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Calculus of Variations

A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension

Huai-Dong Cao · Haizhong Li

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Abstract In this paper, we prove a classification theorem for self-shrinkers of the mean curvature flow with $|A|^2 \leq 1$ in arbitrary codimension. In particular, this implies a gap theorem for self-shrinkers in arbitrary codimension.

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1 Introduction

Let $x : M^n \to \mathbb{R}^{n+p}$ be an n-dimensional submanifold in the (n+p)-dimensional Euclidean space. If we let the position vector x evolve in the direction of the mean curvature $\mathbf{H}$, then it gives rise to a solution to the mean curvature flow:

$$x : M \times [0, T) \to \mathbb{R}^{n+p}, \qquad \frac{\partial x}{\partial t} = \mathbf{H} \tag{1.1}$$

We call the immersed manifold M a self-shrinker if it satisfies the quasilinear elliptic system:

$$\mathbf{H} = -\mathbf{x}^{\perp} \tag{1.2}$$

where $\perp$ denotes the projection onto the normal bundle of M.

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H.-D. Cao
Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA
e-mail: huc2@lehigh.edu

H. Li (✉)
Department of Mathematical Sciences, Tsinghua University,
Beijing 100084, People's Republic of China
e-mail: hli@math.tsinghua.edu.cn

Self-shrinkers are an important class of solutions to the mean curvature flow (1.1). Not only they are shrinking homothetically under mean curvature flow (see, e.g., [5]), but also they describe possible Type I blow ups at a given singularity of the mean curvature flow.

In the curve case, Abresch and Langer [1] gave a complete classification of all solutions to (1.2). These curves are so-called Abresch–Langer curves.

In the hypersurface case (i.e. codimension 1), Ecker and Huisken [8] proved that if an entire graph with polynomial volume growth is a self-shrinker, then it is necessarily a hyperplane. Recently Wang [16] removed the condition of *polynomial volume growth* in Ecker–Huisken's Theorem. Let $|A|^2$ denote the norm square of the second fundamental form of M. In [9] and [10], Huisken proved a classification theorem that n-dimensional self-shrinkers satisfying (1.2) in $\mathbb{R}^{n+1}$ with non-negative mean curvature, bounded $|A|$, and polynomial volume growth are $\Gamma \times \mathbb{R}^{n-1}$, or $\mathbb{S}^m(\sqrt{m}) \times \mathbb{R}^{n-m} (0 \leq m \leq n)$. Here, Γ is a Abresch–Langer curve and $\mathbb{S}^m(\sqrt{m})$ is a m-dimensional sphere of radius $\sqrt{m}$. Recently, Colding and Minicozzi [5] showed that Huisken's classification theorem still holds without the assumption that $|A|$ *is bounded*. Moreover, they showed that the only embedded entropy stable self-shrinkers with polynomial volume growth in $\mathbb{R}^{n+1}$ are hyperplanes, n-spheres, and cylinders.

In arbitrary codimensional case, Smoczyk [15] proved the following two results: (i) For any n-dimensional compact self-shrinker M^n in R^{n+p} satisfying (1.2), if $\mathbf{H} \neq 0$ and unit mean curvature vector field $\nu = \mathbf{H}/|\mathbf{H}|$ is parallel in the normal bundle, then $M^n = \mathbb{S}^n(\sqrt{n})$ in $\mathbb{R}^{n+1}$; (ii) For any n-dimensional non-compact self-shrinker M^n in R^{n+p} satisfying (1.2), if M^n is a complete self-shrinker with $\mathbf{H} \neq 0$ and unit mean curvature vector field $\nu = \mathbf{H}/|\mathbf{H}|$ is parallel in the normal bundle, and having uniformly bounded geometry, then M^n is either $\Gamma \times \mathbb{R}^{n-1}$, or $N^m \times \mathbb{R}^{n-m}$. Here Γ is an Abresch–Langer curve and N^m is a m-dimensional minimal submanifold in $\mathbb{S}^{m+p-1}(\sqrt{m})$. On the other hand, Ding and Wang [6] recently have extended the result of Wang [16] to higher codimensional case under the condition of *flat normal bundle*.

Very recently, based on an identity of Colding and Minicozzi (see (9.42) in [5]), Le and Sesum [11] proved a gap theorem (cf. Theorem 1.7 in [11]) for self-shrinkers of codimension 1: if a hypersurface $M^n \subset \mathbb{R}^{n+1}$ is a smooth complete embedded self-shrinker without boundary and with polynomial volume growth, and satisfies $|A|^2 < 1$, then M^n is a hyperplane. Motivated by this result of Le and Sesum, we prove in this paper the following classification theorem for self-shrinkers in arbitrary codimensions:

Theorem 1.1 *If* $M^n \to \mathbb{R}^{n+p} (p \geq 1)$ *is an n-dimensional complete self-shrinker without boundary and with polynomial volume growth, and satisfies*

$$|A|^2 \leq 1, \tag{1.3}$$

then M *is one of the followings:*

(i) a round sphere $\mathbb{S}^n\left(\sqrt{n}\right)$ *in* $\mathbb{R}^{n+1}$,
(ii) a cylinder $\mathbb{S}^m\left(\sqrt{m}\right) \times \mathbb{R}^{n-m}$, $1 \leq m \leq n-1$, *in* $\mathbb{R}^{n+1}$,
(iii) a hyperplane in $\mathbb{R}^{n+1}$.

Here $|A|^2$ *is the norm square of the second fundamental form of* M.

As an immediate consequence, we have the following gap theorem valid for arbitrary codimensions:

Corollary 1.1 *If $M^n \to \mathbb{R}^{n+p}(p \geq 1)$ is a smooth complete embedded self-shrinker without boundary and with polynomial volume growth, and satisfies*

$$|A|^2 < 1, \tag{1.4}$$

then M is a hyperplane in $\mathbb{R}^{n+1}$.

Remark 1.1 We expect that the condition on volume growth in Theorem 1.1 and Corollary 1.1 can be removed. In fact, it was conjectured by the first author that a complete self-shrinker automatically has polynomial volume growth. Note that Zhou and the first author [4] proved that a complete Ricci shrinker necessarily has at most Euclidean volume growth.

Remark 1.2 Shortly after our work was finished, Ding and Xin [7] proved that any complete non-compact *properly immersed* self-shrinker M^n in $\mathbb{R}^{n+p}$ has at most Euclidean volume growth.

2 Preliminaries

In this section, we recall some formulas and notations for submanifolds in Euclidean space by using the method of moving frames.

Let $x : M^n \to \mathbb{R}^{n+p}$ be an n-dimensional submanifold of the $(n+p)$-dimensional Euclidean space R^{n+p}. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal basis of M with respect to the induced metric, and $\{\theta_1, \ldots, \theta_n\}$ be their dual 1-forms. Let $e_{n+1}, \ldots, e_{n+p}$ be the local unit orthonormal normal vector fields.

In this paper we make the following convention on the range of indices:

$$1 \leq i, j, k \leq n; \qquad n+1 \leq \alpha, \beta, \gamma \leq n+p.$$

Then we have the following structure equations,

$$dx = \sum_i \theta_i e_i, \tag{2.1}$$

$$de_i = \sum_j \theta_{ij} e_j + \sum_{\alpha,j} h^\alpha_{ij} \theta_j e_\alpha, \tag{2.2}$$

$$de_\alpha = -\sum_{i,j} h^\alpha_{ij} \theta_j e_i + \sum_\beta \theta_{\alpha\beta} e_\beta, \tag{2.3}$$

where h^α_{ij} denote the components of the second fundamental form of M and $\theta_{ij}, \theta_{\alpha\beta}$ denote the connection 1-forms of the tangent bundle and normal bundle of M, respectively.

The Gauss equations are given by

$$R_{ijkl} = \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}) \tag{2.4}$$

$$R_{ik} = \sum_\alpha H^\alpha h^\alpha_{ik} - \sum_{\alpha,j} h^\alpha_{ij} h^\alpha_{jk} \tag{2.5}$$

$$R = H^2 - |A|^2 \tag{2.6}$$

where R is the scalar curvature of M, $|A|^2 = \sum_{\alpha,i,j}(h^\alpha_{ij})^2$ is the norm square of the second fundamental form, $\mathbf{H} = \sum_\alpha H^\alpha e_\alpha = \sum_\alpha(\sum_i h^\alpha_{ii})e_\alpha$ is the mean curvature vector field, and $H = |\mathbf{H}|$ is the mean curvature of M.

The Codazzi equations are given by (see, e.g., [12])

$$h^\alpha_{ijk} = h^\alpha_{ikj}, \tag{2.7}$$

where the covariant derivative of h^α_{ij} is defined by

$$\sum_k h^\alpha_{ijk}\theta_k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj}\theta_{ki} + \sum_k h^\alpha_{ik}\theta_{kj} + \sum_\beta h^\beta_{ij}\theta_{\beta\alpha}. \tag{2.8}$$

If we denote by $R_{\alpha\beta ij}$ the curvature tensor of the normal connection $\theta_{\alpha\beta}$ in the normal bundle of $x : M \to \mathbb{R}^{n+p}$, then the Ricci equations are

$$R_{\alpha\beta ij} = \sum_k \left(h^\alpha_{ik}h^\beta_{kj} - h^\alpha_{jk}h^\beta_{ki}\right). \tag{2.9}$$

By exterior differentiation of (2.8), we have the following Ricci identities (see, e.g., [12])

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj}R_{mikl} + \sum_m h^\alpha_{im}R_{mjkl} + \sum_\beta h^\beta_{ij}R_{\beta\alpha kl}. \tag{2.10}$$

We define the first and second covariant derivatives, and Laplacian of the mean curvature vector field $\mathbf{H} = \sum_\alpha H^\alpha e_\alpha$ in the normal bundle $N(M)$ as follows (cf. [3,12])

$$\sum_i H^\alpha_{,i}\theta_i = dH^\alpha + \sum_\beta H^\beta\theta_{\beta\alpha}, \tag{2.11}$$

$$\sum_j H^\alpha_{,ij}\theta_j = dH^\alpha_{,i} + \sum_j H^\alpha_{,j}\theta_{ji} + \sum_\beta H^\beta_{,i}\theta_{\beta\alpha}, \tag{2.12}$$

$$\Delta^\perp H^\alpha = \sum_i H^\alpha_{,ii}, \qquad H^\alpha = \sum_k h^\alpha_{kk}. \tag{2.13}$$

Let f be a smooth function on M, we define the covariant derivatives f_i, f_{ij}, and the Laplacian of f as follows

$$df = \sum_i f_i\theta_i, \qquad \sum_j f_{ij}\theta_j = df_i + \sum_j f_j\theta_{ji}, \qquad \Delta f = \sum_i f_{ii}. \tag{2.14}$$

3 A key lemma

As we mentioned in the introduction, the proof of Le–Sesum's gap theorem relies on an important identity of Colding and Minicozzi [5] for hypersurfaces. The identity, see (9.42) in [5] or (4.1) in [11], is obtained in terms of certain second order linear operator for hypersurfaces (which is part of the Jacobi operator for the second variation). In this section, we derive a similar inequality for arbitrary codimensions.

Let a be any fixed vector in $\mathbb{R}^{n+p}$, we define the following height functions in the a direction on M,

$$f = \langle x, a\rangle, \tag{3.1}$$

and

$$g_\alpha = \langle e_\alpha, a\rangle \tag{3.2}$$

for a fixed normal vector e_α.

From (2.14) for f_i and the structure equation 2.1, we have

$$f_i = \langle e_i, a\rangle. \tag{3.3}$$

Similarly, from (2.14) for f_{ij} and the structure equation 2.2, we have

$$f_{ij} = \sum_\alpha h^\alpha_{ij}\langle e_\alpha, a\rangle. \tag{3.4}$$

Since a can be arbitrary in (3.3) and (3.4), we obtain (see [3])

$$x_i = e_i, \qquad x_{ij} = \sum_\alpha h^\alpha_{ij} e_\alpha. \tag{3.5}$$

Define the first derivative $g_{\alpha,i}$ of g_α by

$$\sum_i g_{\alpha,i}\theta_i = dg_\alpha + \sum_\beta g_\beta \theta_{\beta\alpha}. \tag{3.6}$$

We have, by use of (2.3),

$$g_{\alpha,i} = -\sum_k h^\alpha_{ik}\langle e_k, a\rangle. \tag{3.7}$$

Taking covariant derivatives on both sides of (3.7) in the e_j direction and using (3.5), we have

$$g_{\alpha,ij} = -\sum_k h^\alpha_{ikj}\langle e_k, a\rangle - \sum_{k,\beta} h^\alpha_{ik}h^\beta_{kj}\langle e_\beta, a\rangle, \tag{3.8}$$

where the second derivative $g_{\alpha,ij}$ of g_α is defined by

$$\sum_j g_{\alpha,ij}\theta_j = dg_{\alpha,i} + \sum_j g_{\alpha,j}\theta_{ji} + \sum_\beta g_{\beta,i}\theta_{\beta\alpha}. \tag{3.9}$$

Again, since a is arbitrary in (3.7) and (3.8), we obtain (see [3])

$$e_{\alpha,i} = -\sum_j h^\alpha_{ij} e_j, \qquad e_{\alpha,ij} = -\sum_k h^\alpha_{ikj} e_k - \sum_{k,\beta} h^\alpha_{ik}h^\beta_{kj} e_\beta, \tag{3.10}$$

where the covariant derivative h^α_{ijk} of the second fundamental form h^α_{ij} is defined by (2.8).

Now the self-shrinker equation 1.2 is equivalent to

$$-H^\alpha = \langle x, e_\alpha\rangle, \quad n+1 \le \alpha \le n+p. \tag{3.11}$$

Taking covariant derivative of (3.11) with respect to e_i by use of (3.5) and (3.10), we have

$$-H^\alpha_{,i} = -\sum_j h^\alpha_{ij}\langle x, e_j\rangle, \quad 1 \le i \le n, \quad n+1 \le \alpha \le n+p. \tag{3.12}$$

Taking covariant derivative of (3.12) with respect to e_k by use of (3.5) and (3.11), we have

$$\begin{aligned} -H^{\alpha}_{,ik} &= -\sum_j h^{\alpha}_{ijk}\langle x, e_j\rangle - h^{\alpha}_{ik} - \sum_{\beta,j} h^{\alpha}_{ij}h^{\beta}_{jk}\langle x, e_\beta\rangle \\ &= -\sum_j h^{\alpha}_{ijk}\langle x, e_j\rangle - h^{\alpha}_{ik} + \sum_{\beta,j} H^{\beta} h^{\alpha}_{ij}h^{\beta}_{jk}. \end{aligned} \tag{3.13}$$

Writing

$$\sigma_{\alpha\beta} = \sum_{i,j} h^{\alpha}_{ij}h^{\beta}_{ij}, \tag{3.14}$$

we have

$$\sum_{\alpha,\beta} \sigma_{\alpha\beta} H^{\alpha} H^{\beta} \leq |A|^2|H|^2. \tag{3.15}$$

We are now ready to prove the following key lemma:

Lemma 3.1 *Let M^n be an n-dimensional complete self-shrinker in $\mathbb{R}^{n+p}$ without boundary and with polynomial volume growth, if $|A|^2$ is bounded on M^n, then*

$$\begin{aligned} \int_M |\nabla^{\perp} H|^2 e^{-\frac{|x|^2}{2}} dv &= \int_M \left[\sum_{\alpha,\beta} \sigma_{\alpha\beta} H^{\alpha} H^{\beta} - |H|^2\right] e^{-\frac{|x|^2}{2}} dv \\ &\leq \int_M [|A|^2 - 1]|H|^2 e^{-\frac{|x|^2}{2}} dv. \end{aligned}$$

Proof Letting $i = k$ in (3.13) and summing over i, we get

$$\Delta^{\perp} H^{\alpha} = \sum_j H^{\alpha}_{,j}\langle x, e_j\rangle + H^{\alpha} - \sum_{\beta} \sigma_{\alpha\beta} H^{\beta}. \tag{3.16}$$

Since M^n has polynomial volume growth and $|A|^2$ is bounded on M^n, (3.11), (3.12), (3.14) and (3.16) imply that

$$\int_M |\nabla^{\perp} H|^2 e^{-\frac{|x|^2}{2}} dv < +\infty, \qquad \int_M \sum_{\alpha} H^{\alpha} \Delta^{\perp} H^{\alpha} e^{-\frac{|x|^2}{2}} dv < +\infty,$$

and

$$\int_M \sum_{\alpha,i} H^{\alpha} H^{\alpha}_{,i}\langle x, e_i\rangle e^{-\frac{|x|^2}{2}} dv < +\infty.$$

Let $\varphi_r(x)$ be a smooth cut-off function with compact support in $B_{x_0}(r+1) \subset M$,

$$\varphi_r(x) = \begin{cases} 1, & \text{in } B_{x_0}(r) \\ 0 & \text{in } M \setminus B_{x_0}(r+1) \end{cases} \qquad 0 \leq \varphi_r(x) \leq 1, \quad |\nabla \varphi_r| \leq 1.$$

Then, by integration by parts, we get

$$\int_M \sum_\alpha \Delta^\perp H^\alpha(\varphi_r H^\alpha)e^{-\frac{|x|^2}{2}}dv = \int_M \varphi_r H^\alpha H^\alpha_{,i}\langle x, e_i\rangle e^{-\frac{|x|^2}{2}}dv - \int_M H^\alpha_{,i}(\varphi_r H^\alpha)_{,i}e^{-\frac{|x|^2}{2}}dv$$
$$= \int_M \varphi_r \left(\sum_{\alpha,i} H^\alpha H^\alpha_{,i}\langle x, e_i\rangle - |\nabla^\perp H|^2\right) e^{-\frac{|x|^2}{2}}dv$$
$$- \int_M \sum_{\alpha,i} H^\alpha H^\alpha_{,i}(\varphi_r)_i e^{-\frac{|x|^2}{2}}dv.$$

Letting $r \to +\infty$, the dominated convergence theorem implies that

$$\int_M \sum_\alpha \Delta^\perp H^\alpha H^\alpha e^{-\frac{|x|^2}{2}}dv = \int_M \left(\sum_{\alpha,i} H^\alpha H^\alpha_{,i}\langle x, e_i\rangle - |\nabla^\perp H|^2\right) e^{-\frac{|x|^2}{2}}dv. \quad (3.17)$$

Putting (3.16) into (3.17), we obtain:

$$\int_M |\nabla^\perp H|^2 e^{-\frac{|x|^2}{2}}dv = \int_M \left(\sum_{\alpha,\beta} \sigma_{\alpha\beta}H^\alpha H^\beta - |H|^2\right) e^{-\frac{|x|^2}{2}}dv$$
$$\leq \int_M \left(|A|^2 - 1\right)|H|^2 e^{-\frac{|x|^2}{2}}dv.$$

□

Remark 3.1 From the proof of Lemma 3.1, one can see that the conclusion of Lemma 3.1 is valid even if $|A|^2$ has certain growth in $|x|^2$.

4 Proof of Theorem 1.1

Now we present the proof of Theorem 1.1.

Proof of Theorem 1.1 Under the assumptions of Theorem 1.1, from Lemma 3.1, we know that either $\mathbf{H} \equiv 0$, or $\mathbf{H} \neq 0$ but with $\nabla^\perp \mathbf{H} \equiv 0$ and $|A|^2 \equiv 1$.

If $\mathbf{H} \equiv 0$, we have $\langle x, e_\alpha\rangle \equiv 0$, $n+1 \leq \alpha \leq n+p$, from which we easily conclude from (3.12) that M is totally geodesic, that is, a hyperplane in $\mathbb{R}^{n+1}$.

Next, suppose that $\mathbf{H} \neq 0$, $\nabla^\perp \mathbf{H} \equiv 0$, and $|A|^2 \equiv 1$. In this case, (3.13) becomes

$$\sum_{\beta,j} H^\beta h^\alpha_{ij}h^\beta_{jk} = h^\alpha_{ik} + \sum_j h^\alpha_{ijk}\langle x, e_j\rangle, \quad 1 \leq i,k \leq n; n+1 \leq \alpha \leq n+p. \quad (4.1)$$

Multiplying both sides of (4.1) by h^α_{ik} and summing over α, i, k, we get

$$\sum_{\alpha,\beta,i,j,k} H^\beta h^\alpha_{ij}h^\beta_{jk}h^\alpha_{ik} = |A|^2 + \frac{1}{2}(|A|^2)_{,j}\langle x, e_j\rangle = |A|^2 = 1. \quad (4.2)$$

Next we choose a local orthonormal frame $\{e_\alpha\}$ for the normal bundle of $x : M \to \mathbb{R}^{n+p}$, such that e_{n+p} is parallel to the mean curvature vector $\mathbf{H}$; i.e.,

$$e_{n+p} = \frac{\mathbf{H}}{|\mathbf{H}|}, \quad H^{n+p} = H, \quad H^\alpha = 0, \quad \alpha \neq n+p. \quad (4.3)$$

Because now the equality holds in (3.15), we have

$$h_{ij}^{\alpha}=0,\quad \alpha\neq n+p,\qquad |A|^2=\sum_{i,j}h_{ij}^{n+p}h_{ij}^{n+p}=1. \tag{4.4}$$

Since $\nabla^{\perp}\mathbf{H}\equiv\mathbf{0}$ and $|A|^2\equiv 1$, by the definition of Δ and using (2.7), (2.10), (2.4), (2.5) and (2.9), we have (c.f. [12–14, 17])

$$\begin{aligned}
0 &= \frac{1}{2}\Delta|A|^2\\
&= \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2+\sum_{\alpha,i,j,k}h_{ij}^{\alpha}h_{ijkk}^{\alpha}\\
&= \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2+\sum_{\alpha,i,j,k,m}h_{ij}^{\alpha}h_{mk}^{\alpha}R_{mijk}+\sum_{\alpha,i,j,m}h_{ij}^{\alpha}h_{im}^{\alpha}R_{mj}+\sum_{\alpha,\beta,i,j,k}h_{ij}^{\alpha}h_{ik}^{\beta}R_{\beta\alpha jk}\\
&= \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2+\sum_{\alpha,\beta,i,j,m}H^{\beta}h_{mj}^{\beta}h_{ij}^{\alpha}h_{im}^{\alpha}-\sum_{\alpha,\beta,i,j,k,m}h_{ij}^{\alpha}h_{ij}^{\beta}h_{mk}^{\alpha}h_{mk}^{\beta}+2\sum_{\alpha,\beta,i,j,k}h_{ij}^{\alpha}h_{ik}^{\beta}R_{\beta\alpha jk}.
\end{aligned}$$

Plugging (4.2), (4.3) and (4.4) into the above identity, we conclude that

$$h_{ijk}^{\alpha}=0,\quad n+1\le\alpha\le n+p. \tag{4.5}$$

Because $e_{n+1}\wedge_{n+2}\wedge\cdots\wedge e_{n+p-1}$ is parallel in the normal bundle of M and $h_{ij}^{\alpha}\equiv 0, \alpha\neq n+p$, by Theorem 1 of Yau [18], we see that M is a hypersurface in $\mathbb{R}^{n+1}$. So (4.5) implies that M is an isoparametric hypersurface, thus from $|A|^2=1$ we conclude that M is either a round sphere $\mathbb{S}^n(\sqrt{n})$, or a cylinder $\mathbb{S}^m(\sqrt{m})\times\mathbb{R}^{n-m}$, $1\le m\le n-1$ in $\mathbb{R}^{n+1}$. This completes the proof of Theorem 1.1 □

5 Further remarks

In this section, we make several simple observations:

Proposition 5.1 *If a submanifold $M^n\to\mathbb{R}^{n+p}$ is an n-dimensional complete self-shrinker without boundary and with polynomial volume growth, such that*

$$|H|^2\ge n, \tag{5.1}$$

then $|H|^2\equiv n$ and M is a minimal submanifold in the sphere $\mathbb{S}^{n+p-1}(\sqrt{n})$.

Proof of Proposition 5.1 From (3.5) and (3.11), we have

$$\frac{1}{2}\Delta|x|^2=n+\langle x,\Delta x\rangle=n+\sum_{\alpha}H^{\alpha}\langle x,e_{\alpha}\rangle=n-|H|^2 \tag{5.2}$$

Under the polynomial volume growth assumption, (1.2) and (5.2) guarantee that

$$\int_M(\Delta|x|^2)e^{-\frac{|x|^2}{2}}dv<+\infty\qquad\text{and}\qquad\int_M|\nabla|x|^2|^2e^{-\frac{|x|^2}{2}}dv<+\infty.$$

Then, by integrating by parts and the dominated convergence theorem, it follows that (similar to the proof of Lemma 3.1)

$$\frac{1}{4}\int_M|\nabla|x|^2|^2e^{-\frac{|x|^2}{2}}dv=\frac{1}{2}\int_M(\Delta|x|^2)e^{-\frac{|x|^2}{2}}dv=\int_M(n-|H|^2)e^{-\frac{|x|^2}{2}}dv. \tag{5.3}$$

From (5.1) and (5.3), we get $|H|^2 = n$ and $\langle x, x\rangle = r^2$. Thus by (1.2) we conclude that $r = \sqrt{n}$ and M is a minimal submanifold in the sphere $\mathbb{S}^{n+p-1}(\sqrt{n})$. □

Proposition 5.2 *If a submanifold $M \to \mathbb{R}^{n+p}$ is an n-dimensional compact self-shrinker without boundary and satisfies either $|H|^2 = constant$, or*

$$|H|^2 \leq n, \tag{5.4}$$

then $|H|^2 \equiv n$ and M is a minimal submanifold in the sphere $\mathbb{S}^{n+p-1}(\sqrt{n})$.

Proof of Proposition 5.2 Integrating (5.2) over M and using the Stokes theorem, we have

$$\int_M (n - |H|^2)dv = 0. \tag{5.5}$$

Hence Proposition 5.2 follows from (5.5), (5.4), and (1.2). □

Remark 5.1 Let $x : M \to \mathbb{R}^{n+p}$ be an n-dimensional submanifold. If x satisfies

$$\lambda H^\alpha = \langle x, e_\alpha\rangle, \quad n+1 \leq \alpha \leq n+p \tag{5.6}$$

for some positive constant λ, then we call M a *self-expander* of the mean curvature flow. Observe that for a self-expander, we have

$$\frac{1}{2}\Delta|x|^2 = n + \langle x, \Delta x\rangle = n + n\sum_\alpha H^\alpha\langle x, e_\alpha\rangle = n + n\lambda|H|^2. \tag{5.7}$$

From (5.7), we immediately get

Proposition 5.3 *There exists no n-dimensional compact self-expander without boundary in $\mathbb{R}^{n+p}$.*

Finally, we list some simple examples of self-shrinkers of higher codimensions.

Example 5.1 For any positive integers $m_1, \ldots, m_p$ such that $m_1 + \cdots + m_p = n$, the submanifold

$$M^n = \mathbb{S}^{m_1}(\sqrt{m_1}) \times \cdots \times \mathbb{S}^{m_p}(\sqrt{m_p}) \subset \mathbb{R}^{n+p} \tag{5.8}$$

is an n-dimensional compact self-shrinker in $\mathbb{R}^{n+p}$ with

$$\mathbf{H} = -X, \qquad |\mathbf{H}|^2 = n, \qquad |A|^2 = p \tag{5.9}$$

Here

$$\mathbb{S}^{m_i}(r_i) = \{X_i \in \mathbb{R}^{m_i+1} : |X_i|^2 = r_i^2\}, \qquad i = 1, \ldots, p \tag{5.10}$$

is a m_i-dimensional round sphere with radius r_i.

Example 5.2 For positive integers $m_1, \ldots, m_p, q \geq 1$, with $m_1 + \cdots + m_p + q = n$, the submanifold

$$M^n = \mathbb{S}^{m_1}(\sqrt{m_1}) \times \cdots \times \mathbb{S}^{m_p}(\sqrt{m_p}) \times \mathbb{R}^q \subset \mathbb{R}^{n+p} \tag{5.11}$$

is an n-dimensional complete non-compact self-shrinker in $\mathbb{R}^{n+p}$ with polynomial volume growth which satisfies

$$\mathbf{H} = -X^\perp, \qquad |\mathbf{H}|^2 = \sum_{i=1}^p m_i, \qquad |A|^2 = p. \tag{5.12}$$

Remark 5.2 In Examples 5.1 and 5.2, if we let $p \geq 2$, then we have an n-dimensional self-shrinker of codimension p with $|A|^2 = p \geq 2$, thus not one of the three cases in Theorem 1.1.

Remark 5.3 From Example 5.2, we can see that the condition "$|\mathbf{H}|^2 \geq n$" in Proposition 5.1 is necessary.

Example 5.3 (cf. [2]) Let

$$X : \mathbb{S}^2(\sqrt{m(m+1)}) \hookrightarrow \mathbb{S}^{2m}(\sqrt{2}) \subset \mathbb{R}^{2m+1}, \qquad m \geq 2 \tag{5.13}$$

be a minimal surface in $\mathbb{S}^{2m}(\sqrt{2})$. Consider it as a surface in $\mathbb{R}^{2m+1}$, then it is a self-shrinker with

$$\mathbf{H} = -X, \qquad |\mathbf{H}|^2 = 2, \qquad |A|^2 = 2 - \frac{2}{m(m+1)} < 2, \tag{5.14}$$

Remark 5.4 By choosing local orthogonal frame $\{e_\alpha\}$ for the normal bundle of $x : M^n \to \mathbb{R}^{n+p}$, such that e_{n+p} is parallel to the mean curvature vector $\mathbf{H}$, by Lemma 3.1, if $|A|^2$ is bounded, and

$$\sum_{i,j} h_{ij}^{n+p} h_{ij}^{n+p} \leq 1, \tag{5.15}$$

we have $\nabla^\perp \mathbf{H} = 0$, that is, $|\mathbf{H}|^2 = constant$ and unit mean curvature vector field $\nu = \mathbf{H}/|\mathbf{H}|$ is parallel in the normal bundle. From Proposition 5.2 and Theorem 1.3 of Smoczyk [15], we have

Proposition 5.4 *Let M^n be an n-dimensional complete self-shrinker in $\mathbb{R}^{n+p}$ without boundary and with polynomial volume growth. If $|A|^2$ is bounded on M^n and (5.15) holds, then*

$$M^n = N^m \times \mathbb{R}^{n-m}, \qquad 0 \leq m \leq n,$$

where N^m is a m-dimensional minimal submanifold in $\mathbb{S}^{m+p-1}(\sqrt{m})$.

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