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Calculus of Variations

Regularity of nonlocal minimal cones in dimension 2

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Abstract We show that the only nonlocal s-minimal cones in $\mathbb{R}^2$ are the trivial ones for all $s \in (0, 1)$. As a consequence we obtain that the singular set of a nonlocal minimal surface has at most $n - 3$ Hausdorff dimension.

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1 Introduction

Nonlocal minimal surfaces were introduced in [3] as boundaries of measurable sets E whose characteristic function χ_E minimizes a certain $H^{s/2}$ norm. More precisely, for any$s \in (0, 1)$, the nonlocal s-perimeter functional $\mathrm{Per}_s(E, \Omega)$ of a measurable set E in an open set $\Omega \subset \mathbb{R}^n$ is defined as the Ω-contribution of χ_E in $\|\chi_E\|_{H^{s/2}}$, that is

$$\mathrm{Per}_s(E, \Omega) := L(E \cap \Omega, \mathbb{R}^n \setminus E) + L(E \setminus \Omega, \Omega \setminus E), \tag{1}$$

where $L(A, B)$ denotes the double integral

$$L(A, B) := \int_A \int_B \frac{dx\, dy}{|x - y|^{n+s}}, \qquad A, B \text{ measurable sets.}$$

A set E is *s-minimal* in Ω if $\mathrm{Per}_s(E, \Omega)$ is finite and

$$\mathrm{Per}_s(E, \Omega) \le \mathrm{Per}_s(F, \Omega)$$

for any measurable set F for which $E \setminus \Omega = F \setminus \Omega$.

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We say that E is s-minimal in $\mathbb{R}^n$ if it is s-minimal in any ball B_R for any $R > 0$. The boundary of s-minimal sets are referred to as *nonlocal s-minimal surfaces*.

The theory of nonlocal minimal surfaces developed in [3] is (at least for some features) similar to the theory of standard minimal surfaces. In fact as $s \to 1^-$, the s-minimal surfaces converge to the classical minimal surfaces and the functional in (1) (after a multiplication by a factor of the order of $(1 - s)$) Gamma-converges to the classical perimeter functional (see [5,1]).

In [3] it was shown that nonlocal s-minimal surfaces are $C^{1,\alpha}$ outside a singular set of Hausdorff dimension $n - 2$. The precise dimension of the singular set is determined by the problem of existence in low dimensions of a nontrivial global s-minimal cone (i.e. an s-minimal set E such that $tE = E$ for any $t > 0$). In the case of classical minimal surfaces Simons theorem states that the only global minimal cones in dimension $n \le 7$ must be half-planes, which implies that the Hausdorff dimension of the singular set of a minimal surface in $\mathbb{R}^n$ is $n - 8$. In [6], the authors used these results to show that if s is sufficiently close to 1 the same holds for s-minimal surfaces i.e. global s-minimal cones must be half-planes if $n \le 7$ and the Hausdorff dimension of the singular set is $n - 8$. See also [7] for regularity results in related seetings.

Given the nonlocal character of the functional in (1), it seems more difficult to analyze global s-minimal cones for general values of $s \in (0, 1)$. The purpose of this paper is to show that there are no nontrivial s-minimal cones in the plane. Our theorem is the following.

Theorem 1 *If E is an s-minimal cone in $\mathbb{R}^2$, then E is a half-plane.*

From Theorem 1 above and Theorem 9.4 of [3], we obtain that s-minimal sets in two-dimensional domains are locally $C^{1,\alpha}$ (for further regularity, see [2]). Also, from Theorem 1 and classical blow-up and blow-down arguments[1], we obtain that s-minimal sets in the plane are half-planes. We summarize these observations in the following result:

Corollary 1 *If E is an s-minimal set in $\Omega \subset \mathbb{R}^2$, and $\Omega' \Subset \Omega$, then $(\partial E) \cap \Omega'$ is a $C^{1,\alpha}$-curve.*
If E is an s-minimal set in $\mathbb{R}^2$, then ∂E is a straight line.

In higher dimensions, by combining the result of Theorem 1 here with the dimensional reduction performed in [3], we obtain that any nonlocal s-minimal surface in $\mathbb{R}^n$ is locally $C^{1,\alpha}$ outside a singular set of Hausdorff dimension $n - 3$.

Corollary 2 *Let ∂E be a nonlocal s-minimal surface in $\Omega \subset \mathbb{R}^n$ and let $\Sigma_E \subset \partial E \cap \Omega$ denote its singular set. Then $\mathcal{H}^d(\Sigma_E) = 0$ for any $d > n - 3$.*

The idea of the proof of Theorem 1 is the following. If $E \subset \mathbb{R}^2$ is an s-minimal cone then we construct a set $\tilde{E}$ as a translation of E in $B_{R/2}$ which coincides with E outside B_R. Then the difference between the energies (of the extension) of $\tilde{E}$ and E tends to 0 as $R \to \infty$. This implies that also the energy of $\tilde{E} \cup E$ is arbitrarily close to the energy of E. On the other hand if E is not a half-plane the set $\tilde{E} \cup E$ can be modified locally to decrease its energy by a fixed small amount and we reach a contradiction.

The proof we present is quite robust, and we plan to use this method in a forthcoming paper to obtain monotonicity and symmetry results in a general framework.

In the next section we introduce some notation and obtain the perturbative estimates that are needed for the proof of Theorem 1 in Sect. 3.

[1] For instance, one can use the proof of Theorem 28.17 in [8], where the density estimates, the compactness arguments and the monotonicity formulas for classical minimal surfaces are replaced by the ones in [3]. Of course, in all the results presented, we are implicitly ruling out the trivial case in which either the s-minimal set E or its complement is empty.

2 Perturbative estimates

We start by introducing some notation.

Notation.

We denote points in $\mathbb{R}^n$ by lower case letters, such as $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and points in $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty)$ by upper case letters, such as $X = (x, x_{n+1}) = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_+$.

The open ball in $\mathbb{R}^{n+1}$ of radius R and center 0 is denoted by B_R. Also we denote by $B_R^+ := B_R \cap \mathbb{R}^{n+1}_+$ the open half-ball in $\mathbb{R}^{n+1}$ and by $S^n_+ := S^n \cap \mathbb{R}^{n+1}_+$ the unit half-sphere.

The fractional parameter $s \in (0, 1)$ will be fixed throughout this paper; we also set

$$a := 1 - s \in (0, 1).$$

The standard Euclidean base of $\mathbb{R}^{n+1}$ is denoted by $\{e_1, \dots, e_{n+1}\}$. Whenever there is no possibility of confusion we identify $\mathbb{R}^n$ with the hyperplane $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$.

The transpose of a square matrix A will be denoted by A^T, and the transpose of a row vector V is the column vector denoted by V^T. We denote by I the identity matrix in $\mathbb{R}^{n+1}$.

We introduce the functional

$$\mathcal{E}_R(u) := \int_{B_R^+} |\nabla u(X)|^2 x_{n+1}^a \, dX. \tag{2}$$

which is related to the s-minimal sets by an extension problem, as shown in Section 7 of [3] (this is a version of the extension problem for the fractional Laplacian discussed in [4]). More precisely, given a set $E \subseteq \mathbb{R}^n$ with locally finite s-perimeter, we can associate to it uniquely its extension function $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ whose trace on $\mathbb{R}^n \times \{0\}$ is given by $\chi_E - \chi_{\mathbb{R}^n \setminus E}$ and which minimizes the energy functional in (2) for any $R > 0$.

We recall (see Proposition 7.3 of [3]) that E is s-minimal in $\mathbb{R}^n$ if and only if its extension u is minimal for the energy in (2) under compact perturbations whose trace in $\mathbb{R}^n \times \{0\}$ takes the values ± 1. More precisely, for any $R > 0$,

$$\mathcal{E}_R(u) \le \mathcal{E}_R(v) \tag{3}$$

for any v that coincides with u on $\partial B_R^+ \cap \{x_{n+1} > 0\}$ and whose trace on $\mathbb{R}^n \times \{0\}$ is given by $\chi_F - \chi_{\mathbb{R}^n \setminus F}$ for any measurable set F which is a compact perturbation of E in B_R.

Next we estimate the variation of the functional in (2) with respect to horizontal domain perturbations. For this we introduce a standard cutoff function

$$\phi \in C_0^\infty\left(\mathbb{R}^{n+1}\right), \text{ with } \phi(X) = 1 \text{ if } |X| \le 1/2 \text{ and } \phi(X) = 0 \text{ if } |X| \ge 3/4.$$

Given $R > 0$, we let

$$Y := X + \phi(X/R) e_1. \tag{4}$$

Then we have that $X \mapsto Y = Y(X)$ is a diffeomorphism of $\mathbb{R}^{n+1}_+$ as long as R is sufficiently large (possibly in dependence of ϕ).

Given a measurable function $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$, we define

$$u_R^+(Y) := u(X). \tag{5}$$

Similarly, by switching e_1 with $-e_1$ (or ϕ with $-\phi$ in (4)), we can define $u_R^-(Y)$.

In the next lemma we estimate a discrete second variation for the energy $\mathcal{E}_R(u)$.

Lemma 1 *Suppose that u is homogeneous of degree zero and $\mathcal{E}_R(u) < +\infty$. Then*

$$\left|\mathcal{E}_R(u_R^+) + \mathcal{E}_R(u_R^-) - 2\mathcal{E}_R(u)\right| \leq C R^{n-3+a}, \tag{6}$$

for a suitable $C \geq 0$, depending on ϕ and u.

Proof We start with the following observation. Let us consider the square matrix of order $(n+1)$

$$A := \begin{pmatrix} a_1 & \ldots\ldots & a_{n+1} \\ 0 & \ldots\ldots & 0 \\ & \ddots & \\ 0 & \ldots\ldots & 0 \end{pmatrix}$$

with $1 + a_1 \neq 0$. Then a direct computation shows that

$$(I + A)^{-1} = I - \frac{1}{1 + a_1} A = I - \frac{A}{\det(I + A)}. \tag{7}$$

Now, we define

$$\chi_R(X) := \begin{cases} 1 & \text{if } R/2 \leq |X| \leq R, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathcal{M}(X) := \frac{1}{R} \begin{pmatrix} \partial_1\phi(X/R) & \ldots\ldots & \partial_{n+1}\phi(X/R) \\ 0 & \ldots\ldots & 0 \\ & \ddots & \\ 0 & \ldots\ldots & 0 \end{pmatrix}.$$

Notice that

$$\mathcal{M} = O(1/R)\,\chi_R. \tag{8}$$

Let now

$$\kappa(X) := |\det D_X Y(X)| = \det(I + \mathcal{M}(X)) = 1 + \frac{\partial_1\phi(X/R)}{R} = 1 + \operatorname{tr}\mathcal{M}(X).$$

By (7), we see that

$$(D_X Y)^{-1} = (I + \mathcal{M})^{-1} = I - \frac{\mathcal{M}}{\kappa}. \tag{9}$$

Also, $1/\kappa = 1 + O(1/R)$, therefore, by (8),

$$\frac{\mathcal{M}\,\mathcal{M}^T}{\kappa} = O(1/R^2)\chi_R. \tag{10}$$

Now, we perform some chain rule differentiation of the domain perturbation. For this, we take X to be a function of Y; also, the functions u, Y, χ_R, $\mathcal{M}$ and κ will be evaluated at X, while u_R^+ will be evaluated at Y (e.g., the row vector $\nabla_X u$ is a short notation for $\nabla_X u(X)$, while $\nabla_Y u_R^+$ stands for $\nabla_Y u_R^+(Y)$). We use (5) and (9) to obtain

$$\nabla_Y u_R^+ = \nabla_X u\, D_Y X = \nabla_X u\, (D_X Y)^{-1} = \nabla_X u \left(I - \frac{\mathcal{M}}{\kappa}\right).$$

Also, by changing variables,

$$dY = |\det D_X Y|\, dX = \kappa\, dX.$$

Accordingly

$$\begin{aligned} \left|\nabla_Y u_R^+\right|^2 y_{n+1}^a\, dY &= \nabla_X u \left(I - \frac{\mathcal{M}}{\kappa}\right)\left(I - \frac{\mathcal{M}}{\kappa}\right)^T (\nabla_X u)^T\, x_{n+1}^a\, \kappa\, dX \\ &= \nabla_X u \left(\kappa\, I - \mathcal{M} - \mathcal{M}^T + \frac{\mathcal{M}\mathcal{M}^T}{\kappa}\right)(\nabla_X u)^T\, x_{n+1}^a\, dX \\ &= \nabla_X u \left((1 + \operatorname{tr}\mathcal{M})\, I - \mathcal{M} - \mathcal{M}^T + \frac{\mathcal{M}\mathcal{M}^T}{\kappa}\right)(\nabla_X u)^T\, x_{n+1}^a\, dX. \end{aligned}$$

Hence, from (10),

$$\begin{aligned} &\left|\nabla_Y u_R^+\right|^2 y_{n+1}^a\, dY \\ &\quad = \nabla_X u \left((1 + \operatorname{tr}\mathcal{M})\, I - \mathcal{M} - \mathcal{M}^T + O\left(1/R^2\right)\chi_R\right)(\nabla_X u)^T\, x_{n+1}^a\, dX. \end{aligned}$$

The similar term for $\nabla_Y u_R^-$ may be computed by switching ϕ to $-\phi$ (which makes $\mathcal{M}$ switch to $-\mathcal{M}$): thus we obtain

$$\begin{aligned} &\left|\nabla_Y u_R^-\right|^2 y_{n+1}^a\, dY \\ &\quad = \nabla_X u \left((1 - \operatorname{tr}\mathcal{M})\, I + \mathcal{M} + \mathcal{M}^T + O\left(1/R^2\right)\chi_R\right)(\nabla_X u)^T\, x_{n+1}^a\, dX. \end{aligned}$$

By summing up the last two expressions, after simplification we conclude that

$$\left(\left|\nabla_Y u_R^+\right|^2 + \left|\nabla_Y u_R^-\right|^2\right) y_{n+1}^a\, dY \;=\; 2\left(1 + O\left(1/R^2\right)\chi_R\right)\, |\nabla_X u|^2\, x_{n+1}^a\, dX. \qquad (11)$$

On the other hand, the function$g(X) := |\nabla_X u(X)|^2\, x_{n+1}^a$ is homogeneous of degree $a-2$, hence

$$\begin{aligned} \int_{B_R^+} \chi_R\, |\nabla_X u|^2\, x_{n+1}^a\, dX &= \int_{B_R^+ \setminus B_{R/2}^+} g\, dX = \int_{R/2}^{R} \left[\int_{S_+^n} g(\vartheta\varrho)\, d\vartheta\right] \varrho^n\, d\varrho \\ &= \int_{R/2}^{R} \varrho^{n+a-2} \left[\int_{S_+^n} g(\vartheta)\, d\vartheta\right] d\varrho = C R^{n+a-1}, \end{aligned}$$

for a suitable $C \geq 0$ depending on u. This and (11) give that

$$\begin{aligned} &\int_{B_R^+} \left(\left|\nabla_Y u_R^+\right|^2 + \left|\nabla_Y u_R^-\right|^2\right) y_{n+1}^a\, dY - 2\int_{B_R^+} |\nabla_X u|^2\, x_{n+1}^a\, dX \\ &\quad = O(1/R^2) \int_{B_R^+} \chi_R\, |\nabla_X u|^2\, x_{n+1}^a\, dX \\ &\quad = O(1/R^2) \cdot C R^{n+a-1}, \end{aligned}$$

which completes the proof of the lemma. □

Lemma 1 turns out to be particularly useful when $n = 2$. In this case (6) yields

$$\mathcal{E}_R\left(u_R^+\right) + \mathcal{E}_R\left(u_R^-\right) - 2\mathcal{E}_R(u) \le \frac{C}{R^s}, \tag{12}$$

and the right hand side becomes arbitrarily small for large$R$. As a consequence, we also obtain the following corollary.

Corollary 3 *Suppose that E is an s-minimal cone in $\mathbb{R}^2$ and that u is the extension of $\chi_E - \chi_{\mathbb{R}^2\setminus E}$. Then*

$$\mathcal{E}_R\left(u_R^+\right) \le \mathcal{E}_R(u) + \frac{C}{R^s}. \tag{13}$$

Proof Since E is a cone, we know that u is homogeneous of degree zero (see Corollary 8.2 in [3]): thus, the assumptions of Lemma 1 are fulfilled and so (12) holds true.

From the minimality of u (see (3)), we infer that

$$\mathcal{E}_R(u) \le \mathcal{E}_R(u_R^-),$$

which together with (12) gives the desired claim. □

3 Proof of Theorem 1

We argue by contradiction, by supposing that $E \subset \mathbb{R}^2$ is an s-minimal cone different than a half-plane. By Theorem 10.3 in [3], E is the disjoint union of a finite number of closed sectors. Then, up to a rotation, we may suppose that a sector of E has angle less than π and is bisected by e_2. Thus, there exist $M > 0$ and $p \in B_M$, on the e_2-axis, such that p lies in the interior of E, and $p + e_1$ and $p - e_1$ lie in the exterior of E.

Let $R > 4M$ be sufficiently large. Using the notation of Lemma 1 we have

$$\begin{aligned} u_R^+(Y) &= u\left(Y - e_1\right), \text{ for all } Y \in B_{2M}^+, \text{ and} \\ u_R^+(Y) &= u(Y) \text{ for all } Y \in \mathbb{R}^3_+ \setminus B_R^+, \end{aligned} \tag{14}$$

where u is the extension of $\chi_E - \chi_{\mathbb{R}^2\setminus E}$. We define

$$v_R(X) := \min\{u(X),\, u_R^+(X)\} \quad \text{and} \quad w_R(X) := \max\{u(X),\, u_R^+(X)\}.$$

Denote $P := (p, 0) \in \mathbb{R}^3$. We claim that

$$\begin{aligned} u_R^+ < w_R &= u \text{ in a neighborhood of } P, \text{ and} \\ u < w_R &= u_R^+ \text{ in a neighborhood of } P + e_1. \end{aligned} \tag{15}$$

Indeed, by (14)

$$u_R^+(P) = u\left(P - e_1\right) = \left(\chi_E - \chi_{\mathbb{R}^2\setminus E}\right)(p - e_1) = -1$$

while

$$u(P) = \left(\chi_E - \chi_{\mathbb{R}^2\setminus E}\right)(p) = 1.$$

Similarly, $u_R^+(P + e_1) = u(P) = 1$ while $u(P + e_1) = -1$. This and the continuity of the functions u and u_R^+ at P, respectively $P + e_1$, give (15).

We point out that $\mathcal{E}_R(u) \le \mathcal{E}_R(v_R)$, thanks to (14) and the minimality of u. This and the identity

$$\mathcal{E}_R(v_R) + \mathcal{E}_R(w_R) = \mathcal{E}_R(u) + \mathcal{E}_R\left(u_R^+\right)$$

imply that

$$\mathcal{E}_R(w_R) \le \mathcal{E}_R\left(u_R^+\right). \tag{16}$$

Now we observe that w_R is not a minimizer for $\mathcal{E}_{2M}$ with respect to compact perturbations in B_{2M}^+. Indeed, if w_R were a minimizer we use $u \le w_R$ and the first fact in (15) to conclude $u = w_R$ in B_{2M}^+ from the strong maximum principle. However this contradicts the second inequality in (15).

Therefore, we can modify w_R inside a compact set of B_{2M}^+ and obtain a competitor u_* such that

$$\mathcal{E}_{2M}(u_*) + \delta \le \mathcal{E}_{2M}(w_R),$$

for some $\delta > 0$, independent of R (since w_R restricted to B_{2M}^+ is independent of R, by (14)).

The inequality above implies

$$\mathcal{E}_R(u_*) + \delta \le \mathcal{E}_R(w_R), \tag{17}$$

since u_* and w_R coincide outside B_{2M}^+. Thus, we use (13), (16) and (17) to conclude that

$$\mathcal{E}_R(u_*) + \delta \le \mathcal{E}_R(w_R) \le \mathcal{E}_R(u_R^+) \le \mathcal{E}_R(u) + \frac{C}{R^s}.$$

Accordingly, if R is large enough we have that $\mathcal{E}_R(u_*) < \mathcal{E}_R(u)$, which contradicts the minimality of u. This completes the proof of Theorem 1.

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