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Calculus of Variations

Layered solutions with multiple asymptotes for non autonomous Allen–Cahn equations in $\mathbb{R}^3$

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Abstract We consider a class of semilinear elliptic equations of the form

$$-\Delta u(x, y, z) + a(x)W'(u(x, y, z)) = 0, \quad (x, y, z) \in \mathbb{R}^3, \tag{0.1}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a periodic, positive, even function and, in the simplest case, $W : \mathbb{R} \to \mathbb{R}$ is a double well even potential. Under non degeneracy conditions on the set of minimal solutions to the one dimensional heteroclinic problem

$$-\ddot{q}(x) + a(x)W'(q(x)) = 0,\ x \in \mathbb{R}, \quad q(x) \to \pm 1 \text{ as } x \to \pm\infty,$$

we show, via variational methods the existence of infinitely many geometrically distinct solutions u of (0.1) verifying $u(x, y, z) \to \pm 1$ as $x \to \pm\infty$ uniformly with respect to $(y, z) \in \mathbb{R}^2$ and such that $\partial_y u \not\equiv 0$, $\partial_z u \not\equiv 0$ in $\mathbb{R}^3$.

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1 Introduction

In this article we deal with a class of semilinear elliptic equations of the form

$$-\Delta u(x) + a(x)W'(u(x)) = 0, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where, briefly, we assume that $a(x)$ is a continuous, positive, periodic function and $W(s)$ is modeled on the double well Ginzburg–Landau potential $W(s) = (s^2 - 1)^2$ or to the

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Sine-Gordon potential, $W(s) = 1 + \cos(\pi s)$. Potentials of this kind are widely used in phase transitions and condensed state Physics. The introduction of an oscillatory factor $a(x)$ can be used to study inhomogeneous materials.

The problem of existence and multiplicity of entire bounded solutions of (1.1) has been intensively investigated in the last years. In particular a long standing problem concerning (1.1) in the case in which $a(x)$ is *constant*, is to characterize the set of the solutions $u \in C^2(\mathbb{R}^n)$ of (1.1) with $n \geq 2$, satisfying $|u(x)| \leq 1$ and $\partial_{x_1} u(x) > 0$ in $\mathbb{R}^n$. This problem was pointed out by Ennio De Giorgi in [24], where he conjectured that if $a(x) = a_0 > 0$, when $n \leq 8$ and $W(s) = (s^2 - 1)^2$, the whole set of these solutions of (1.1) can be obtained by the action of the group of space roto-translations on the solutions of the one dimensional heteroclinic problem

$$\begin{cases} -\ddot{q}(x) + a_0 W'(q(x)) = 0, & x \in \mathbb{R}, \\ \lim\limits_{x \to \pm\infty} q(x) = \pm 1, \end{cases} \tag{1.2}$$

The conjecture has been firstly proved in the planar case by Ghoussoub and Gui in [37] also for a general (not necessarily even) double well potential $W(s)$. We refer also to [16,17,31] where studying a weaker version of the De Giorgi conjecture, known as *Gibbons conjecture*, the same conclusion is obtained in all the dimensions n for solutions of (1.1) satisfying the asymptotic condition

$$\lim_{x_1 \to \pm\infty} u(x) = \pm 1, \text{ uniformly w.r.t. } (x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n-1}. \tag{1.3}$$

The De Giorgi's conjecture has been proved for a general potential $W(s)$ in dimension $n = 3$ in [14] (see also [2]), and for the Ginzburg–Landau potential in dimension $n \leq 8$ in [52,53], assuming that the solutions satisfy (1.3) pointwise with respect to $(x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n-1}$ (see [32,33,35] for further developments and references). In [27,30] a counterexample is given in dimension $n > 8$.

Note that when $a(x)$ is constant, the set of solutions of (1.2) is a continuum homeomorphic to $\mathbb{R}$, being constituted by the translations of a single heteroclinic solution. These results tell us in particular that for autonomous equations the set of the solutions to (1.1) satisfying (1.3) reduces, modulo space translation, to this unique one dimensional solution and the problem is in fact *one dimensional*.

On the other hand, if $a(x)$ is *not constant and periodic* the solutions can exhibit different (and more complicated) asymptotic behaviour in different directions of $\mathbb{R}^n$. Indeed the oscillations of the function $a(x)$ allows the existence of "gaps" in the ordered families of minimal entire solutions of the problem and under these gap conditions different families of solutions do exist (e.g. heteroclinic, multibump, multitransition, slope changing and mountain pass type solutions), see [9,15,18,19,25,42,45,47–50] and the recent comprehensive monograph [51].

In particular, as showed in [4,6,8], when $a(x)$ is not constant and periodic, the one dimensional symmetry of the problem generically disappears even if the potential depends only on the single variable x_1 (see also [5] for the case $a(x)$ almost periodic and [1,55] for related results in the case of systems of autonomous Allen–Cahn equations). Indeed, in these papers, under a discreteness assumption on the set of minimal one dimensional solutions, it is obtained the existence of infinitely many entire solutions on $\mathbb{R}^2$ to (1.1)–(1.3) exhibiting different behaviours with respect to x_2 (periodic, of the homoclinic or heteroclinic type).

In other words the oscillation of the potential in the x_1 variable generically implies the existence of complex classes of two-dimensional entire solutions of (1.1)–(1.3). A natural problem in this setting is to understand when the oscillation of the potential in the x_1 variable can imply the existence of entire solutions of (1.1)–(1.3) which depends in a non trivial way on more than two variables.

In this article we tackle this problem studying the existence of entire solutions of (1.1)–(1.3) in $\mathbb{R}^3$ assuming that the positive even function $a(x)$ depends periodically on the single variable x_1 in such a way that the minimal set of one dimensional solutions satisfies suitable discreteness and non degeneracy conditions. Under such assumptions we prove the existence of infinitely many geometrically distinct bounded solutions depending in a non trivial way on all the variables.

To be more precise, we assume

(H_1) $a : \mathbb{R} \to \mathbb{R}$ is Hölder continuous, 1-periodic, even, not constant and positive,
(H_2) $W \in \mathcal{C}^3(\mathbb{R}, \mathbb{R})$ is even and satisfies $W(s) \geq 0$ for all $s \in \mathbb{R}$, $W(s) > 0$ for all $s \in (-1, 1)$, $W(\pm 1) = W'(\pm 1) = 0$ and $W''(\pm 1) > 0$,

and we consider the one dimensional problem

$$\begin{cases} -\ddot{q}(x) + a(x)W'(q(x)) = 0, \quad x \in \mathbb{R}, \\ \lim_{x\to\pm\infty} q(x) = \pm 1. \end{cases} \tag{1.4}$$

In particular we are interested in the structure of the minimal set of the Action

$$\varphi(q) = \int_{\mathbb{R}} \tfrac{1}{2}|\dot{q}(x)|^2 + a(x)W(q(x))\,dx$$

on the class

$$\mathcal{H} = p_0 + H^1(\mathbb{R})$$

where $p_0 \in C^\infty(\mathbb{R})$ is a fixed odd and increasing function such that $|p_0(x)| = 1$ for all $|x| \geq 1$. We define

$$m = \inf_{\mathcal{H}} \varphi \quad \text{and} \quad \mathcal{M} = \{q \in \mathcal{H} \,/\, \varphi(q) = m\},$$

recalling that $\mathcal{M}$ is a non empty and ordered set, consisting of solutions to (1.4).

We first assume that the following discreteness condition holds

($*$) for all $q \in \mathcal{M}$ there results $q(0) \neq 0$.

Since $\mathcal{M}$ is an ordered set, making use of the terminology in [51], condition ($*$) implies the existence of a *gap pair* in $\mathcal{M}$ around the origin. More precisely there exist two adjacent members $q_\pm \in \mathcal{M}$, such that

$$q_-(0) < 0 < q_+(0)$$

and for every $q \in \mathcal{M} \setminus \{q_\pm\}$ there results either, $q(x) < q_-(x)$ or $q(x) > q_+(x)$ for all $x \in \mathbb{R}$. The symmetry of the problem allows us also to show that q_- and q_+ are related one to the other by the symmetry relation $q_-(x) = -q_+(-x)$.

On this extremal solutions $q_\pm$ we furthermore require that they are not degenerate, i.e., we assume

($**$) there exists $\omega^* > 0$ such that

$$\varphi''(q_\pm)h \cdot h = \int_{\mathbb{R}} \dot{h}^2 + a(x)W''(q_\pm)h^2\,dx \geq \omega^* \|h\|^2_{L^2}, \quad \forall h \in H^1(\mathbb{R}).$$

We remark that the assumptions $(*)$–$(**)$ excludes the autonomous case. On the other hand, Melnikov–Poincaré methods (see [11–13] and the references therein) allows to construct examples in which $(*)$ and $(**)$ hold true in cases in which $a(x)$ is a small periodic perturbation of a positive constant. We refer moreover to [8] where, following [7], $(*)$ is verified when $a(x)$ is a *slowly oscillating* function (or equivalently for the singularly perturbed equation) with a maximum in the origin. In particular, using the argument in [39], $(*)$ and $(**)$ can be verified when $\varepsilon > 0$ is small enough, for the singularly perturbed equation

$$-\varepsilon^2\ddot{q}(x) + a(x)W'(q(x)) = 0, \quad x \in \mathbb{R},$$

when $a(x)$ has not degenerate critical points and assumes maximum value in 0.

The genericity of the existence of "gaps" in $\mathcal{M}$ is proved in a more general setting in [51], see Proposition 3.56 and Theorem 4.58. In this respect we have to note that our assumption $(*)$ requires not only the existence of a gap in $\mathcal{M}$ but also that this gap exists in correspondence of a symmetry point of the function a.

We can now state our main result

Theorem 1.1 *Assume* (H_1)–(H_2) *and* $(*)$–$(**)$. *Then, there exist infinitely many solutions of the problem*

$$\begin{cases} -\Delta v(x,y,z) + a(x)W'(v(x,y,z)) = 0, & (x,y,z) \in \mathbb{R}^3, \\ \lim\limits_{x\to\pm\infty} v(x,y,z) = \pm 1 & \textit{uniformly w.r.t. } (y,z) \in \mathbb{R}^2. \end{cases} \tag{1.5}$$

More precisely, for every $j \in \mathbb{N}$, $j \geq 2$ *there exists a solution* $v_j \in C^2(\mathbb{R}^3)$ *of* (1.5) *such that, denoting* $\tilde{v}_j(x,\rho,\theta) = v_j(x, \rho\cos\theta, \rho\sin\theta)$, *it satisfies*

(i) $q_-(x) \leq v_j(x,y,z) \leq q_+(x)$ *on* $\mathbb{R}^3$,
(ii) $\tilde{v}_j$ *is periodic in* θ *with period* $\frac{2\pi}{j}$,
(iii) $\lim_{\rho\to+\infty} \tilde{v}_j(x,\rho,\frac{\pi}{2} + \frac{\pi}{j}(\frac{1}{2}+k)) = \begin{cases} q_+(x), & \textit{if } k \textit{ is odd}, \\ q_-(x), & \textit{if } k \textit{ is even}, \end{cases}$ *uniformly w.r.t.* $x \in \mathbb{R}$.

Note that, by (iii), the solution $\tilde{v}_j(x,\rho,\theta)$ is asymptotic as $\rho \to +\infty$ to q_+ or q_- whenever the angle θ is respectively equal to $\frac{\pi}{2} + \frac{\pi}{j}(\frac{1}{2}+k)$ with $k \in \{0,\ldots,2j-1\}$ odd or even. Then, Theorem 1.1 provides the existence of infinitely many geometrically distinct bounded solutions of (1.5), all depending in a non trivial way on both the variables $y, z \in \mathbb{R}$.

Moreover, we remark that the property (iii) characterizes the function v_j as being asymptotic in $2j$ directions, orthogonal to the x-axes, to one dimensional solutions to the problem. The existence of solutions asymptotic in different directions to other prescribed solutions has been recently study in different papers. In particular in [28] (see also [41]) the existence of *multiple end* solutions on $\mathbb{R}^2$ for the Allen–Cahn equation was given. Multiple end solutions for Allen–Cahn equation, again in the planar case, are also the saddle solutions found in [23] and the saddle type solutions found in [10] (see also [21,22,38,40,54]). Entire solutions with prescribed asymptotes in different direction are moreover found in [43,44] and then in [29] in the case of Nonlinear Schrödinger equations (we refer also to [26] for a survey on the topic). Accordingly with these results we can say that Theorem 1.1 states the existence and multiplicity of multiple end solutions for non autonomous Allen–Cahn equations in $\mathbb{R}^3$.

Following a strategy already used in [10] (see also [3]), the proof of Theorem 1.1 uses variational methods to study an auxiliary problem. Indeed, given $j \in \mathbb{N}$, $j \geq 2$, setting $\bar{z} = \tan(\frac{\pi}{2j})z$, we consider the infinite prism

$$\mathcal{P}_j = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in \mathbb{R} \times (-\bar{z},\bar{z}),\ z \geq 0\},$$

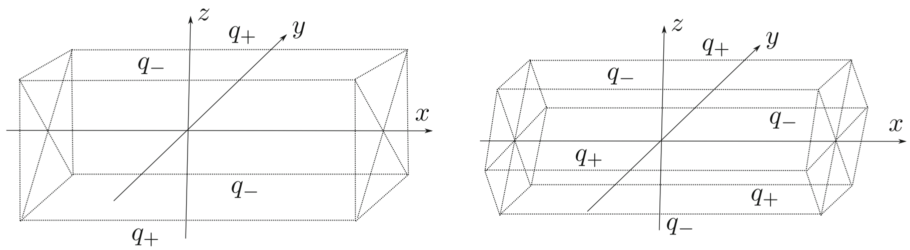

Fig. 1 The asymptotes of v_j for $j = 2$ and $j = 3$

and we look for a minimal (with respect to $\mathcal{C}_0^\infty(\mathbb{R}^3)$ perturbations) solution of

$$\begin{cases} -\Delta v(x,y,z) + a(x)W'(v(x,y,z)) = 0, & (x,y,z) \in \mathcal{P}_j, \\ \lim\limits_{z\to+\infty} v(x,\bar z,z) = q_+(x) & \text{uniformly w.r.t } x \in \mathbb{R} \\ v(x,y,z) = -v(-x,-y,z), & (x,y,z) \in \mathcal{P}_j, \\ \partial_\nu v(x,y,z) = 0 & (x,y,z) \in \partial\mathcal{P}_j, \\ \lim_{x\to\pm\infty} v(x,y,z) = \pm 1, & \text{uniformly w.r.t. } (y,z). \end{cases} \tag{1.6}$$

If v solves (1.6) then as $z \to +\infty$ we have

$$v(x,-\bar z,z) = -v(-x,\bar z,z) \to -q_+(-x) = q_-(x).$$

Then the entire solution v_j on $\mathbb{R}^3$ is obtained from v by recursive reflections of the prism $\mathcal{P}_j$ about its faces (see Fig. 1).

To solve (1.6) we build up a renormalized variational procedure which takes into account the informations we have on the lower dimensional problems. More precisely, solutions of (1.6) are found as minima of the double renormalized functional

$$\varphi_{3,j}(u) = \int_0^{+\infty} \left[\int_{-\bar z}^{\bar z} \left[\int_{\mathbb{R}} \tfrac{1}{2}|\nabla u(x,y,z)|^2 + a(x)W(u(x,y,z))\,dx - m \right] dy - m_{2,\bar z} \right] dz$$

on the class

$$\mathcal{Z}_j = \{u \in H^1_{loc}(\mathcal{P}_j) \,|\, u(x,y,z) = -u(-x,-y,z) \text{ for a.e. } (x,y,z) \in \mathcal{P}_j\}.$$

In the definition of $\varphi_{3,j}$ enter the two renormalizing terms m and $m_{2,\bar z}$, where m is the minimum of φ on $\mathcal{H}$, while, for a fixed $z > 0$,

$$m_{2,z} = \inf_{\mathcal{X}_z} \int_{-z}^{z} \int_{\mathbb{R}} \tfrac{1}{2}|\nabla u(x,y)|^2 + a(x)W(u(x,y))\,dx - m\,dy$$

where, setting $S_z = \mathbb{R} \times (-z, z)$,

$$\mathcal{X}_z = \{u \in H^1_{loc}(S_z) \,/\, u(x,y) = -u(-x,-y) \text{ for a.e.} (x,y) \in S_z\}.$$

As a preliminary step in studying $\varphi_{3,j}$ on $\mathcal{Z}_j$, in Sects. 3 and 4 we need to provide the existence and a variational characterization of the *heteroclinic type* and *periodic* bidimensional solutions of (1.5), studying the asymptotic properties of the minimal values $m_{2,z}$.

This global variational approach allows us to directly control the asymptotes as $z \to +\infty$ of the minima of $\varphi_{3,j}$ on $\mathcal{Z}_j$. Indeed, if v is such a minimum then (up to (x,y) reflection) $v(x,y,z) \to v_+(x,y)$ as $z \to +\infty$ uniformly w.r.t. $(x,y) \in S_{\bar z}$, where v_+ is the unique antisymmetric planar solution of (1.5) such that $v_+(x, y) \to q_\pm(x)$ as $y \to \pm\infty$ uniformly w.r.t $x \in \mathbb{R}$.

We finally remark that even if the solution v_j is obtained by recursive reflections of a minimal solution of (1.6), the solution v_j is no more minimal for the action with respect to $\mathcal{C}_0^\infty(\mathbb{R}^3)$ perturbations. Indeed, since we know that $\partial_y v_j \not\equiv 0$ and $\partial_z v_j \not\equiv 0$, by Theorem 9 and Corollary 10 in [34], the minimality of v_j would imply that $\partial_y v_j$ and $\partial_z v_j$ are strictly positive or negative on $\mathbb{R}^3$, while, by (ii) and (iii) in Theorem 1.1, we recognize that both the derivatives have to change their sign.

This article is organized as follows. After recalling in Sect. 2 variational properties of the minimal solutions of (1.4), in Sects. 3 and 4 we consider *heteroclinic type* and *periodic* bidimensional solutions of (1.5), studying the asymptotic properties of the minimal values $m_{2,z}$. The double renormalized functional $\varphi_{3,j}$ is introduced and studied in Sect. 5, where we finally prove Theorem 1.1.

2 One dimensional solutions

In this section we consider the one dimensional problem

$$\begin{cases} -\ddot{q}(x) + a(x)W'(q(x)) = 0, \quad x \in \mathbb{R}, \\ \lim_{x\to\pm\infty} q(x) = \pm 1, \end{cases} \tag{P_1}$$

recalling some well known results and displaying some consequences of the assumptions $(*)$ and $(**)$.

Remark 2.1 We note that, by (H_2), there exists $\overline{\delta} \in (0, \frac{1}{4})$ and $\overline{w} > \underline{w} > 0$ such that

$$\overline{w} \geq W''(s) \geq \underline{w} \text{ for every } |s| \in [1 - 2\overline{\delta}, 1 + 2\overline{\delta}]. \tag{2.1}$$

Fixed an increasing function $p_0 \in C^\infty(\mathbb{R})$ such that $|p_0(x)| = 1$ for all $|x| \geq 1$, we consider on the space

$$\mathcal{H} = p_0 + H^1(\mathbb{R}),$$

the functional

$$\varphi(q) = \frac{1}{2}\|\dot{q}\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} a(x)W(q(x))\,dx.$$

Remark 2.2 Endowing $\mathcal{H}$ with the hilbertian structure induced by the map $p \in H^1(\mathbb{R}) \mapsto p_0 + p \in \mathcal{H}$, it is classical to prove that $\varphi \in \mathcal{C}^2(\mathcal{H})$ with Frechet differential

$$\varphi'(q)h =< \dot{q}, \dot{h} >_{L^2(\mathbb{R})} + \int_{\mathbb{R}} a(x)W'(q(x))h(x)\,dx, \quad q \in \mathcal{H},\ h \in H^1(\mathbb{R}),$$

and that critical points of φ are classical solutions to (P_1).

We are interested in the minimal properties of φ on $\mathcal{H}$ and we set

$$m = \inf_{\mathcal{H}} \varphi \quad \text{and} \quad \mathcal{M} = \{q \in \mathcal{H} \,/\, \varphi(q) = m\}.$$

We note that φ is weakly lower semicontinuous with respect to the $H^1_{loc}(\mathbb{R})$ convergence. In particular there results (see e.g. Lemma 2.1 in [4])

Lemma 2.1 *If $(q_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ and $\varphi(q_n) \le m+\lambda_0$ for all $n \in \mathbb{N}$ and some $\lambda_0 > 0$, then there exists $q \in H^1_{loc}(\mathbb{R})$ such that, along a subsequence, $q_n \to q$ weakly in $H^1_{loc}(\mathbb{R})$, $\dot{q}_n \to \dot{q}$ weakly in $L^2(\mathbb{R})$ and moreover $\varphi(q) \le \liminf_{n\to\infty} \varphi(q_n)$.*

On $\mathcal{H}$ is well defined with value in $\mathbb{R}$ the function

$$X(q) = \sup\{x \in \mathbb{R} \,|\, q(x) = 0\},$$

which associates to every function $q \in \mathcal{H}$ the supremum of the zero-set of q. As in Lemma 2.3 in [8], we can prove that if $(q_n)_{n\in\mathbb{N}}$ is a minimizing sequence of φ with $X(q_n)$ bounded, then $(q_n)_{n\in\mathbb{N}}$ is precompact with respect to the $H^1(\mathbb{R})$ topology. Precisely

Lemma 2.2 *Let $(q_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ with $X(q_n) \to X_0$ and $\varphi(q_n) \to m$ as $n \to +\infty$. Then, there exists $q \in \mathcal{M}$ such that $X(q) = X_0$ and, up to a subsequence, $\|q_n - q\|_{H^1} \to 0$ as $n \to +\infty$.*

Since the problem (P_1) is invariant under integer translations, given any minimizing sequence $(q_n)_{n\in\mathbb{N}}$ of φ, translating it if necessary, we can always assume that $X(q_n)$ is bounded, obtaining by Lemma 2.2 that $\mathcal{M} \ne \emptyset$. Moreover, by Remark 2.2, if $q \in \mathcal{M}$ then $q \in C^2(\mathbb{R})$ is a classical solution to (P_1) and since q minimizes φ we have $\|q\|_{L^\infty} \le 1$.

Again using Lemma 2.2 we obtain that if $\varphi(q_n) \to m$ then $\inf_{\bar{q}\in\mathcal{M}} \|q_n - \bar{q}\|_{H^1(\mathbb{R})} \to 0$ as $n \to +\infty$. Hence, we can say that for all $d > 0$ there exists $\nu_d > 0$ such that

$$\text{if} \quad q \in \mathcal{H} \quad \text{is such that} \quad \inf_{\bar{q}\in\mathcal{M}} \|q - \bar{q}\|_{H^1(\mathbb{R})} \ge d \quad \text{then} \quad \varphi(q) \ge m + \nu_d. \tag{2.2}$$

Another remarkable property of the functions $q \in \mathcal{M}$ is that they are uniformly exponentially asymptotic to the points ± 1 (see e.g. Proposition 2.2 in [10])

Lemma 2.3 *There exist $\omega, \ell > 0$ and $K > 0$ such that if $q \in \mathcal{M}$, then*

$$\begin{aligned} &0 \le 1 - q(x - X(q)) \le K e^{-\sqrt{\omega} x}, \quad \forall x \ge X(q) + \ell \quad \text{and} \\ &0 \le 1 + q(x - X(q)) \le K e^{\sqrt{\omega} x}, \quad \forall x \le X(q) - \ell \end{aligned}$$

The minimality of the functions $q \in \mathcal{M}$ implies moreover that $\mathcal{M}$ is an ordered set.

Lemma 2.4 *If $q_1, q_2 \in \mathcal{M}$ then, either $q_1(x) > q_2(x)$, $q_1(x) < q_2(x)$ or $q_1(x) = q_2(x)$ for all $x \in \mathbb{R}$.*

Proof The proof is based on the fact that since the functions in $\mathcal{M}$ are minima of the action φ on $\mathcal{H}$, then if $q_1 \ne q_2 \in \mathcal{M}$ their graphs cannot intersect. Indeed if there exist $x_0 \in \mathbb{R}$ such that $q_1(x_0) = q_2(x_0)$ then we can compare the values $m_1 = \varphi_{(-\infty,x_0)}(q_1)$ and $m_2 = \varphi_{(-\infty,x_0)}(q_2)$. We can simply exclude the case $m_1 \ne m_2$ since in this situation we would have that, defined

$$\tilde{q}(x) = \begin{cases} q_1(x) & x \le x_0 \\ q_2(x) & x > x_0 \end{cases} \quad \text{if } m_1 < m_2 \ \text{ or } \ \tilde{q}(x) = \begin{cases} q_2(x) & x \le x_0 \\ q_1(x) & x > x_0 \end{cases} \quad \text{if } m_1 > m_2,$$

then $\tilde{q} \in \mathcal{H}$ but $\varphi(\tilde{q}) < m = \min_{\mathcal{H}} \varphi$, a contradiction. Then $m_1 = m_2$ and hence the function

$$\tilde{q}(x) = \begin{cases} q_1(x), & x \le x_0, \\ q_2(x), & x < x_0, \end{cases}$$

belongs to $\mathcal{M}$ and solves (P_1). By uniqueness of the solution of the Cauchy problem, we must have that $\tilde{q} \equiv q_1$ and $\tilde{q} \equiv q_2$ and hence $q_1 \equiv q_2$. □

Remark 2.3 Arguing as in the proof of the previous Lemma, one can show that if $q_0 \in \mathcal{M}$ then for every (eventually unbouded) interval $I \subset \mathbb{R}$, setting

$$\varphi_I(q) = \frac{1}{2}\|\dot q\|^2_{L^2(I)} + \int_I a(x)W(q(x))\,dx,$$

q_0 is also a minimum for φ_I on the set $\{q \in H^1_{loc}(I)\,|\,q(\partial I) = q_0(\partial I)\}$. In particular we derive that if $q \in \mathcal{H}$ and $q_0 \in \mathcal{M}$ are such that $q(\partial I) = q_0(\partial I)$ for some interval $I \subset \mathbb{R}$, then, defining

$$\hat q(x) = \begin{cases} q(x) & \text{if } x \notin I, \\ q_0(x) & \text{if } x \in I, \end{cases}$$

there results $\hat q \in \mathcal{H}$ and $\varphi(\hat q) \le \varphi(q)$.

We finally discuss some consequences of the assumptions $(*)$ and $(**)$. Recall first our discreteness assumption $(*)$ on $\mathcal{M}$:

$(*)$ for all $q \in \mathcal{M}$ there results $q(0) \neq 0$.

We note that assumption $(*)$ is equivalent to require that there are not odd function in $\mathcal{M}$. Indeed, note that by the symmetric assumptions on the potential, if $q \in \mathcal{H}$, setting

$$q^*(x) = -q(-x), \quad x \in \mathbb{R},$$

there results $q^* \in \mathcal{H}$ and $\varphi(q) = \varphi(q^*)$. Therefore if $q \in \mathcal{M}$ is such that $q(0) = 0$, then $q^* \in \mathcal{M}$ and $q^*(0) = 0$. Hence, by the ordering property given by Lemma 2.4, $q \equiv q^*$, i.e., q is an odd function.

By condition $(*)$ we clearly have $X(q) \neq 0$ for all $q \in \mathcal{M}$ and so, using Lemmas 2.2 and 2.4, we obtain

Lemma 2.5 *There exist $q_\pm \in \mathcal{M}$ such that $q_- = (q_+)^*$, $q_-(0) < 0 < q_+(0)$ and for every $q \in \mathcal{M} \setminus \{q_\pm\}$ there results either $q(x) < q_-(x)$ or $q(x) > q_+(x)$ for all $x \in \mathbb{R}$.*

Proof Since $X(q) \neq 0$ for all $q \in \mathcal{M}$, by Lemma 2.2 we obtain that there exists $\delta > 0$ such that $|X(q)| > \delta$ for all $q \in \mathcal{M}$. By Lemma 2.2, there exists $q_+ \in \mathcal{M}$ such that $X(q_+) = \max\{X(q) < 0\,|\,q \in \mathcal{M}\}$. By Lemma 2.4 we have $0 < q_+(0) = \min\{q(0) > 0\,|\,q \in \mathcal{M}\}$. By symmetry, letting $q_- = (q_+)^*$ we obtain $q_- \in \mathcal{M}$, $X(q_-) = \min\{X(q) > 0\,|\,q \in \mathcal{M}\}$ and $\max\{q(0) < 0\,|\,q \in \mathcal{M}\} = q_-(0) < 0$. Then, by Lemma 2.4, the lemma follows.

Remark 2.4 By Lemma 2.4, we have $q_-(x) < q_+(x)$ for all $x \in \mathbb{R}$ and we set

$$\|q_+ - q_-\|_{L^2} = 2d_0 > 0. \tag{2.3}$$

Moreover note that if $q \in \mathcal{H}$ is such that $q_- \le q \le q_+$ in $\mathbb{R}$, then

$$\inf_{\bar q \in \mathcal{M}} \|q - \bar q\|_{H^1} = \inf_{\bar q = q_\pm} \|q - \bar q\|_{H^1}.$$

The assumption $(**)$ requires that the extremal functions $q_\pm$ given by Lemma 2.5 satisfy the following non degenerate condition

$(**)$ there exist $\omega^* > 0$ such that

$$\varphi''(q_\pm)h\cdot h = \int_{\mathbb{R}} |\dot h(x)|^2 + a(x)W''(q_\pm(x))|h(x)|^2\,dx \ge \omega^*\|h\|^2_{L^2} \quad \forall h \in H^1(\mathbb{R}).$$

As consequence we obtain

Lemma 2.6 *There exists* $\nu^* > 0$ *such that if* $q \in \mathcal{H}$ *is such that* $q_-(x) \le q(x) \le q_+(x)$ *for a.e.* $x \in \mathbb{R}$ *and* $\varphi(q) \le m + \nu^*$ *then*

$$\varphi(q) - m \ge \frac{\omega^*}{4} \inf_{\bar{q} \in \mathcal{M}} \|q - \bar{q}\|^2_{L^2}.$$

Proof Setting $\overline{W} = \frac{1}{6} \max_{s \in [-1,1]} |W'''(s)|$, note first that by the compactness property of Lemma 2.2 there exists $\nu^* > 0$ such that if $q \in \mathcal{H}$, $q_-(x) \le q(x) \le q_+(x)$ for a.e. $x \in \mathbb{R}$ and $\varphi(q) \le m + \nu^*$ then

$$\inf_{\bar{q}=q_\pm} \|q - \bar{q}\|_{H^1} \le \frac{\omega^*}{4\overline{a}\,\overline{W} c_0}, \tag{2.4}$$

where c_0 is the immersion constant $H^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ and $\overline{a} = \max_{\mathbb{R}} a(x)$. Now, assuming that $\inf_{\bar{q}=q_\pm} \|q - \bar{q}\|_{H^1} = \|q - q_+\|_{H^1}$, using the Taylor Formula, we infer that

$$\begin{aligned}
\varphi(q) - \varphi(q_+) &= \varphi'(q_+)(q - q_+) + \frac{1}{2}\varphi''(q_+)(q - q_+)(q - q_+) \\
&\quad + \int_{\mathbb{R}} a(x)\Big(W(q) - W(q_+) - W'(q_+)(q - q_+) - \frac{1}{2}W''(q_+)(q - q_+)^2\Big)\, dx \\
&\ge \frac{\omega^*}{2} \|q - q_+\|^2_{L^2} - \overline{a}\,\overline{W} \|q - q_+\|^3_{L^3} \\
&\ge \left(\frac{\omega^*}{2} - \overline{a}\,\overline{W} c_0 \|q - q_+\|_{H^1}\right) \|q - q_+\|^2_{L^2}.
\end{aligned}$$

Then the lemma follows using (2.4). □

3 Two dimensional heteroclinic type solutions

In this section we display some results concerning the solutions of the two dimensional problem

$$\begin{cases} -\Delta v(x,y) + a(x) W'(v(x,y)) = 0, & (x,y) \in \mathbb{R}^2 \\ \lim_{x \to \pm\infty} v(x,y) = \pm 1, & \text{uniformly w.r.t. } y \in \mathbb{R}, \end{cases} \tag{P_2}$$

which are asymptotic as $y \to \pm\infty$ to the functions q_- and q_+. Some of the results in this section are known or can be recovered from the general ones concerning the minimality and the asymptotic behaviour described in Sect. 2 in [51]. However, since the technical setting is different, for the sake of clarity and completeness we give here most of the details.

Let us consider the renormalized functional

$$\begin{aligned}
\varphi_2(v) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \tfrac{1}{2} |\nabla v(x,y)|^2 + a(x) W(v(x,y))\, dx - m \right) dy \\
&= \int_{\mathbb{R}} \tfrac{1}{2} \|\partial_y v(\cdot, y)\|^2_{L^2} + \varphi(v(\cdot, y)) - m\, dy
\end{aligned}$$

which is well defined on the space

$$\mathcal{X} = \{ v \in H^1_{loc}(\mathbb{R}^2) \mid v(\cdot, y) \in \mathcal{H} \text{ for a.e. } y \in \mathbb{R} \}.$$

Note that $\varphi_2(v) \geq 0$ for all $v \in \mathcal{X}$ and if $q \in \mathcal{M}$, then the function $v(x, y) = q(x)$ belongs to $\mathcal{X}$ and $\varphi_2(v) = 0$, i.e., the minimal solutions of (P_1) are global minima of φ_2 on $\mathcal{X}$.

We will look for bidimensional solutions of (P_2) as minima of φ_2 on suitable subspaces of $\mathcal{X}$. We recall (see e.g. [8]) that φ_2 is weakly lower semicontinuous on $\mathcal{X}$ with respect to the $H^1_{loc}(\mathbb{R}^2)$. Concerning the coerciveness of φ_2, we display here below some basic estimates which will be useful to characterize the compactness properties of sublevels of φ_2.

First we note that if $v \in \mathcal{X}$ then $\varphi(v(\cdot, y)) \geq m$ for a.e. $y \in \mathbb{R}$ and so

$$\|\partial_y v\|^2_{L^2(\mathbb{R}^2)} \leq 2\varphi_2(v) \qquad \forall v \in \mathcal{X}. \tag{3.1}$$

Moreover, if $v \in \mathcal{X}$ then $v(x, \cdot) \in H^1_{loc}(\mathbb{R})$ for a.e. $x \in \mathbb{R}$. Therefore, if $y_1 < y_2 \in \mathbb{R}$ then $v(x, y_2) - v(x, y_1) = \int_{y_1}^{y_2} \partial_y v(x, y)\, dy$ holds for all $v \in \mathcal{X}$ and a.e. $x \in \mathbb{R}$. So, if $v \in \mathcal{X}$, by (3.1), for $y_1 < y_2 \in \mathbb{R}$ we obtain

$$\begin{aligned} \|v(\cdot, y_2) - v(\cdot, y_1)\|^2_{L^2} &= \int_{\mathbb{R}} \left| \int_{y_1}^{y_2} \partial_y v(x, y)\, dy \right|^2 dx \leq |y_2 - y_1 \left| \int_{\mathbb{R}} \int_{\mathbb{R}} |\partial_y v(x, y) \right|^2 dy\, dx \\ &\leq 2\varphi_2(v)|y_2 - y_1|. \end{aligned} \tag{3.2}$$

Given any interval $I \subset \mathbb{R}$, let us denote

$$\varphi_{I,2}(v) = \int_I \tfrac{1}{2}\|v(\cdot, y)\|^2_{L^2} + \varphi(v(\cdot, y)) - m\, dy, \quad v \in \mathcal{X}.$$

By (2.2), if $(y_1, y_2) \subset \mathbb{R}$ and $v \in \mathcal{X}$ are such that $\inf_{q \in \mathcal{M}} \|v(\cdot, y) - q\|_{H^1(\mathbb{R})} \geq d > 0$ for a.e. $y \in (y_1, y_2)$, then there exists $\nu_d > 0$ such that

$$\begin{aligned} \varphi_2(v) \geq \varphi_{(y_1,y_2),2}(v) &\geq \int_{y_1}^{y_2} \tfrac{1}{2} \int_{\mathbb{R}} |\partial_y v(x, y)|^2\, dx\, dy + \nu_d(y_2 - y_1) \\ &\geq \tfrac{1}{2(y_2 - y_1)} \int_{\mathbb{R}} \left(\int_{y_1}^{y_2} |\partial_y v(x, y)|\, dy \right)^2 dx + \nu_d(y_2 - y_1) \\ &\geq \tfrac{1}{2(y_2 - y_1)} \|v(\cdot, y_1) - v(\cdot, y_2)\|^2_{L^2} + \nu_d(y_2 - y_1) \geq \sqrt{2\nu_d}\|v(\cdot, y_1) - v(\cdot, y_2)\|_{L^2}. \end{aligned} \tag{3.3}$$

As consequence of (3.2) and (3.3) we get information on the asymptotic behavior as $y \to \pm\infty$ of the functions in the sublevels of φ_2. More precisely we have

Lemma 3.1 *If $v \in \mathcal{X}$ and $\varphi_2(v) < +\infty$, then $\mathsf{d}_{L^2}(v(\cdot, y), \mathcal{M}) \to 0$ as $y \to \pm\infty$.*

Proof We only sketch the proof (see Lemma 3.3 in [8] for more details). First of all note that since $\varphi_2(v) < +\infty$, we have $\liminf_{y \to \pm\infty} \varphi(v(\cdot, y)) - m = 0$, and so by (2.2) we obtain

$$\liminf_{y \to \pm\infty} \mathsf{d}_{L^2}(v(\cdot, y), \mathcal{M}) = 0.$$

Considering the case $y \to +\infty$ we assume that $\limsup_{y \to +\infty} \mathsf{d}_{L^2}(v(\cdot, y), \mathcal{M}) > 0$. Then, by (3.2) the path $y \to v(\cdot, y)$ crosses infinitely many times an annulus of positive thickness $d > 0$ around $\mathcal{M}$ in the L^2 metric. This allows us to use (3.3) to conclude that $\varphi_2(u) = +\infty$, a contradiction. ◻

We are interested in solutions of (P_2) which connect $q_\pm$ as $y \to \pm\infty$, in particular on the minima of φ_2 on the symmetric classes

$$\mathcal{H}_2^- = \{v \in \mathcal{X} \mid \lim_{y\to\pm\infty} \|v(\cdot, y) - q_\mp\|_{L^2} = 0\},$$
$$\mathcal{H}_2^+ = \{v \in \mathcal{X} \mid \lim_{y\to\pm\infty} \|v(\cdot, y) - q_\pm\|_{L^2} = 0\}.$$

Note that if $v \in \mathcal{X}$, setting $v^*(x, y) = -v(-x, y)$, we get $\|v^*(\cdot, y) - q_\pm\|_{L^2} = \|v(\cdot, y) - q_\mp\|_{L^2}$ hence, if $v \in \mathcal{H}_2^+$ then $v^* \in \mathcal{H}_2^-$ and viceversa. Moreover, by the symmetric assumptions on the functions a and W, $\varphi_2(v) = \varphi_2(v^*)$ and we derive

$$m_2 \equiv \inf_{v\in\mathcal{H}_2^+} \varphi_2(v) = \inf_{v\in\mathcal{H}_2^-} \varphi_2(v).$$

Using suitable test functions one plainly see that $m_2 < +\infty$. Moreover we have

Lemma 3.2 *For every $v \in \mathcal{H}_2^\pm$ there exists $\hat{v} \in \mathcal{H}_2^\pm$ such that $\varphi_2(\hat{v}) \le \varphi_2(v)$ and*

$$q_-(x) \le \hat{v}(x, y) \le q_+(x) \quad \textit{for a.e. } (x, y) \in \mathbb{R}^2.$$

Proof Let $v \in \mathcal{H}_2^+$ and setting

$$\hat{v}(x, y) = \max\{\min\{v(x, y); q_+(x)\}; q_-(x)\},$$

note that $\hat{v} \in \mathcal{H}_2^+$ and $q_-(x) \le \hat{v}(x, y) \le q_+(x)$ for a.e. $(x, y) \in \mathbb{R}^2$. We claim that $\varphi_2(\hat{v}) \le \varphi_2(v)$.

First note that if $y \in \mathbb{R}$ is such that $v(\cdot, y) \in \mathcal{H}$ and $v(x_0, y) > q_+(x_0)$ for some $x_0 \in \mathbb{R}$, by continuity, there exists an interval I (eventually unbounded) such that $v(x, y) > q_+(x)$ for all $x \in I$ and $v(\partial I, y) = q_+(\partial I)$. Hence, setting $B_+(y) = \{x \in \mathbb{R} \mid v(x, y) > q_+(x)\}$, we have that $B_+(y) = \cup_{\alpha\in A} I_\alpha$ with I_α disjoint intervals such that $v(\partial I_\alpha, y) = q_+(\partial I_\alpha)$. Then, setting $\underline{v}(x, y) = \min\{v(x, y); q_+(x)\}$, we have

$$\underline{v}(x, y) = \begin{cases} v(x, y) & \text{if } x \notin B_+(y), \\ q_+(x) & \text{if } x \in B_+(y), \end{cases}$$

and, by Remark 2.3, we obtain $\varphi(\underline{v}(\cdot, y)) \le \varphi(v(\cdot, y))$. This holds true for almost every $y \in \mathbb{R}$ and so we deduce that

$$\int_{\mathbb{R}} \varphi(\underline{v}(\cdot, y)) - m\, dy \le \int_{\mathbb{R}} \varphi(v(\cdot, y)) - m\, dy.$$

Moreover, see e.g. Lemma 7.6 in [36], for a.e. $y \in \mathbb{R}$ there results

$$\partial_y \underline{v}(x, y) = \begin{cases} \partial_y v(x, y) & \text{if } x \notin B_+(y), \\ 0 & \text{if } x \in B_+(y), \end{cases}$$

and hence $\|\partial_y \underline{v}(\cdot, y)\|^2_{L^2(\mathbb{R}^2)} \le \|\partial_y v(\cdot, y)\|^2_{L^2(\mathbb{R}^2)}$. We conclude that $\varphi_2(\underline{v}) \le \varphi_2(v)$ and $\underline{v}(x, y) \le q_+(x)$ for a.e. $(x, y) \in \mathbb{R}^2$. Finally, since $\hat{v}(x, y) = \max\{\underline{v}(x, y); q_-(x)\}$, we obtain analogously that $\varphi_2(\hat{v}) \le \varphi_2(\underline{v}) \le \varphi_2(v)$ and $q_-(x) \le \hat{v}(x, y) \le \underline{v}(x, y) \le q_+(x)$ for a.e. $(x, y) \in \mathbb{R}^2$. □

By Lemma 3.2, to find a minimum of φ_2 in $\mathcal{H}_2^\pm$, we can restrict ourselves to consider minimizing sequences $(v_n)_{n\in\mathbb{N}}$ which satisfy the condition $q_-(x) \le v_n(x, y) \le q_+(x)$ for a.e. $(x, y) \in \mathbb{R}^2$. We have the following first compactness property.

Lemma 3.3 *Let $(v_n)_{n\in\mathbb{N}} \subset \mathcal{H}_2^{\pm}$ be such that $q_-(x) \le v_n(x,y) \le q_+(x)$ a.e. $\mathbb{R}^2$ and $\varphi_2(v_n) \le C$ for all $n \in \mathbb{N}$ and some $C > 0$.*
Then, there exist $v \in \mathcal{X}$ and a subsequence of $(v_n)_{n\in\mathbb{N}}$ (still denoted by v_n), such that $v_n - v \to 0$ weakly in $H^1(\mathbb{R}\times[-L,L])$ for all $L \in \mathbb{N}$.
Moreover, $\|v(\cdot,y) - q_\pm\|_{L^2} \le \liminf_{n\to\infty} \|v_n(\cdot,y) - q_\pm\|_{L^2}$ for a.e. $y \in \mathbb{R}$.

Proof We first show that there exists $v \in H^1_{loc}(\mathbb{R}^2)$ such that, along a subsequence, $v_n - p_0 \to v - p_0$ weakly in $H^1(\mathbb{R}\times[-L,L])$ for any $L \in \mathbb{N}$. This will imply also that $v \in \mathcal{X}$. To this aim, note that since $q_- \le v_n \le q_+$ a.e. in $\mathbb{R}^2$, we have that $\sup_{y\in\mathbb{R}} \|v_n(\cdot,y) - q_+\|_{L^2} \le \|q_- - q_+\|_{L^2} = 2d_0$, for every $n \in \mathbb{N}$. Then, $\|v_n - q_+\|^2_{L^2(\mathbb{R}\times[-L,L])} \le 8Ld_0^2$ for any $n \in \mathbb{N}$ and $L \in \mathbb{N}$. Since moreover $\|\nabla v_n\|^2_{L^2(\mathbb{R}\times[-L,L])} \le 2(\varphi_2(v_n) + 2Lm)$ we conclude that the sequence $(v_n - q_+)_{n\in\mathbb{N}}$, and so the sequence $(v_n - p_0)_{n\in\mathbb{N}}$, is bounded in $H^1(\mathbb{R}\times[-L,L])$ for any $L \in \mathbb{N}$. Then, a diagonal argument implies the existence of a function $v \in H^1_{loc}(\mathbb{R}^2)$ and a subsequence $(v_{n_j})_{j\in\mathbb{N}}$, such that $v_{n_j} - v \to 0$ weakly in $H^1(\mathbb{R}\times[-L,L])$ for all $L \in \mathbb{N}$.

To conclude the proof note that since $v_{n_j} \to v$ in $L^2_{loc}(\mathbb{R}^2)$ there exists $A \subset \mathbb{R}$ with $\text{meas}(A) = 0$ such that $\|v_{n_j}(\cdot,y) - v(\cdot,y)\|_{L^2(-M,M)} \to 0$ for every $M > 0$ and $y \in \mathbb{R}\setminus A$. Then for all $M > 0$ and $y \in \mathbb{R}\setminus A$

$$\|v(\cdot,y) - q_\pm\|_{L^2(-M,M)} \le o(1) + \|v_{n_j}(\cdot,y) - q_\pm\|_{L^2(-M,M)} \le o(1) + \|v_{n_j}(\cdot,y) - q_\pm\|_{L^2}$$

and the lemma follows.

Moreover we have the following concentration result.

Lemma 3.4 *There exist $\delta_0 \in (0, \frac{d_0}{2})$ and $\bar\lambda > 0$ such that if $v \in \mathcal{H}_2^+$, $q_-(x) \le v(x,y) \le q_+(x)$ for a.e. $(x,y) \in \mathbb{R}^2$ and $\varphi_2(v) \le m_2 + \bar\lambda$ then*

(i) *if $\|v(\cdot,y_0) - q_-\|_{H^1} \le \delta_0$ then $\|v(\cdot,y) - q_-\|_{L^2} \le d_0$ for a.e. $y \le y_0$,*
(ii) *if $\|v(\cdot,y_0) - q_+\|_{H^1} \le \delta_0$ then $\|v(\cdot,y) - q_+\|_{L^2} \le d_0$ for a.e. $y \ge y_0$.*

Proof We set $\bar\lambda = \frac{d_0}{4}\sqrt{\frac{\nu_0}{2}}$, where ν_0 is given by (3.3) corresponding to $d = \frac{d_0}{2}$, and let $\delta_0 \in (0, \min\{\frac{d_0}{2}, \sqrt{2\bar\lambda}\})$ be such that

$$\sup\{\varphi(q) \mid \|q - q_-\|_{H^1(\mathbb{R})} \le \delta_0\} \le m + \bar\lambda.$$

Let $v \in \mathcal{H}_2^+$ be such that $q_-(x) \le v(x,y) \le q_+(x)$ for a.e. $(x,y) \in \mathbb{R}^2$, $\varphi_2(u) \le m_2 + \bar\lambda$ and assume that $y_0 \in \mathbb{R}$ is such that $\|v(\cdot,y_0) - q_-\|_{H^1(\mathbb{R})} \le \delta_0$. We define

$$\hat v(x,y) = \begin{cases} q_-(x) & \text{if } y \le y_0 - 1, \\ v(x,y_0)(y - y_0 + 1) + q_-(x)(y_0 - y) & \text{if } y_0 - 1 \le y \le y_0, \\ v(x,y) & \text{if } y \ge y_0. \end{cases}$$

We have $\hat v \in \mathcal{H}_2^+$ and so $\varphi_2(\hat v) \ge m_2$. Then, we obtain

$$\begin{aligned} m_2 \le \varphi_2(\hat v) &= \varphi_2(u) - \varphi_{(-\infty,y_0),2}(v) + \int_{y_0-1}^{y_0} \frac{1}{2}\int_{\mathbb{R}} |v(x,y_0) - q_-(x)|^2 dx\, dy + \\ &\quad + \int_{y_0-1}^{y_0} \varphi(v(x,y_0)(y - y_0 + 1) + q_-(x)(y_0 - y)) - m\, dy \\ &\le \varphi_2(v) - \varphi_{(-\infty,y_0),2}(v) + \tfrac{1}{2}\delta_0^2 + \bar\lambda \le \varphi_2(v) - \varphi_{(-\infty,y_0),2}(v) + 2\bar\lambda \end{aligned}$$

from which, since $\varphi_2(u) \leq m_2 + \bar{\lambda}$ we conclude that $\varphi_{(-\infty,y_0),2}(v) \leq 3\bar{\lambda}$. Assume by contradiction that there exists $y_1 < y_0$ such that $\|v(\cdot, y_1) - q_-\|_{L^2} \geq d_0$. Then by (3.2) there exists $(y_1', y_0') \subset (y_1, y_0)$ such that $\|v(\cdot, y_1') - v(\cdot, y_0')\|_{L^2} \geq \frac{d_0}{2}$ and, for any $y \in (y_1', y_0')$, $\|v(\cdot, y) - q_-\|_{L^2} \in (\frac{d_0}{2}, d_0)$. In particular, since $q_-(x) \leq v(x, y) \leq q_+(x)$ for a.e. $(x, y) \in \mathbb{R}^2$, we get that

$$\inf_{q\in\mathcal{M}} \|v(\cdot, y) - q\|_{H^1(\mathbb{R})} = \|v(\cdot, y) - q_-\|_{L^2} \geq \frac{d_0}{2}$$

for a.e. $y \in (y_1', y_0')$ and using (3.3) we get the contradiction $3\bar{\lambda} \geq \varphi_{(-\infty,y_0),2}(v) \geq \sqrt{\frac{\nu_0}{2}} d_0 = 4\bar{\lambda}$.

Similarly one can show that if $\|v(\cdot, y_0) - q_+\|_{H^1} \leq \delta_0$ then $\|u(\cdot, y) - q_+\|_{L^2} \leq d_0$ for all $y \geq y_0$. □

Lemma 3.4 has the following key consequence.

Lemma 3.5 *There exists $\bar{\ell} > 0$ such that if $v \in \mathcal{H}_2^+$, $q_-(x) \leq v(x, y) \leq q_+(x)$ for a.e. $(x, y) \in \mathbb{R}^2$, $\varphi_2(v) \leq m_2 + \bar{\lambda}$ and $\min_{q=q_\pm} \|v(\cdot, 0) - q\|_{L^2} \geq \frac{d_0}{2}$ then*

$$\|v(\cdot, y) - q_-\|_{L^2} \leq d_0 \ \textit{for} \ y \leq -\bar{\ell} \ \textit{and} \ \|v(\cdot, y) - q_+\|_{L^2} \leq d_0 \ \textit{for} \ y \geq \bar{\ell}.$$

Proof Considering δ_0 as defined in Lemma 3.4, by (3.3) we can fix $\bar{\ell} > 0$ such that if I is any real interval with length greater than or equal to $\bar{\ell}$ and $v \in \mathcal{X}$ is such that $\inf_{q\in\mathcal{M}} \|v(\cdot, y) - q\|_{H^1} > \delta_0$ for a.e. $y \in I$ then $\varphi_2(v) \geq m_2 + 2\bar{\lambda}$. Since $\min_{q=q_\pm} \|v(\cdot, 0) - q\|_{L^2} \geq \frac{d_0}{2}$, we derive that $\inf_{q\in\mathcal{M}} \|v(\cdot, 0) - q\|_{H^1} > \delta_0$ and since $\varphi_2(v) \leq m_2 + \bar{\lambda}$, by definition of $\bar{\ell}$ there exist $y_- \in (-\bar{\ell}, 0)$ and $y_+ \in (0, \bar{\ell})$ such that $\inf_{q\in\mathcal{M}} \|v(\cdot, y_\pm) - q\|_{H^1} < \delta_0$. By Lemma 3.4, since $v \in \mathcal{H}_2^+$ and $q_-(x) \leq v(x, y) \leq q_+(x)$ for a.e. $(x, y) \in \mathbb{R}^2$, we derive that necessarily $\|v(\cdot, y_-) - q_-\|_{H^1} \leq \delta_0$ and $\|v(\cdot, y_+) - q_+\|_{H^1} \leq \delta_0$ and the lemma follows applying again Lemma 3.4. □

Lemma 3.5 together with Lemma 3.3 allow us to use the direct method of the Calculus of Variation to show that the functional φ_2 admits a minimum in the class $\mathcal{H}_2^\pm$. Setting

$$\mathcal{M}_2^\pm = \{v \in \mathcal{H}_2^\pm \,/\, \varphi_2(v) = m_2\}$$

we have

Proposition 3.1 *There exists $v_\pm \in \mathcal{M}_2^\pm$ such that $q_-(x) \leq v_\pm(x, y) \leq q_+(x)$, for all $(x, y) \in \mathbb{R}^2$.*

Proof By symmetry, it is sufficient to prove that there exists $v \in \mathcal{M}_2^+$. Let $(v_n)_{n\in\mathbb{N}}$ be a minimizing sequence for φ_2 in $\mathcal{H}_2^+$ which, by Lemma 3.2, we can assume such that $q_-(x) \leq v_n(x, y) \leq q_+(x)$ for a.e. $(x, y) \in \mathbb{R}^2$. Since $v_n \in \mathcal{H}_2^+$ we know that $\lim_{y\to\pm\infty} \|v_n(\cdot, 0) - q_\pm\|_{L^2} = 0$ and so, since $\|q_+ - q_-\|_{L^2} = 2d_0$ and since by (3.2) the map $y \mapsto v_n(\cdot, y) \in \mathcal{H}_2^+$ is continuous with respect to the L^2 metric, we deduce that there exists $y_{0,n} \in \mathbb{R}$ such that $\min_{q=q_\pm} \|v_n(\cdot, y_{0,n}) - q\|_{L^2} \geq \frac{d_0}{2}$. By y-translation invariance we can assume that $y_{0,n} = 0$ so that $\min_{q=q_\pm} \|v_n(\cdot, 0) - q\|_{L^2} \geq \frac{d_0}{2}$ holds true for all $n \in \mathbb{N}$. Hence, by Lemma 3.5, there exists $\bar{\ell} > 0$ such that for all $n \in \mathbb{N}$

$$\|v_n(\cdot, y) - q_-\|_{L^2} \leq d_0 \quad \text{for a.e.} \quad y \leq -\bar{\ell} \quad \text{and} \quad \|v_n(\cdot, y) - q_+\|_{L^2} \leq d_0 \quad \text{for a.e.} \quad y \geq \bar{\ell}.$$

Now, by Lemma 3.3, there exists $v \in \mathcal{X}$ such that, along a subsequence, $v_n \to v$ weakly in $H^1_{loc}(\mathbb{R}^2)$. By semicontinuity we have $\varphi_2(v) \leq m_2$ and moreover, again by Lemma 3.3,

$$\|v(\cdot, y) - q_-\|_{L^2} \leq d_0 \quad \text{for a.e.} \quad y \leq -\bar{\ell} \quad \text{and} \quad \|v(\cdot, y) - q_+\|_{L^2} \leq d_0 \quad \text{for a.e.} \quad y \geq \bar{\ell}.$$

Then, since by pointwise convergence $q_-(x) \leq v(x, y) \leq q_+(x)$ for a.e. $(x, y) \in \mathbb{R}^2$, by Lemma 3.1 and the previous estimate we obtain $\|v(\cdot, y) - q_\pm\|_{L^2} \to 0$ as $y \to \pm\infty$. Hence $v \in \mathcal{H}_2^+$ and $\varphi_2(v) = m_2$ follows. □

Remark 3.1 Since $v_\pm$ are minima of φ_2 on $\mathcal{H}_2^\pm$, it is classical to prove that they are $\mathcal{C}^2$ solution of $-\Delta v + aW'(v) = 0$ on $\mathbb{R}^2$. Since moreover $q_-(x) \leq v_\pm(x, y) \leq q_+(x)$, for all $(x, y) \in \mathbb{R}^2$, we obtain that $v_\pm(x, y) \to \pm 1$ as $x \to \pm\infty$ uniformly with respect to $y \in \mathbb{R}$ and so that $v_\pm$ are classical solutions to (P_2).

Using the maximum principle as in [45,47], we prove that $\mathcal{M}_2^\pm$ are ordered sets and that every $v \in \mathcal{M}_2^\pm$ is strictly monotone w.r.t. $y \in \mathbb{R}$.

Lemma 3.6 *If $v_1, v_2 \in \mathcal{M}_2^\pm$, then either $v_1 > v_2$, $v_1 < v_2$ or $v_1 \equiv v_2$ in $\mathbb{R}^2$. Moreover, if $v \in \mathcal{M}_2^+$, then $v(x, y_1) < v(x, y_2)$ for all $x \in \mathbb{R}$, $y_1 < y_2$.*

Proof Given $v_1, v_2 \in \mathcal{M}_2^+$, to prove the Lemma it is sufficient to show that if $v_1(x_0, y_0) = v_2(x_0, y_0)$ for some $(x_0, y_0) \in \mathbb{R}^2$, then $v_1(x, y) = v_2(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

We define $\bar{v}(x, y) = \max\{v_1(x, y), v_2(x, y)\}$ and $\underline{v}(x, y) = \min\{v_1(x, y), v_2(x, y)\}$. Then $q_+(x) \geq \bar{v}(x_0, y_0) \geq \underline{v}(x_0, y_0) \geq q_-(x)$ and $\bar{v}$, $\underline{v} \in \mathcal{H}_2^+$. Then

$$2m_2 \leq \varphi_2(\bar{v}) + \varphi_2(\underline{v}) = \varphi_2(v_1) + \varphi_2(v_2) = 2m_2$$

and we derive $\varphi_2(\bar{v}) = \varphi_2(\underline{v}) = m_2$. Hence $\bar{v}$, $\underline{v} \in \mathcal{M}_2^+$ and, by Remark 3.1, they are classical solution of $-\Delta v + aW'(v) = 0$ on $\mathbb{R}^2$. Setting $V = \bar{v} - \underline{v}$ we obtain that $V(x, y) \geq 0$ on $\mathbb{R}^2$, $V(x_0, y_0) = 0$ and V is a solution of the linear elliptic equation $-\Delta V + A(x)V = 0$ on $\mathbb{R}^2$ where

$$A(x) = \begin{cases} a(x)\frac{W'(\bar{v}(x,y)) - W'(\underline{v}(x,y))}{\bar{v}(x,y) - \underline{v}(x,y)}, & \text{if } \bar{v}(x, y) > \underline{v}(x, y) \\ a(x)W''(\bar{v}(x, y)) & \text{if } \bar{v}(x, y) = \underline{v}(x, y) \end{cases}$$

Then, $-\Delta V + \max\{A(x), 0\}V \geq 0$ and the maximum principle implies $V \equiv 0$, that is $\bar{v} \equiv \underline{v}$ and so $v_1 \equiv v_2$.

To prove the monotonicity property let us consider a function $v \in \mathcal{M}_2^+$. Given $y_1 < y_2$, by y-translations invariance, setting $v_1(x, y) = v(x, y + y_1)$ and $v_2(x, y) = v(x, y + y_2)$, we have $v_1, v_2 \in \mathcal{M}_2^+$. Assuming by contradiction that $v(x, y_1) = v(x, y_2)$ for some $x \in \mathbb{R}$, we get $v_1(x, 0) = v_2(x, 0)$ and hence, as proved above, $v_1 \equiv v_2$ in $\mathbb{R}^2$, i.e., $v(x, y + y_1) = v(x, y + y_2)$ for all $(x, y) \in \mathbb{R}^2$. This implies that $v(x, y + y_2 - y_1) = v(x, y)$, i.e., $v(x, y)$ is periodic in y of period $y_2 - y_1$, in contradiction with the fact that for a.e. $x \in \mathbb{R}$ we have

$$\lim_{y \to +\infty} v(x, y) = q_+(x) \neq q_-(x) = \lim_{y \to -\infty} v(x, y).$$

□

Remark 3.2 By Lemma 3.6 we have that every $v \in \mathcal{M}_2^\pm$ is monotone with respect to the y variable. Since $v(\cdot, y)$ is asymptotic to $q_\pm$ as $|y| \to +\infty$ we deduce that if $v \in \mathcal{M}_2^\pm$ then the condition $q_-(x) \leq v(x, y) \leq q_+(x)$ holds true for a.e. $(x, y) \in \mathbb{R}^2$.

By definition, since $v_\pm \in \mathcal{M}_2^\pm$, we have $v_+(\cdot, y) \to q_\pm$ and $v_-(\cdot, y) \to q_\mp$ as $y \to \pm\infty$ with respect to the L^2 metric. We can in fact say more

Lemma 3.7 *If $v_\pm \in \mathcal{M}_2^\pm$ then*

$$\|v_+(\cdot, y) - q_\pm\|_{L^\infty(\mathbb{R})} \to 0 \text{ and } \|v_-(\cdot, y) - q_\mp\|_{L^\infty(\mathbb{R})} \to 0 \text{ as } y \to \pm\infty.$$

Proof Letting $v_+ \in \mathcal{M}_2^+$ we know that v_+ solves (P_2) and by Remark 3.2 that $|v_+(x, y)| \le 1$ on $\mathbb{R}^2$. Then, by Schauder estimates, we have $\|v_+\|_{\mathcal{C}^2(\mathbb{R}^2)} < +\infty$. If by contradiction $\limsup_{y\to+\infty} \|v_+(\cdot, y) - q_\pm\|_{L^\infty(\mathbb{R})} \ge 2\rho_0 > 0$ then there exists a sequence $(x_n, y_n) \in \mathbb{R}^2$ with $y_n \to +\infty$ such that $|v_+(x_n, y_n) - q_\pm(x_n)| \ge \rho_0$. Then, since $\|v_+\|_{\mathcal{C}^2(\mathbb{R}^2)} < +\infty$, there exists $r_0 > 0$ such that $|v_+(x, y_n) - q_\pm(x)| \ge \rho_0/2$ whenever $|x - x_n| \le r_0$ and $n \in \mathbb{N}$. Then $\|v_+(\cdot, y_n) - q_\pm\|^2_{L^2(\mathbb{R})} \ge r_0\rho_0^2/2$ for all $n \in \mathbb{N}$, a contradiction. □

We now prove that all the functions in $\mathcal{M}_2^\pm$ are odd in $\mathbb{R}^2$ modulo y-translation. In fact, setting

$$\tilde{\mathcal{X}} = \{v \in \mathcal{X} \,/\, v(-x, -y) = -v(x, y)\}$$

we have

Lemma 3.8 *For all $v \in \mathcal{M}_2^\pm$ there exists $y_0 \in \mathbb{R}$ such that $\tilde{v} \equiv v(\cdot, \cdot + y_0) \in \mathcal{M}_2^\pm \cap \tilde{\mathcal{X}}$.*

Proof By symmetry, it is enough to prove the statement for a given $v \in \mathcal{M}_2^+$. Since $v(0, y) \to q_\pm(0)$ as $y \to \pm\infty$ and since by definition $q_-(0) < 0 < q_+(0)$, there exists $y_0 \in \mathbb{R}$ such that $v(0, y_0) = 0$. By y-translation invariance we have that $\tilde{v} = v(\cdot, \cdot + y_0) \in \mathcal{M}_2^+$ and moreover $\tilde{v}(0, 0) = 0$. Considering $\tilde{v}^*(x, y) = -\tilde{v}(-x, -y)$ we have $\tilde{v}^* \in \mathcal{H}_2^\pm$ and, by the symmetric assumption on the functions a and W we have $\varphi_2(\tilde{v}^*) = \varphi_2(\tilde{v}) = m_2$. Hence, $\tilde{v}, \tilde{v}^* \in \mathcal{M}_2^+$ with $\tilde{v}(0, 0) = 0 = \tilde{v}^*(0, 0)$ and by Lemma 3.6 we conclude $\tilde{v} \equiv \tilde{v}^*$, i.e., $\tilde{v} \in \tilde{\mathcal{X}}$. □

Collecting the results above obtained we can conclude.

Proposition 3.2 *There exists a unique function $\tilde{v}^+ \in \mathcal{M}_2^+ \cap \tilde{\mathcal{X}}$. It is a classical solution to problem (P_2), it is monotone increasing w.r.t. $y \in \mathbb{R}$, it satisfies $q_-(x) \le \tilde{v}^+(x, y) \le q_+(x)$ for all $(x, y) \in \mathbb{R}^2$ and the asymptotic conditions $\tilde{v}^+(x, y) \to q_\pm(x)$ as $y \to \pm\infty$ uniformly on $\mathbb{R}$. Moreover, $\mathcal{M}_2^+ = \{\tilde{v}^+(\cdot, \cdot + y_0) \mid y_0 \in \mathbb{R}\}$.*

Proof By Lemma 3.8 we know that $\mathcal{M}_2^+ \cap \tilde{\mathcal{X}} \ne \emptyset$. To prove the uniqueness just note that by the antisymmetric condition we have that if $v_1, v_2 \in \mathcal{M}_2^+ \cap \tilde{\mathcal{X}}$, then $v_1(0, 0) = 0 = v_2(0, 0)$. Hence, since $v_1, v_2 \in \mathcal{M}_2^+$, by Lemma 3.6 we get $v_1(x, y) = v_2(x, y)$ for all $(x, y) \in \mathbb{R}^2$. The other assertions follow directly by the results above proved. □

Remark 3.3 By Proposition 3.2 we obtain a symmetric analogous statement for the function $\tilde{v}^-(x, y) = \tilde{v}^+(x, -y) = -\tilde{v}^+(-x, y)$, the unique element in $\mathcal{M}_2^- \cap \tilde{\mathcal{X}}$. In the sequel we will denote $\tilde{m}_{2,\infty} = m_2 = \varphi_2(\tilde{v}^\pm)$.

As last step in this section we use condition $(**)$ to obtain L^2 compactness of the minimizing sequences in $\tilde{\mathcal{H}}_2^\pm = \mathcal{H}_2^\pm \cap \tilde{\mathcal{X}}$.

Remark 3.4 We note that if $v \in \tilde{\mathcal{H}}_2^\pm$ and $q_-(x) \le v(x, y) \le q_+(x)$ for a.e. $(x, y) \in \mathbb{R}^2$, then

$$\inf_{q\in\mathcal{M}} \|v(\cdot, 0) - q\|_{L^2} \ge d_0.$$

Indeed if $v \in \tilde{\mathcal{H}}_2^\pm$ then the function $x \mapsto v(x, 0)$ is odd. Moreover since $q_-(x) \le v(x, 0) \le q_+(x)$ we have that $\inf_{q\in\mathcal{M}} \|v(\cdot, 0) - q\|_{L^2} = \min_{q=q_\pm} \|v(\cdot, 0) - q\|_{L^2}$. Since $q_-(x) =$

$-q_+(-x)$, we get $4d_0^2 = \|q_+ - q_-\|_{L^2}^2 = 2\int_0^{+\infty} |q_\pm(-x) + q_\pm(x)|^2\,dx$ and since $v(x,0)$ is odd we obtain $\|v(\cdot,0) - q_\pm\|_{L^2}^2 = \int_0^{+\infty} |v(x,0) - q_\pm(x)|^2 + |v(x,0) + q_\pm(-x)|^2\,dx \geq \frac{1}{2}\int_0^{+\infty} |q_\pm(-x) + q_\pm(x)|^2\,dx = d_0^2$.

Moreover, note that by Lemma 3.5, if $v \in \tilde{\mathcal{H}}_2^+, q_-(x) \leq v(x,y) \leq q_+(x)$ for a.e. $(x,y) \in \mathbb{R}^2$ and $\varphi_2(v) \leq \tilde{m}_{2,\infty} + \bar{\lambda}$ then $\|v(\cdot,y) - q_+\|_{L^2} \leq d_0$ for a.e. $y \geq \bar{\ell}$.

Lemma 3.9 *Let* $(v_n)_{n\in\mathbb{N}} \subset \tilde{\mathcal{H}}_2^\pm$ *be a minimizing sequence for* φ_2 *such that* $q_-(x) \leq v_n(x,y) \leq q_+(x)$ *for a.e.* $(x,y) \in \mathbb{R}^2$ *and every* $n \in \mathbb{N}$. *Then* $\|v_n - \tilde{v}^\pm\|_{L^2(\mathbb{R}^2)} \to 0$ *as* $n \to +\infty$.

Proof Let $(v_n) \subset \tilde{\mathcal{H}}_2^+$ be such that $\varphi_2(v_n) \to \tilde{m}_{2,\infty}$ and $q_-(x) \leq v_n(x,y) \leq q_+(x)$ for a.e. $(x,y) \in \mathbb{R}^2$ and all $n \in \mathbb{N}$. Assuming $\varphi_2(v_n) \leq \tilde{m}_{2,\infty} + \bar{\lambda}$, by Remark 3.4 we have

$$\|v_n(\cdot,y) - q_+\|_{L^2} \leq d_0 \text{ for all } n \in \mathbb{N} \text{ and } y \geq \bar{\ell}. \tag{3.4}$$

Then, arguing as in the proof of Lemma 3.3 and Proposition 3.1, given any subsequence, we can extract a sub-subsequence, denoted again by (v_n), such that $v_n - \tilde{v}^+ \to 0$ weakly in $H^1(S_L)$ for any $L \in \mathbb{N}$, where $S_L = \mathbb{R} \times [-L, L]$. Since $|v_n(x,y) - \tilde{v}^+(x,y)| \leq q_+(x) - q_-(x)$ and since $v_n - \tilde{v}^+ \to 0$ weakly in $H^1(S_L)$ for any $L \in \mathbb{N}$, we have that $v_n - \tilde{v}^+ \to 0$ strongly in $L^2(S_L)$ for any $L \in \mathbb{N}$. To conclude the proof it remains to show that

$$\forall \varepsilon > 0\, \exists L_\varepsilon > 0,\ \bar{n} \in \mathbb{N} \text{ such that } \|v_n - \tilde{v}^+\|_{L^2(\mathbb{R}^2\setminus S_{L_\varepsilon})} \leq \varepsilon, \quad \forall n \geq \bar{n}. \tag{3.5}$$

First note that, by semicontinuity, given any $L > 0$, we have

$$\int_L^{+\infty} \varphi(v_n(\cdot,y)) - m\,dy \to \int_L^{+\infty} \varphi(\tilde{v}^+(\cdot,y)) - m\,dy. \tag{3.6}$$

Indeed $\int_L^{+\infty} \varphi(v_n(\cdot,y)) - m\ dy = \varphi_2(v_n) - \frac{1}{2}\|\partial_y v_n\|_{L^2(\mathbb{R}^2)}^2 - \int_0^L \varphi(v_n(\cdot,y)) - m\ dy$. By semicontinuity $\|\partial_y \tilde{v}^+\|_{L^2(\mathbb{R}^2)} \leq \liminf \|\partial_y v_n\|_{L^2(\mathbb{R}^2)}$ and $\int_0^L \varphi(\tilde{v}^+(\cdot,y)) - m\ dy \leq \liminf \int_0^L \varphi(v_n(\cdot,y)) - m\,dy$. Since $\varphi_2(\tilde{v}^+) = \lim \varphi_2(v_n)$ we deduce

$$\limsup \int_L^{+\infty} \varphi(v_n(\cdot,y)) - m\,dy \leq \int_L^{+\infty} \varphi(\tilde{v}^+(\cdot,y)) - m\,dy$$

and (3.6) follows.

Then, given any $\varepsilon > 0$, we fix $L_\varepsilon \geq \bar{\ell}$ such that $\int_{L_\varepsilon}^{+\infty} \varphi(\tilde{v}^+(\cdot,y)) - m\,dy < \frac{\varepsilon}{2}$ and, by (3.6), we fix also $n_\varepsilon \in \mathbb{N}$ such that

$$\int_{L_\varepsilon}^{+\infty} \varphi(v_n(\cdot,y)) - m\,dy < \varepsilon \text{ for all } n \geq n_\varepsilon. \tag{3.7}$$

Then, letting $\nu^* > 0$ be given by Lemma 2.6 and setting

$$\mathcal{A}_n = \{y > L_\varepsilon \,/\, \varphi(v_n(\cdot,y)) - m > \nu^*\},$$

by (3.7), we obtain meas $(\mathcal{A}_n) \leq \frac{\varepsilon}{\nu^*}$ for any $n \geq n_\varepsilon$. Since $L_\varepsilon \geq \bar{\ell}$, by (3.4) we get

$$\int_{\mathcal{A}_n} \|v_n(\cdot,y) - q_+\|_{L^2}^2 dy \leq \frac{\varepsilon d_0}{\nu^*} \quad \text{for any} \quad n \geq n_\varepsilon. \tag{3.8}$$

Then we note that if $y \in (L_\varepsilon, +\infty) \setminus \mathcal{A}_n$, by definition we have $\varphi(v_n(\cdot, y)) - m \leq \nu^*$. Hence, by Lemma 2.6, noting that by (3.4) $\inf_{q \in \mathcal{M}} \|v_n(\cdot, y) - q\|_{L^2} = \|v_n(\cdot, y) - q_+\|_{L^2}$ for $y > L_\varepsilon$, we recover that

$$\|v_n(\cdot, y) - q_+\|_{L^2}^2 \leq \frac{4}{\omega^*}(\varphi(v_n(\cdot, y)) - m) \text{ for } y \in (L_\varepsilon, +\infty) \setminus \mathcal{A}_n \text{ and } n \geq n_\varepsilon. \quad (3.9)$$

Then, integrating (3.9) on $(L_\varepsilon, +\infty) \setminus \mathcal{A}_n$, by (3.7) we conclude that

$$\int\limits_{(L_\varepsilon, +\infty)\setminus \mathcal{A}_n} \|v_n(\cdot, y) - q_+\|_{L^2}^2 dy \leq \frac{4\varepsilon}{\omega^*} \text{ for any } n \geq n_\varepsilon. \quad (3.10)$$

By (3.8) and (3.10) for every $n \geq n_\varepsilon$ we recover

$$\int\limits_{(L_\varepsilon, +\infty)} \|v_n(\cdot, y) - q_+\|_{L^2}^2 dy \leq \varepsilon \left(\frac{4}{\omega^*} + \frac{d_0}{\nu^*} \right)$$

and, by semicontinuity, the same estimate holds with $\tilde{v}^+$ instead of v_n. Therefore we conclude

$$\int\limits_{(L_\varepsilon, +\infty)} \|v_n(\cdot, y) - \tilde{v}^+(\cdot, y)\|_{L^2}^2 dy \leq 4\varepsilon \left(\frac{4}{\omega^*} + \frac{d_0}{\nu^*} \right).$$

Since v_n and $\tilde{v}^+$ are odd functions the same holds true for $\int_{(-\infty, -L_\varepsilon)} \|v_n(\cdot, y) - \tilde{v}^+(\cdot, y)\|_{L^2}^2 dy$ and (3.5) follows. Since the original subsequence of v_n was arbitrary, the Lemma follows. □

4 Two dimensional periodic solutions

A second series of observations on the two dimensional problem regards, given $L > 0$, some variational properties of the solutions to the problem

$$\begin{cases} -\Delta v(x, y) + a(x) W'(v(x, y)) = 0, & (x, y) \in S_L, \\ v(x, y) = -v(-x, -y), & (x, y) \in S_L \\ \partial_y v(x, \pm L) = 0, & x \in \mathbb{R} \end{cases} \quad (\tilde{P}_{L,2})$$

where $S_L = \mathbb{R} \times [-L, L]$. Given $L > 0$ we consider on the space

$$\tilde{\mathcal{X}}_L = \{v_{|S_L} \mid v \in \tilde{\mathcal{X}}\},$$

the functional

$$\varphi_{[-L,L],2}(v) = \int_{-L}^{L} \tfrac{1}{2}\|v(\cdot, y)\|_{L^2}^2 + \varphi(v(\cdot, y)) - m \, dy.$$

We look for minima of $\varphi_{[-L,L],2}$ on $\tilde{\mathcal{X}}_L$ and we set

$$\tilde{m}_{2,L} = \inf_{\tilde{\mathcal{X}}_L} \varphi_{[-L,L],2} \quad \text{and} \quad \tilde{\mathcal{M}}_{2,L} = \{v \in \tilde{\mathcal{X}}_L \mid \varphi_{[-L,L],2}(v) = \tilde{m}_{2,L}\}.$$

Remark 4.1 Note that the map $L \mapsto \tilde{m}_{2,L}$ is not decreasing and $\tilde{m}_{2,L} \leq \tilde{m}_{2,\infty} \equiv m_2$ for all $L > 0$. Indeed, let $\tilde{v}^+ \in \tilde{\mathcal{M}}_2^+$ be given by Lemma 3.8. Then $\tilde{v}_L^+ = \tilde{v}|_{S_L} \in \tilde{\mathcal{X}}_L$ and

$$\tilde{m}_{2,L} \leq \varphi_{[-L,L],2}(\tilde{v}_L^+) \leq \varphi_2(\tilde{v}^+) = \tilde{m}_{2,\infty}$$

The application of the direct method of the calculus of variation allows us to show that $\tilde{\mathcal{M}}_{2,L}$ is not empty for every $L > 0$.

Proposition 4.1 *For every $L > 0$, there exists $v \in \tilde{\mathcal{M}}_{2,L}$ with $q_-(x) \le v(x, y) \le q_+(x)$ for a.e. $(x, y) \in S_L$.*

Proof Fixed $L > 0$, let $(v_n)_{n\in\mathbb{N}} \subset \tilde{\mathcal{X}}_L$ be such that $\varphi_{[-L,L],2}(v_n) \to \tilde{m}_{2,L}$ and note that, arguing as in Lemma 3.2, it is not restrictive to assume that $q_-(x) \le v_n(x, y) \le q_+(x)$ for a.e. $(x, y) \in S_L$ and all $n \in \mathbb{N}$. Then, arguing as in Lemma 3.3, we have that there exists $v \in \mathcal{X}$ such that, up to a subsequence, $v_n \to v$ weakly in $H^1(S_L)$. Moreover, by pointwise convergence we have that $q_-(x) \le v(x, y) \le q_+(x)$ for a.e. $(x, y) \in S_L$ and $v \in \tilde{\mathcal{X}}$, i.e. $v \in \tilde{\mathcal{X}}_L$. Then, by semicontinuity, we conclude $\varphi_{[-L,L],2}(v) = \tilde{m}_{2,L}$. □

Arguing as in Lemma 3.3 in [10] (a similar argument is used also in the proof of Lemma 5.2 below), we have

Lemma 4.1 *If $v \in \tilde{\mathcal{M}}_{2,L}$ then $v \in \mathcal{C}^2(S_L)$ and it is a classical solution to $(\tilde{P}_{L,2})$.*

From every function $v \in \tilde{\mathcal{M}}_{2,L}$ we can recover a two dimensional periodic solution of $(\tilde{P}_2)$. Indeed, if $v \in \tilde{\mathcal{M}}_{2,L}$, reflecting with respect to the axes $y = \pm L$ and then continuing by periodicity in the y-direction we obtain a solution on $\mathbb{R}^2$, that we again denote with v, which is y-periodic of period $4L$ and which satisfies $\partial_y v(x, \pm L) = 0$ for all $x \in \mathbb{R}$. Since $\|v\|_{L^\infty} \le 1$ for all $v \in \tilde{\mathcal{M}}_{2,L}$, by Schauder estimates we obtain the existence of a constant $C > 0$ such that

$$\|v\|_{\mathcal{C}^2(\mathbb{R}^2)} \le C \ \text{ for all } \ v \in \tilde{\mathcal{M}}_{2,L} \quad \text{and} \quad L > 0. \tag{4.1}$$

Remark 4.2 We recall that considered δ_0 as defined in Lemma 3.4, $\bar{\ell}$ was correspondingly fixed in the proof of Lemma 3.5 such that if $v \in \mathcal{X}$, $\inf_{q\in\mathcal{M}} \|v(\cdot, y) - q\|_{H^1} > \delta_0$ for a.e. y in an interval $I \subset \mathbb{R}$ of length $\bar{\ell}$ then by (3.3) we have $\varphi_{I,2}(v) \ge \tilde{m}_{2,\infty} + \bar{\lambda}$ with $\bar{\lambda}$ given by Lemma 3.4.

We can use these observations to say in particular that if $L \ge \bar{\ell}$, $v \in \tilde{\mathcal{X}}_L$ and $\varphi_{[-L,L],2}(v) < \tilde{m}_{2,L} + \bar{\lambda}$ then there exists $y_0 \in (0, \bar{\ell})$ such that $\inf_{q\in\mathcal{M}} \|v(\cdot, y_0) - q\|_{H^1} \le \delta_0$.

Then the argument in the proof of Lemma 3.4 applies here to recover that if $L > \bar{\ell}$ and $v \in \tilde{\mathcal{X}}_L$ is such that $q_-(x) \le v(x, y) \le q_+(x)$ for a.e. $(x, y) \in S_L$ and $\varphi_{[-L,L],2}(v) \le \tilde{m}_{2,L} + \bar{\lambda}$, then either

$$\|v(\cdot, y) - q_+\|_{L^2} \le d_0 \quad \text{for all } y \in [\bar{\ell}, L] \quad \text{or} \quad \|v(\cdot, y) - q_-\|_{L^2} \le d_0 \quad \text{for all } y \in [\bar{\ell}, L].$$

For $L \ge \bar{\ell}$, we will denote

$$\tilde{\mathcal{X}}_L^{\pm} = \{v \in \tilde{\mathcal{X}}_L \mid \|v(\cdot, y) - q_\pm\|_{L^2} \le d_0, \quad \forall y \in [\bar{\ell}, L]\}$$

and note that if $v \in \tilde{\mathcal{X}}_L^+$ then the symmetric function $v^*(x, y) = -v(-x, y)$ belongs to $\tilde{\mathcal{X}}_L^-$ and viceversa. By Remark 4.2, we have that for every $L \ge \bar{\ell}$ there results $\tilde{\mathcal{M}}_{2,L} \subset \tilde{\mathcal{X}}_L^+ \cup \tilde{\mathcal{X}}_L^-$. We use again Remark 4.2 to prove the following concentration property.

Lemma 4.2 (Concentration: L^∞ estimate) *For all $\varepsilon > 0$ there exists $L_\varepsilon > \bar{\ell}$ such that if $L > L_\varepsilon$ and $v \in \tilde{\mathcal{M}}_{2,L} \cap \tilde{\mathcal{X}}_L^+$ is such that $q_-(x) \le v(x, y) \le q_+(x)$ for a.e. $(x, y) \in S_L$, then*

$$\|v(\cdot, y) - q_+\|_{L^\infty} < \varepsilon, \quad \forall y \in (L_\varepsilon, L].$$

Proof To prove the lemma, we argue by contradiction assuming that there exist $\varepsilon_0 \in (0, \min\{\frac{1}{2}; d_0\})$, two increasing sequences $0 < \bar{\ell} < y_n \le L_n$ such that $y_n \to +\infty$ and a sequence $(v_n) \subset \tilde{\mathcal{M}}_{2,L_n} \cap \tilde{\mathcal{X}}^+_{L_n}$ such that for every $n \in \mathbb{N}$ there results $\|v_n(\cdot, y_n) - q_+\|_{L^\infty} > 2\varepsilon_0$.

First, by (4.1), note that $\sup_{n\in\mathbb{N}} \|v_n - q_+\|_{\mathcal{C}^1(S_{L_n})} < +\infty$ and so

$$\exists\, y_0 \in (0, y_1 - \bar{\ell}) \text{ such that } \|v_n(\cdot, y) - q_+\|_{L^\infty} > \varepsilon_0 \quad \text{for all } y \in (y_n - y_0, y_n).$$

Again by (4.1) we then recover that there exists $\eta_0 > 0$ such that

$$\|v_n(\cdot, y) - q_+\|_{L^2} > \eta_0 \quad \text{for all } y \in (y_n - y_0, y_n).$$

Since $v_n \in \tilde{\mathcal{X}}^+_{L_n}$ we then have $\mathsf{d}_{L^2}(v_n(\cdot, y), \mathcal{M}) = \|v_n(\cdot, y) - q_+\|_{L^2} > \eta_0$ for all $y \in (y_n - y_0, y_n)$ and so by the compactness property (2.2) we obtain that there exists $\bar{\nu} > 0$ such that for all $n \in \mathbb{N}$ there results $\varphi(v_n(\cdot, y)) - m \ge \bar{\nu}, \forall\, y \in (y_n - y_0, y_n)$. Then we conclude

$$\varphi_{(y_n - y_0, y_n),2}(v_n) \ge \bar{\nu} y_0 \text{ for all } n \ge \bar{n}. \tag{4.2}$$

On the other hand, since $\varphi_{(\bar{\ell}, y_n - y_0),2}(v_n) \le \tilde{m}_{2,L_n} \le \tilde{m}_{2,\infty}$ for all $n \in \mathbb{N}$, we obtain that there exists $\bar{y}_n \in (\bar{\ell}, y_n - y_0)$ for which $\varphi(v_n(\cdot, \bar{y}_n)) \to m$ and so by the compactness property in Lemma 2.2, since $\|v_n(\cdot, \bar{y}_n) - q_+\|_{L^2} \le d_0$, we derive $\|v_n(\cdot, \bar{y}_n) - q_+\|_{H^1} \to 0$. Now, setting

$$\bar{v}_n(x, y) = \begin{cases} q_+(x) & y \ge \bar{y}_n + 1, \\ (\bar{y}_n + 1 - y) v_n(x, \bar{y}_n) + (y - \bar{y}_n) q_+(x) & \bar{y}_n \le y \le \bar{y}_n + 1 \\ v_n(x, y), & -\bar{y}_n \le y \le \bar{y}_n \\ (\bar{y}_n + 1 + y) v_n(x, -\bar{y}_n) + (-y - \bar{y}_n) q_-(x) & -\bar{y}_n \ge y \ge -\bar{y}_n - 1 \\ q_-(x) & y \le -\bar{y}_n - 1, \end{cases}$$

for all $(x, y) \in S_{L_n}$, we recognize that $\bar{v}_n \in \tilde{\mathcal{X}}_{L_n}$ and so $\varphi_{(-L_n, L_n),2}(\bar{v}_n) \ge \tilde{m}_{2,L_n}$. Moreover, since $\|v_n(\cdot, \bar{y}_n) - q_+\|_{H^1} \to 0$, we derive that

$$\|\bar{v}_n(\cdot, y) - q_+\|_{H^1} \to 0 \text{ and } \|\partial_y \bar{v}_n(\cdot, y)\|_{L^2} \to 0,$$

uniformly w.r.t. $y \in (\bar{y}_n, \bar{y}_n + 1)$ as $n \to +\infty$. Hence also $\varphi(\bar{v}_n(\cdot, y)) \to m$ uniformly w.r.t. $y \in (\bar{y}_n, \bar{y}_n + 1)$ and we conclude

$$\varphi_{(\bar{y}_n, \bar{y}_n + 1),2}(\bar{v}_n) = \int_{\bar{y}_n}^{\bar{y}_n + 1} \tfrac{1}{2} \|\partial_y \bar{v}_n(\cdot, y)\|^2_{L^2} + (\varphi(\bar{v}_n(\cdot, y)) - m)\, dy \to 0.$$

Finally, since $\varphi_{(\bar{y}_n + 1, L_n),2}(\bar{v}_n) = 0$ for all $n \in \mathbb{N}$, by symmetry, we derive that as $n \to +\infty$ there results

$$\begin{aligned}\varphi_{(-\bar y_n,\bar y_n),2}(v_n) &= \varphi_{(-\bar y_n,\bar y_n),2}(\bar v_n) = \varphi_{(-L_n,L_n),2}(\bar v_n) - 2\varphi_{(\bar y_n,\bar y_n+1),2}(\bar v_n)\\ &\geq \tilde m_{2,L_n} + o(1),\end{aligned}$$

and so, by (4.2), we reach the contradiction

$$\begin{aligned}\tilde m_{2,L_n} &= \varphi_{(-L_n,L_n),2}(v_n) \geq \varphi_{(-\bar y_n,\bar y_n),2}(v_n) + \varphi_{(y_n-y_0,y_n),2}(v_n)\\ &\geq \tilde m_{2,L_n} + o(1) + \bar\nu y_0.\end{aligned}$$

□

Remark 4.3 If $v \in \tilde{\mathcal{M}}_{2,L} \cap \tilde{\mathcal{X}}_L^+$, since v and q_+ solves (P_2) on S_L we can use Lemma 4.2 to derive via the Schauder estimates that, for every $\varepsilon > 0$ there exists $L_\varepsilon > \bar\ell$ such that for $L > L_\varepsilon$ we have

$$\text{if } v \in \tilde{\mathcal{M}}_{2,L} \cap \tilde{\mathcal{X}}_L^+ \text{ then } \|v(\cdot,y) - q_+\|_{C^2(\mathbb{R})} < \varepsilon, \quad \forall y \in (L_\varepsilon, L]. \tag{4.3}$$

Here below we use $(**)$ and Lemma 2.3 to give a refined version of (4.3).

We set $\tilde\omega = \min\{\sqrt{\omega^*}, \sqrt{\omega}\}$, where ω^* is given by $(**)$ and ω by Lemma 2.3.

Lemma 4.3 (Exponential decay) *There exist $\tilde L \geq \bar\ell$ and a constant $C > 0$ such that if $L > \tilde L$, $v \in \tilde{\mathcal{M}}_{2,L} \cap \tilde{\mathcal{X}}_L^+$ and $q_-(x) \leq v(x,y) \leq q_+(x)$ for all $(x,y) \in S_L$ then*

$$\|v(\cdot,y) - q_+\|_{L^2} \leq C\, e^{-\frac{\tilde\omega}{2}y} \text{ for all } y \in [\tilde L, L] \tag{4.4}$$

and

$$\|v(\cdot,y) - q_+\|_{H^1} \leq C \sqrt[4]{y}\, e^{-\frac{\tilde\omega}{2}y} \text{ for all } y \in [\tilde L, L]. \tag{4.5}$$

Proof Given $v \in \tilde{\mathcal{M}}_{2,L} \cap \tilde{\mathcal{X}}_L^+$, with abuse of notation, we denote again with v the corresponding periodic prolongation in $\mathbb{R}^2$. Consider the function $\phi(y) = \|v(\cdot,y) - q_+\|_{L^2}^2$ for $y > 0$. Since $q_-(x) \leq v(x,y) \leq q_+(x)$, we obtain $\phi(y) \leq \|q_+ - q_-\|_{L^2}^2 = 4d_0^2$ for all y. Moreover, we have that $\phi \in C^2((0,+\infty))$ and denoted by $\langle\cdot,\cdot\rangle$ the bracket symbol for the scalar product in $L^2(\mathbb{R})$, for every $y > 0$ we obtain

$$\ddot\phi(y) = 2\|\partial_y v(\cdot,y)\|_{L^2}^2 + 2\langle v(\cdot,y) - q_+, \partial_{yy}u(\cdot,y)\rangle \geq 2\langle v(\cdot,y) - q_+, \partial_{yy}v(\cdot,y)\rangle \tag{4.6}$$

Now, to estimate $\langle v(\cdot,y) - q_+, \partial_{yy}v(\cdot,y)\rangle$, note that using the L^∞-concentration estimate given by Lemma 4.2, we can choose $\tilde L \geq \bar\ell$ such that for any $y \in [\tilde L, L]$ and for any $x \in \mathbb{R}$ there results

$$W'(v(x,y)) - W'(q_+(x)) = W''(q_+(x))(v(x,y) - q_+(x)) + r(x,y)$$

with $|r(x,y)| \leq \frac{\omega^*}{4}|v(x,y) - q_+(x)|$. Then, since v and q_+ solve $-\Delta u + aW'(u) = 0$ on $\mathbb{R}^2$, for all $y \in [\tilde L, L]$ we have

$$\begin{aligned}\langle v(\cdot,y) - q_+, \partial_{yy}v(\cdot,y)\rangle &= \langle v(\cdot,y) - q_+, -\partial_{xx}(v(\cdot,y) - q_+)\\ &\quad + aW'(v(\cdot,y)) - aW'(q_+)\rangle\\ &\geq \langle v(\cdot,y) - q_+, -\partial_{xx}(v(\cdot,y) - q_+) + aW''(q_+)(v(\cdot,y) - q_+)\rangle - \tfrac{\omega^*}{4}\|v(\cdot,y) - q_+\|_{L^2}^2\\ &= \varphi''(q_+)(v(\cdot,y) - q_+)\cdot(v(\cdot,y) - q_+) - \tfrac{\omega^*}{4}\|v(\cdot,y) - q_+\|_{L^2}^2.\end{aligned}$$

and by $(**)$ we obtain

$$\langle v(\cdot,y) - q_+, \partial_{yy}v(\cdot,y)\rangle \geq \tfrac{\omega^*}{2}\|v(\cdot,y) - q_+\|_{L^2}^2 \tag{4.7}$$

Then, by (4.6) and (4.7), since by definition $\tilde{\omega} \leq \sqrt{\omega^*}$, we obtain that

$$\ddot{\phi}(y) \geq \omega^* \phi(y) \geq \tilde{\omega}^2 \phi(y) \text{ for all } y \in [\tilde{L}, L]. \tag{4.8}$$

By symmetry, we have that the function ϕ verifies $\phi(\tilde{L}) = \phi(2L - \tilde{L}) \leq 4d_0^2$ and moreover that (4.8) holds true also for all $y \in (\tilde{L}, 2L - \tilde{L})$. Setting

$$\psi(y) = 4d_0^2 \frac{\cosh(\tilde{\omega}(L-y))}{\cosh(\tilde{\omega}(L-\tilde{L}))}$$

we get that $\ddot{\psi}(y) = \tilde{\omega}^2 \psi(y)$ and $\psi(\tilde{L}) = \psi(2L - \tilde{L}) = 4d_0^2$. Hence, by the maximum principle, we obtain that $\phi(y) \leq \psi(y)$ for all $y \in [\tilde{L}, 2L - \tilde{L}]$, that is

$$\|v(\cdot, y) - q_+\|_{L^2}^2 \leq \bar{\delta} \frac{\cosh(\tilde{\omega}(L-y))}{\cosh(\tilde{\omega}(L-\tilde{L}))} \quad \forall y \in [\tilde{L}, L].$$

Since for $y \in [\tilde{L}, L]$ we have $\cosh(\tilde{\omega}(L-y)) \leq \mathrm{e}^{\tilde{\omega}(L-y)}$ and $2\cosh(\tilde{\omega}(L-\tilde{L})) \geq \mathrm{e}^{\tilde{\omega}(L-\tilde{L})}$, we conclude

$$\|v(\cdot, y) - q_+\|_{L^2}^2 \leq 8d_0^2 \frac{\mathrm{e}^{\tilde{\omega}(L-y)}}{\mathrm{e}^{\tilde{\omega}(L-\tilde{L})}} = 8d_0^2\, \mathrm{e}^{\tilde{\omega}(\tilde{L}-y)} \quad \forall y \in [\tilde{L}, L]$$

which proves (4.4).

To recover the estimate in $H^1(\mathbb{R})$ we set $\phi_y(x) = v(x, y) - q_+(x)$ and note that since $\phi_y(x) \to 0$ as $|x| \to +\infty$ we have

$$\|\partial_x \phi_y\|_{L^2}^2 = \int_{\mathbb{R}} \partial_x(\phi_y \partial_x \phi_y)\, dx - \int_{\mathbb{R}} \phi_y \partial_x^2 \phi_y\, dx = -\int_{\mathbb{R}} \phi_y \partial_x^2 \phi_y\, dx.$$

Now, since by the Schauder estimates $|\partial_x^2 v(x, y) - \ddot{q}_+(x)| \leq C$ on $\mathbb{R}^2$ for some constant $C > 0$ and since, by Lemma 2.3, choosing $\tilde{L}$ bigger if necessary, for $|x| > \tilde{L}$ we have $|q_+(x) - q_-(x)| \leq 2K\, \mathrm{e}^{-\sqrt{\omega}|x|}$, using (4.4) we obtain

$$\begin{aligned} -\int_{\mathbb{R}} \phi_y \partial_x^2 \phi_y\, dx &= \int_{-y}^{y} (v(x, y) - q_+(x))(\partial_x^2 v(x, y) - \ddot{q}_+(x)) dx \\ &\quad + \int_{|x|>y} (v(x, y) - q_+(x))(\partial_x^2 v(x, y) - \ddot{q}_+(x)) dx \\ &\leq C\sqrt{2y}\|v(\cdot, y) - q_+\|_{L^2} + C \int_{|x|>y} |q_+(x) - q_-(x)| dx \\ &\leq C\sqrt{2y}\sqrt{2\bar{\delta}}\, \mathrm{e}^{\tilde{\omega}(\tilde{L}-y)} + 4\frac{CK}{\sqrt{\omega}}\, \mathrm{e}^{-\sqrt{\omega}y} \leq \tilde{C}\sqrt{2y}\, \mathrm{e}^{-\tilde{\omega}y}. \end{aligned}$$

from which (4.5) follows. □

Thanks to (4.5) we can now estimate the asymptotic behaviour of $\tilde{m}_{2,L}$ as $L \to +\infty$.

Lemma 4.4 (Asymptotic behaviour of $\tilde{m}_{2,L}$) *There exists a constant $C > 0$ such that for all $L \geq \tilde{L}$ there results*

$$\tilde{m}_{2,\infty} - \tilde{m}_{2,L} \leq C\sqrt{L}\, e^{-\frac{\tilde{\omega}}{2}L}. \tag{4.9}$$

Proof Letting $u \in \tilde{\mathcal{M}}_{2,L} \cap \tilde{\mathcal{X}}_L^+$, we define

$$v(x,y) = \begin{cases} q_+(x) & y \geq L+1, \\ (L+1-y)u(x,L) + (y-L)q_+(x) & L \leq y \leq L+1 \\ u(x,y), & -L \leq y \leq L \\ (L+1+y)u(x,-L) + (-y-L)q_-(x) & -L \geq y \geq -L-1 \\ q_-(x) & y \leq -L-1, \end{cases}$$

We recognize that $v \in \tilde{\mathcal{H}}_2^+$ and so $\varphi_2(v) \geq \tilde{m}_{2,\infty}$. Hence, since $\varphi_2(v) = \tilde{m}_{2,L} + 2\varphi_{(L,L+1),2}(v)$, we get

$$\tilde{m}_{2,\infty} - \tilde{m}_{2,L} \leq 2\varphi_{(L,L+1),2}(v) = 2\int_L^{L+1} \tfrac{1}{2}\|\partial_y v(\cdot,y)\|_{L^2}^2 + \varphi(v(\cdot,y)) - m\, dy. \quad (4.10)$$

To evaluate $\varphi_{(L,L+1),2}(v)$ we use (4.4). First observe that by (4.4) there exists a constant $C_1 > 0$ such that for every $L \geq \tilde{L}$ and $y \in (L, L+1)$ there results

$$\|\partial_y v(\cdot,y)\|_{L^2}^2 = \|u(\cdot,L) - q_+\|_{L^2}^2 \leq C_1\, \mathrm{e}^{-\tilde{\omega}L}. \quad (4.11)$$

To estimate the term $\varphi(v(\cdot,y)) - m = \varphi(v(\cdot,y)) - \varphi(q_+)$, note that for all $y \in (L, L+1)$ we have

$$\begin{aligned} ||\partial_x v(x,y)|^2 - |\dot{q}_+(x)|^2| &= ||(L+1-y)\partial_x u(x,L) + (y-L)\dot{q}_+(x)|^2 - |\dot{q}_+(x)|^2| \\ &= ||\dot{q}_+(x) + (L+1-y)(\partial_x u(x,L) - \dot{q}_+(x))|^2 - |\dot{q}_+(x)|^2| \\ &\leq (L+1-y)^2|\partial_x u(x,L) - \dot{q}_+(x)|^2 + 2(L+1-y)|\partial_x u(x,L) - \dot{q}_+(x)||\dot{q}_+(x)| \\ &\leq |\partial_x u(x,L) - \dot{q}_+(x)|^2 + 2|\partial_x u(x,L) - \dot{q}_+(x)||\dot{q}_+(x)| \end{aligned}$$

then, by (4.5), there exists $C_2 > 0$ such that for all $y \in (L, L+1)$ we have

$$\begin{aligned} |\|\partial_x v(\cdot,y)\|_{L^2}^2 - \|\dot{q}_+\|_{L^2}^2| &\leq \|\partial_x u(\cdot,L) - \dot{q}_+\|_{L^2}^2 + 2\|\partial_x u(\cdot,L) - \dot{q}_+\|_{L^2}\|\dot{q}_+\|_{L^2} \\ &\leq C_2(\sqrt{L}e^{-\tilde{\omega}L} + \sqrt[4]{L}e^{-\frac{\tilde{\omega}}{2}L}) \end{aligned}$$

Moreover, setting $\tilde{w} = \sup_{s\in[-1,1]} W'(s)$, we get for all $y \in (L, L+1)$

$$\begin{aligned} |W(v(x,y)) - W(q_+(x))| &= |W((L+1-y)u(x,L) + (y-L)q_+(x)) - W(q_+(x))| \\ &\leq \tilde{w}|(L+1-y)u(x,L) + (y-L)q_+(x) - q_+(x)| \\ &= \tilde{w}(L+1-y)|u(x,L) - q_+(x)| \leq \tilde{w}|u(x,L) - q_+(x)| \end{aligned}$$

and so, by (4.4) and since, by Lemma 2.3, for $|x| > L$, $|u(x,L) - q_+(x)| \leq |q_-(x) - q_+(x)| \leq 2K\, \mathrm{e}^{-\sqrt{\omega}|x|} \leq 2K\, \mathrm{e}^{-\tilde{\omega}|x|}$, we obtain for all $y \in (L, L+1)$

$$\begin{aligned} \int_{\mathbb{R}} a(x)W(v(x,y)) - a(x)W(q_+(x))dx &\leq \tilde{w}\bar{a}\int_{\mathbb{R}} |u(x,L) - q_+(x)|dx \\ &\leq \tilde{w}\bar{a}\left(\int_{-L}^{L} |u(x,L) - q_+(x)|dx + \int_{|x|>L} |u(x,L) - q_+(x)|dx\right) \\ &\leq \tilde{w}\bar{a}\left(\sqrt{2L}\|u(\cdot,L) - q_+\|_{L^2} + \int_{|x|>L} |q_-(x) - q_+(x)|dx\right) \end{aligned}$$

$$\le \tilde{w}\bar{a}\left(\sqrt{2L}e^{-\frac{\tilde{\omega}}{2}L} + 2\frac{K}{\tilde{\omega}}e^{-\tilde{\omega}L}\right)$$
$$\le C_3(\sqrt{L}e^{-\frac{\tilde{\omega}}{2}L} + e^{-\frac{\tilde{\omega}}{2}L}).$$

Gathering the above estimates, for every $y \in (L, L+1)$ there results

$$\varphi(u(\cdot, y)) - m \le \frac{C_2}{2}(\sqrt{L}e^{-\tilde{\omega}L} + \sqrt[4]{L}e^{-\frac{\tilde{\omega}}{2}L}) + C_3(\sqrt{L}e^{-\frac{\tilde{\omega}}{2}L} + e^{-\frac{\tilde{\omega}}{2}L}) \le C_4\sqrt{L}e^{-\frac{\tilde{\omega}}{2}L}$$

and hence using (4.11) we conclude

$$\tilde{m}_{2,\infty} - \tilde{m}_{2,L} \le 2\varphi_{(L,L+1),2}(v) \le C_1 e^{-\frac{\tilde{\omega}}{2}L} + 2C_4\sqrt{L}e^{-\frac{\tilde{\omega}}{2}L}$$

and the Lemma follows. □

As final property in this section we state a compactness property which will be useful in the construction of the three dimensional solutions.

Lemma 4.5 *Let $L_n \to +\infty$ and $v_n \in \tilde{\mathcal{X}}^+_{L_n}$ with $q_-(x) \le v_n(x, y) \le q_+(x)$ on S_{L_n} be such that $\varphi_{(-L_n,L_n),2}(v_n) - \tilde{m}_{2,L_n} \to 0$ as $n \to \infty$. Then, for all $L > 0$*

$$\|v_n - \tilde{v}^+\|_{L^2(S_L)} \to 0.$$

Moreover, if $\bar{L}_n \in (0, L_n)$ is such that $\bar{L}_n \to +\infty$ then $\|v_n(\cdot, \bar{L}_n) - q^+\|_{L^2(\mathbb{R})} \to 0$.

Proof Since $\tilde{m}_{2,L_n} \to \tilde{m}_{2,\infty}$ and $\varphi_{(-L_n,L_n),2}(v_n) - \tilde{m}_{2,L_n} \to 0$ as $n \to \infty$, we have that the sequence $(\varphi_{(-L_n,L_n),2}(v_n))_{n\in\mathbb{N}}$ is bounded. Then, letting $\bar{L}_n \in (0, L_n)$ such that $\bar{L}_n \to +\infty$, we have that $(\varphi_{(\bar{L}_n/2,\bar{L}_n),2}(v_n))_{n\in\mathbb{N}}$ is bounded too and since $\bar{L}_n \to +\infty$ we recover that there exists $y_n \in (\bar{L}_n/2, \bar{L}_n - 1)$ such that $\varphi(v_n(\cdot, y_n)) \to m$. By Lemma 2.2, since $v_n \in \tilde{\mathcal{X}}^+_{L_n}$ we obtain that, up to a subsequence, $\|v_n(\cdot, y_n) - q_+\|_{H^1} \to 0$. Defining

$$\bar{v}_n(x, y) = \begin{cases} q_+(x) & y \ge y_n + 1, \\ (y_n + 1 - y)v_n(x, y_n) + (y - y_n)q_+(x) & y_n \le y \le y_n + 1 \\ v_n(x, y) & -y_n \le y \le y_n \\ (y_n + 1 + y)v_n(x, -y_n) + (-y - y_n)q_-(x) & -y_n \ge y \ge -y_n - 1 \\ q_-(x) & y \le -y_n - 1, \end{cases}$$

we recognize that

(i) $\bar{v}_n \in \tilde{\mathcal{H}}^+_2$ and $q_-(x) \le \bar{v}_n(x, y) \le q_+(x)$ on $\mathbb{R}^2$,
(ii) $\bar{v}_n \in \tilde{\mathcal{X}}^+_{L_n}$ (with abuse of notation) and so $\varphi_{(-L_n,L_n),2}(\bar{v}_n) \ge \tilde{m}_{2,L_n}$.
Arguing as in the proof of Lemma 4.2, we recover that
(iii) $\varphi_{(y_n,y_n+1),2}(\bar{v}_n) \to 0$ as $n \to +\infty$.

Since $\varphi_{(y_n+1,L_n),2}(\bar{v}_n) = \varphi_{(y_n+1,L_n),2}(q_+) = 0$, by (iii) there results

$$\varphi_2(\bar{v}_n) = \varphi_{(-y_n,y_n),2}(\bar{v}_n) + 2\varphi_{(y_n,y_n+1),2}(\bar{v}_n) = \varphi_{(-y_n,y_n),2}(\bar{v}_n) + o(1). \tag{4.12}$$

Moreover by (ii) and (iii)

$$\varphi_{(-y_n,y_n),2}(\bar{v}_n) \ge \varphi_{(-L_n,L_n),2}(\bar{v}_n) - 2\varphi_{(y_n,y_n+1),2}(\bar{v}_n) \ge \tilde{m}_{2,L_n} + o(1),$$

and since

$$o(1) = \varphi_{(-L_n,L_n),2}(v_n) - \tilde{m}_{2,L_n} \ge \varphi_{(-y_n,y_n),2}(v_n) - \tilde{m}_{2,L_n} = \varphi_{(-y_n,y_n),2}(\bar{v}_n) - \tilde{m}_{2,L_n},$$

we conclude that $\varphi_{(-y_n,y_n),2}(\bar{v}_n) - \tilde{m}_{2,L_n} \to 0$. Since $\tilde{m}_{2,L_n} \to \tilde{m}_{2,\infty}$ by (4.12) we finally obtain

$$\varphi_2(\bar{v}_n) \to \tilde{m}_{2,\infty} \text{ as } n \to \infty.$$

By Lemma 3.9 we conclude that $\|\bar{v}_n - \tilde{v}^+\|_{L^2(\mathbb{R}^2)} \to 0$ and so since $\bar{v}_n(x, y) = v_n(x, y)$ on S_{y_n} we derive $\|v_n - \tilde{v}^+\|_{L^2(S_{y_n})} \to 0$ and so, since $y_n \to +\infty$ that $\|v_n - \tilde{v}^+\|_{L^2(S_L)} \to 0$ for any $L > 0$.

To conclude the proof note that if $\|v_n(\cdot, \bar{L}_n) - q^+\|_{L^2(\mathbb{R})} \not\to 0$ then there exists $\delta \in (0, d_0/2)$ and a subsequence of $(\bar{L}_n)$, denoted again with $\bar{L}_n$, such that $\|v_n(\cdot, \bar{L}_n) - q^+\|_{L^2(\mathbb{R})} \geq 2\delta$. Since $v_n \in \mathcal{X}^+_{L_n}$ we deduce $\inf_{q\in\mathcal{M}} \|v_n(\cdot, \bar{L}_n) - q\|_{L^2(\mathbb{R})} \geq 2\delta$ and since $\|v_n(\cdot, y_n) - q^+\|_{L^2(\mathbb{R})} \to 0$ we recover by (3.2) that there exists $(\sigma_n, \tau_n) \subset (y_n, \bar{L}_n)$ such that $\|v_n(\cdot, \tau_n) - v_n(\cdot, \sigma_n)\|_{L^2(\mathbb{R})} = \delta$ and $\inf_{q\in\mathcal{M}} \|v_n(\cdot, y) - q\|_{L^2(\mathbb{R})} \geq \delta$ for a.e. $y \in (\sigma_n, \tau_n)$. Then by (2.2) there exists $\nu > 0$ such that $\varphi(v_n(\cdot, y)) \geq m + \nu$ for a.e. $y \in (\sigma_n, \tau_n)$ and by (3.3) we conclude $\varphi_{(\sigma_n,\tau_n),2}(v_n) \geq \sqrt{2\delta}\nu > 0$ for any $n \in \mathbb{N}$. This is in contradiction with the fact that, as proved above,

$$\varphi_{(\sigma_n,\tau_n),2}(v_n) \leq \varphi_{(y_n,L_n),2}(v_n) = \tfrac{1}{2}(\varphi_{(-L_n,L_n),2}(v_n) - \varphi_{(-y_n,y_n),2}(v_n)) \to 0$$

and the lemma follows. □

Remark 4.4 Clearly by Lemma 4.5 we symmetrically obtain that if $L_n \to +\infty$ and $u_n \in \tilde{\mathcal{X}}^-_{L_n}$ with $q_-(x) \leq v_n(x, y) \leq q_+(x)$ on S_{L_n} are such that $\varphi_{(-L_n,L_n),2}(v_n) - \tilde{m}_{2,L_n} \to 0$ as $n \to \infty$, then, for all $L > 0$ there results

$$\|v_n - \tilde{v}^-\|_{L^2(S_L)} \to 0.$$

We then conclude that if $L_n \to +\infty$ and $v_n \in \tilde{\mathcal{X}}_{L_n}$ are such that $q_-(x) \leq v_n(x, y) \leq q_+(x)$ on S_{L_n} and $\varphi_{(-L_n,L_n),2}(v_n) - \tilde{m}_{2,L_n} \to 0$ as $n \to \infty$, then, for all $L > 0$ there exists a subsequence $(v_{n_j})_{j\in\mathbb{N}} \subset (v_n)_{n\in\mathbb{N}}$ such that

$$\|v_{n_j} - \tilde{v}^-\|_{L^2(S_L)} \to 0 \quad \text{or} \quad \|v_{n_j} - \tilde{v}^+\|_{L^2(S_L)} \to 0.$$

In particular we obtain that there exists $\bar{L} > \tilde{L}$ such that for all $\delta > 0$ there exists $\mu_\delta > 0$ such that

$$\text{if } v \in \tilde{\mathcal{X}}_L, L \geq \bar{L}, \inf_{\bar{v}=\tilde{v}^\pm} \|v - \bar{v}\|_{L^2(S_L)} > \delta \quad \text{then} \quad \varphi_{(-L,L),2}(v) \geq \tilde{m}_{2,L} + \mu_\delta. \tag{4.13}$$

5 Three dimensional solutions

In this section we will complete the proof of Theorem 1.1.

First, fixed $\theta \in (0, \frac{\pi}{4}]$ and denoted $\bar{z} = z\tan\theta$, we consider the infinite prism

$$\mathcal{P}_\theta = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in S_{\bar{z}},\ z \geq 0\}$$

and the existence of solutions to the problem

$$\begin{cases} -\Delta v(x, y, z) + a(x)W'(v(x, y, z)) = 0, & (x, y, z) \in \mathcal{P}_\theta, \\ v(x, y, z) = -v(-x, -y, z), & (x, y, z) \in \mathcal{P}_\theta \\ \partial_\nu v(x, y, z) = 0 & (x, y, z) \in \partial\mathcal{P}_\theta \\ \lim\limits_{x\to\pm\infty} v(x, y, z) = \pm 1, & \text{uniformly w.r.t. } (y, z) \end{cases} \tag{$\tilde{P}_{3,\theta}$}$$

To this aim let us consider the space

$$\mathcal{Z}_\theta = \{u \in H^1_{loc}(\mathcal{P}_\theta) \mid u(\cdot, \cdot, z) \in \tilde{\mathcal{X}}_{\bar{z}} \text{ for a.e. } z \in (0, +\infty)\}$$

on which we will look for a minima of the functional

$$\varphi_3(u) = \int_0^{+\infty}\int_{-\bar z}^{\bar z}\int_{\mathbb{R}} \tfrac{1}{2}|\nabla u(x,y,z)|^2 + a(x)W(u(x,y,z))\,dx - m\,dy - \tilde m_{2,\bar z}\,dz$$

$$= \int_0^{+\infty} \frac{1}{2}\|\partial_z u(\cdot,\cdot,z)\|^2_{L^2(S_z)} + \varphi_{(-\bar z,\bar z),2}(u(\cdot,\cdot,z)) - \tilde m_{2,\bar z}\,dz.$$

Note that if $u \in \mathcal{Z}_\theta$ then $u(\cdot,\cdot,z) \in \tilde{\mathcal{X}}_{\bar z}$ for a.e. $z > 0$ and then $\varphi_{(-\bar z,\bar z),2}(u(\cdot,\cdot,z)) \geq \tilde m_{2,\bar z}$ for a.e. $z > 0$. Hence we recover that φ_3 is well defined on $\mathcal{Z}_\theta$ and non negative. It is standard to show that φ_3 is weakly lower semicontinuous with respect to the $H^1_{loc}(\mathcal{P}_\theta)$ topology. We set

$$\tilde m_3(\theta) = \inf_{u\in\mathcal{Z}_\theta} \varphi_3(u).$$

Remark 5.1 The problem is well posed since $\tilde m_3(\theta) < +\infty$. Indeed, the function $v(x,y,z) = \tilde v^+_{|\mathcal{P}_\theta}(x,y) \in \mathcal{Z}_\theta$ and by (4.9) there results

$$\varphi_3(v) = \int_0^{+\infty} \varphi_{(-\bar z,\bar z),2}(\tilde v^+) - \tilde m_{2,\bar z}\,dz$$

$$\leq \int_0^{+\infty} \tilde m_{2,\infty} - \tilde m_{2,\bar z}\,dz \leq C\int_0^{+\infty} \sqrt{\tan\theta z}\,\mathrm{e}^{-\frac{\tilde\nu}{2}\tan\theta z}\,dz < +\infty.$$

An important remark for our construction, is an estimate concerning the functional φ_3, analogous to the one we gave in (3.3) for the two dimensional functional φ_2.

First we note that if $u \in \mathcal{Z}_\theta$ then $\varphi_{(-\bar z,\bar z),2}(v(\cdot,\cdot,z)) \geq \tilde m_{2,\bar z}$ for a.e. $z > 0$ and so

$$\|\partial_z u\|^2_{L^2(\mathcal{P}_\theta)} \leq 2\varphi_3(u) \quad \forall u \in \mathcal{Z}_\theta. \tag{5.1}$$

Therefore, as in (3.2), note that if $u \in \mathcal{Z}_\theta$ then $u(x,y,\cdot) \in H^1_{loc}(z,+\infty)$ for a.e. $(x,y) \in S_{\bar z}$ and hence, if $0 < z_1 < z_2$, then $u(x,y,z_2) - u(x,y,z_1) = \int_{z_1}^{z_2} \partial_z u(x,y,z)\,dz$ holds for all $u \in \mathcal{Z}_\theta$ and a.e. $(x,y) \in S_{\bar z_1}$. So, if $u \in \mathcal{Z}_\theta$, by (5.1), for $z_1 < z_2 \in \mathbb{R}^+$, we obtain that

$$\begin{aligned}\|u(\cdot,\cdot,z_2) - u(\cdot,\cdot,z_1)\|^2_{L^2(S_{\bar z_1})} &= \int_{S_{\bar z_1}} \Big|\int_{z_1}^{z_2} \partial_z u(x,y,z)\,dz\,\Big|^2 dx\,dy \\ &\leq |z_2 - z_1|\int_{S_{\bar z_1}}\int_{\mathbb{R}^+} |\partial_z u(x,y,z)|^2\,dz\,dx\,dy \\ &\leq 2\varphi_3(u)|z_2 - z_1|.\end{aligned} \tag{5.2}$$

Finally, given an interval $I \subset \mathbb{R}_+$ and $u \in \mathcal{Z}_\theta$ we set

$$\varphi_{I,3}(u) = \int_I \|\partial_z u(\cdot,\cdot,z)\|^2_{L^2(S_{\bar z})} + (\varphi_{(-\bar z,\bar z),2}(u(\cdot,\cdot,z)) - \tilde m_{2,\bar z})\,dz.$$

and note that if $u \in \mathcal{Z}_\theta$ is such that $\varphi_{(-\bar z,\bar z),2}(u(\cdot,\cdot,z)) - \tilde m_{2,\bar z} \geq \nu > 0$ for a.e. $z \in (\sigma,\tau) \subset \mathbb{R}^+$, then

$$\varphi_{(\sigma,\tau),3}(u) \geq \frac{1}{2(\tau-\sigma)}\|u(\cdot,\cdot,\tau)-u(\cdot,\cdot,\sigma)\|^2_{L^2(S_{\bar\sigma})}+\nu(\tau-\sigma)$$
$$\geq \sqrt{2\nu}\|u(\cdot,\cdot,\tau)-u(\cdot,\cdot,\sigma)\|_{L^2(S_{\bar\sigma})}. \tag{5.3}$$

Remark 5.2 Arguing as in Lemma 3.2 we can prove that if $u \in \mathcal{Z}_\theta$ then, setting

$$\hat u(x,y,z)=\max\{\min\{u(x,y,z);q_+(x)\};q_-(x)\},$$

we have $\hat u \in \mathcal{Z}_\theta$, $q_-(x) \leq \hat u(x,y,z) \leq q_+(x)$ for a.e. $(x,y,z) \in \mathcal{P}_\theta$ and $\varphi_3(\hat u) \leq \varphi_3(u)$.

As in Lemma 3.1, the estimate (5.3), together with Lemma 4.5, allow us to characterize the asymptotic behaviour, as $z \to +\infty$, of the functions $u \in \mathcal{Z}_\theta$ such that $\varphi_3(u) < +\infty$. Precisely, we have

Lemma 5.1 *If $u \in \mathcal{Z}_\theta$ is such that $q_-(x) \leq u(x,y,z) \leq q_+(x)$ for a.e. $(x,y,z) \in \mathcal{P}_\theta$ and $\varphi_3(u) < +\infty$ then, fixed any $L > 0$, we have either*

$$\|u(\cdot,\cdot,z)-\tilde v^+\|_{L^2(S_L)} \to 0 \quad or \quad \|u(\cdot,\cdot,z)-\tilde v^-\|_{L^2(S_L)} \to 0 \text{ as } z \to +\infty.$$

Proof Assume $u \in \mathcal{Z}_\theta$ and $\varphi_3(u) < +\infty$. Since $\varphi_{(-\bar z,\bar z),2}(u(\cdot,\cdot,z)) - \tilde m_{2,\bar z} \geq 0$ for a.e. $z > 0$, we plainly derive that there exists an increasing sequence $z_n \to +\infty$ such that $\varphi_{(-\bar z_n,\bar z_n),2}(u(\cdot,\cdot,z_n)) - \tilde m_{2,\bar z_n} \to 0$.

Fixed any $L > 0$, by Remark 4.4, we obtain that there exists an increasing subsequence $(z_{n_k}) \subset (z_n)$ with $z_{n_1} \geq \tilde L$ such that $\|u(\cdot,\cdot,z_{n_k})-\tilde v^+\|_{L^2(S_L)} \to 0$ or $\|u(\cdot,\cdot,z_{n_k})-\tilde v^-\|_{L^2(S_L)} \to 0$ as $k \to +\infty$. Possibly considering the function $u^*(x,y,z)=u(x,-y,z)$, it is not restrictive to assume that $\|u(\cdot,\cdot,z_{n_k})-\tilde v^+\|_{L^2(S_L)} \to 0$ as $k \to +\infty$. We claim that in fact $\|u(\cdot,\cdot,z)-\tilde v^+\|_{L^2(S_L)} \to 0$ as $z \to +\infty$.

Indeed, arguing by contradiction, setting $4\delta_0 = \|\tilde v^- - \tilde v^+\|_{L^2(\mathbb{R}^2)}$, by (5.2) we obtain the existence of a sequence of intervals $(\sigma_k,\tau_k) \subset \mathbb{R}^+$, a positive number $\delta \in (0,\delta_0)$ for which

(i) $\sigma_{k+1} > \tau_k \to +\infty$,
(ii) $\|u(\cdot,\cdot,\tau_k)-u(\cdot,\cdot,\sigma_k)\|_{L^2(S_L)} = \delta$,
(iii) $2\delta \geq \|u(\cdot,\cdot,z)-\tilde v^+\|_{L^2(S_L)} \geq \delta$, for every $z \in (\sigma_k,\tau_k)$.

By (4.13) and (iii) we recover that there exists $\mu > 0$ and $\bar k \in \mathbb{N}$ such that $\varphi_{(-\bar z,\bar z),2}(u(\cdot,\cdot,z)) - \tilde m_{2,\bar z} \geq \mu$ for all $z \in (\sigma_k,\tau_k)$ and $k \geq \bar k$. Using now (5.3) and (ii) we obtain $\varphi_{(\sigma_k,\tau_k),3}(u) \geq \sqrt{2\mu}\delta > 0$ for all $k \geq \bar k$ and so, by (i), we conclude $\varphi_3(u) \geq \sum_{k\geq\bar k}\varphi_{(\sigma_k,\tau_k),3}(u) = +\infty$, a contradiction. □

We are now able to prove the existence of a minimum of φ_3 on $\mathcal{Z}_\theta$.

Proposition 5.1 *For all $\theta \in (0,\frac{\pi}{4}]$, there exist $u^\pm_\theta \in \mathcal{Z}_\theta$ such that $\varphi_3(u^\pm_\theta) = \tilde m_3(\theta)$ and $q_-(x) \leq u^\pm_\theta(x,y,z) \leq q_+(x)$ a.e. on $\mathcal{P}_\theta$. Moreover, $u^-_\theta(x,y,z) = u^+_\theta(x,-y,z)$ a.e. on $\mathcal{P}_\theta$ and for every $L > 0$*

$$\|u^\pm_\theta(\cdot,\cdot,z)-\tilde v^\pm\|_{L^2(S_L)} \to 0 \quad as\, z \to +\infty.$$

Proof Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{Z}_\theta$ be a minimizing sequence for φ_3 which, by Remark 5.2, can be assumed such that $q_-(x) \leq u_n(x,y,z) \leq q_+(x)$ for a.e. $(x,y,z) \in \mathcal{P}_\theta$, $n \in \mathbb{N}$. It is not difficult to recognize that, fixed any $r > 0$, if $T_r = \mathcal{P}_\theta \cap \{z < r\}$, then $(u_n - p_0)$ is a bounded sequence on $H^1(T_r)$. Indeed, since $q_-(x) \leq u_n(x,y,z) \leq q_+(x)$ a.e. in $\mathcal{P}_\theta$ for every $n \in \mathbb{N}$ we have $\|u_n - p_0\|_{L^2(T_r)} < +\infty$ for any $n \in \mathbb{N}$. Moreover

$$\|\nabla u_n\|^2_{L^2(T_r)} = \int_0^r \int_{-\bar z}^{\bar z} \int_{\mathbb{R}} |\nabla u_n(x, y, z)|^2 \, dx\, dy\, dz$$

$$\leq 2\varphi_3(u_n) + 2 \int_0^r \tilde m_{2,\bar z}\, dz + 2 \int_0^r \int_{-\bar z}^{\bar z} m \, dy\, dz$$

$$\leq 2\varphi_3(u_n) + 2r\tilde m_{2,\infty} + 2mr^2 \tan\theta = 2(\tilde m_3(\theta) + r\tilde m_{2,\infty} + mr^2 \tan\theta) + o(1)$$

and our claim follows.

Thus, by a classical diagonal argument, there exists $u \in p_0 + \cap_{r>0} H^1(T_r)$ and a subsequence of (u_n), still denoted (u_n), such that $u_n - u \to 0$ weakly in $H^1(T_r)$ for any $r > 0$ and for a.e. $(x, y, z) \in \mathcal{P}_\theta$.

Note that, by pointwise convergence, we have $q_-(x) \leq u(x, y, z) \leq q_+(x)$ a.e. on $\mathcal{P}_\theta$. Moreover since $u - p_0 \in \cap_{r>0} H^1(T_r)$ we have that $u(\cdot, y, z) \in p_0 + H^1(\mathbb{R}) = \mathcal{H}$ a.e. on $\{(y, z) \,/\, z > 0,\ y \in (-\bar z, \bar z)\}$. Finally, since $u_n(x, y, z) = -u_n(-x, -y, z)$, by the pointwise convergence, there results also that $u(x, y, z) = -u(-x, -y, z)$ for a.e. $(x, y, z) \in \mathcal{P}_\theta$. This proves that $u \in \mathcal{Z}_\theta$ and by semicontinuity we recover that $\varphi_3(u) = \tilde m_3(\theta)$. By Lemma 5.1, it follows that fixed any $L > 0$, we have either

$$\|u(\cdot, \cdot, z) - \tilde v^+\|_{L^2(S_L)} \to 0 \quad \text{or} \quad \|u(\cdot, \cdot, z) - \tilde v^-\|_{L^2(S_L)} \to 0 \quad \text{as} \quad z \to +\infty.$$

If the first case occurs we set $u^+(x, y, z) = u(x, y, z)$, otherwise $u^+(x, y, z) = u(x, -y, z)$. Finally, setting $u^-(x, y, z) = u^+(x, -y, z)$, the Lemma follows. □

By Lemma 5.1 we have that if $\psi \in C_0^\infty(\mathbb{R}^3)$ verifies $\psi(x, y, z) = -\psi(-x, -y, z)$ then $\varphi_3(u^\pm + \psi) \geq \varphi_3(u^\pm)$. From this we derive that in fact $u^\pm$ are weak solution on $\mathcal{P}_\theta$ of the equation $-\Delta u + a(x)W'(u) = 0$ satisfying Neumann boundary condition on $\partial\mathcal{P}_\theta$. Indeed we have

Lemma 5.2 *If u is a minimum of φ_3 on $\mathcal{Z}_\theta$ then we have*

$$\int_{P_\theta} \nabla u \cdot \nabla\psi + a(x)W'(u)\psi \, dx\, dy\, dz = 0 \quad \textit{for all} \quad \psi \in C_0^\infty(\mathbb{R}^3)$$

Proof Given any $\psi \in C_0^\infty(\mathbb{R}^3)$, we set $\psi_o(x, y, z) = \frac{1}{2}(\psi(x, y, z) - \psi(-x, -y, z))$, $\psi_e(x, y, z) = \frac{1}{2}(\psi(x, y, z) + \psi(-x, -y, z))$. Since $\psi_o(x, y, z) = -\psi_o(-x, -y, z)$ we have $\varphi_3(u + t\psi_o) \geq \varphi_3(u)$ and so

$$\varphi_3(u + t\psi) - \varphi_3(u) \geq \varphi_3(u + t\psi) - \varphi_3(u + t\psi_o).$$

Observing that the functions $\nabla u \cdot \nabla\psi_e$ and $\nabla\psi_o \cdot \nabla\psi_e$ change sign under the transformation $(x, y, z) \to (-x, -y, z)$, we recover

$$\varphi_3(u + t\psi) - \varphi_3(u + t\psi_o) = \int_{P_\theta} \tfrac{t^2}{2}|\nabla\psi_e|^2 + a(x)(W(u + t\psi) - W(u + t\psi_o))\, dx\, dy\, dz.$$

Finally, since $W'(u(x, y, z))\psi_e(x, y, z) = -W'(u(-x, -y, z))\psi_e(-x, -y, z)$, we conclude

$$\int_{P_\theta} \nabla u \cdot \nabla \psi + a(x)W'(u)\psi \, dx\, dy\, dz = \lim_{t\to 0^+} \tfrac{1}{t}(\varphi_3(u+t\psi) - \varphi_3(u))$$

$$\geq \lim_{t\to 0^+} \int_{P_\theta} a(x)\tfrac{W(u+t\psi)-W(u)}{t} + a(x)\tfrac{W(u)-W(u+t\psi_o)}{t}\, dx\, dy\, dz$$

$$= \int_{P_\theta} a(x)W'(u)\psi_e \, dx\, dy\, dz = 0.$$

This proves that $\int_{P_\theta} \nabla u \cdot \nabla \psi + a(x)W'(u)\psi \, dx\, dy\, dz \geq 0$ for all $\psi \in C_0^\infty(\mathbb{R}^3)$, which is actually equivalent to our statement. □

By Proposition 5.1 and Lemma 5.2, by classical arguments, we obtain in particular that for every $\theta \in (0, \frac{\pi}{4}]$ there exist at least two solution $u_\theta^\pm \in \mathcal{Z}_\theta$ of problem $(\tilde{P}_{3,\theta})$. Since $u_\theta^\pm$ are classical solutions to $(\tilde{P}_{3,\theta})$ and since $|u_\theta^\pm(x, y, z)| \leq 1$ on $\mathcal{P}_\theta$ by Schauder estimates we obtain the existence of a constant $C > 0$ such that

$$\|u_\theta^\pm\|_{C^2(\mathcal{P}_\theta)} \leq C \quad \text{for any } \theta \in (0, \pi/4]. \tag{5.4}$$

This can be used to prove the following asymptotic property of the functions $u_\theta^\pm$.

Lemma 5.3 *There results* $\|u_\theta^\pm(\cdot, \cdot, z) - \tilde{v}^\pm\|_{L^\infty(S_{\bar{z}})} \to 0$ *as* $z \to +\infty$.

Proof Assume by contradiction that there exists $\rho_0 > 0$ and $(x_n, y_n, z_n) \in \mathcal{P}_\theta$ with $z_n \to +\infty$ such that $|u_\theta^+(x_n, y_n, z_n) - v^+(x_n, y_n)| \geq 4\rho_0$. By (5.4) we obtain that there exists $r_0 > 0$ such that if $n \in \mathbb{N}$ and $(x, y, z) \in \mathcal{P}_\theta$ is such that $|x - x_n|, |y - y_n|, |z - z_n| \leq r_0$ then $|u_\theta^+(x, y, z) - v^+(x, y)| \geq 2\rho_0$. Since by Proposition 5.1 we have $\|u_\theta^+(\cdot, \cdot, z) - v^+\|_{L^2(S_L)} \to 0$ as $z \to +\infty$ for every $L > 0$, we recover that $|y_n| \to +\infty$. By symmetry we can assume that $y_n \to +\infty$. Since by Lemma 3.7 we know that $v^+(\cdot, y) \to q^+$ as $y \to +\infty$ uniformly on $\mathbb{R}$, we deduce that there exists $\bar{n} > 0$ such that if $n \geq \bar{n}$ and $(x, y, z) \in \mathcal{P}_\theta$ is such that $|x - x_n|, |y - y_n|, |z - z_n| \leq r_0$ then $|u_\theta^+(x, y, z) - q^+(x)| \geq \rho_0$. In particular this implies that if $n \geq \bar{n}$, $|z - z_n| \leq r_0$, $y \in [y_n - r_0, \min\{y_n + r_0, \bar{z}\}]$, then

$$\|u_\theta^+(\cdot, y, z) - q^+\|_{L^2(\mathbb{R})} \geq \sqrt{2r_0}\rho_0. \tag{5.5}$$

Denoting $A = \cup_{n\geq\bar{n}}[z_n - r_0, z_n + r_0]$, since $\varphi_3(u_\theta^+) < +\infty$ we recover in particular that $\int_A \varphi_{(-\bar{z},\bar{z}),2}(u_\theta^+(\cdot, \cdot, z)) - \tilde{m}_{\bar{z},2}\, dz < +\infty$. This implies that there exists a sequence $\xi_j \to +\infty, \xi_j \in A$, such that $\varphi_{(-\bar{\xi}_j,\bar{\xi}_j),2}(u_\theta^+(\cdot, \cdot, \xi_j)) - \tilde{m}_{\bar{\xi}_j,2} \to 0$. By (5.5), if $\xi_j \in [z_n - r_0, z_n + r_0]$ for a certain $n \in \mathbb{N}$ we can pick a $\eta_j \in [y_n - r_0, \min\{y_n + r_0, \xi_j \tan\theta\}]$ such that

$$\|u_\theta^+(\cdot, \eta_j, \xi_j) - q^+\|_{L^2(\mathbb{R})} \geq \sqrt{2r_0}\rho_0. \tag{5.6}$$

Note that $0 < \eta_j < \xi_j$ and $\eta_j \to +\infty$.

Since $\varphi_{(-\bar{\xi}_j,\bar{\xi}_j),2}(u_\theta^+(\cdot, \cdot, \xi_j)) - \tilde{m}_{\bar{\xi}_j,2} \to 0$, we have that there exists $j_0 \in \mathbb{N}$ such that $\varphi_{(-\bar{\xi}_j,\bar{\xi}_j),2}(u_\theta^+(\cdot, \cdot, \xi_j)) \leq \tilde{m}_{\bar{\xi}_j,2} + \bar{\lambda}$ for every $j \geq j_0$. By Remark 4.2 we deduce that for every $j \geq j_0$ we have either

$$\|u_\theta^+(\cdot, y, \xi_j) - q_+\|_{L^2} \leq d_0 \quad \text{or} \quad \|u_\theta^+(\cdot, y, \xi_j) - q_-\|_{L^2} \leq d_0, \quad \forall y \in [\bar{\ell}, \xi_j \tan\theta].$$

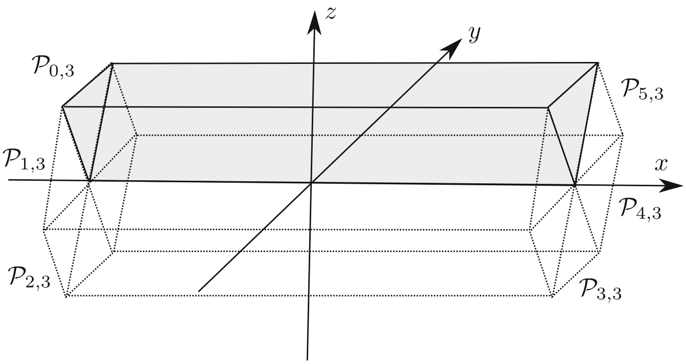

Fig. 2 The family $\{\mathcal{P}_{k,j} \mid k = 0, \dots, 2j-1\}$ for $j = 3$

Since by Proposition 5.1 we know that for every $L > 0$ we have $\|u_\theta^+(\cdot,\cdot,\xi_j) - \tilde{v}_+\|_{L^2(S_L)} \to 0$ as $j \to +\infty$, we deduce that the first case occurs for $j \geq j_0$ (taking j_0 bigger if necessary), resulting that

$$u_\theta^+(\cdot,\cdot,\xi_j) \in \tilde{\mathcal{X}}^+_{\bar{\xi}_j} \text{ for all } j \geq j_0 \text{ and } \varphi_{(-\bar{\xi}_j,\bar{\xi}_j),2}(u_\theta^+(\cdot,\cdot,\xi_j)) - \tilde{m}_{\bar{\xi}_j,2} \to 0.$$

Since $\eta_j \to +\infty$ and $0 < \eta_j < \xi_j$, we can then apply Lemma 4.5 to conclude that $\|u_\theta^+(\cdot,\eta_j,\xi_j) - q^+\|_{L^2(\mathbb{R})} \to 0$, in contradiction with (5.6). □

Now, choosing $\theta_j = \frac{\pi}{2j}$, $j \in \mathbb{N}$, $j \geq 2$, from the corresponding solutions $u^\pm_{\theta_j} : \mathcal{P}_{\theta_j} \to \mathbb{R}$ given by Proposition 5.1, we can construct an entire solution $v_j : \mathbb{R}^3 \to \mathbb{R}$ to (1.5) just recursively reflecting $u^\pm_{\theta_j}$ with respect to the faces of $\mathcal{P}_{\theta_j}$. In this way we will obtain solutions which depends in a non trivial way on $(y,z) \in \mathbb{R}^2$ as stated in Theorem 1.1.

Fixed $j \in \mathbb{N}$, $j \geq 2$, consider the rotation matrix

$$A_j = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{j} & \sin\frac{\pi}{j} \\ 0 & -\sin\frac{\pi}{j} & \cos\frac{\pi}{j} \end{Vmatrix}$$

Setting $\mathcal{P}_{k,j} = A_j^k \mathcal{P}_{\frac{\pi}{2j}}$, for every $k = 0, \dots, 2j-1$, we have $\mathbb{R}^3 = \cup_{k=0}^{2j-1} \mathcal{P}_{k,j}$ and that if $k_1 \neq k_2$ then $\operatorname{int}(\mathcal{P}_{k_1,j}) \cap \operatorname{int}(\mathcal{P}_{k_2,j}) = \emptyset$ (Fig. 2).

Now considering the minimum $u_j^\pm$ given in Proposition 5.1 corresponding to $\theta = \frac{\pi}{2j}$, recalling that $u_j^-(x,y,z) = u_j^+(x,-y,z)$ for all $(x,y,z) \in \mathcal{P}_{\frac{\pi}{2j}}$, since $A_j^{-k}\mathcal{P}_{k,j} = \mathcal{P}_{\frac{\pi}{2j}}$, for $(x,y,z) \in P_{k,j}$, $k = 0, \dots, 2j-1$, we define

$$v_j(x,y,z) = u_j^+(A_j^{-k}(x,(-1)^k y,z)) \begin{cases} = & u_j^+(A_j^{-k}(x,y,z)), & \text{if } k \text{ is even}, \\ & u_j^-(A_j^{-k}(x,y,z)), & \text{if } k \text{ is odd}. \end{cases}$$

As one can recognize, we have that $v_j|_{\mathcal{P}_{1,j}}$ is the reflection of $v_j|_{\mathcal{P}_{0,j}} = u_j^+$ w.r.t. the plane which separates $\mathcal{P}_{0,j} = \mathcal{P}_{\frac{\pi}{2j}}$ from $\mathcal{P}_{1,j}$ and, in general, for every $k \in \{1, \dots, 2j-1\}$ we have that $v_j|_{\mathcal{P}_{k,j}}$ is the reflection of $v_j|_{\mathcal{P}_{k-1,j}}$ w.r.t. the plane separating $\mathcal{P}_{k-1,j}$ from $\mathcal{P}_{k,j}$. Then, since $u_j^\pm \in H^1_{loc}(\mathcal{P}_{\frac{\pi}{2j}})$, we recover that $v_j \in H^1_{loc}(\mathbb{R}^3)$ (see e.g. [20, Lemma IX.2]). Moreover, note that if $\psi \in C_0^\infty(\mathbb{R}^3)$ and $k \in \{1, \dots, 2j-1\}$ then, trivially, $\psi \circ A_j^k \in C_0^\infty(\mathbb{R}^3)$

and so by Lemma 5.2, we obtain

$$\int_{\mathcal{P}_{k,j}} \nabla v_j(x,y,z) \cdot \nabla\psi(x,y,z) + a(x)W'(v_j(x,y,z))\psi(x,y,z)\,dx\,dy\,dz$$
$$= \int_{\mathcal{P}_{\frac{\pi}{2j}}} \nabla u_j^+(x,(-1)^k y,z)\cdot\nabla\psi\circ A_j^k(x,(-1)^k y,z)$$
$$+a(x)W'(u_j^+(x,(-1)^k y,z))\psi\circ A_j^k(x,(-1)^k y,z)\,dx\,dy\,dz = 0.$$

Hence, for any $\psi \in C_0^\infty(\mathbb{R}^3)$, we recover

$$\int_{\mathbb{R}^3} \nabla v_j\cdot\nabla\psi + a(x)W'(v_j)\psi\,dx\,dy = \sum_{k=0}^{2j-1}\int_{\mathcal{P}_{k,j}} \nabla v_j\cdot\nabla\psi + a(x)W'(v_j)\psi\,dx\,dy = 0,$$

i.e., v_j is a weak and so, by standard bootstrap arguments, a classical solution of $-\Delta v + a(x)W'(v) = 0$ on $\mathbb{R}^3$. Moreover, by definition and Proposition 5.1, we have that every v_j satisfies the conditions:

(i) $q_-(x) \le v_j(x,y,z) \le q_+(x)$ on $\mathbb{R}^3$ and so $v_j(x,y,z) \to \pm 1$ as $x \to \pm\infty$ uniformly w.r.t. $(y,z) \in \mathbb{R}^2$,

(ii) $v_j(x,y,z) - \tilde{v}^+(x,y) \to 0$ as $z \to +\infty$ for all $(x,y) \in \mathbb{R}^2$.
Moreover, denoting $\tilde{v}_j(x,\rho,\theta) = v_j(x,\rho\cos\theta,\rho\sin\theta)$, by construction and Lemma 5.3 we recover that

(iii) $\tilde{v}_j$ is periodic in θ with period $\frac{2\pi}{j}$

(iv) $\lim_{\rho\to+\infty}\tilde{v}_j(x,\rho,\frac{\pi}{2}+\frac{\pi}{j}(\frac{1}{2}+k)) = \begin{cases} q_+(x) & \text{if } k \text{ is odd} \\ q_-(x) & \text{if } k \text{ is even}\end{cases}$ uniformly for $x \in \mathbb{R}$.
Finally, since $u_j^\pm(0,0,z) = 0$, every $v_j|_{x=0}$ has $2j$ nodal lines being that

(v) $v_j(0,\rho\cos(\frac{\pi}{2}+\frac{k\pi}{j}),\rho\sin(\frac{\pi}{2}+\frac{k\pi}{j})) = 0$ for $\rho \ge 0$, $k = 0,\ldots,2j-1$.

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