

$$\mathcal{L}_\infty^- u\,(x_0) = \frac{-u(x_0)}{\operatorname{dist}(x_0,\partial N)^\alpha},$$

or, in other words, for all $y \in \mathbb{R}^n$

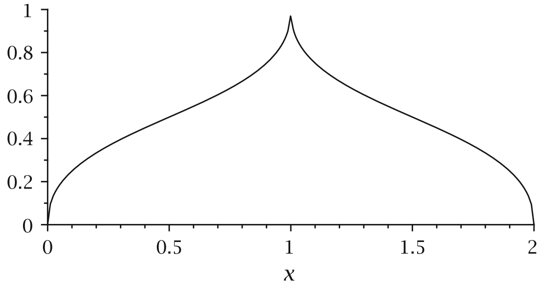
$$u(y) \ge u(x_0)\left(1 - \frac{|y-x_0|^\alpha}{\operatorname{dist}(x_0,\partial N)^\alpha}\right)$$

with equality if $y = x_0$ or if $y = y_0$ with $|y_0 - x_0| = \operatorname{dist}(x_0, \partial N)$. Consequently, with R large enough the function

$$w(y) = u(x_0) - \frac{u(x_0)}{\operatorname{dist}(x_0,\partial N)^\alpha} C_{x_0,R}(y)$$

touches u from below at y_0. From Eq. (17), $\mathcal{L}_\infty w\,(y_0) \le 0$. But on the other hand, Lemma 24 implies $\mathcal{L}_\infty w > 0$, a contradiction. □

Fig. 1 The first eigenfunction on (0, 2), for $\alpha = 1/2$



12 One dimensional examples

Certain aspects of this non-local problem differ from the situation in the eigenvalue problem (3) for the infinity Laplacian. In the case $\alpha < 1$, these differences appear explicitly in one-dimensional examples.

12.1 The first eigenfunction

Consider the interval (0, 2). Its High Ridge consists only of the midpoint, and by Corollary 37 the first eigenfunction is unique and given by the representation formula in Lemma 36 (Fig. 1). In the case of the interval (0, 2) it reduces to

$$u(x) = \frac{\min(|x|^\alpha, |2-x|^\alpha)}{\min(|x|^\alpha, |2-x|^\alpha) + |x-1|^\alpha}.$$

12.2 The second eigenfunction

Consider the interval (0, 2). Assuming that the function u is anti-symmetric around the point $x = 1$ one can construct a solution having two nodal domains:

$$u(x) = \begin{cases} \dfrac{x^\alpha}{x^\alpha + (a-x)^\alpha} & \text{for } x \in (0, a), \\ \dfrac{(2-a-x)^\alpha - (x-a)^\alpha}{(2-a-x)^\alpha + (x-a)^\alpha} & \text{for } x \in (a, 2-a), \\ -\dfrac{(2-x)^\alpha}{(2-x)^\alpha + (x-(2-a))^\alpha} & \text{for } x \in (2-a, 2), \end{cases}$$

here

$$a = \frac{2}{2^{\frac{1}{\alpha}} + 2}, \quad \lambda = (2^{\frac{1}{\alpha}-1} + 1)^\alpha$$

and the nodal domains are the two intervals (0, 1) and (1, 2). The maximum is at $x = a$ and the minimum at $x = 2 - a$. For $\alpha \neq 1$, one can see that $a < 1/2$. The remarkable feature is that the maximum is not attained at the midpoint of the nodal interval (0, 1) but to the left (Fig. 2). In this example $\lambda > \Lambda_\infty^\alpha(\{u > 0\}) = \Lambda_\infty^\alpha((0, 1)) = 1$.

Fig. 2 One eigenfunction on (0, 2) with two nodal domains for $\alpha = 1/2$ and $\lambda = \sqrt{3}$

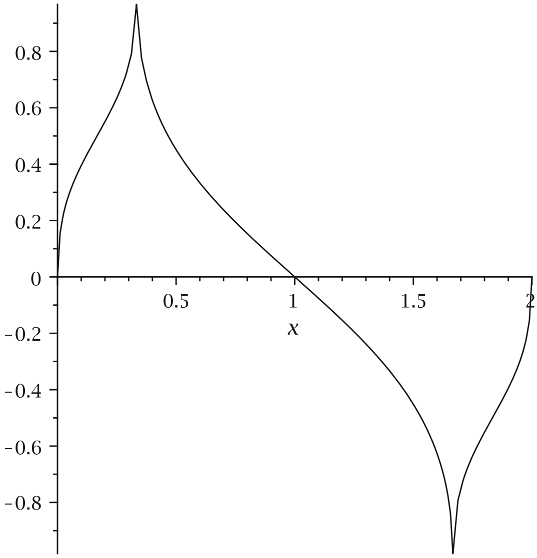
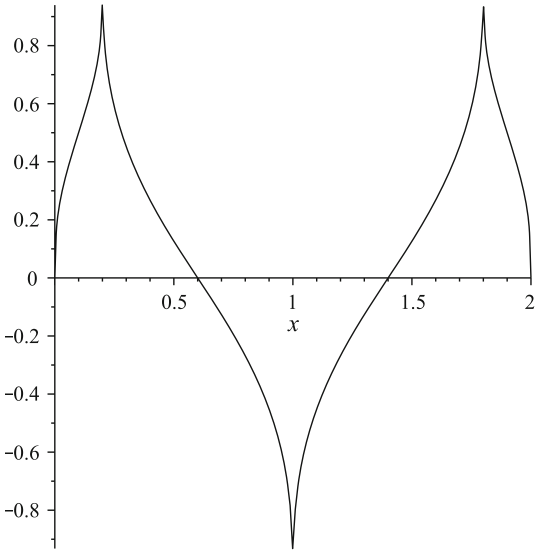


Fig. 3 One eigenfunction on (0, 2) with three nodal domains for $\alpha = 1/2$ and $\lambda = \sqrt{5}$



12.3 A function with three nodal domains

Consider the interval (0, 2). Assuming that the solution is symmetric around the point $x = 1$, we obtain one eigenfunction with three nodal intervals:

$$u(x) = \begin{cases} \dfrac{x^\alpha}{x^\alpha + (a-x)^\alpha} & \text{when } x \in (0, a), \\ \dfrac{(1-x)^\alpha - (x-a)^\alpha}{(1-x)^\alpha + (x-a)^\alpha} & \text{when } x \in (a, 1), \end{cases}$$

and $u(2-x) = u(x)$. Here

$$a = \frac{1}{2^{\frac{1}{\alpha}} + 1}, \quad \lambda = (1 + 2^{\frac{1}{\alpha}})^{\alpha}$$

and the nodal intervals are $(0, \frac{1+a}{2})$, $(\frac{1+a}{2}, \frac{3-a}{2})$ and $(\frac{3-a}{2}, 2)$. The remarkable feature is that the nodal intervals do not have the same length. The middle interval is the longest. This illustrates that nodal domains (coming from the same eigenfunction) can have different first ∞-eigenvalues (Fig. 3).

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