Calc. Var. (2013) 47:589–610
DOI 10.1007/s00526-012-0529-9

Calculus of Variations

The stress intensity factor for non-smooth fractures in antiplane elasticity

Antonin Chambolle · Antoine Lemenant

Received: 28 October 2011 / Accepted: 20 April 2012 / Published online: 22 May 2012

Abstract Motivated by some questions arising in the study of quasistatic growth in brittle fracture, we investigate the asymptotic behavior of the energy of the solution u of a Neumann problem near a crack in dimension 2. We consider non smooth cracks K that are merely closed and connected. At any point of density 1/2 in K, we show that the blow-up limit of u is the usual "cracktip" function $C\sqrt{r}\sin(\theta/2)$, with a well-defined coefficient (the "stress intensity factor" or SIF). The method relies on Bonnet's monotonicity formula (Bonnet, Variational methods for discontinuous structures, pp. 93–103. Birkhäuser, Basel, 1996) together with Γ-convergence techniques.

Mathematics Subject Classification Primary: 35J20 · Secondary: 74R10

1 Introduction

According to Griffith's theory, the propagation of a brittle fracture in an elastic body is governed by the competition between the energy spent to produce a crack, proportional to its length, and the corresponding release of bulk energy. An energetic formulation of this idea is the core of variational models for crack propagation, which were introduced by Francfort and Marigo in [10] and are based on a Mumford–Shah-type [17] functional.

In this work, we will restrict ourselves to the case of *anti-plane shear*, where the domain is a cylinder $\Omega \times \mathbb{R}$, with $\Omega \subset \mathbb{R}^2$, which is linearly elastic with Lamé coefficients λ and μ. Moreover we assume the crack to be vertically invariant, while the displacement is vertical

Communicated by L. Ambrosio.

A. Chambolle
CMAP, Ecole polytechnique, CNRS, Palaiseau, France
e-mail: antonin.chambolle@polytechnique.fr

A. Lemenant (✉)
LJLL, Université Paris-Diderot, CNRS, Paris, France
e-mail: lemenant@ann.jussieu.fr

only. Under those assumptions, the problem reduces to a purely 2D, scalar problem. Extending our result to (truely 2D) planar elasticity requires a finer knowledge of monotonicity formulas for the bilaplacian and is still out of reach, it is the subject of future study.

Given a loading process $g : t \mapsto g(t) \in H^1(\Omega)$, and assuming that $K(t) \subset \Omega$ (a closed set) is the fracture at time t, the bulk energy at the time t_0 is given by

$$E(t_0) := \min_u \int\limits_{\Omega \setminus K(t_0)} (A\nabla u) \cdot \nabla u \, dx, \tag{1}$$

where the minimum is taken among all functions $u \in H^1(\Omega \setminus K(t_0), \mathbb{R})$ satisfying $u = g(t_0)$ on $\partial\Omega \setminus K(t_0)$, and the surface energy, for any fracture $K \supseteq K(t_0)$ is proportional to $\kappa\mathcal{H}^1(K)$, where $\mathcal{H}^1$ denotes the one dimensional Hausdorff measure and κ is a constant which is known as the *toughness* of the material. Here the matrix A which appears in the integral in (1) is $(\mu/2)Id$, however in the paper we will also address the case of more general matrices $A(x)$, which will be assumed to be uniformly elliptic and spatially Hölder-continuous.

The proof of existence for a crack $K(t)$ satisfying the propagation criterions of brittle fracture as postulated by Francfort and Marigo [10], was first proved by Dal Maso and Toader [8] in the simple 2D linearized anti-plane setting, then extended in various directions by other authors [4,7,11,1].

In this paper we will freeze the "time" at a certain fixed value t_0, and therefore do not really matter exactly which model of existence we use. We will only need to know that such fractures exist, as a main motivation for our results.

In the quasistatic model, the fracture $K(t)$ is in equilibrium at any time, which means that the total energy cannot be improved at time t_0 by extending the crack. Precisely, for any closed set $K \supseteq K(t_0)$ such that $K(t_0) \cup K$ is connected, and for any $u \in H^1(\Omega \setminus (K(t_0) \cup K))$ satisfying $u = g(t_0)$ on $\partial\Omega \setminus (K(t_0) \cup K)$, one must have that

$$E(t_0) + \kappa\mathcal{H}^1(K(t_0)) \leq \int\limits_{\Omega \setminus (K(t_0) \cup K)} (A\nabla u) \cdot \nabla u \, dx + \kappa\mathcal{H}^1(K).$$

This implies that the propagation of the crack is totally dependent on the external force g, and a necessary condition for K to propagate is that of the first order limit of the bulk energy, namely

$$\limsup_{h \to 0^+} \frac{E(t_0 + h) - E(t_0)}{h}, \tag{2}$$

to be greater or equal to κ. The limit in (2) can be interpreted as an *energy release rate* along the growing crack, which is the central object of many recent works [3,5,6,13,14].

In all the aforementioned papers, a strong regularity assumption is made on the fracture $K(t)$: it is assumed to be a segment near the tip in [3,5,13]; to our knowledge the weakest assumption is the $C^{1,1}$ regularity in [14]. The main reason for this is the precise knowledge of the asymptotic development of the displacement u near the tip of the crack, when it is straight. Indeed the standard elliptic theory in polygonal domains (see e.g. Grisvard [12]) says that in a small ball $B(0, \varepsilon)$ (we assume the crack tip is the origin), if u denotes the minimizer for the problem (1), then there exists $\tilde{u} \in H^2(B(0, \varepsilon) \setminus K(t_0))$ such that

$$u = C\sqrt{r}\sin(\theta/2) + \tilde{u}, \tag{3}$$

(in polar coordinates, assuming the crack is $\{\theta = \pm\pi\}$). In fracture theory, the constant C in front of the sinus is usually referred as the *stress intensity factor* (SIF). In [14], G. Lazzaroni and R. Toader proved that (3) is still true if $K(t_0)$ is a $C^{1,1}$ regular curve, up to a change of coordinates, and they base their study of the energy release rate upon this fact.

The main goal of this paper is to extend (3) to fractures that are merely closed and connected sets, and asymptotic to a half-line at small scales. (We will need the technical assumption that the Hausdorff density is $1/2$ at the origin, that is, the length in small balls is roughly the radius—which basically means that $K(t_0)$ admits a tangent, up to suitable rotations.) Our main result is as follows:

Theorem 1.1 *Assume that $K := K(t_0) \subset \Omega \subset \mathbb{R}^2$ is closed and connected, and let u be a solution for the minimizing problem in (1) with some α-hölderian coefficients $A : \Omega \to \mathcal{S}^{2\times 2}$. Assume that $x_0 \in K \cap \Omega$ is a point of density $1/2$, that is,*

$$\limsup_{r\to 0} \frac{\mathcal{H}^1(K \cap B(x_0, r))}{2r} = \frac{1}{2}$$

and that $A(x_0) = Id$. Then the limit

$$\lim_{r\to 0} \frac{1}{r} \int\limits_{B(x_0,r)\setminus K} (A\nabla u) \cdot \nabla u \, dx \tag{4}$$

exists and is finite. Moreover denoting C_0 the value of this limit, considering R_r a suitable family of rotations, and taking

$$v_0(r, \theta) := u(0) + \sqrt{\frac{2C_0 r}{\pi}} \sin(\theta/2), \quad (r, \theta) \in [0, 1] \times [-\pi, \pi],$$

then the blow up sequence $u_r := r^{-\frac{1}{2}} u(r R_r(x - x_0))$ converges to v_0 and ∇u_r converges to ∇v_0 both strongly in $L^2(B(0, 1))$ when $r \to 0$.

If $A(x_0) \neq Id$ we obtain a similar statement by applying the change of variable $x \mapsto \sqrt{A(x_0)}x$ (see Theorem 4.2). We also stress that a rigourous sense to the value of $u(0)$ has to be given, and this will be done in Lemma 4.1. Besides, the exact definition of the rotations R_r will be given in Remark 2.

Theorem 1.1 is a first step toward understanding the energy release rate for non-smooth fractures, and study qualitative properties of the crack path. It provides also the existence of a generalized stress intensity factor, that we can define as being the limit in (4), and which always exists without any regularity assumptions on $K(t_0)$ of that of being closed and connected (see Proposition 5).

Our main motivation is the study of brittle fracture, but of course Theorem 1.1 contains a general result about the regularity of solutions for a Neumann Problem in rough domains, that could be interesting for other purpose.

The proof of Theorem 1.1 will be done in two main steps, presented in Sects. 3 and 4, which will come just after some preliminaries (Sect. 2). The first step is to prove the existence of limit in (4). For this we will use the monotonicity argument of Bonnet [2], which was used to prove existence of blow up limits for the minimizers of the Mumford–Shah functional. We will adapt here the argument to more general energies as the one with coefficient $A(x)$, and also with a second member f. Notice that when $f = 0$ we need only K to be closed and connected, whereas when $f \neq 0$ we need furthermore that K is of finite length.

The second step is to prove the convergence strongly in L^2 of the blow-up limit $u_r := r^{-\frac{1}{2}} u(r R_r(x - x_0))$ and its gradient, to the function $C\sqrt{r}\sin(\theta/2)$. This is the purpose of

Theorem 4.2, and the existence of limit in (4) is the first step, because it implies that ∇u_r is bounded in $L^2(B(0,1))$ which helps us to extract subsequences.

Notice that Bonnet [2] already had a kind of blow-up convergence for u_r, analogue to ours in his paper on regularity for Mumford–Shah minimizers. The main difference with the result of Bonnet, is that here the set K is any given set whereas for Bonnet, K was a minimizer for the Mumford–Shah functional, which allowed him to modify it at his convenience to create competitors and prove some results on u. Here we cannot argue by the same way and this brings some interesting technical difficulties in the proof of convergence of u_r.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^2$, $K \subset \Omega$ be a closed and connected set satisfying $\mathcal{H}^1(K) < +\infty$ (here $\mathcal{H}^1$ denotes the one dimensional Hausdorff measure), $f \in L^\infty(\Omega)$, $\lambda \geq 0$ and $g \in H^1(\Omega) \cap L^\infty(\Omega)$. If K and K' are two closed sets of $\mathbb{R}^2$ we will denote the Hausdorff distance by

$$d_H(K, K') := \max\left(\sup_{x\in K} \mathrm{dist}(x, K'), \sup_{x\in K'} \mathrm{dist}(x, K)\right).$$

We also consider some α-Hölder regular coefficients $A : x \mapsto A(x) \in \mathcal{S}^{2\times 2}$, uniformly bounded and uniformly coercive (with constant γ). We will use the following series of notations

$$\|X\|_A^2 :={}^t XAX = (AX)\cdot X = \left\|\sqrt{A}X\right\|_{Id}^2 = \left\|\sqrt{A}X\right\|^2.$$

For simplicity we will assume without loss of generality that $\kappa = 1$. We consider a slight more general energy than the one in (1) with a second member f, namely

$$F(u) := \int_{\Omega\setminus K} \|\nabla u\|_A^2 dx + \frac{1}{\lambda}\int_{\Omega} |\lambda u - f|^2. \tag{5}$$

We will also allow the case $\lambda = 0$ and then we ask also $f = 0$ and F is simply

$$F(u) := \int_{\Omega\setminus K} \|\nabla u\|_A^2 dx.$$

We consider a minimizer u for F among all functions $v \in H^1(\Omega \setminus K)$ such that $v = g$ on $\partial\Omega$, i.e. u is a weak solution for the problem

$$\begin{cases} \lambda u - \mathrm{div} A\nabla u = f & \text{in } \Omega \setminus K \\ (A\nabla u)\cdot \nu = 0 & \text{on } K \\ u = g & \text{on } \partial\Omega \end{cases} \tag{6}$$

It is well known that such a minimizer exists and is unique (up to additional constant if necessary in connected components of $\Omega \setminus K$ when eventually $f = 0$), which provides a week solution for the problem (6).

We begin with some elementary geometrical facts.

Proposition 1 *Let $K \subset \mathbb{R}^2$ be a closed and connected set such that*

$$\limsup_{r\to 0} \frac{\mathcal{H}^1(K\cap B(x_0, r))}{2r} = \frac{1}{2}. \tag{7}$$

For all $r > 0$ small enough, let x_r be any chosen point in $K \cap \partial B(x_0, r)$. Then we have that

$$\lim_{r\to 0} \frac{1}{r} d_H(K \cap B(x_0, r), [x_r, x_0]) = 0. \tag{8}$$

Proof Since K is closed, connected and not reduced to one point (because of (7)) we have that $K \cap \partial B(x_0, r)$ is nonempty for all r small enough. Moreover since K is connected, there exists a simple connected curve $\Gamma_r \subset K$ that starts from x_0 and hits $\partial B(x_0, r)$ for the first time at some point $y_r \in K \cap \partial B(x_0, r)$. Since Γ_r is connected we have that $\mathcal{H}^1(\Gamma) \geq \mathcal{H}^1([y_r, x_0]) = r$ and using (7) we get $\mathcal{H}^1(\Gamma_r) \leq r + o(r)$. From the last two inequalities, since Γ_r is a connected curve, it is then very classical using Pythagoras inequality to prove that

$$d_H(\Gamma_r, [y_r, x_0]) = o(r). \tag{9}$$

Indeed, let z be the point in Γ_r of maximal distance to $[y_r, x_0]$, and let h be this distance. Now let w be a point at distance h to $[y_r, x_0]$, whose orthogonal projection onto $[y_r, x_0]$ is exactly the middle of $[y_r, x_0]$. Then the triangle (y_r, x_0, w) is isocel, and in particular minimizes the perimeter among all triangle of same basis and same height. Therefore,

$$2\sqrt{(r/2)^2 + h^2} = |w - y_r| + |w - x_0| \leq |z - y_r| + |z - x_0| \leq \mathcal{H}^1(\Gamma_r) \leq r + o(r)$$

which implies that $h = o(r)$ and proves (9).

Now (7) also implies that

$$\mathcal{H}^1(K \cap B(x_0, r) \setminus \Gamma_r) = o(r),$$

from which we deduce that

$$\sup\{\mathrm{dist}(x, \Gamma);\ x \in K \cap B(x_0, r)\} = o(r)$$

which implies $d_H(K \cap B(x_0, r), [y_r, x_0]) = o(r)$. Finally (8) follows from the fact that $\mathrm{dist}(x_r, y_r) = o(r)$ for any other point $x_r \in K \cap \partial B(x_0, r)$. □

Remark 1 The density condition (7) does not imply the existence of tangent at the origin. One of such example can be found in [5, Remark 2.7.], as being a curve with oscillating tangent at the origin: $\exp(-t^2)(\cos(t)\mathbf{e}_1 + \sin(t)\mathbf{e}_2)$, $t \in [0, +\infty]$.

Remark 2 (Definition of R_r) As noticed in the preceding remark, the existence of tangent, i.e. the existence of a limit for the sequence of rescaled set $\frac{1}{r}(K \cap B(x_0, r) - x_0)$, is not always guaranteed by the density condition. On the other hand if R_r denotes for each $r > 0$, the rotation that maps x_r on the negative part of the first axis, then $R_r(\frac{1}{r}(K \cap B(x_0, r) - x_0))$ converges to the segment $[-1, 0] \times \{0\}$. In the sequel, R_r will always refer to this rotation.

Remark 3 There exists some connected sets such that $\frac{1}{r_n} K \cap B(0, r_n)$ converges to some radius in $B(0, 1)$ for some sequence $r_n \to 0$, and such that $\frac{1}{t_n} K \cap B(0, t_n)$ converges to a diameter for another sequence $t_n \to 0$. Such a set can be constructed as follows. Take a sequence $q_n \to 0$ such that

$$q_{n+1}/q_n \longrightarrow 0 \tag{10}$$

The idea relies on the observation that thanks to (10), while looking at the scale of size q_n, that is, in the ball $B(0, q_n)$, all the piece of set contained in $B(0, q_{n+1})$ is negligible in terms of Hausdorff distance. Therefore we can build two subsequences, one at the scales q_{2n}, and the other one at the scales q_{2n+1}, that will not be seen by each other.

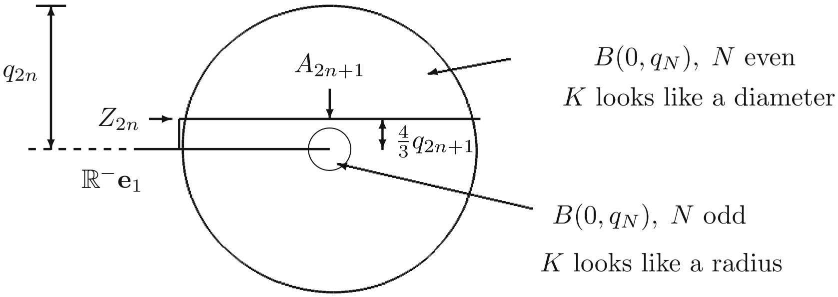

Fig. 1 A crack tip with two different limits along different subsequences

To do so, we consider the points $A_n := (0, \frac{4}{3}q_n)$ on the second axis of $\mathbb{R}^2$ and we define K, as being $\mathbb{R}^- \times \{0\}$ union of all horizontal diameters of $B(A_{2n+1}, q_{2n})$, that are connected to the first axis by their left extremities. In other words, denoting $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ the two unit canonical vectors of $\mathbb{R}^2$,

$$K := \mathbb{R}^-\mathbf{e}_1 \cup \left(\bigcup_{n\in\mathbb{N}^*} (\mathbb{R}\mathbf{e}_1 + A_{2n+1}) \cap B(A_{2n+1}, q_{2n}) \right) \cup \bigcup_{n\in\mathbb{N}^*} \left(\left[0, \frac{4}{3}q_{n+1}\right] \mathbf{e}_2 + Z_{2n} \right),$$

where Z_{2n} is the left extremity point of the segment $(\mathbb{R}e_1 + A_{2n+1}) \cap B(A_{2n+1}, q_{2n})$ (which is actually the horizontal diameter of $B(A_{2n+1}, q_{2n})$), see Fig. 1.

Then it is easy to see that, for the Hausdorff distance,

$$\frac{1}{q_{2n}} K \cap B(x, q_{2n}) \longrightarrow \mathbb{R}\mathbf{e}_1 \cap B(0, 1)$$
$$\text{and} \quad \frac{1}{q_{2n+1}} (K \cap U(x, q_{2n+1}) \longrightarrow \mathbb{R}^-\mathbf{e}_1 \cap B(0, 1)$$

as desired.

Remark 4 Notice that a consequence of Theorem 1.1 for the example exhibited in Remark 3 is the following curious fact: even if $\frac{1}{r}K \cap B(x, r)$ has no limit when $r \to 0$, the limit of $\frac{1}{r}\int_{B(0,r)} \|\nabla u\|^2$ as $r \to 0$ exists thus has same value C_0 for any subsequences of r. Now, since K has density $1/2$ along the odd subsequence $r_n = q_{2n+1}$, applying the proof of Theorem 1.1 for this subsequence we infer that the limit of the blow up sequence $r_n^{-1/2}u(r_n x)$ converges to $\sqrt{2C_0 r/\pi}\,\sin(\theta/2)$. But now regarding the limit in the even scales, $r_n = q_{2n}$, as K is converging to a diameter, a similar proof as the one used to prove Theorem 1.1 would imply that the blow up sequence is converging to the solution of a Neumann problem in a domain which is a ball, cut into two pieces by a diameter. This implies $C_0 = 0$ (because of the decomposition of u in spherical harmonics), so that actually returning to the odd subsequence, for which K is converging to a radius, we can conclude that $r_n^{-1/2}u(r_n x)$ must converge to 0 as well.

It is well known that any closed and connected set K is arcwise connected, namely for any $x \neq y$ in K one can find an injective Lipschitz curve inside K going from x to y (see e.g. [9, Proposition 30.14]). This allows us to talk about geodesic curve inside K, that connects x to y, which stands to be the curve with that property which support has minimal length.

Definition 2.1 We say that K is locally-chord-arc at x_0 if there exists a constant C and a radius r_0 such that for every $r \leq r_0$ and for any couple of points y and z lying on $K \cap \partial B(x_0, r)$ the geodesic curve inside K connecting y and z has length less than Cr.

Proposition 2 *Let $K \subset \mathbb{R}^2$ be a closed and connected set satisfying the density condition*

$$\limsup_{r\to 0} \frac{1}{2r}\mathcal{H}^1(K \cap B(x_0, r)) = \frac{1}{2}. \tag{11}$$

Then K is locally-chord-arc at x_0.

Proof The density condition (11) together with the fact that K is closed and connected guarantees that K is non reduced to one point, contains x_0, and that $\partial B(x_0, r) \cap K$ is nonempty for r small enough. Let $r_0 > 0$ be one of this radius small enough such that moreover

$$\mathcal{H}^1(K \cap B(x_0, r)) \leq \left(1 + \frac{1}{10}\right) r \quad \forall r \leq 3r_0. \tag{12}$$

Let now y and z be two points in $K \cap \partial B(x_0, r)$ for any $r \leq r_0$ and let $\Gamma \subset K$ be the geodesic curve connecting y and z. Then Γ is injective (by definition since it is a geodesic) and in addition we claim that $\Gamma \subset B(x_0, 3r)$. Indeed, otherwise there would be a point $x \in \Gamma \setminus B(x_0, 3r)$ which would imply $\mathcal{H}^1(\Gamma \cap B(x_0, 3r)) \geq 4r$ (because y and z are lying on $\partial B(x_0, r)$) and this contradicts (12). But now that $\Gamma \subset B(x_0, 3r)$, condition (12) again implies that $\mathcal{H}^1(\Gamma) \leq \mathcal{H}^1(K \cap B(x_0, 3r)) \leq 4r$ which proves the proposition. □

In the sequel we will need to know that a minimizer of F is bounded.

Proposition 3 *Let K be closed and connected, u be a minimizer for the functional F defined in* (5) *with $f \in L^\infty(\Omega)$ and $\lambda > 0$. Then*

$$\|u\|_\infty \leq \frac{1}{\min(1, \lambda)} \max(\|f\|_\infty, \|g\|_\infty).$$

Proof It suffice to fix $M := (\min(1, \lambda))^{-1} \max(\|f\|_\infty, \|g\|_\infty)$ and notice that the function

$$w := \max(-M, \min(u, M))$$

is a competitor for u and has less energy. By uniqueness of the minimizer we deduce that $u = w$. □

3 Bonnet's monotonicity Lemma and variants

In this section we prove the existence of the limit

$$\lim_{r\to 0} \frac{1}{r} \int_{\sqrt{A(0)}B(x,r)} \|\nabla u\|_A^2 dx,$$

for any $x \in \Omega$ when u is a minimizer of F. Of course when $x \in \Omega \setminus K$ this is clear by the interior regularity of solution for the Problem (6), and the value of the limit in this case is zero. Therefore it is enough to consider a point $x \in K$. The case of harmonic functions need no further assumptions on K than being just closed and connected, and the argument comes from [2] and [9]. Our aim is to consider the case of a non zero second member f and more general second order operator of divergence form.

We begin with some technical tools.

3.1 Technical tools

We will need the following 2 versions of the Gauss-Green formula.

Lemma 3.1 (Integration by parts, first version) *Let K be closed and connected, u be a minimizer for the functional F defined in (5). Then for any $x \in \Omega$ and for a.e. r such that $B(x,r) \subset \Omega$ it holds*

$$\int_{B(x,r)\setminus K} \|\nabla u\|_A^2 \, dy = \int_{B(x,r)\setminus K} (f - \lambda u)u \, dy + \int_{\partial B(x,r)\setminus K} u(A\nabla u)\cdot \nu \, d\mathcal{H}^1.$$

Proof If u is a minimizer, then comparing the energy of u with the one of $u + t\varphi$ and using a standard variational argument we get

$$\int_{\Omega\setminus K} (A\nabla u)\cdot \nabla\varphi \, dx = \int_{\Omega} (\lambda u - f)\varphi \, dx \tag{13}$$

must hold for any function $\varphi \in H^1(\Omega \setminus K)$ compactly supported inside Ω. Let us choose φ to be equal to $\psi_\varepsilon u(x)$, where $\psi_\varepsilon(x) = g_\varepsilon(\|x\|)$ is radial, and g_ε is equal to 1 on $[0, r-\varepsilon]$, equal to 0 on $[r+\varepsilon, +\infty[$ and linear on $[r-\varepsilon, r+\varepsilon]$. Applying (13) with $\varphi = \psi_\varepsilon u$ gives

$$\int_{\Omega\setminus K} (A\nabla u)\cdot u\nabla\psi_\varepsilon + \int_{\Omega\setminus K} (A\nabla u)\cdot \psi_\varepsilon \nabla u = \int_{\Omega} (\lambda u - f)\psi_\varepsilon u. \tag{14}$$

It is clear that ψ_ε converges to $\mathbf{1}_{B(x,r)}$ strongly in $L^2(\Omega)$, which gives the desired convergence for the second term and last term in (14). Now for the first term, we notice that ψ_ε is Lipschitz and its derivative is equal a.e. to $\frac{x}{2\varepsilon\|x\|}\mathbf{1}_{B(x,r+\varepsilon)\setminus B(x,r-\varepsilon)}$ so that

$$\int_{\Omega\setminus K} (A\nabla u)\cdot u\nabla\psi_\varepsilon = \frac{1}{2\varepsilon}\int_{(B(x,r+\varepsilon)\setminus B(x,r-\varepsilon))\setminus K} (A\nabla u)\cdot u\frac{x}{\|x\|}$$

which converges to $\int_{\partial B(x,r)\setminus K}(A\nabla u)\cdot u \;\nu \; d\mathcal{H}^1$ for a.e. r by Lebesgue's differentiation theorem applied to the L^1 function $r \mapsto \int_{\partial B(x,r)\setminus K}(A\nabla u)\cdot \nu \, d\mathcal{H}^1$. □

The first part of the next Lemma comes from a topological argument in [9] (see p. 299).

Lemma 3.2 (Integration by parts, second version) *Let $K \subset \Omega$ be closed and connected, $x \in K$ and $r_0 > 0$ be such that $B(x,r_0) \subset \Omega$. For all $r \in (0, r_0)$ we decompose $\partial B(x,r) \setminus K = \bigcup_{j\in J(r)} I_j(r)$ where $I_j(r)$ are disjoints arcs of circles. Then for each $j \in J(r)$ there exists a connected component $U_j(r)$ of $\Omega \setminus (I_j(r) \cup K)$ such that*

$$\partial U_j(r) \setminus K = I_j(r).$$

Moreover if u is a minimizer for the functional F defined in (5), then for a.e. $r \in (0, r_0)$ and for every $j \in I_j(r)$ we have

$$\int_{I_j(r)} (A\nabla u)\cdot \nu \, d\mathcal{H}^1 = \int_{U_j} (\lambda u - f)dx, \tag{15}$$

where ν is the inward normal vector in U_j, i.e. pointing inside U_j.

Proof We assume without loss of generality that $x = 0$. By assumption K is closed, so that $\partial B(0,r) \setminus K$ is a relatively open set in $\partial B(0,r)$ which is one dimensional. Therefore we can decompose $\partial B(0,r) \setminus K$ as a union of arc of circles as in the statement of the Lemma, namely

$$\partial B(0,r) \setminus K = \bigcup_{j\in J} I_j.$$

(we avoid the dependance in r to lighten the notations). Let us denote by U_j^+ the connected component of $\Omega \setminus (K \cup I_j)$ containing the points of $B(0,r)\setminus K$ very close to I_j, and similarly U_j^- is the one containing the points of $\Omega \setminus (K \cup B(0,r))$ very close to I_j. Then there is one between $U_j^\pm$, that we will denote by U_j, which satisfies

$$\partial U_j \setminus K = I_j. \tag{16}$$

The proof of (16) relies on the connectedness of K [see [9] pp. 299 and 300 for details: in our case the connectedness of K implies the topological assumption denoted by (8) in [9] that is used to prove (16) (which is actually (14) in [9])].

Then we want to prove (15) by an argument similar to Lemma 3.1 applied in U_j. For this purpose we consider as before a radial function but we need to separate two cases: if $U_j \subset B(0,r)$ then we take the same function $\psi_\varepsilon(x) = g_\varepsilon(\|x\|)$ where g_ε is equal to 1 on $[0, r-\varepsilon]$, equal to 0 on $[r+\varepsilon, +\infty[$ and linear on $[r-\varepsilon, r+\varepsilon]$. Now if $U_j \subset \Omega \setminus B(0,r)$ we define $\psi_\varepsilon := 1 - \psi_\varepsilon$.

Then we take as a competitor for u the function $u + t\varphi$, with $\varphi = \mathbf{1}_{\hat{U}_j}\psi_\varepsilon$, where $\hat{U}_j$ is the connected component of $\Omega \setminus K$ containing U_j. Notice that this is an admissible choice, namely $\varphi \in H^1(\Omega \setminus K)$ and $\varphi = 0$ on $\partial\Omega$.

Applying (13) with $\varphi = \mathbf{1}_{\hat{U}_j}\psi_\varepsilon$ gives

$$\int_{\hat{U}_j} (A\nabla u)\cdot \nabla\psi_\varepsilon = \int_{\hat{U}_j} (\lambda u - f)\psi_\varepsilon. \tag{17}$$

As in the proof of Lemma 3.1, it is clear that $\mathbf{1}_{\hat{U}_j}\psi_\varepsilon$ converges to $\mathbf{1}_{\hat{U}_j}$ strongly in $L^2(\Omega)$, which gives the desired convergence for the right hand side term in (17). Now for the left hand side term, we use as before that ψ_ε is Lipschitz and its derivative is equal a.e. to $\pm\frac{x}{2\varepsilon\|x\|}\mathbf{1}_{B(x,r+\varepsilon)\setminus B(x,r-\varepsilon)}$ (with the correct sign depending on which side of I_j lies U_j) so that

$$\int_{\Omega\setminus K} (A\nabla u)\cdot\nabla\psi_\varepsilon = \pm\frac{1}{2\varepsilon}\int_{\hat{U}_j\cap(B(x,r+\varepsilon)\setminus B(x,(1-\varepsilon)r))} (A\nabla u)\cdot\frac{x}{\|x\|}$$

which converges to $\int_{I_j}(A\nabla u)\cdot\nu d\mathcal{H}^1$ for a.e. r by Lebesgue's differentiation theorem applied to the L^1 function $r \mapsto \int_{\partial B(x,r)\cap\hat{U}_j}(A\nabla u)\cdot\nu \, d\mathcal{H}^1$. □

3.2 Monotonicity result

So we arrive now to the monotonicity results. The starting point is the following proposition, which is the key ingredient in Bonnet's proof of the classification of global minimizers for the Mumford–Shah functional [2] (see also Sect. 47 of Guy David's book [9] for a more detailled proof with slightly weaker assumptions than [2]). The same argument was also used in Lemma 2.2. of [16] to prove a monotonicity result for the energy of a harmonic function in the complement of minimal cones in $\mathbb{R}^3$. Notice also that a similar argument with the elastic energy (i.e. L^2 norm of the symmetric gradient) of a vectorial function $u : \Omega \to \mathbb{R}^2$ seems not to be working.

Proposition 4 ([2,9] Monotonicity of Energy, the harmonic case) *Let K be a closed and connected set and let u be a solution for the problem (6) with $A = Id$, $f = 0$ and $\lambda = 0$ (therefore u is harmonic in $\Omega \setminus K$). For any point $x_0 \in K$ we denote*

$$E(r) := \int_{B(x_0,r)\setminus K} \|\nabla u\|^2 dx.$$

Then $r \mapsto E(r)/r$ is an increasing function of r on $(0, r_0)$. As a consequence, the limit $\lim_{r\to 0} E(r)/r$ exists and is finite.

Remark 5 Notice that our statement is slightly different than the one in [9], where the assumption $\mathcal{H}^1(K) < \infty$ is needed whereas K is not necessarily connected. Here we do not suppose $\mathcal{H}^1(K) < \infty$ but we ask K to be connected which is a stronger topological assumption but weaker regularity assumption than [9] (see Remark 47.42 in [9] for the precise role of $\mathcal{H}^1(K) < \infty$). The result as stated in Proposition 4 follows as a particular case of Proposition 5 proved below.

Now we prove a variant of Bonnet's monotonicity Lemma, which will be the starting point for our main Theorem. We consider α-Hölder regular coefficients $A : x \mapsto A(x) \in \mathcal{S}^{2\times 2}$, uniformly bounded and γ-coercive with $\gamma > 0$. For any $x \in \Omega$ and $r > 0$ we define the ellipsoid

$$B_A(x, r) := \sqrt{A(x)}(B(x, r)).$$

Remark 6 (Change of variable) Let u be a solution for Problem 6 in Ω, that we assume to contain the origin, and fix $A_0 := A(0)$. Then $u \circ \sqrt{A_0}$ is the solution of a similar problem in $(\sqrt{A_0})^{-1}(\Omega \setminus K)$ with coefficient $\tilde{A} := (\sqrt{A_0})^{-1} \circ A \circ (\sqrt{A_0})^{-1}$ instead of A. In particular $\tilde{A}(0) = Id$, and

$$\int_{B(0,r)} \|\nabla v\|^2_{\tilde{A}} dx = \int_{B_A(0,r))} \|\nabla u\|^2_A \det\left(\sqrt{A_0}\right)^{-1} dx \tag{18}$$

We will need the following Lemma of Gronwall type. The proof is given in the appendix.

Lemma 3.3 (Gronwall) *Assume that $E(r)$ admits a derivative a.e. on $[0, r_0]$, is absolutely continuous, and satisfies the following inequality for some $\alpha \in (0, 1)$*

$$E(r) \leq \left(r + Cr^{1+\alpha}\right) E'(r) + CN(r)r^2, \quad \forall r \in [0, r_0], \tag{19}$$

with N integrable on $(0, r_0)$. Then the limit $\lim_{r\to 0} E(r)/r$ exists and is finite. Moreover if $N = 0$ then

$$r \mapsto \frac{E(r)}{r}\left(1 + Cr^{\alpha}\right)^{\frac{1}{\alpha}}$$

is nondecreasing.

We now prove a variant of Proposition 4 in the context of general coefficients and second member.

Proposition 5 (Energy estimate for general coefficients and second member) *Let u be a solution for the problem (6) with α-Hölderian coefficients A, $\lambda > 0$ and $f, g \in L^{\infty}$, and assume that K is a closed and connected set of finite length. For any point $x_0 \in K$ we denote*

$$E(r) := \int_{B_A(x_0,r)\setminus K} \|\nabla u\|_A^2 dx.$$

We assume in addition that K is locally-chord-arc at point x_0. Then the limit $\lim_{r\to 0}(E(r)/r)$ exists and is finite.

Moreover if $f = 0$ and $\lambda = 0$, then the assumptions on K of being locally-chord-arc and of finite length can be both removed, and in this case the function

$$r \mapsto \frac{E(r)}{r}\left(1 + Cr^{\frac{\alpha}{2}}\right)^{\frac{2}{\alpha}}$$

is nondecreasing, where $C > 0$ is a constant which is equal to 0 when $A = Id$ (the harmonic case).

Proof We follow the proof of Bonnet. We begin with the end of the statement, namely we assume first that K is only closed and connected, and that $\lambda = 0$ and $f = 0$. Let us also assume without loss of generality that x_0 is the origin. Up to the change of coordinates $x \mapsto \sqrt{A(0)}x$ and thank to Remark 6, we can furthermore assume that $A(0) = Id$. In this case $B_A(x,r) = B_{Id}(x,r) = B(x,r)$.

The Gauss-Green formula (Lemma 3.1) applied in $B(0,r)$ yields

$$\int_{B(0,r)\setminus K} \|\nabla u\|_A^2 dx = \sum_j \int_{I_j} u(A\nabla u)\cdot \nu \, d\mathcal{H}^1, \tag{20}$$

where $\partial B(0,r) \setminus K = \cup_j I_j$. On the other hand Lemma 3.2 gives for each j,

$$\int_{I_j} (A\nabla u)\cdot \nu \, dx = 0.$$

Denoting by m_j the average of u on I_j we deduce that

$$\int_{I_j} u(A\nabla u)\cdot \nu \, d\mathcal{H}^1 = \int_{I_j} (u - m_j)(A\nabla u)\cdot \nu \, d\mathcal{H}^1. \tag{21}$$

Thus

$$\int_{B(0,r)\setminus K} \|\nabla u\|_A^2 dx \le \sum_{j=1}^{N} \int_{I_j} |u - m_j||(A\nabla u)\cdot \nu| \, d\mathcal{H}^1. \tag{22}$$

Then by use of Cauchy-Schwarz inequality, $ab \leq \frac{1}{4r}a^2 + rb^2$, and Wirtinger we can write

$$\begin{aligned}\int_{I_j} |u - m_j||(A\nabla u)\cdot \nu| d\mathcal{H}^1 &\leq \left(\int_{I_j} |u-m_j|^2\right)^{\frac{1}{2}} \left(\int_{I_j} |(A\nabla u)\cdot \nu|^2\right)^{\frac{1}{2}} \\ &\leq \frac{1}{4r}\int_{I_j} |u-m_j|^2 + r\int_{I_j} |(A\nabla u)\cdot \nu|^2 \\ &\leq r\int_{I_j} |\nabla u\cdot \tau|^2 + r\int_{I_j} |(A\nabla u)\cdot \nu|^2. \qquad (23)\end{aligned}$$

Now we want to recover the full norm $\|\nabla u\|_A$ from the partial norms $|\nabla u \cdot \tau|$ and $|(A\nabla u)\cdot \nu|$. For this purpose we write

$$|\nabla u\cdot \tau|^2 = |(A\nabla u)\cdot \tau|^2 + |(Id - A)\nabla u \cdot \tau|^2 + [2(A\nabla u)\cdot \tau][((Id-A)\nabla u)\cdot \tau],$$

and we notice that, by Hölder regularity of A and γ-coerciveness we have (the constant C can vary from line to line)

$$\begin{aligned}|(Id-A)\nabla u\cdot \tau|^2 &\leq \|(Id-A)\nabla u\|^2 \\ &\leq \|Id - A\|^2_{L^\infty(B(0,r),\mathbb{R}^2)} \|\nabla u\|^2 \\ &\leq Cr^{2\alpha}\|\nabla u\|^2 \\ &\leq \gamma C r^{2\alpha}\|\nabla u\|^2_A\ , \qquad (24)\end{aligned}$$

and

$$\begin{aligned}2|A\nabla u\cdot \tau(Id-A)\nabla u\cdot \tau| &= 2|A\nabla u\cdot \tau||(Id-A)\nabla u\cdot \tau| \\ &\leq 2\|A\nabla u\| C r^\alpha \|\nabla u\|_A \\ &\leq Cr^\alpha \|A\|_\infty \|\nabla u\|^2_A.\end{aligned}$$

Therefore summing over j and putting all the estimates together we have proved that for r small enough,

$$\begin{aligned}\int_{B(0,r)\setminus K} \|\nabla u\|^2_A dx &\leq r\int_{\partial B(0,r)} \left(|A\nabla u\cdot \tau|^2 + Cr^\alpha \|\nabla u\|^2_A\right) + r\int_{\partial B(0,r)} |(A\nabla u)\cdot \nu|^2 \\ &= r\int_{\partial B(0,r)} \|A\nabla u\|^2 + Cr^{1+\alpha}\int_{\partial B(0,r)} \|\nabla u\|^2_A. \qquad (25)\end{aligned}$$

By Hölder regularity of A we infer that

$$\left\|\sqrt{A}\right\|^2_{L^\infty(B(0,r),\mathbb{R}^2)} \leq 1 + Cr^{\alpha/2},$$

which implies

$$\begin{aligned}\int_{\partial B(0,r)} \|A\nabla u\|^2 &\leq \int_{\partial B(0,r)} \left\|\sqrt{A}\right\|^2 \|\sqrt{A}\nabla u\|^2 \\ &\leq (1 + Cr^{\alpha/2}) \int_{\partial B(0,r)} \left\|\sqrt{A}\nabla u\right\|^2 = \left(1 + Cr^{\alpha/2}\right)\int_{\partial B(0,r)} \|\nabla u\|^2_A\,.\end{aligned}$$

Therefore, since $E'(r) = \int_{\partial B(0,r)} \|\nabla u\|_A^2$ we have proved for r small enough,

$$E(r) \leq \left(r + Cr^{1+\alpha/2}\right) E'(r), \tag{26}$$

and we conclude with Lemma 3.3, applied with the exponent $\alpha/2 \in (0, 1)$.

Next we assume that $\lambda \neq 0$ and $f \in L^\infty$. Furthermore we now assume that K is locally-chord-arc at the origin and K is of finite length. If we reproduce the above argument, one sees that the second member f is just a perturbation under control which does not affect the limit of $E(r)/r$. Precisely, this time we will prove the inequality

$$E(r) \leq \left(r + Cr^{1+\alpha/2}\right) E'(r) + CN(r)r^2 \quad \text{for a.e. } r \leq r_0, \tag{27}$$

with $N(r) \in L^1([0, r_0])$. This implies the proposition thank to Lemma 3.3. The exact definition of N is given by

$$N(r) = \sharp K \cap \partial B(0, r),$$

which is known to be finite for a.e. $r \in (0, r_0)$ due to the fact that K has a finite length. Actually by [9, Lemma 26.1.] we know that N is Borel mesurable on $(0, r_0)$ and that

$$\int_0^t N(s)ds \leq \mathcal{H}^1(K \cap B(0, t)). \tag{28}$$

This will be needed later. For now, take a radius r a.e. in $(0, r_0)$ such that $N(r) < +\infty$ and decompose $S_r := \partial B(0, r) \setminus K$ into a finite number of arcs of circle denoted I_j, for $j = 1..N(r)$. Moreover since K is closed and connected, for each j there exists a geodesic curve $F_j \subset K$ connecting the two endpoints of I_j. We denote D_j the domain delimited by I_j and F_j. Since K is locally-chord-arc at the origin we infer that $|D_j| \leq Cr^2$. Notice also that D_j corresponds to the set U_j of Lemma 3.2.

The Gauss-Green formula (Lemma 3.1) applied in $B(0, r)$ yields

$$\int_{B(0,r)\setminus K} \|\nabla u\|_A^2 dx = \int_{B(0,r)\setminus K} (f - \lambda u)\, u\, dx + \sum_{i=1}^{N(r)} \int_{I_j} u(A\nabla u) \cdot \nu \, d\mathcal{H}^1, \tag{29}$$

and applied in D_j (Lemma 3.2) gives

$$\int_{I_j} u(A\nabla u) \cdot \nu \, d\mathcal{H}^1 = \pm \int_{D_j} (f - \lambda u) dx,$$

the sign depending on the relative position of D_j with respect to $\partial B(0, r)$. Denoting by m_j the average of u on I_j we deduce that

$$\int_{I_j} u(A\nabla u) \cdot \nu \, d\mathcal{H}^1 = \int_{I_j} (u - m_j)(A\nabla u) \cdot \nu \, d\mathcal{H}^1 \pm \int_{D_j} m_j\, (f - \lambda u)\, dx. \tag{30}$$

Now since u is bounded it comes $|m_j| \leq C$, and we also have $\sum_{j=1}^{N(r)} |D_j| \leq CN(r)r^2$. Moreover f is also bounded thus returning to (29) and plugging (30) we get

$$\int_{B(0,r)\setminus K} \|\nabla u\|^2 dx \leq CN(r)r^2 + \sum_{j=1}^{N(r)} \int_{I_j} |u - m_j| \Big|(A\nabla u) \cdot \nu \Big| d\mathcal{H}^1. \tag{31}$$

Then the same computations as for proving (26) (i.e. using Cauchy-Schwarz inequality, $ab \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$, Wirtinger and estimating the rest by the Hölder regularity of A), we obtain

$$\int_{I_j} |u - m_j| \left|\frac{\partial u}{\partial \nu}\right| d\mathcal{H}^1 \leq \left(r + Cr^{1+\alpha/2}\right) \int_{I_j} \|\nabla u\|_A^2 dx \ ,$$

and after summing over j, (27) is proved, as claimed, and we conclude using Lemma 3.3. □

Remark 7 A careful look at the proof of Proposition 5 and Lemma 3.3 would also provide monotonicity results when $\lambda \neq 0$. The particular case when $A = Id$ is simpler, and yields the monotonicity of

$$r \mapsto \frac{E(r)}{r} + CP(r),$$

where $P(r)$ is a primitive of $N(r)$ and $C > 0$ is a constant. The exact monotonicity formula when $A \neq Id$ is more complicate but could be easily derived from our estimates.

4 Blow up

Here we prove the second part of Theorem 1.1 concerning the blow up sequence. Before going on with blow up limits at the origin, we start with a rigorous definition of $u(0)$. Indeed, let u be a solution for the problem (6) with $g \in L^\infty$, ($\lambda = f = 0$) or ($f \in L^\infty$ and $\lambda > 0$). We suppose that K is a closed and connected set satisfying the density condition (7) at 0. Let R_r the family of rotations given by remark 2 so that $r^{-1}R_r(K \cap B(0, r))$ converges to the segment $[-1, 0] \times \{0\}$ when r goes to 0. For any r small enough we define $D_r := R_r^{-1}(B((r/2, 0), r/4))$ and

$$m_r := \frac{1}{|D_r|} \int_{D_r} u(x)dx.$$

Lemma 4.1 (Definition of $u(0)$) *The sequence m_r converges to some finite number that we will denote by $u(0)$.*

Proof We begin with a discrete sequence $r_n := 2^{-n}r_0$ for some r_0 small, $n \in \mathbb{N}$. In particular we assume r_0 small enough to have

$$\frac{1}{r_n} \int_{B(0,r_n)} \|\nabla u\|^2 dx \leq C \quad \forall n \in \mathbb{N}, \tag{32}$$

for some constant C that surely exists thank to Sect. 3. Since $r^{-1}R_r(K \cap B(0, r))$ converges to the segment $[-1, 0] \times \{0\}$, we are sure that for r_0 small enough and for every n, the ball $B_n := R_{r_n}^{-1}(B(r_n/2, 0), 3r_n/8)$ does not meet K and contains both D_{r_n} and $D_{r_{n+1}}$. We denote by m_n the average of u on B_n. Applying Poincaré inequality in B_n yields

$$|m_{r_n} - m_n| = \left|\frac{1}{|D_{r_n}|} \int_{D_{r_n}} (u - m_n)dx\right| \leq \frac{1}{|D_{r_n}|} \int_{B_n} |u - m_n| \leq C\frac{1}{r_n} \int_{B_n} \|\nabla u\| dx$$

and the same for $m_{r_{n+1}}$ so that at the end

$$|m_{r_n} - m_{r_{n+1}}| \leq C\frac{1}{r_n}\int_{B_n} \|\nabla u\| dx \leq C\left(\int_{B_n} \|\nabla u\|^2 dx\right)^{\frac{1}{2}} \leq C r_n^{1/2} \tag{33}$$

because of (32). In particular this implies that m_{r_n} is a Cauchy sequence, thus converges to some limit $\ell \in \mathbb{R}$. Now if r_k is any other sequence converging to zero, we claim that the limit of m_{r_k} is still equal to ℓ. To see this it suffice to find a subsequence r_{n_k} of r_n such that $r_{n_k}/2 \leq r_k \leq r_{n_k}$ and compare m_{r_k} with $m_{r_{n_k}}$ by the same way as we obtained (33) and conclude that they must have same limit. □

Remark 8 In the future it will be convenient to introduce another type of averages on circles, namely

$$\tilde{m}_r := \frac{1}{\tilde{D}_r}\int_{\tilde{D}_r} u \, d\mathcal{H}^1,$$

with

$$\tilde{D}_r := B\left((r,0), \frac{r}{4}\right) \cap \partial B(0,r)$$

It is easily checked that the sequence of $\tilde{m}_r$ are aslo converging to $u(0)$, i.e. has same limit as m_r.

We are now ready to prove the last part of Theorem 1.1.

Theorem 4.2 (Convergence of the blow-up sequence) *Let u be a solution for the problem (6) with $g \in L^\infty$, ($\lambda = f = 0$) or ($f \in L^\infty$ and $\lambda > 0$). We suppose that K is a closed and connected set satisfying the density condition (7) at the origin. We denote by $u(0)$ the real number given by Lemma (4.1). Let R_r the family of rotations given by Remark 2 so that $r^{-1}R_r(K \cap B(0,r))$ converges to $\Sigma_0 = [-1,0] \times \{0\}$ when r goes to 0. If (r,θ) are the polar coordinates such that $(\sqrt{A(0)})^{-1}(\Sigma_0) = (\mathbb{R}^- \times \{\pi\})$ we denote by v_0 the function defined in polar coordinates by*

$$v_0(r,\theta) := \sqrt{\frac{2C_0 r}{\pi}} \sin(\theta/2).$$

Then

$$u_r := r^{-\frac{1}{2}}\left(u\left(rR_r^{-1}x\right) - u(0)\right) \underset{r\to 0}{\longrightarrow} v_0 \circ \sqrt{A(0)},$$

where the constant C_0 is given by

$$C_0 = \lim_{r\to 0}\left(\frac{1}{\det(\sqrt{A(0)})r}\int_{B_A(0,r)\setminus K} \|\nabla u\|_A^2 dx\right),$$

and the convergence holds strongly in $L^2(B(0,1))$ for both u_r and ∇u_r.

Proof We know that $K_r := \frac{1}{r}R_r(K)$ converges to the half-line $\mathbb{R}^- \times \{0\}$ locally in $\mathbb{R}^2$ for the Hausdorff distance. To simplify the notations and without loss of generality, in the sequel we will identify u with $u \circ R_r^{-1}$, and K with $R_r(K)$ so that we can assume that $R_r = Id$ for all r. We can also assume that $u(0) = 0$ and as before, it is enough to consider the case when $A(0) = Id$ because the general case follows using the change of variable of Proposition 6.

As in the proof of Lemma 4.1, for any r we denote by m_r the average of u on the ball $B((r/2, 0), r/4)$. Thank to Lemmaé1 4.1, the function $x \mapsto r^{-\frac{1}{2}}(u(rx) - m_r)$ has same limit as u_r, thus we will now use a new definition of u_r, being $u_r := r^{-\frac{1}{2}}(u(rx) - m_r)$. The function u_r is defined in $\frac{1}{r}(\Omega \backslash K)$. The domain $\frac{1}{r}(\Omega \backslash K)$ converges to $\mathbb{R}^2 \setminus K_0$ with $K_0 := \mathbb{R}^- \times \{0\}$.

We will prove that u_r converges, in some sense that will be given later, to function in $\mathbb{R}^2 \backslash K_0$ that satisfies a certain Neumann problem. In the sequel we will work up to subsequences, but this will not be restrictive in the end by uniqueness of the limit.

The starting point is that ∇u_r is uniformly bounded in $L^2(B(0, 2))$ (we start working in $B(0, 2)$ for security but the real interesting ball will be $B(0, 1)$). Indeed, $\nabla u_r(x) = \sqrt{r} \nabla u(rx)$,

$$\int_{B(0,2)\backslash K_r} \|\nabla u_r\|^2 dx = \int_{B(0,2)\backslash K_r} r \|\nabla u(rx)\|^2 dx = \frac{1}{r} \int_{B(0,2r)\backslash K} \|\nabla u(x)\|^2 dx.$$

From Proposition 4, we know that $\frac{1}{r} \int_{B(0,r)} \|\nabla u(x)\|_A^2 dx$ converges to C_0 and we deduce (using the coerciveness of A), that ∇u_r is uniformly bounded in $L^2(B(0, 2))$.

Therefore we can extract a subsequence such that ∇u_r converges to some h, weakly in $L^2(B(0, 2))$, and

$$\int_{B(0,1)} \|h\|^2 \leq \liminf_{r \to 0} \int_{B(0,1)} \|\nabla u_r\|^2 dx \leq C. \tag{34}$$

Next we want to prove that in compact sets of $B(0, 2) \setminus K_0$, the convergence is much better. For this purpose we introduce for any $a > 0$

$$U(a) := \{x \in B(0, 2); d(x, K_0) > a\}.$$

The sequence u_r is uniformly bounded in $H^1(U(a))$ for any a. Therefore taking a sequence $a_n \to 0$, extracting some subsequence of u_r and using a diagonal argument we can find a subsequence of u_r, not relabeled, that converges weakly in H^1 and strongly in L^2 in any of the domains $U(a)$. In other words, this subsequence u_r converges weakly in $H^1_{loc}(B(0, 2) \setminus K_0)$ and strongly in $L^2_{loc}(B(0, 2), \setminus K_0)$ to some function $u_0 \in H^1_{loc}(B(0, 2) \setminus K_0)$. By uniqueness of the limit we must have that $\nabla u_0 = h$ a.e. in $B(0, 2)$ and therefore (34) reads

$$\int_{B(0,1)} \|\nabla u_0\|^2 \leq \liminf_{r \to 0} \int_{B(0,1)} \|\nabla u_r\|^2 dx \leq C. \tag{35}$$

Now we want to prove that u_0 is a minimizer for the Dirichlet energy, and at the same time prove that the convergence hold strongly in $L^2(B(0, 1))$ both for u_r and ∇u_r. To do this we consider any function $v \in H^1(B(0, 1) \setminus K_0)$ with $v \equiv u_0$ in $B(0, 1) \setminus B(0, 1 - \delta)$ and $v \equiv 0$ in $B(0, \eta)$, for some small $\delta > 0$. The family of all such functions v is dense in the space of functions of $H^1(B(0, 1) \setminus K_0)$ with trace equal to u_0 on $\partial B(0, 1) \setminus K_0$ and therefore to prove that u_0 is a minimizer, it is enough to prove that

$$\int_{B(0,1)} \|\nabla u_0\|^2 dx \leq \int_{B(0,1)} \|\nabla v\|^2 dx$$

for all such functions v.

We denote by $N_r(s)$ the number of points of $K_r \cap \partial B(0,s)$. As already used before, since by assumption $\mathcal{H}^1(K_r \cap B(0,1))$ converges to 1 and

$$1 \leq \int_0^1 N_r(s) ds \leq \mathcal{H}^1(K_r \cap B(0,1)),$$

we can extract a subsequence such that $N_r(s) \to 1$ a.e. Then Fatou's lemma yields

$$\int_0^1 \liminf_r \int_{\partial B_s} \|\nabla u_r\|^2 d\mathcal{H}^1\, ds \;\leq\; \liminf_r \int_0^1 \int_{\partial B_s} \|\nabla u_r\|^2 d\mathcal{H}^1\, ds = C_1, \tag{36}$$

where C_1 is closely related to C_0. This will allows us later to find a good radius s for which both $N(s) = 1$ and $\int_{\partial B_s} \|\nabla u_r\|^2 d\mathcal{H}^1$ is uniformly bounded.

At this stage we only know that ∇u_r converges weakly in L^2 to ∇u_0. On the other hand, up to a further subsequence, we can find a measure μ such that $|\nabla u_r|^2 dx$ weakly-$\star$ converges to μ. Let $x \in B(0,2)$, $\rho > 0$ such that $\overline{B(x,\rho)} \subset B(0,2) \setminus K_0$. Let ψ be a smooth cutoff, with support in $B(x,\rho)$, and equal to 1 in $B(x,\rho/2)$. Then we can write that

$$\int_{B(x,\rho)} (A_r \nabla u_r) \cdot \nabla(\psi(u_r - u_0)) + r^2 \lambda u_r (u_r - u_0)\psi - r^{3/2} f_r (u_r - u_0)\psi = 0 \tag{37}$$

where $A_r(x) = A(rx)$, $f_r(x) = f(rx) - \lambda m_r$, and (taking the limit in the "first" u_r while freezing the test function $(u_r - u_0)\psi$, and using the weak convergence in $H^1(B(x,\rho)$ of u_r to u_0):

$$\int_{B(x,\rho)} (\nabla u_0) \cdot \nabla(\psi(u_r - u_0)) \;=\; 0. \tag{38}$$

Taking the difference of (37) and (38), and using the fact that $u_r \to u_0$ strongly in $L^2(B_r)$, ∇u_r is uniformly bounded in $L^2(B_r)^2$, and $A_r \to Id$ uniformly, we obtain that

$$\lim_{r \to 0} \int_{B(x,\rho/2)} \|\nabla u_r - \nabla u_0\|^2\, dx \;=\; 0$$

so that clearly, $\mu \llcorner (B(0,2) \setminus K_0) = \|\nabla u_0\|^2\, dx$: if μ has a singular part it must be concentrated on K_0. Moreover, we have $\mu(\{(-s,0)\}) = 0$ for all $s \in [0,2)$ but a countable number. (Observe that using any other test function in (37) and passing to the limit, we easily deduce that u_0 is harmonic in $B(0,2) \setminus K_0$, but this will also be a consequence of the minimality of the Dirichlet energy which will soon be shown).

Now from (36) we may choose s, $1 - \delta < s < 1$, so that $\mu(\{-s,0\}) = 0$, $N_r(s) = 1$ for all r large enough, and

$$\liminf_r \int_{\partial B_s} \|\nabla u_r\|^2 d\mathcal{H}^1 \;<\; +\infty$$

In particular, upon extracting a further subsequence, we may assume that

$$\sup_r \int_{\partial B_s} \|\nabla u_r\|^2 d\mathcal{H}^1 \ < \ +\infty.$$

Then, by Sobolev's embedding, and using the fact that the averages $\tilde{m}_r$ are uniformly bounded (see Remark 8), we deduce that there exists $C > 0$ such that

$$\|u_r\|_{L^\infty(\partial B_s)} \ \leq \ C.$$

We now consider any constant $M > C$ and define

$$u_r^M \ = \ (-M \vee (u_r \wedge M))$$

we have that $u_r^M \to u_0^M$ in $L^2_{loc}(B(0,1) \setminus K_0)$, where u_0^M is naturally defined as being $u_0^M := (-M \vee (u_0 \wedge M))$. Up to a subsequence the convergence holds almost everywhere. But now, since the functions are uniformly bounded, it converges also strongly in $L^2(B(0,1) \setminus K_0)$.

Now, from the original function $v \in H^1(B(0,1) \setminus K_0)$, we want to construct a function $v_r \in H^1(B(0,1) \setminus K_r)$ not much different from v. We denote by $C_r^\pm$ the connected components of $(B(0,1) \setminus K_r) \cap \{x \leq 0\}$ containing $(-1/2, \pm 1/2)$ and we define $v_r(x,y)$ as follows. In $B(0,1) \cap \{x > 0\}$ we set $v_r(x,y) = v(x,y)$.

$$\text{In } C_r^+,\, v_r(x,y) = \begin{cases} v(x,y) & \text{if } y \geq 0 \\ v(x,-y) & \text{otherwise.} \end{cases}$$

$$\text{In } C_r^-,\, v_r(x,y) = \begin{cases} v(x,y) & \text{if } y \leq 0 \\ v(x,-y) & \text{otherwise.} \end{cases}$$

And finally $v_r = 0$ everywhere else (i.e. in $B(0,1) \cap \{x \leq 0\} \setminus (C_r^+ \cap C_r^-)$). Then it is easy to see that $v_r \in H^1(B(0,1) \setminus K_r)$, converges strongly to v in L^2 and $\mathbf{1}_{B(0,1)\setminus K_r} \nabla v_r$ converges strongly to $\mathbf{1}_{B(0,1)\setminus K_0} \nabla v$ in $L^2(B(0,1))$. However, by this procedure the trace on $\partial B(0,1)$ is not necessarily preserved.

To get rid of that we let $\varepsilon < s - (1-\delta)$, we pick a smooth cut-off ψ_ε with compact support in B_s, $0 \leq \psi_\varepsilon \leq 1$ and $\psi_\varepsilon \equiv 1$ in $B_{s-\varepsilon}$, and we let

$$v_r^\varepsilon \ = \ \psi_\varepsilon v_r + (1-\psi_\varepsilon) u_r^M$$

which converges strongly in L^2 to $\psi_\varepsilon v + (1-\psi_\varepsilon) u_0^M$ as $r \to 0$. Next we write, since $v_r^\varepsilon = u_r^M = u_r$ on ∂B_s and u_r is a minimizer,

$$\int_{B_s} (A_r \nabla u_r) \cdot \nabla u_r + \lambda r^2 u_r^2 - 2r^{3/2} f_r u_r \, dx \leq \int_{B_s} (A_r \nabla v_r^\varepsilon) \cdot \nabla v_r^\varepsilon + \lambda r^2 (v_r^\varepsilon)^2 - 2r^{3/2} f_r v_r^\varepsilon \, dx.$$

Recall that $|f_r| \leq C$ and $|u_r| \leq C/\sqrt{r}$ (by definition) so that $2r^{3/2} f_r u_r = o(r)$, and we also easily check that

$$\int_{B_s} \lambda r^2 (v_r^\varepsilon)^2 - 2r^{3/2} f_r v_r^\varepsilon \, dx \ = \ o(r)$$

hence we focus on the other terms: we write for $\delta > 0$ small,

$$\begin{aligned}\int_{B_s} & (A_r \nabla u_r) \cdot \nabla u_r \, dx \\ & \leq (1+\eta) \int_{B_s} \psi_\varepsilon^2 (A_r \nabla v_r) \cdot \nabla v_r \, dx + C'/\eta \int_{B_s} \|\nabla \psi_\varepsilon\|^2 |v_r - u_r^M|^2 \, dx \\ & \quad + C'/\eta \int_{B_s} (1-\psi_\varepsilon)^2 \|\nabla u_r\|^2 \, dx + o(r)\end{aligned}$$

Then sending $r \to 0$ we obtain

$$\begin{aligned}\int_{B_s} \|\nabla u_0\|^2 \leq (1+\eta) \int_{B_s} \psi_\varepsilon^2 \|\nabla v\|^2 \, dx \\ + C'/\eta \int_{B_s} \|\nabla \psi_\varepsilon\|^2 |v - u_0^M|^2 \, dx + C'/\eta \mu(B_\varepsilon(-s, 0))\end{aligned}$$

but on the support of $\nabla \psi$, $v - u_0^M$ is equal to $(1-\psi_\varepsilon)(u_0^M - u_0)$. Therefore letting M tend to $+\infty$ we get

$$\int_{B_s} \|\nabla u_0\|^2 \leq (1+\eta) \int_{B_s} \psi_\varepsilon^2 \|\nabla v\|^2 \, dx + C'/\eta \mu(B_\varepsilon(-s, 0))$$

finaly letting $\varepsilon \to 0$, then $\eta \to 0$, and adding the integral over $B(0,1) \setminus B(0,s)$ on both sides (where v and u_0 actually coincide) we get the desired inequality, namely

$$\int_{B(0,1)} \|\nabla u_0\|^2 dx \leq \int_{B(0,1)} \|\nabla v\|^2 \, dx$$

which proves that u_0 is a minimizer. Moreover taking the particular choice $v = u_0$ in the same argument as before would give

$$\limsup_{r \to 0} \int_{B(0,1)} \|\nabla u_r\|^2 dx \leq \int_{B(0,1)} \|\nabla u_0\|^2 dx$$

and this toghether with (35), implies the convergence of norms, which by the weak convergence yields the strong convergence in L^2 for the gradients, as desired.

Finally all that we did in $B(0,1)$ could be done in any $B(0,R)$ for R as large as we want, which gives a definition of u_0 in $\mathbb{R}^2 \setminus K_0$. Moreover u_0 is of constant normalized energy. In other words we claim that $s \mapsto \frac{1}{s} \int_{B(0,s)} \|\nabla u_0\|^2$ is constant in s, identically equal to C_0. Indeed, by the strong convergence in L^2 of ∇u_r, the value of $\frac{1}{s} \int_{B(0,s)} \|\nabla u_0\|^2$ is given by

$$\lim_{r \to 0} \frac{1}{s} \int_{B(0,s)} \|\nabla u_r\|^2,$$

which we actually claim to be equal to C_0: a change of variable gives

$$\int_{B(0,s)} \|\nabla u_r\|^2 = \frac{1}{r}\int_{B(0,rs)} \|\nabla u\|^2$$
$$= \frac{1}{r}\int_{B(0,rs)} \|\nabla u\|_A^2 + \frac{1}{r}\int_{B(0,rs)} \langle (Id-A)\nabla u, \nabla u\rangle. \tag{39}$$

The first term in (39) converges to sC_0 and the second term converges to zero because less than $\|Id-A\|_{L^\infty(B(0,sr))}$ times something bounded.

The latter implies that u_0 is the cracktip function. More precisely, we claim now that

$$u_0 = \sqrt{\frac{2C_0 r}{\pi}}\sin(\theta/2). \tag{40}$$

We shall give two different arguments for (40). The first one is very nice and due to Bonnet: returning to the proof of the monotonicity Lemma applied to u_0, which says that $s \mapsto \frac{1}{s}\int_{B(0,s)} \|\nabla u_0\|^2$ must be increasing (Proposition 4), since $s \mapsto \frac{1}{s}\int_{B(0,s)} \|\nabla u_0\|^2$ is actually constant in s, all the inequalities in the proof are equalities. In particular u_0 must be the optimal function in Wirtinger inequality, thus it is the famous $C\sqrt{r}\sin(\theta/2)$ function.

The second argument is to decompose u_0 in spherical harmonics, i.e. as a sum of homogeneous harmonic functions in the complement of the half line K_0, which Neumann boundary conditions on K_0. Now using that $s \mapsto \frac{1}{s}\int_{B(0,s)} \|\nabla u_0\|^2$ is constant we can kill all the terms of degree different from $1/2$ by taking blow-up and blow-in limits. This implies that u_0 must be homogeneous of degree $1/2$, and from this information it is not difficult to deduce (40) by looking at u_0 on the unit circle (see Theorem 15 in [15] for a similar argument).

Then, the exact constant $C := \sqrt{\frac{2C_0}{\pi}}$ in front of the sinus can be easily computed by hand with the formulas

$$\frac{\partial u_0}{\partial \tau} = \frac{1}{r}\frac{\partial u_0}{\partial \theta} = C\frac{1}{2\sqrt{r}}\cos(\theta/2) \quad \text{and} \quad \frac{\partial u_0}{\partial r} = C\frac{1}{2\sqrt{r}}\sin(\theta/2).$$

It comes

$$RC_0 = \int_{B(0,R)} \|\nabla u_0\|^2 = \int_{B(0,R)} \left|\frac{\partial u_0}{\partial \tau}\right|^2 + \left|\frac{\partial u_0}{\partial r}\right|^2 = \int_0^R\int_{-\pi}^{\pi} \frac{C^2}{4} dr d\theta = C^2 R\frac{\pi}{2}$$

thus $C = \sqrt{\frac{2C_0}{\pi}}$.

Finally, originally u_r was converging to $\sqrt{\frac{2C_0}{\pi}r}\sin(\theta/2)$ up to subsequences, but by uniqueness of the limit we conclude that the whole sequence converges to this function and this achieves the proof. □

Acknowledgments The authors wish to thank the anonymous referee for giving precious remarks and corrections on the first version of this paper.

Appendix A: Proof of Lemma 3.3

Proof of Lemma 3.3 We first focus on the homogeneous case, i.e. when $N = 0$. Observe that a primitive of $1/(r + Cr^{1+\alpha})$ is

$$\int \frac{1}{r + Cr^{1+\alpha}} dr \ = \ \ln\left(\frac{r}{(Cr^\alpha + 1)^{\frac{1}{\alpha}}}\right) =: \ h(r). \tag{41}$$

Hence (19) yields that

$$\left(E(r)e^{-h(r)}\right)' \ = \ (-h'(r)E(r) + E'(r))e^{-h(r)} \ \geq \ 0\,,$$

in other words,

$$r \mapsto \frac{E(r)}{r}\left(1 + Cr^\alpha\right)^{\frac{1}{\alpha}}$$

is nondecreasing. Therefore the limit of $E(r)(1 + Cr^\alpha)^{\frac{1}{\alpha}}/r$ exists when r goes to zero, and since $(1 + Cr^\alpha)^{\frac{1}{\alpha}}$ converges to 1, we obtain the existence of limit also for $E(r)/r$. Now by monotonicity, this limit is necessarily finite since less than $\frac{E(r_0)}{r_0}(1 + Cr_0^\alpha)^{\frac{1}{\alpha}}$ which is finite for some r_0 fixed.

Now we consider the inhomogeneous case : $N \neq 0$. Using the method of "variation of the constant" one finds that a particular solution of the inhomogeneous equation

$$G(r) = \left(r + Cr^{1+\alpha}\right) G'(r) + CN(r)r^2. \tag{42}$$

is given by

$$G(r) = \lambda(r)\frac{r}{(Cr^\alpha + 1)^{1/\alpha}}\ ,$$

with $\lambda(r) = -C\int_0^r N(t)(Ct^\alpha+1)^{\frac{1-\alpha}{\alpha}}\,dt$ (notice that $N(t)(Ct^\alpha+1)^{\frac{1-\alpha}{\alpha}}$ is integrable because N is).

Observe in particular that

$$\lim_{r\to 0}\frac{G(r)}{r} = 0. \tag{43}$$

Now let us return to $E(r)$, which is assumed to satisfy (19). If we subtract $G(r)$ in the equation (19) we get

$$H(r) \leq \left(r + Cr^{1+\alpha}\right) H'(r)\ ,$$

where $H(r) = E(r) - G(r)$. Therefore we can apply the first part of the proof (homogeneous case) to H which gives the existence of the limit

$$\lim_{r\to 0}\left(\frac{H(r)}{r}\right) < +\infty,$$

and we conclude using (43). □

References

1. Babadjian, J.-F., Giacomini, A.: Existence of strong solutions for quasi-static evolution in brittle fracture. 2012 (preprint)
2. Bonnet, A.: On the regularity of the edge set of Mumford–Shah minimizers. In Variational methods for discontinuous structures (Como, 1994), vol. 25 of Progr. Nonlinear Diff. Eqs. Appl., pp. 93–103. Birkhäuser, Basel (1996)
3. Chambolle, A., Francfort, G.A., Marigo, J.-J.: When and how do cracks propagate? J. Mech. Phys. Solids **57**(9), 1614–1622 (2009)

4. Chambolle, A.: A density result in two-dimensional linearized elasticity, and applications. Arch. Ration. Mech. Anal. **167**(3), 211–233 (2003)
5. Chambolle, A., Francfort, G.A., Marigo, J.-J.: Revisiting energy release rates in brittle fracture. J. Nonlinear Sci. **20**(4), 395–424 (2010)
6. Chambolle, A., Giacomini, A., Ponsiglione, M.: Crack initiation in brittle materials. Arch. Ration. Mech. Anal. **188**(2), 309–349 (2008)
7. Dal Maso, G., Francfort, G.A., Toader, R.: Quasistatic crack growth in nonlinear elasticity. Arch. Ration. Mech. Anal., **176**(2), 165–225 (2005)
8. Dal Maso, G., Toader, R.: A model for the quasi-static growth of brittle fractures: existence and approximation results. Arch. Ration. Mech. Anal. **162**(2), 101–135 (2002)
9. David, G.: Singular sets of minimizers for the Mumford–Shah functional, volume 233 of Progress in Mathematics. Birkhäuser Verlag, Basel (2005)
10. Francfort, G.A., Marigo, J.-J.: Revisiting brittle fracture as an energy minimization problem. J. Mech. Phys. Solids **46**(8), 1319–1342 (1998)
11. Francfort, G.A., Larsen, C.J.: Existence and convergence for quasi-static evolution in brittle fracture. Comm. Pure Appl. Math. **56**(10), 1465–1500 (2003)
12. Grisvard, P.: Elliptic Problems in Nonsmooth Domains, vol. 24 of Monographs and Studies in Mathematics. Pitman (Advanced Publishing Program), Boston (1985)
13. Khludnev, A.M., Kovtunenko, V.A., Tani, A.: On the topological derivative due to kink of a crack with non-penetration. Anti-plane model. J. Math. Pures Appl. (9) **94**(6), 571–596 (2010)
14. Lazzaroni, G., Toader, R.: Energy release rate and stress intensity factor in antiplane elasticity. J. Math. Pures Appl. **95**(6), 565–584 (2011).
15. Lemenant, A.: On the homogeneity of global minimizers for the Mumford–Shah functional when K is a smooth cone. Rend. Semin. Mat. Univ. Padova **122**, 129–159 (2009)
16. Lemenant, A.: Energy improvement for energy minimizing functions in the complement of generalized Reifenberg-flat sets. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **9**(2), 351–384 (2010)
17. Mumford, D., Shah, J.: Optimal approximation by piecewise smooth functions and associated variational problems. Comm. Pure Appl. Math. **42**, 577–685 (1989)