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Calculus of Variations

On perturbation of a functional with the mountain pass geometry

Applications to the nonlinear Schrödinger–Poisson equations and the nonlinear Klein–Gordon–Maxwell equations

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Abstract Consider a functional I_0 with the mountain-pass geometry and a critical point u_0 of mountain-pass type. In this paper, we discuss about the existence of critical points u_ε around u_0 for functionals I_ε perturbed from I_0 in a suitable sense. As applications, we show the existence of a solution to the nonlinear Schrödinger–Poisson equations and the nonlinear Klein–Gordon–Maxwell equations with quite general class of nonlinearity.

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1 Introduction and statement of main result

We divide this introductory section into two parts. The first part is devoted to present an abstract critical point theory inspired by the work of Byeon and Jeanjean [12]. In the remaining part, we introduce its applications to the nonlinear Schrödinger–Poisson equations and the nonlinear Klein–Gordon–Maxwell equations.

1.1 Abstract critical point theory

Let H be a separable Hilbert space with norm $\|\cdot\|$ and let $I_0 : H \to \mathbb{R}$ be a C^1 functional of the form

$$I_0(u) = \frac{1}{2}\|u\|^2 - P(u), \text{ where } P : H \to \mathbb{R} \text{ and } P' : H \to H^* \text{ are compact.} \quad (1.1)$$

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If we assume $P'(0) = 0$, then $0 \in H$ is a critical point of I_0. This type of functionals has been received much attention by a large number of authors because, in many applications, their critical points are weak solutions of various semilinear elliptic partial differential equations arising from diverse areas of mathematics and physics.

Suppose that I_0 has the mountain pass geometry and also has a critical point of mountain pass type. More precisely, suppose that I_0 satisfies the following:

(M1) there exist $c, r > 0$ such that if $\|u\| = r$, then $I_0(u) \geq c > I_0(0)$ and there exists $v_0 \in H$ such that $\|v_0\| > r$ and $I_0(v_0) < I_0(0)$;
(M2) there exists a critical point $u_0 \in H$ of I_0 such that

$$I_0(u_0) = C_0 := \min_{\gamma \in \Gamma} \max_{s \in [0,1]} I_0(\gamma(s)),$$

where $\Gamma = \{\gamma \in C([0, 1], H) \mid \gamma(0) = 0,\ \gamma(1) = v_0\}$.

Consider a functional I_ε perturbed from the limit functional I_0:

$$I_\varepsilon(u) := I_0(u) + J_\varepsilon(u) \tag{1.2}$$

where $\varepsilon > 0$ denotes a small parameter and $J_\varepsilon : H \to \mathbb{R}$ is a C^1 functional such that

(J) (i) $J_\varepsilon(u)$ and $J'_\varepsilon(u)$ converge to 0 locally uniformly for u, i.e., for any $M > 0$,

$$\lim_{\varepsilon \to 0} \sup_{\|u\| \leq M} |J_\varepsilon(u)| = \lim_{\varepsilon \to 0} \sup_{\|u\| \leq M} \|J'_\varepsilon(u)\| = 0;$$

(ii) $J_\varepsilon : H \to \mathbb{R}$ and $J'_\varepsilon : H \to H^*$ are compact.

The first aim of this paper is to develop a variational method for finding a critical point u_ε of I_ε around u_0 when ε is small. From a variational point of view, there are two main issues for obtaining critical points of a functional, that is,

(I1) the geometric structure of a functional for generating Palais–Smale sequences;
(I2) the compactness of Palais–Smale sequences.

By (J), we can easily see the functional I_ε inherits the mountain pass geometry from I_0 if ε is sufficiently small, and thus the existence of a Palais–Smale sequence is straightforward from the Ekeland variational principle. Also, the compactness of Palais–Smale sequences comes from the boundedness of them because $P(u)$ and $J_\varepsilon(u)$ are compact. However, unless quite strong restriction on I_ε is made, it is difficult in general to check whether Palais–Smale sequences of I_ε are bounded even when the limit functional I_0 has the compactness of every Palais–Smale sequences and $\varepsilon > 0$ is small.

To resolve this difficulty, one thus need to develop more sophisticated critical point theories which assert the existence of a bounded Palais–Smale sequence instead of using the Ekeland's variational principle. One powerful method to this direction is the well-known Struwe's monotonicity trick (See [17,27]). Let $I_\mu : H \to \mathbb{R}$ be of the form

$$I_\mu(u) = \alpha(u) - \mu\beta(u),$$

where $\mu > 0$, α is a C^1 functional which is coercive, i.e., $\lim_{\|u\| \to \infty} \alpha(u) = \infty$ and β is a C^1 functional such that $\beta(u) \geq 0$ and β, β' map bounded sets into bounded sets. Then, Struwe's monotonicity technique says that if I_μ has the mountain pass geometry, there is a bounded Palais–Smale sequence corresponding to mountain pass level for almost every $\mu > 0$. However, we point out that this method is not perturbative in nature; neither α is a limit functional nor $\mu > 0$ is a small parameter so this technique is not suitable for achieving our goal.

As the perturbation scheme, there is a critical point theory recently developed by Azzollini, d'Avenia and Pomponio [6]. Let $H_r^1(\mathbb{R}^N)$ be the set of radial functions in the standard Sobolev space $W^{1,2}(\mathbb{R}^N)$ with norm

$$\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + \omega u^2 \, dx, \quad \omega > 0.$$

Choose the limit functional I_0 as

$$I_0(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \omega u^2 \, dx - \int_{\mathbb{R}^N} F(u) \, dx, \tag{1.3}$$

where $F(t) = \int_0^t f(s) \, ds$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following three conditions:

(ii.1) (Superlinearity near zero) $\lim_{s \to 0^+} f(s)/s = 0$;
(ii.2) (Subcriticality near infinity) $\limsup_{s \to \infty} f(s)/s^p < \infty$ for some $p \in (1, \frac{N+2}{N-2})$;
(ii.3) (The Berestycki-Lions condition) there exists $T > 0$ such that $\frac{1}{2}\omega T^2 < F(T)$,

Here we note that above three conditions are almost optimal for (1.3) to admit at least one nontrivial critical point. The perturbation term $J_\varepsilon(u)$ is defined by

$$J_\varepsilon(u) := \varepsilon \sum_{i=1}^{k} R_i(u) \quad \text{for some } k,$$

where each R_i satisfies the following:

(R1) R_i is a nonnegative even C^1 functional on $H_r^1(\mathbb{R}^N)$;
(R2) there exists $\delta_i > 0$ such that $R_i'(u)[u] \le C\|u\|^{\delta_i}$ for any $u \in H_r^1(\mathbb{R}^N)$;
(R3) if $u_j \rightharpoonup u$ weakly in $H_r^1(\mathbb{R}^N)$, then

$$\limsup_{j \to \infty} R_i'(u_j)[u - u_j] \le 0;$$

(R4) there exist $\alpha_i, \beta_i \ge 0$ such that if $u \in H_r^1(\mathbb{R}^N)$, $t > 0$ and $u_t = u(\cdot/t)$, then

$$R_i(u_t) = t^{\alpha_i} R_i(t^{\beta_i} u).$$

Then, it was proved in [6] that there exists a bounded Palais–Smale sequence of I_ε for sufficiently small $\varepsilon > 0$. To obtain this result, the authors applied new ideas developed by Hirata et al. [15]. In fact, the ideas in [15] are equally applicable when I_ε has the symmetric mountain pass geometry so the existence of bounded Palais–Smale sequences corresponding to the symmetric mountain pass energy level of I_ε was also proved in [6]. On the other hand, this result requires the Hilbert space H, the limit functional I_0 and the perturbation term $J_\varepsilon(u)$ to be of some specific form as well as $J_\varepsilon(u)$ to fulfill a kind of scaling property, which does not seem essential.

Inspired by the work of Byeon and Jeanjean [12], we remove these restrictions. In fact, we will develop a critical point theory for functionals defined on abstract Hilbert space. Our result says that if we assume that the mountain pass type critical point of I_0 from (M2) is a ground state, i.e., it has the lowest energy level and the set of ground state critical points of I_0 is compact in H(these are accomplished by I_0 of [6]), then there exists a Palais–Smale sequence of I_ε near the set of ground state critical points of I_0 for sufficiently small $\varepsilon > 0$. More precisely, we will show that there is a bounded Palais–Smale sequence of I_ε away from the origin. We also point out that our result is related to the stability of the mountain pass type critical points under small perturbation of functionals. Recall that there is a non-variational theorem, the implicit function theorem, which tells that if I_ε is C^2 and u_0 is a non-degenerate critical point of I_0, i.e., $I_0'(u_0) = 0$ and $I_0''(u_0)$ is non-singular, then there is a curve of critical points u_ε of I_ε converging to u_0 as $\varepsilon \to 0$. This means that the non-degeneracy condition provides a critical point u_0 with stability under small perturbation. Also, this kind of existence result can be generalized via degree theory when the conditions for I_ε and u_0 are weakened as follows:

(i) I_ε is C^1;
(ii) u_0 is an isolated critical point of I_0 and $\mathrm{ind}(I_0'(u_0), 0) \neq 0$.

However, the non-degeneracy or even the isolatedness of a critical point obtained variationally is quite hard to check in general so that these methods can be applied to a very restricted class of I_ε. We stress that when applying our methods we do not need to worry about any non-degeneracy issue at all.

Now, to state our main theorem, we list the conditions which should be fulfilled by I_0:

(M1) $I_0(0) = 0$, there exist $c, r > 0$ such that if $\|u\| = r$, then $I_0(u) \geq c$ and there exists $v_0 \in H$ such that $\|v_0\| > r$ and $I_0(v_0) < 0$;
(M2) there exists a critical point $u_0 \in H$ of I_0 such that

$$I_0(u_0) = C_0 := \min_{\gamma \in \Gamma} \max_{s \in [0,1]} I_0(\gamma(s)),$$

where $\Gamma = \{\gamma \in C([0,1], H) \mid \gamma(0) = 0,\ \gamma(1) = v_0\}$;
(M3) it holds that

$$C_0 = \inf\{I_0(u) \mid u \in H \setminus \{0\},\ I_0'(u) = 0\};$$

(M4) the set $S := \{u \in H \mid I_0'(u) = 0,\ I_0(u) = C_0\}$ is compact in H;
(M5) there exists a curve $\gamma_0(s) \in \Gamma$ passing through u_0 at $s = s_0$ and satisfying

$$I_0(u_0) > I_0(\gamma_0(s)) \quad \text{for all } s \neq s_0.$$

Then, our main result is the following:

Theorem 1 *Assume that* (M1)–(M5) *hold for a C^1 functional I_0 with the form* (1.1) *and* (J) *holds for one parameter family of C^1 functionals J_ε. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the functional $I_\varepsilon := I_0 + J_\varepsilon$ admits a nontrivial critical point $u_\varepsilon \in H$. In addition, for any sequence $\{\varepsilon_j\}$ converging to* 0*, the sequence of critical points $\{u_{\varepsilon_j}\}$ found above converges to some $W \in S$ up to a subsequence.*

Remark 1 Observe that if I_0 satisfies the Palais–Smale condition, the conditions (M2) and (M4) are automatically implied by (M1) and the well-known mountain pass theorem [1]. For the sake of generality, we assume that I_0 satisfies (M1)–(M5) instead of assuming the (PS) condition, (M1), (M3) and (M5).

Remark 2 It is natural to anticipate that the critical point u_ε found above is of mountain pass type. Also, we think that the technical condition (M5) is not essential because (M5) never contribute to make a mountain pass type critical point stable. It seems interesting to prove Theorem 1 without (M) for wider application.

1.2 Applications to some systems of PDEs

As a first application, we consider the following system of equations:

$$\begin{cases} -\Delta u + \omega u - \lambda \phi u + F_u(x,u) = 0 & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, \quad \lim_{|x|\to\infty} \phi(x) = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\omega \in (0,\infty)$, $\lambda \in \mathbb{R}$ and $u : \mathbb{R}^3 \to \mathbb{R}$, $\phi : \mathbb{R}^3 \to \mathbb{R}$ are unknown functions and $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a given potential function. The system (1.4) is called the nonlinear Schrödinger–Poisson equations because a single nonlinear Schrödinger equation is coupled with a Poisson-term.

A great deal of work has been devoted to the Eq. (1.4) especially in case of $\lambda < 0$. See [2,3,5,6,8,13,14,18,26]. On the other hand, Mugnai first dealt with the case of $\lambda > 0$ in [25], where he showed that there are infinitely many triples $(\lambda, u, \phi) \in \mathbb{R}^+ \times H_r^1(\mathbb{R}^3) \times D_r^1(\mathbb{R}^3)$ which solve (1.4) when the potential function F satisfies the following conditions:

- $\hat{F}1$ $F : \mathbb{R}^3 \times \mathbb{R} \to [0,\infty]$ is such that the derivative $F_u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $F(x,s) = F(|x|,s)$ and $F(x,0) = F_u(x,0) = 0$;
- $\hat{F}2$ there exist C_1, C_2 and $1 < q < p < 5$ such that $|F_u(x,s)| \le C_1|s|^q + C_2|s|^p$;
- $\hat{F}3$ there exists $k \ge 2$ such that $0 \le sF_u(x,s) \le kF(x,s)$;
- $\hat{F}4$ it holds that $F(x,s) = F(x,-s)$.

Here $H_r^1(\mathbb{R}^3)$ denoted the set of radial functions in the standard Sobolev space $W^{1,2}(\mathbb{R}^3)$ and $D_r^1(\mathbb{R}^3)$ the set of radial functions in the space $D^1(\mathbb{R}^3)$, the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^1}^2 = \int_{\mathbb{R}^3} |\nabla u|^2 \, dx.$$

Applying Theorem 1, we considerably generalize the class of the nonlinearity $F(x,s)$ by just assuming that

- (F1) $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is such that the derivative $F_u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $F(x,s) = F(|x|,s)$ and $F(x,0) = F_u(x,0) = 0$;
- (F2) $\lim_{s\to 0} F_u(x,s)/s = 0$ uniformly for $x \in \mathbb{R}^3$ and $\limsup_{|s|\to\infty} |F_u(x,s)|/|s|^p < \infty$ uniformly for $x \in \mathbb{R}^3$ for some $p \in (1,5)$.

Under these assumptions on F, we obtain the following result:

Theorem 2 *Suppose that* (F1) *and* (F2) *hold. Then, for sufficiently large* $\lambda > 0$*, there exists a solution* $(u,\phi) \in H_r^1(\mathbb{R}^3) \times D_r^1(\mathbb{R}^3)$ *of* (1.4).

Observe that we drop the conditions ($\hat{F}3$), ($\hat{F}4$) and nonnegativity of $F(x,s)$. Also, our theorem covers not only infinitely many λ's but also a continuum of them.

Next, as a second application, we consider the following system of equations:

$$\begin{cases} -\Delta u - \omega^2(1 - q\phi)^2 u + F'(u) = 0 & \text{in } \mathbb{R}^3, \\ -\Delta \phi = q(1 - q\phi)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

where $q > 0$ and the potential F is supposed to be

$$F(s) = \frac{m^2}{2}s^2 - G(s)$$

and $G : \mathbb{R} \to \mathbb{R}$ is continuous. This system is called the nonlinear Klein–Gordon–Maxwell equations and also has received a great interest by various authors [4,7,9,10,13,14,22–24]. When $G(s)$ is a power nonlinearity, i.e.,

$$G(s) = \frac{1}{p}|s|^p,$$

D'Aprile and Mugnai proved in [13] that there is no nontrivial finite energy solution $(u, \phi) \in H^1(\mathbb{R}) \times D^1(\mathbb{R})$ and proved in [14] that there are infinitely many finite energy radial solutions $(u, \phi) \in H_r^1(\mathbb{R}) \times D_r^1(\mathbb{R})$ if we assume one of the following conditions:

(i) $m > \omega > 0$ and $p \in [4, 6)$;
(ii) $m\sqrt{(p-2)/2} > \omega > 0$ and $p \in (2, 4)$.

In [7], Azzollini, Pisani and Pomponio improved the existence range of (m, ω) for $p \in (2, 4)$ as follows:

$$0 < \omega < mg(p), \quad g(p) = \begin{cases} \sqrt{(p-2)(4-p)} & \text{if } 2 < p < 3, \\ 1 & \text{if } 3 \le p < 4. \end{cases}$$

They also dealt with a limit case $m = \omega$ in [7]. If $q > 0$ is sufficiently small, the existence of solution can be shown for all $m > \omega > 0$, $p \in (2, 4)$ by a work of Long [22]. More precisely, for sufficiently small q, the existence of solutions to (1.5) was shown in [22] when G is $C^2(\mathbb{R})$ such that $G(0) = G'(0) = 0$ and satisfies the following:

$\hat{\text{G}}1$ the equation

$$-\Delta u + (m^2 - \omega^2)u - G'(u) = 0 \tag{1.6}$$

has a unique positive solution $u_0 \in H_r^4(\mathbb{R}^3)$;
$\hat{\text{G}}2$ u_0 is non-degenerate, i.e., the kernel of linearized operator of (1.6) at u_0,

$$L_{u_0}[\phi] := -\Delta\phi + (m^2 - \omega^2)\phi - G''(u_0)\phi, \quad \phi \in H_r^2(\mathbb{R}^3)$$

is trivial on $H_r^2(\mathbb{R}^3)$.

We note that in case of $G(u) = |u|^p/p$, $2 < p < 6$, ($\hat{\text{G}}1$) and ($\hat{\text{G}}2$) hold for all $0 < \omega < m$ by the well-known work of Kwong [19]. This result is based on the fact that as $q \to 0$ the first equation in (1.5) tends to (1.6) so the conditions ($\hat{\text{G}}1$) and ($\hat{\text{G}}2$) are needed for applying the implicit function theorem as mentioned in the first part of introduction.

The final result of this paper is to eliminate these uniqueness and non-degeneracy assumptions, which significantly restrict the class of G. We will only assume that

(G1) $G \in C^1(\mathbb{R})$, $G(0) = 0$, and $\lim_{s\to 0} G'(s)/s = 0$;
(G2) $\limsup_{s\to\infty} |G'(s)/s^p| < \infty$ for some $p \in (1, 5)$;
(G3) there exists $T > 0$ such that $\frac{1}{2}(m^2 - \omega^2)T^2 < G(T)$,

which are believed to be optimal conditions that can be given to (1.6). Under these assumptions on G, we have the following result:

Theorem 3 *For sufficiently small $q > 0$ and any ω with $0 < \omega < m$, there exists a solution $(u, \phi) \in H_r^1(\mathbb{R}^3) \times D_r^1(\mathbb{R}^3)$ of (1.5).*

Before closing this section, we refer to the results on [10,24], in which the authors considered the nonnegative potential case, i.e., $F(s) \ge 0$ for all s. Based on the paper [10], it was proved in [25] that for sufficiently small $q > 0$, there exists a triple $(\omega, u, \phi) \in \mathbb{R}^+ \times H_r^1(\mathbb{R}^3) \times D_r^1(\mathbb{R}^3)$ satisfying (1.5) if we assume the following:

(F1) $F \in C^2(\mathbb{R})$, $F \geq 0$ and $F(0) = F'(0) = 0$;
(F2) $F''(0) = m^2 > 0$;
(F3) there exist $C_1, C_2 > 0$ and $p \in (0, 4)$ such that $|F''(s)| \leq C_1 + C_2|s|^p$ for every $s \in \mathbb{R}$;
(F4) $0 \leq sF'(s) \leq 2F(s)$ for every $s \in \mathbb{R}$;
(F5) there exist $m_1, c > 0$ with $m_1 < m^2/2$ such that $F(s) \leq m_1 s^2 + c$ for every $s \in \mathbb{R}$.

It was also proved in [25] that the same result holds if (F4) and (F5) are replaced with (F4)' and (F5)', which admit potentials F with super quadratic growth rate as the following:

(F4)' there exists $k \geq 2$ such that $0 \leq sF'(s) \leq kF(s)$ for every $s \in \mathbb{R}$;
(F5)' there exist $m_1, c > 0$ and $\theta > 2$ such that $F(s) \leq c|s| + m_1|s|^\theta$ for every $s \in \mathbb{R}$ and

$$m_1 < \min\left\{\frac{m^2}{k(\theta-1)}, 5^{2-\theta}\frac{(\theta-2)^{\theta-2}}{(\theta-1)^{\theta-1}}\left(1+\frac{c}{2}\right)^{2-\theta}k^{1-\theta}m^{2(\theta-1)}\right\}.$$

We point out that Theorem 3 completely includes the former result. It is easy to see that if we set

$$F(s) = \frac{1}{2}m^2 s^2 - G(s),$$

(G1)–(G3) follow from (F1), (F2), F(3) and (F5) for $\omega > 0$ satisfying $m_1 < \omega^2/2 < m^2/2$. Observe that Theorem 3 covers a continuum of $\omega > 0$ as Theorem 2. Moreover, we can also see that Theorem 3 admits a wide class of potentials $F \geq 0$ with super quadratic growth since, for given G satisfying (G1)–(G3) and $G(s) \leq m^2 s^2/2$, we can modify G as $\tilde{G}$ by defining that $\tilde{G}(s)$ is the same with $G(s)$ until $s < T_1$ for some $T_1 > T$ and interpolate it continuously with some negative function with super quadratic growth. It seems that Theorem 3 however does not cover the latter result entirely and vise versa because (G1)–(G3) don't admit potentials

$$F(s) = \frac{m}{2}s^2 + m_1|s|^p, \quad 2 < p < 6$$

while they are free from the Ambrosetti and Rabinowitz type global condition (F4)'.

The rest of paper is organized as follows. In Sect. 2, we give a proof of Theorem 1. In Sect. 3, we prove Theorems 2 and 3 by making use of Theorem 1.

2 Proof of Theorem 1

In this section, we give a proof of Theorem 1. Our approach is based on the idea developed by Byeon and Jeanjean in [12]. The strategy is to search for a Palais–Smale sequence of I_ε near the set of the least energy critical points of I_0.

Before proceeding, we define a modified mountain pass energy level of I_ε

$$C_\varepsilon = \min_{\gamma \in \Gamma_M} \max_{s \in [0,1]} I_\varepsilon(\gamma(s)),$$

where

$$\Gamma_M = \left\{\gamma \in \Gamma \mid \sup_{s \in [0,1]} \|\gamma(s)\| \leq M\right\}, \quad M := 2\max\left\{\sup_{u \in S}\|u\|, \sup_{s \in [0,1]}\|\gamma_0(s)\|\right\}.$$

By the choice of M, we see that $\gamma_0 \in \Gamma_M$, and thus

$$C_0 = \min_{\gamma \in \Gamma_M} \max_{s \in [0,1]} I_0(\gamma(s)).$$

Now, we shall prove a series of propositions, whose combination gives a complete proof of Theorem 1.

Proposition 1 *The mountain pass energy level C_ε is continuous at* 0*, i.e.,*

$$\lim_{\varepsilon \to 0} C_\varepsilon = C_0.$$

Proof By (M5), there is a curve $\gamma_0 \in \Gamma$ satisfying $\max_{s\in[0,1]} I_0(\gamma_0(s)) = I_0(u_0) = C_0$. Then we obtain from (J) that

$$\begin{aligned} C_\varepsilon &\le \max_{s\in[0,1]} I_\varepsilon(\gamma_0(s)) \\ &\le \max_{s\in[0,1]} I_0(\gamma_0(s)) + \max_{s\in[0,1]} J_\varepsilon(\gamma_0(s)) \\ &= I_0(u_0) + o(1) = C_0 + o(1) \qquad \text{as } \varepsilon \to 0. \end{aligned}$$

This shows $\limsup_{\varepsilon\to 0} C_\varepsilon \le C_0$.

On the other hand, the definition of C_ε gives that for any $\delta > 0$, there exists a curve $\gamma_{\varepsilon,\delta} \in \Gamma_M$ such that $C_\varepsilon + \delta \ge \max_{s\in[0,1]} I_\varepsilon(\gamma_{\varepsilon,\delta}(s))$. Moreover by (J), it follows that for any $\eta > 0$ and any small $\varepsilon > 0$ depending on η,

$$|J_\varepsilon(\gamma_{\varepsilon,\delta}(s))| \le \eta \quad \text{uniformly for } s \in [0, 1] \text{ and } \delta > 0.$$

Then we see that

$$\begin{aligned} C_\varepsilon + \delta &\ge \max_{s\in[0,1]} \left\{ I_0(\gamma_{\varepsilon,\delta}(s)) + J_\varepsilon(\gamma_{\varepsilon,\delta}(s)) \right\} \\ &\ge \max_{s\in[0,1]} I_0(\gamma_{\varepsilon,\delta}(s)) - \eta \\ &\ge C_0 - \eta. \end{aligned}$$

By taking $\delta, \eta \to 0$, we have that $\liminf_{\varepsilon\to 0} C_\varepsilon \ge C_0$, and thus this completes the proof. □

Next, we define

$$B_d(u) := \{v \in H \mid \|v - u\| \le d\}$$

and

$$S^d := \bigcup_{u\in S} B_d(u).$$

Proposition 2 *Suppose that there exist sequences $\{\varepsilon_j\} \to 0$ and $\{u_j\} \subset S^d$ satisfying*

$$\lim_{j\to\infty} I_{\varepsilon_j}(u_j) \le C_0 \qquad \text{and} \qquad \lim_{j\to\infty} I'_{\varepsilon_j}(u_j) = 0. \tag{2.1}$$

Then there is $d_0 > 0$ such that for $0 < d < d_0$, $\{u_j\}$ converges to some $u \in S$, up to a subsequence.

Proof From the definition of S^d, there exists a sequence $\{w_j\} \subset S$ such that $u_j \in B_d(w_j)$. Then, by $(M4)$, we can assume w_j converges to some $w \in S$ in H by taking a subsequence if it is necessary. This gives that $u_j \in B_{2d}(w)$ for j large, and so $\{u_j\}$ converges weakly to some u in H. Here we notice the set $B_{2d}(w)$ is weakly closed in H because it is convex and closed in H. Then $u \in B_{2d}(w)$, which implies u is nontrivial for $d > 0$ small enough.

Now by using (2.1), (J) and the fact that P' is compact, we see that

$$I'_{\varepsilon_j}(u_j)\varphi \to I'_0(u)\varphi = 0 \qquad \text{for all } \varphi \in H,$$

which shows u is a nontrivial critical point of I_0. Moreover, we have that

$$C_0 \geq \lim_{j\to\infty} I_{\varepsilon_j}(u_j) = \lim_{j\to\infty} I_0(u_j) + \lim_{j\to\infty} J_{\varepsilon_j}(u_j)$$
$$\geq \frac{1}{2}\|u\|^2 - P(u) = I_0(u) \geq C_0,$$

where the latter inequality holds due to (M3). It leads to $I_0(u) = C_0$ and consequently $u \in S$.

Finally we use again (2.1), (J) and the fact that P is compact to see that

$$\begin{aligned}\liminf_{j\to\infty} \|u_j\|^2 &\geq \|u\|^2 = 2\{I_0(u) + P(u)\} = 2\{C_0 + P(u)\} \\ &\geq 2\left\{\limsup_{j\to\infty} I_{\varepsilon_j}(u_j) + P(u)\right\} \\ &= \limsup_{j\to\infty}\|u_j\|^2 - 2\left\{\lim_{j\to\infty} P(u_j) - P(u)\right\} + 2\lim_{j\to\infty} J_{\varepsilon_j}(u_j) \\ &= \limsup_{j\to\infty}\|u_j\|^2.\end{aligned}$$

Thus $\{u_j\}$ converges strongly to u in H, up to a subsequence. This completes the proof. □

Now we let

$$D_\varepsilon := \max_{s\in[0,1]} I_\varepsilon(\gamma_0(s)).$$

Then by Proposition 1, we see that $C_\varepsilon \leq D_\varepsilon$ and

$$\lim_{\varepsilon\to 0} C_\varepsilon = \lim_{\varepsilon\to 0} D_\varepsilon = C_0.$$

Also we define

$$I_\varepsilon^{D_\varepsilon} = \{u \in H \mid I_\varepsilon(u) \leq D_\varepsilon\}.$$

Hereafter, the letter d_0 always means the real number in Proposition 2.

Proposition 3 *For any $d_1, d_2 > 0$ satisfying $d_2 < d_1 < d_0$, there are constants $\alpha > 0$ and $\varepsilon_0 > 0$ depending on d_1 and d_2 such that for $\varepsilon \in (0, \varepsilon_0)$, the following is true:*

$$\|I'_\varepsilon(u)\| \geq \alpha \qquad \text{forall } u \in I_\varepsilon^{D_\varepsilon} \cap (S^{d_1} \setminus S^{d_2}).$$

Proof To the contrary, suppose that for some $d_1, d_2 > 0$ satisfying $d_0 > d_1 > d_2$, there exist sequences $\{\varepsilon_j\}$ with $\lim_{j\to\infty} \varepsilon_j = 0$ and $\{u_j\} \subset S^{d_1} \setminus S^{d_2}$ such that

$$\lim_{j\to\infty} I_{\varepsilon_j}(u_j) \leq C_0 \qquad \text{and} \qquad \lim_{j\to\infty} I'_{\varepsilon_j}(u_j) = 0.$$

Then by Proposition 2, there exists $u \in S$ such that $\|u_j - u\| \to 0$ as $j \to \infty$, and thus we have $\mathrm{dist}(u_j, S) \to 0$ as $j \to \infty$. It contradicts with $u_j \notin S^{d_2}$ for all j.

Proposition 4 *For $d > 0$, there exists $\delta > 0$ such that if $\varepsilon > 0$ is sufficiently small,*

$$I_\varepsilon(\gamma_0(s)) \geq C_\varepsilon - \delta \text{ implies } \gamma_0(s) \in S^d.$$

Proof We also argue by contradiction. Suppose that for some $d > 0$, there are sequences $\{\delta_j\} \to 0$, $\{\varepsilon_j\} \to 0$ and $\{s_j\} \subset [0, 1]$ such that

$$I_{\varepsilon_j}(\gamma_0(s_j)) \geq C_{\varepsilon_j} - \delta_j \qquad \text{but} \qquad \gamma_0(s_j) \notin S^d.$$

Then $\{s_j\}$ converges to some $s_\infty \in [0,1]$, up to a subsequence, and by taking a limit, we obtain that

$$C_0 \ge I_0(\gamma_0(s_\infty)) \ge C_0 \quad \text{and} \quad \gamma_0(s_\infty) \notin S^{d/2}.$$

However, $\gamma_0(s_\infty)$ should belong to S by (M5), and therefore we see a contradiction.

Proposition 5 *For any $d < d_0$ and sufficiently small $\varepsilon > 0$ depending on d, there exists a sequence $\{u_j\} \subset S^d \cap I_\varepsilon^{D_\varepsilon}$ such that*

$$I'_\varepsilon(u_j) \to 0 \quad \text{as } j \to \infty.$$

Proof To the contrary, suppose that there is $d < d_0$ and sequences $\{\varepsilon_j\} \to 0$ and $\{c_j\} \subset (0,\infty)$ such that

$$\|I'_{\varepsilon_j}(u)\| \ge c_j > 0 \quad \text{for all } u \in S^d \cap I_{\varepsilon_j}^{D_{\varepsilon_j}}.$$

By Proposition 3, we see that there exists $\alpha > 0$, independent of j, such that

$$\|I'_{\varepsilon_j}(u)\| \ge \alpha \quad \text{for all } u \in I_{\varepsilon_j}^{D_{\varepsilon_j}} \cap (S^d \setminus S^{d/2}).$$

Take a large j such that Proposition 4 holds for d and ε_j. Hereafter, we shall denote this ε_j just as ε.

Now, consider a pseudo-gradient vector field V_ε of I_ε (for definition, see [28]) and take a neighborhood N_ε of $S^d \cap I_\varepsilon^{D_\varepsilon}$ satisfying $N_\varepsilon \subset B_M(0)$. Let η_ε be a Lipschitz continuous function on H such that $0 \le \eta_\varepsilon \le 1$ everywhere and

$$\eta_\varepsilon = \begin{cases} 1 \text{ on } S^d \cap I_\varepsilon^{D_\varepsilon}, \\ 0 \text{ on } H \backslash N_\varepsilon. \end{cases}$$

Also let ξ_ε be a Lipschitz continuous function on $\mathbb{R}$ such that $0 \le \xi_\varepsilon \le 1$ everywhere and

$$\xi_\varepsilon(s) = \begin{cases} 1 \text{ if } |s - C_\varepsilon| \le \delta/2, \\ 0 \text{ if } |s - C_\varepsilon| \ge \delta. \end{cases}$$

Then there exists a global solution $\psi_\varepsilon : H \times \mathbb{R} \to H$ of the initial value problem

$$\begin{cases} \dfrac{\partial}{\partial \tau}\psi_\varepsilon(u,\tau) = -\eta_\varepsilon(\psi_\varepsilon(u,\tau))\xi_\varepsilon(I_\varepsilon(\psi_\varepsilon(u,\tau)))V_\varepsilon(\psi_\varepsilon(u,\tau)), \\ \psi_\varepsilon(u,0) = u. \end{cases}$$

By recalling Proposition 4, $\lim_{\varepsilon\to 0}(C_\varepsilon - D_\varepsilon) = 0$ and using the fact that

$$\frac{d}{d\tau} I_\varepsilon(\psi_\varepsilon(u,\tau)) \le -\eta_\varepsilon(\psi_\varepsilon(u,\tau))\xi_\varepsilon(I_\varepsilon(\psi_\varepsilon(u,\tau)))\|I'_\varepsilon(\psi_\varepsilon(u,\tau))\|^2,$$

a standard argument gives that for some large $\tau_\varepsilon > 0$,

$$I_\varepsilon(\psi_\varepsilon(\gamma_0(s),\tau_\varepsilon)) \le C_\varepsilon - \delta/4 \quad \text{for any } s \in [0,1].$$

From this, setting $\widetilde{\gamma}_0(s) := \psi_\varepsilon(\gamma_0(s),\tau_\varepsilon)$, we have that $\widetilde{\gamma}_0(s) \in \Gamma_M$ and $I_\varepsilon(\widetilde{\gamma}_0(s)) < C_\varepsilon$ for any $s \in [0,1]$. This is a contradiction to the definition of C_ε.

Proposition 6 *For any $d > 0$, there exists $\varepsilon_0(d) > 0$ such that for $0 < \varepsilon < \varepsilon_0(d)$, the functional I_ε admits a critical point $u_\varepsilon \in S^d$.*

Proof By Proposition 5, there exists a Palais–Smale sequence $\{u_j\} \subset S^{d/2}$ corresponding to a fixed small $\varepsilon > 0$. Since $\{u_j\}$ is bounded in H, it holds that $u_j \rightharpoonup u_\varepsilon$ in H for some $u_\varepsilon \in H$. Then we obtain $I'_\varepsilon(u_\varepsilon) = 0$ from the compactness of P' and J'_ε and the fact $I'_\varepsilon(u_j) \to 0$ as $j \to \infty$. Hence u_ε is a critical point of I_ε.

Now we claim that $u_\varepsilon \in S^d$. Indeed, by the fact $u_j \in S^{d/2}$, there is $v_j \in S$ satisfying $\|v_j - u_j\| \le d/2$. Then from the compactness of S, there exists $v \in S$ such that $v_j \to v$ in S, up to a subsequence, as $j \to \infty$. This means that for all j, $u_j \in B_d(v)$, a weakly closed set in H. Thus, it follows that $u_\varepsilon \in B_d(v)$. This completes the proof.

Completion of the proof of Theorem 1 Take $d \in (0, d_0)$ sufficiently small such that S^d does not contain the origin. Proposition 6 says that there is a critical point $u_\varepsilon \in S^d$ of I_ε if ε is small, so that each u_ε is nontrivial. Note that for any sequence $\{\varepsilon_j\}$ converging to 0, the sequence $\{u_{\varepsilon_j}\}$ satisfies all assumptions of Proposition 2. Thus $\{u_{\varepsilon_j}\}$ converges, up to a subsequence, to some $W \in S$. This completes the proof of Theorem 1.

3 Proofs of Theorems 2 and 3

In this section, we shall give proofs of Theorems 2 and 3. We first recall the definition of the function spaces $H^1(\mathbb{R}^3)$ and $D^1(\mathbb{R}^3)$.

For $\alpha > 0$, let $H^1 = H^1(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} |\nabla u|^2 + \alpha\, u^2\, dx$$

which is equivalent to the usual Sobolev norm. Moreover, we denote the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^1}^2 = \int_{\mathbb{R}^3} |\nabla u|^2\, dx$$

by $D^1 = D^1(\mathbb{R}^3)$.

To avoid a lack of compactness, in our applications, we mainly consider the set of radial functions as follows:

$$H_r^1 = H_r^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) \mid u(x) = u(|x|)\}$$

and

$$D_r^1 = D_r^1(\mathbb{R}^3) = \{u \in D^1(\mathbb{R}^3) \mid u(x) = u(|x|)\}.$$

Here we note that the continuous embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ is compact for any $q \in (2, 6)$.

3.1 Proof of Theorem 2

We are concerned with the existence of a solution (u, ϕ) satisfying the nonlinear Schrödinger–Poisson equations

$$\begin{cases} -\Delta u + \omega u - \lambda \phi u + F_u(x, u) = 0 & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, \quad \lim_{|x|\to\infty} \phi(x) = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{3.1}$$

where ω, λ are positive and the potential function $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ satisfies (F1)–(F2). In this subsection, we shall verify Theorem 2 saying that for $\lambda > 0$ large, the Eq. (3.1) admit a nontrivial solution $(u, \phi) \in H_r^1 \times D_r^1$. To obtain this result, we look for a critical point of the associated energy functional $I : H_r^1 \times D_r^1 \to \mathbb{R}$ given by

$$I(u,\phi) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} |\nabla\phi|^2\,dx - \frac{\lambda}{2}\int_{\mathbb{R}^3} \phi\, u^2\,dx + \int_{\mathbb{R}^3} F(x,u)\,dx.$$

We observe that by the Lax–Milgram theorem, for given $u \in H^1$, there exists a unique solution $\phi = \phi_u \in D^1$ satisfying $-\Delta\phi_u = u^2$ in a weak sense. The function ϕ_u is represented by

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\,dy,$$

and it has the following properties (see [26]):

Proposition 7 *The followings hold:*

(i) there exists $C > 0$ such that for any $u \in H^1(\mathbb{R}^3)$,

$$\|\phi_u\|_{D^1} \le C\|u\|_{L^{12/5}}^2 \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla\phi_u|^2\,dx = \int_{\mathbb{R}^3} \phi_u u^2\,dx \le C\|u\|^4;$$

(ii) $\phi_u \ge 0$ for all $u \in H^1$;
(iii) if u is radially symmetric, then ϕ_u is radial;
(iv) $\phi_{tu} = t^2\phi_u$ for all $t > 0$ and $u \in H^1$;
(v) if $u_j \rightharpoonup u$ weakly in H_r^1, then, up to a subsequence, $\phi_{u_j} \to \phi_u$ in D^1 and

$$\int_{\mathbb{R}^3} \phi_{u_j} {u_j}^2\,dx \to \int_{\mathbb{R}^3} \phi_u u^2\,dx.$$

Now by plugging ϕ_u into the first equation of (3.1), we obtain the following Schrödinger equation with a nonlocal term:

$$-\Delta u + \omega u - \lambda\phi_u u + F_u(x,u) = 0 \qquad \text{in } \mathbb{R}^3. \tag{3.2}$$

By defining $u(x) = \varepsilon v(x)$ with $\varepsilon = 1/\sqrt{\lambda}$, (3.2) is equivalent to

$$-\Delta v + \omega v - \phi_v v + F_{v,\varepsilon}(x,v) = 0 \qquad \text{in } \mathbb{R}^3, \tag{3.3}$$

where

$$F_{v,\varepsilon}(x,v) = \frac{1}{\varepsilon} F_v(x,\varepsilon v),$$

and we easily see that $F_{v,\varepsilon}(x,v) \to 0$ as $\varepsilon \to 0$ uniformly in x from (F2). By the above proposition and (F2), the functional $I_\varepsilon : H_r^1 \to \mathbb{R}$ given by

$$I_\varepsilon(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx + \int_{\mathbb{R}^3} F_\varepsilon(x,u)\,dx,$$

where $F_\varepsilon(x,u) = \frac{1}{\varepsilon^2}F(x,\varepsilon u)$, is well-defined. Moreover, a standard calculation shows that it is a C^1 functional with derivative given by

$$I'_\varepsilon(u)v = \int_{\mathbb{R}^3} \nabla u \nabla v + \omega uv\,dx - \int_{\mathbb{R}^3} \phi_u uv\,dx + \int_{\mathbb{R}^3} F_{u,\varepsilon}(x,u)v\,dx$$

for all $v \in H_r^1$. Thus it suffices to find a critical point of I_ε in order to obtain a solution for (3.1).

For the purpose of applying Theorem 1 to this problem, we need to consider the following limit equation of (3.3) which is called Choquard equation (see [20,21]):

$$-\Delta v + \omega v - \phi_v v = 0 \qquad \text{in } \mathbb{R}^3. \tag{3.4}$$

For $u \in H_r^1$, we define an energy functional for (3.4) by

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx. \tag{3.5}$$

Proposition 7 says that the functional

$$P(u) = \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx$$

is compact. The following lemma asserts that its derivative $P'(u)$ is also compact.

Lemma 1 *The derivative $P' : u \in H_r^1 \mapsto \int_{\mathbb{R}^3} \phi_u u \cdot dx \in (H_r^1)^*$ is compact.*

Proof Let $\{u_j\}$ be a bounded sequence in H_r^1. Then along a subsequence, we may assume that $u_j \rightharpoonup u$ in H_r^1. We observe that for all $\varphi \in H_r^1$, using Hölder's inequality,

$$\begin{aligned}
\left|P'(u_j)\varphi - P'(u)\varphi\right| &= \left|\int_{\mathbb{R}^3} \phi_{u_j} u_j\varphi\,dx - \int_{\mathbb{R}^3} \phi_u u\varphi\,dx\right| \\
&\le \left|\int_{\mathbb{R}^3} \phi_{u_j}(u_j - u)\varphi\,dx\right| + \left|\int_{\mathbb{R}^3} (\phi_{u_j} - \phi_u)u\varphi\,dx\right| \\
&\le \|\phi_{u_j}\|_{L^6}\|u_j - u\|_{L^{12/5}}\|\varphi\|_{L^{12/5}} \\
&\quad + \|\phi_{u_j} - \phi_u\|_{L^6}\|u\|_{L^{12/5}}\|\varphi\|_{L^{12/5}} = o(1)\|\varphi\|
\end{aligned}$$

because $\phi_{u_j} \to \phi_u$ in L^6 and $u_j \to u$ in $L^{12/5}$ up to a subsequence. Then we have that

$$P'(u_j) \to P'(u) \qquad \text{in } (H_r^1)^*,$$

which proves the compactness of P'.

In the remaining part of this subsection, we will check that the conditions (M1)–(M5) for $I_0(u)$ and (J) for

$$J_\varepsilon(u) := \int_{\mathbb{R}^3} F_\varepsilon(x,u)\,dx$$

are satisfied. First we prove the following lemma:

Lemma 2 *The functional I_0 satisfies* (M1).

Proof By Proposition 7 (i), we easily see that

$$I_0(u) \geq \frac{1}{2}\|u\|^2 - \frac{C}{4}\|u\|^4 = \frac{1}{4C} > 0$$

if $\|u\|^2 = 1/C$. Moreover, if we take $u \neq 0$ and $t > 0$, then by Proposition 7 (iv),

$$I_0(tu) = \frac{t^2}{2}\|u\|^2 - \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx,$$

which goes to $-\infty$ as $t \to \infty$.

Now we define

$$C_0 = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} I_0(\gamma(s)),$$

where $\Gamma = \{\gamma \in C([0,1], H_r^1) \mid \gamma(0) = 0, \quad I_0(\gamma(1)) < 0\}$. Then we deduce the following lemma:

Lemma 3 *The functional I_0 satisfies the Palais–Smale condition at any energy level C.*

Proof Let $\{u_j\} \subset H_r^1$ be a sequence satisfying

$$I_0(u_j) \to C \qquad \text{and} \qquad I_0'(u_j) \to 0 \qquad \text{as} \quad j \to \infty.$$

Then we have that for j large,

$$4C + o(1)(1 + \|u_j\|) \geq 4I_0(u_j) - I_0'(u_j)u_j = \|u_j\|^2.$$

Hence $\{u_j\}$ is bounded in H_r^1. Observe that by the Riesz representation theorem, u_j is represented as

$$u_j = I_0'(u_j) + P'(u_j).$$

Since P' is compact and $I_0'(u_j)$ converges to 0, $\{u_j\}$ converges in H_r^1 to some $u \in H_r^1$. This completes the proof.

Therefore by Mountain Pass Theorem, we deduce that for any $\omega > 0$, Eq. (3.4) admits a mountain pass solution u_0 in H_r^1 satisfying $I_0(u_0) = C_0$, so that (M2) holds.

Now we need to check that (M3) holds. For this, we let

$$N = \{u \in H_r^1 \mid I_0'(u) = 0, \ u \neq 0\}.$$

Then we obtain the following lemma:

Lemma 4 *The mountain pass solution u_0 of* (3.4) *is a least energy solution, i.e.,*

$$C_0 = \inf_{u\in N} I_0(u) =: C_1.$$

Proof First, we easily see that $u_0 \in N$, and therefore $I_0(u_0) = C_0 \geq C_1$.

Before showing that $C_0 \leq C_1$, we need to see that the value C_1 is attained by some $v \in N$. Choose a minimizing sequence $\{v_j\} \subset N$. Then it satisfies $I_0'(v_j) = 0$ for all j and $I_0(v_j) \to C_1$ as $j \to \infty$. Thus by Lemma 3, $\{v_j\}$ converges to some $v \in H_r^1$ up to a subsequence, and therefore $I_0(v) = 0$, that is, $v \in N$ and $I_0'(v) = C_1$.

Now for this $v \in N$, we set $\gamma_1(s) = st_1 v$, where $t_1 > 0$ is such that $I_0(t_1 v) < 0$ (this exists by Lemma 2). Since $\gamma_1 \in \Gamma$ and it holds that

$$\frac{d}{dt} I_0'(\gamma_1(s))|_{s=1/t_1} = 0,$$

we have that

$$\max_{s\in[0,1]} I_0(\gamma_1(s)) = I_0(v) \geq C_0$$

by the definition of C_0. Therefore, $C_0 = C_1$.

By the above argument, we notice that there exists $t_0 > 0$ such that $I_0(tu_0) < 0$ for all $t \geq t_0$. Then setting $\gamma_0(s) = st_0 u_0$, we see that the curve $\gamma_0 \in \Gamma$ has maximum value at $s = 1/t_0$ and satisfies

$$\max_{s\in[0,1]} I_0(\gamma_0(s)) = I_0(\gamma_0(1/t_0)) = I_0(u_0) = C_0.$$

Hence (M5) is also satisfied.

Finally, by defining

$$S_\omega := \{u \in H_r^1 \mid I_0'(u) = 0, I_0(u) = C_0\},$$

we arrive at the following result which clearly holds due to the Palais–Smale condition of I_0 (Lemma 3):

Lemma 5 *For any $\omega > 0$, the set S_ω is compact in H_r^1, that is,* (M4) *holds.*

On the other hand, from (F1)–(F2), we obtain the following result:

Lemma 6 *The functional J_ε given by*

$$J_\varepsilon(u) := \int_{\mathbb{R}^3} F_\varepsilon(x, u)\, dx$$

satisfies the condition (J).

Proof Given $M > 0$, choose any $u \in H_r^1$ with $\|u\| \leq M$. By (F2), for given $\delta > 0$, there is $C_{\delta,M} > 0$ depending on δ and M such that

$$|F(x, u)| \leq \frac{\delta}{M^2} u^2 + C_{\delta,M} |u|^{p+1}.$$

Thus we have that

$$\begin{aligned} |J_\varepsilon(u)| &\leq \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} |F(x, \varepsilon u)|\, dx \\ &\leq \frac{\delta}{M^2} \int_{\mathbb{R}^3} u^2\, dx + C_{\delta,M}\, \varepsilon^{p-1} \int_{\mathbb{R}^3} |u|^{p+1}\, dx. \end{aligned}$$

Taking $\varepsilon \to 0$, it follows that

$$\lim_{\varepsilon\to 0} |J_\varepsilon(u)| \leq \frac{\delta}{M^2} \|u\|^2 \leq \delta,$$

which yields that

$$\lim_{\varepsilon\to 0}\sup_{\|u\|\le M}|J_\varepsilon(u)|=0.$$

Similarly, using the fact that for any $\delta>0$

$$|F_u(x,u)|\le\delta|u|+C_\delta|u|^p,$$

it follows that for $v\in H_r^1$

$$|J_\varepsilon'(u)v|\le\frac{1}{\varepsilon}\int_{\mathbb{R}^3}|F_u(x,\varepsilon u)v|\,dx$$
$$\le\delta\int_{\mathbb{R}^3}|u||v|\,dx+C_\delta\,\varepsilon^{p-1}\int_{\mathbb{R}^3}|u|^p|v|\,dx.$$

Therefore we deduce that

$$\lim_{\varepsilon\to 0}\sup_{\|u\|\le M}\|J_\varepsilon'(u)\|=0.$$

The compactness of J_ε and J_ε' is well-known in literature(see [28]). This completes the proof. □

In conclusion, this problem satisfies the conditions (M1)–(M5) and (J) in Theorem 1, and thus we conclude that for $\varepsilon>0$ small, the Eq. (3.3) admits a nontrivial solution u_ε in H_r^1 and $u_\varepsilon\in S_\omega^d$. In other words, for sufficiently large $\lambda>0$, the Eq. (3.2) has a nontrivial solution u_λ in H_r^1. We complete the proof of Theorem 2.

3.2 Proof of Theorem 3

We are interested in the existence of a nontrivial solution of the stationary system of Klein–Gordon–Maxwell type:

$$\begin{cases}-\Delta u+\left[m^2-\omega^2(1-q\phi)^2\right]u=G'(u) & \text{in } \mathbb{R}^3,\\ -\Delta\phi=q(1-q\phi)u^2 & \text{in } \mathbb{R}^3,\end{cases}\tag{3.6}$$

where $\omega\in\mathbb{R}$, $q\in(0,\infty)$ and the potential $G:\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying the so-called Beresticki-Lions conditions (G1)–(G3). In this subsection, we shall give a proof of Theorem 3 in which for q small enough and any ω with $m^2>\omega^2$, the system (3.6) admits a solution $(u,\phi)\in H_r^1\times D_r^1$.

In order to find a solution for (3.6), for any ω with $m^2>\omega^2$, let us consider the functional $I:H^1\times D^1\to\mathbb{R}$ defined by

$$I(u,\phi)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2+(m^2-\omega^2)u^2\,dx-\int_{\mathbb{R}^3}G(u)\,dx$$
$$+\frac{\omega^2}{2}\left(2q\int_{\mathbb{R}^3}\phi u^2\,dx-q^2\int_{\mathbb{R}^3}\phi^2u^2\,dx-\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx\right).$$

We easily see that the functional I belongs to $C^1(H^1\times D^1,\mathbb{R})$ and its critical points solve the system (3.6).

On the other hand, we find that by the Lax–Milgram theorem, for any $u \in H^1$, there exists a unique $\phi = \phi_u \in D^1$ which solves $-\Delta\phi = q(1-q\phi)u^2$ in a weak sense (see [14]). Moreover, the function ϕ_u satisfies the following properties (see [24]):

Proposition 8 *The following properties hold:*

(i) if u is radially symmetric, then ϕ_u is radial.
(ii) if $u_j \rightharpoonup u$ weakly in H_r^1, then, up to a subsequence, $\phi_{u_j} \to \phi_u$ in D^1 and

$$\int_{\mathbb{R}^3} \phi_{u_j} {u_j}^2\,dx \to \int_{\mathbb{R}^3} \phi_u u^2\,dx.$$

Then by inserting the function ϕ_u into the first equation of (3.6), we obtain the following equation:

$$-\Delta u + \left[m^2 - \omega^2(1-q\phi_u)^2\right]u = G'(u) \quad \text{in } \mathbb{R}^3. \tag{3.7}$$

For any ω with $m^2 > \omega^2$, we define the energy functional $I_q : H_r^1 \to \mathbb{R}$ for (3.7) by

$$I_q(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + (m^2-\omega^2)u^2\,dx - \int_{\mathbb{R}^3} G(u)\,dx + \frac{\omega^2}{2} J_q(u),$$

where

$$J_q(u) := 2q\int_{\mathbb{R}^3} \phi_u u^2\,dx - q^2\int_{\mathbb{R}^3} {\phi_u}^2 u^2\,dx - \int_{\mathbb{R}^3} |\nabla\phi_u|^2\,dx.$$

It is not hard to see that the functional I_q is of class C^1 and

$$I_q'(u)v = \int_{\mathbb{R}^3} \nabla u\nabla v + (m^2-\omega^2)uv\,dx - \int_{\mathbb{R}^3} G'(u)v\,dx$$
$$+\omega^2\left(2q\int_{\mathbb{R}^3} \phi_u uv\,dx - q^2\int_{\mathbb{R}^3} {\phi_u}^2 uv\,dx\right)$$

for all $v \in C_0^\infty(\mathbb{R}^3)$ (for calculation, see [24]). In other words, we have the following proposition:

Proposition 9 *The following statements are equivalent:*

(i) $(u,\phi) \in H^1 \times D^1$ is a critical point of I;
(ii) u is a critical point of I_q and $\phi = \phi_u$.

Let us consider the following limit equation of (3.7):

$$-\Delta u + (m^2-\omega^2)u = G'(u) \quad \text{in } \mathbb{R}^3. \tag{3.8}$$

Also for any ω with $m^2 > \omega^2$, we define the functional $I_0 : H_r^1 \to \mathbb{R}$ by

$$I_0(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + (m^2-\omega^2)u^2\,dx - \int_{\mathbb{R}^3} G(u)\,dx$$
$$:= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} G(u)\,dx.$$

We let

$$C_0 = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} I_0(\gamma(s)),$$

where $\Gamma = \{\gamma \in C([0,1], H_r^1) \mid \gamma(0) = 0,\ I_0(\gamma(1)) < 0\}$. Here as for the previous application, we consider the functional I_0 defined on H_r^1. Recall that $P(u) = \int_{\mathbb{R}^3} G(u)\,dx$ and its derivative P' are compact on H_r^1. From now on, we will look for a critical point of I_q as a solution of (3.7) by applying Theorem 1. Thus we have to check that the conditions (M1)–(M5) for I_0 and (J) for J_q.

Concerning Eq. (3.8), Berestycki and Lions [11] proved that for any ω with $m^2 > \omega^2$, (3.8) has a radially symmetric least energy solution $u_0 \in H_r^1(\mathbb{R}^3)$ under the the conditions (G1)–(G3). Moreover, Jeanjean and Tanaka [16] verified that I_0 has mountain pass geometry and a least energy solution u_0 of (3.8) is a mountain pass solution, that is, $I_0(u_0) = C_0$. Therefore, (M1)–(M3) are clearly satisfied.

Furthermore, we obtain from the Pohozaev identity that for $u_{0,t}(x) = u_0(x/t)$,

$$\begin{aligned} I_0(u_{0,t}) &= \frac{t}{2}\int_{\mathbb{R}^3} |\nabla u_0|\,dx + \frac{t^3}{2}(m^2-\omega^2)\int_{\mathbb{R}^3} {u_0}^2\,dx - t^3\int_{\mathbb{R}^3} G(u_0)\,dx \\ &= \left(\frac{t}{2} - \frac{t^3}{6}\right)\int_{\mathbb{R}^3} |\nabla u_0|\,dx. \end{aligned}$$

Thus there exists $t_0 > 0$ such that $I_0(u_{0,t}) < 0$ for all $t \geq t_0$. Now setting $\gamma_0(s) = u_{0,st_0}$, we have that the curve $\gamma_0 \in \Gamma$ has maximum value at $s = 1/t_0$ and satisfies

$$\max_{s\in[0,1]} I_0(\gamma_0(s)) = I_0(\gamma_0(1/t_0)) = I_0(u_0) = C_0.$$

Hence (M5) is also satisfied.

For any ω with $m^2 > \omega^2$, we let

$$S_{m,\omega} = \{u \in H_r^1 \mid I_0'(u) = 0,\ I_0(u) = C_0\}.$$

Then the following result holds (see [12]):

Lemma 7 *For any ω with $m^2 > \omega^2$, the set $S_{m,\omega}$ is compact in H_r^1, that is,* (M4) *holds.*

Finally we prove the following result:

Lemma 8 *The functional J_q satisfies condition (J).*

Proof Multiplying the equation $-\Delta\phi_u = q(1-q\phi_u)u^2$ by ϕ_u and integrating by parts, we find that

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2\,dx = q\int_{\mathbb{R}^3} \phi_u u^2\,dx - q^2\int_{\mathbb{R}^3} {\phi_u}^2 u^2\,dx. \tag{3.9}$$

Then for given $M > 0$, by choosing any $u \in H_r^1$ with $\|u\| \leq M$, we deduce from the Hölder and Sobolev inequalities that

$$\begin{aligned} \|\nabla\phi_u\|_{L^2}^2 &\leq q\|\phi_u\|_{L^6}\|u^2\|_{L^{6/5}} + q^2\|\phi_u^2\|_{L^3}\|u^2\|_{L^{3/2}} \\ &\leq qC\|\nabla\phi_u\|_{L^2}\|u\|^2 + q^2C\|\nabla\phi_u\|_{L^2}^2\|u\|^2, \end{aligned}$$

which implies

$$\|\nabla\phi_u\|_{L^2} \le \frac{CM^2q}{1-CM^2q}$$

for small $q > 0$ and some constant $C > 0$. By (3.9), we have that

$$J_q(u) = q \int_{\mathbb{R}^3} \phi_u u^2 \, dx,$$

and thus

$$|J_q(u)| \le q \, \|\phi_u\|_{L^6} \|u\|^2_{L^{12/5}} \le CM^2q \frac{CM^2q}{1-CM^2q}$$

by Hölder's inequality, and consequently $|J_q(u)| \to 0$ as $q \to 0$. It is similar to show that $\|J_q'(u)\| \to 0$ as $q \to 0$. We omit the proof.

Finally, using the same argument as in the proof of Lemma 1, the compactness of J_q and J_q' comes from Proposition 8 and compactness of the Sobolev embedding $H_r^1 \hookrightarrow L^p$ for $2 < p < 6$. We also omit the proof.

At this point, the functional $I_q(u)$ satisfies the conditions (M1)–(M5) and (J) in Theorem 1. Therefore we conclude that for $q > 0$ small and any ω with $m^2 > \omega^2$, the Eq. (3.7) admits a solution u_q in H_r^1. This completes the proof of Theorem 3.

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References

1. Ambrosetti, A., Rabinowitz, P.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**, 349–381 (1973)
2. Ambrosetti, A., Ruiz, D.: Multiple bound states for the Schrödinger-Poisson problem. Commun. Contemp. Math. **10**, 391–404 (2008)
3. Azzollini, A., Pomponio, A.: Ground state solutions for the nonlinear Schrödinger-Maxwell equations. J. Math. Anal. Appl. **345**, 90–108 (2008)
4. Azzollini, A., Pomponio, A.: Ground state solutions for the nonlinear Klein-Gordon-Maxwell equations. Topol. Methods Nonlinear Anal. **35**, 33–42 (2010)
5. Azzollini, A., d'Avenia, P., Pomponio, A.: On the Schrödinger-Maxwell equations under the effect of a general nonlinear term. **27**, 779–791 (2010)
6. Azzollini, A., d'Avenia, P., Pomponio, A.: Multiple critical points for a class of nonlinear functionals. Ann. Mat. Pura Appl. **190**, 507–523 (2011)
7. Azzollini, A., Pisani, L., Pomponio, A.: Improved estimates and a limit case for the electrostatic Klein-Gordon-Maxwell system. Proc. R. Soc. Edinburgh Sect. A **141**, 449–463 (2011)
8. Benci, V., Fortunato, D.: An eigenvalue problem for the Schrödinger-Maxwell equations. Topol. Methods Nonlinear Anal. **11**, 283–293 (1998)
9. Benci, V., Fortunato, D.: Solitary waves of the nonlinear Klein-Gordon field equation coupled with the Maxwell equations. Rev. Math. Phys. **14**, 409–420 (2002)
10. Benci, V., Fortunato, D.: Solitary waves in abelian gauge theories. Adv. Nonlinear Stud. **8**, 327–352 (2008)
11. Berestycki, H., Lions, P.-L.: Nonlinear scalar field equations, I, existence of a ground state. Arch. Ration. Mech. Anal. **82**, 313–345 (1983)

12. Byeon, J., Jeanjean, L.: Standing waves for nonlinear Schrödinger equations with a general nonlinearity. Arch. Ration. Mech. Anal. **185**, 185–200 (2007)
13. D'Aprile, T., Mugnai, D.: Non-existence results for the coupled Klein-Gordon-Maxwell equations. Adv. Nonlinear Stud. **4**, 307–322 (2004)
14. D'Aprile, T., Mugnai, D.: Solitary Waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations. Proc. R. Soc. Edinb. Sect. A **134**, 1–14 (2004)
15. Hirata, J., Ikoma, N., Tanaka, K.: Nonlinear scalar field equations in $\mathbb{R}^N$: mountain pass and symmetric mountain pass approaches. Topol. Methods Nolinear Anal. **35**, 253–276 (2010)
16. Jeanjean, L., Tanaka, K.: A remark on least energy solutions in $\mathbb{R}^N$. Proc. Am. Math. Soc. **131**, 2399–2408 (2003)
17. Jeanjean, L., Toland, J.: Bounded Palais-Smale mountain-pass sequences. C. R. Acad. Sci. Paris Sér. I Math 327, 23–28 (1998).
18. Kikuchi, H.: On the existence of a solution for elliptic system related to the Maxwell-Schrödinger equations. Nonlinear Anal. **67**, 1445–1456 (2007)
19. Kwong, M.: Uniqueness of positive solutions of $\Delta u - u + u^p$ in $\mathbb{R}^n$. Arch. Ration. Mech. Anal. **3**, 243–266 (1989)
20. Lieb, E. H. : Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. Studies in, Appl. Math. 57, 93–105 (1976/77).
21. Lions, P.L.: The Choquard equation and related questions. Nonlinear Anal. **6**, 1063–1072 (1980)
22. Long, E.: Existence and stability of solitary waves in nonlinear Klein-Gordon-Maxwell equations. Rev. Math. Phys. **18**, 747–779 (2006)
23. Long, E., Stuart, D.: Effective dynamics for solitons in the nonlinear Klein-Gordon-Maxwell system and the Lorentz force law. Rev. Math. Phys. **21**, 459–510 (2009)
24. Mugnai, D.: Solitary waves in abelian gauge theories with strongly nonlinear potentials. Ann. I. H. Poincaré Anal. Non Linéaire 27, 1055–1071 (2010).
25. Mugnai, D.: The Schrödinger-Poisson system with positive potential. Commun. Partial Differ. Equ. **36**, 1099–1117 (2011)
26. Ruiz, D.: The Schödinger-Poisson equation under the effect of a nonlinear local term. J. Funct. Anal. **237**, 655–674 (2006)
27. Struwe, M.: The existence of surfaces of constant mean curvature with free boundaries. Acta. Math. **160**, 19–64 (1988)
28. Stuwe, M.: Variational Methods: Application to Nonlinear Partial Differential Equations and Hamiltonian Systems. Springer (1990).