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Calculus of Variations

Radial solutions of Neumann problems involving mean extrinsic curvature and periodic nonlinearities

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Abstract We show that if $\mathcal{A} \subset \mathbb{R}^N$ is an annulus or a ball centered at zero, the homogeneous Neumann problem on $\mathcal{A}$ for the equation with continuous data

$$\nabla \cdot \left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) = g(|x|, v) + h(|x|)$$

has at least one radial solution when $g(|x|, \cdot)$ has a periodic indefinite integral and $\int_{\mathcal{A}} h(|x|)\, \mathrm{d}x = 0$. The proof is based upon the direct method of the calculus of variations, variational inequalities and degree theory.

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1 Introduction

The study of quasilinear differential equations involving ϕ-Laplacian differential operators

$$[\phi(u')]' = f(x, u, u')$$

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submitted to various boundary conditions has been the source of many contributions. Most of them deal with the case where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and the paradigm is the p-Laplacian associated to $\phi(s) = |s|^{p-2}s$ with $p > 1$. References can be found in [15]. Another class of problems, motivated by the curvature operator associated to $\phi(s) = s/\sqrt{1+s^2}$, corresponds to homeomorphisms $\phi : \mathbb{R} \to (-a, a)$. One can consult for example the papers [2,3,12,9,8,14] and their references. Finally, the class of ϕ we shall deal with here is that of homeomorphisms $\phi : (-a, a) \to \mathbb{R}$ motivated by the relativistic acceleration, for which $\phi(s) = s/\sqrt{1-s^2}$. This class already appears in [11], where nonlinearities depending upon the derivative are treated, and in [7] in the general case and Neumann boundary conditions. Slightly more general classes of equations, corresponding to the radial solutions on a ball or an annulus of quasilinear partial differential equations associated to the mean extrinsic curvature in Minkowski space [1], have been first considered in [4].

In a recent paper [6], the authors have used topological degree techniques to obtain existence and multiplicity results for the radial solutions of the Neumann problem

$$\nabla \cdot \left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \mu \sin v = h(|x|) \quad \text{in} \quad \mathcal{A}, \quad \partial_\nu v = 0 \quad \text{on} \quad \partial\mathcal{A}, \tag{1}$$

on the ball or annulus

$$\mathcal{A} = \{x \in \mathbb{R}^N : R_1 \le |x| \le R_2\} \quad (0 \le R_1 < R_2)$$

i.e., for the equivalent one-dimensional problem

$$\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' + r^{N-1}\mu \sin u = r^{N-1}h(r), \quad u'(R_1) = 0 = u'(R_2).$$

They have proved the existence of at least two radial solutions not differing by a multiple of 2π when

$$2(R_2 - R_1) < \pi \quad \text{and} \quad \left|\frac{N}{R_2^N - R_1^N}\int_{R_1}^{R_2} h(r)\, r^{N-1}\mathrm{d}r\right| < \mu\cos(R_2 - R_1),$$

and the existence of at least one radial solution when $2(R_2 - R_1) = \pi$ and

$$\int_{R_1}^{R_2} h(r)\, r^{N-1}\mathrm{d}r = 0. \tag{2}$$

Condition (2) is easily seen to be necessary for the existence of a radial solution to (1) for any $\mu > 0$ and a natural question is to know if condition

$$2(R_2 - R_1) \le \pi \tag{3}$$

can be dropped.

In the analogous problem of the forced pendulum equation

$$u'' + \mu \sin u = h(t)$$

with periodic or Neumann homogeneous boundary conditions on $[0, T]$, it has been shown that the corresponding necessary condition

$$\int_0^T h(t)\,\mathrm{d}t = 0 \tag{4}$$

is also sufficient for the existence of at least two solutions not differing by a multiple of 2π. But, in this case, all the known proofs are of variational or symplectic nature (see e.g., the survey [13]).

Recently, it has been proved in [10] that the "relativistic forced pendulum equation"

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + \mu \sin u = h(t)$$

has at least one T-periodic solution for any $\mu > 0$ when the (necessary) condition (4) is satisfied. The approach is essentially variational, but combined with some topological arguments. The aim of this paper is to adapt the methodology introduced in [10] to the radial Neumann problem for (1) and prove that, for the existence part, condition (3) can be dropped.

The results are stated and proved, like in [10] but in a slightly different functional framework, for the more general class of equations of the form

$$[r^{N-1}\phi(u')]' = r^{N-1}[g(r,u) + h(r)], \quad u'(R_1) = 0 = u'(R_2) \tag{5}$$

where $\phi : (-a, a) \to \mathbb{R}$ is a suitable homeomorphism and g belongs to some class of functions 2π-periodic with respect to its second variable.

2 Hypotheses and function spaces

In what follows, we assume that $\Phi : [-a, a] \to \mathbb{R}$ satisfies the following hypothesis:

($\mathbf{H}_\Phi$) Φ is continuous, of class C^1 on $(-a, a)$, *with* $\phi := \Phi' : (-a, a) \to \mathbb{R}$ an increasing homeomorphism such that $\phi(0) = 0$.

Consequently, $\Phi : [-a, a] \to \mathbb{R}$ is strictly convex.

Given $0 \le R_1 < R_2$, the function $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ satisfies the following hypothesis:

($\mathbf{H}_\mathbf{g}$) g is continuous and its indefinite integral

$$G(r, x) := \int_0^x g(r, \xi)\mathrm{d}\xi, \quad (r, x) \in [R_1, R_2] \times \mathbb{R}$$

is 2π–periodic for each $r \in [R_1, R_2]$.

We set $C := C[R_1, R_2]$, $L^1 := L^1(R_1, R_2)$, $L^\infty := L^\infty(R_1, R_2)$ and $W^{1,\infty} := W^{1,\infty}(R_1, R_2)$. The usual norm $\|\cdot\|_\infty$ is considered on L^∞ and $W^{1,\infty}$ is endowed with the norm

$$\|v\| = \|v\|_\infty + \|v'\|_\infty \quad (v \in W^{1,\infty}).$$

Each $v \in L^1$ can be written $v(r) = \overline{v} + \tilde{v}(r)$, with

$$\overline{v} := \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} v(r)\, r^{N-1} \mathrm{d}r, \quad \int_{R_1}^{R_2} \tilde{v}(r)\, r^{N-1} \mathrm{d}r = 0.$$

If $v \in W^{1,\infty}$ then $\tilde{v}$ vanishes at some $r_0 \in (R_1, R_2)$ and

$$|\tilde{v}(r)| = |\tilde{v}(r) - \tilde{v}(r_0)| \le \int_{R_1}^{R_2} |v'(t)| \mathrm{d}t \le (R_2 - R_1)\|v'\|_\infty. \tag{6}$$

We set

$$K = \{v \in W^{1,\infty} \ : \ \|v'\|_\infty \le a\}.$$

K is closed and convex.

Lemma 1 *If $\{u_n\} \subset K$ and $u \in C$ are such that $u_n(r) \to u(r)$ for all $r \in [R_1, R_2]$, then*

(i) $u \in K$;
(ii) ${u_n}' \to u'$ *in the w^*–topology* $\sigma(L^\infty, L^1)$.

Proof From the relation

$$|u_n(r_1) - u_n(r_2)| = \left| \int_{r_2}^{r_1} u'_n(r)\, \mathrm{d}r \right| \le a|r_1 - r_2|,$$

letting $n \to \infty$, we get

$$|u(r_1) - u(r_2)| \le a|r_1 - r_2| \quad (r_1, r_2 \in [R_1, R_2]),$$

which yields $u \in K$.

Next, we show that that if $\{u'_k\}$ is a subsequence of $\{u'_n\}$ with $u'_k \to v \in L^\infty$ in the w^*–topology $\sigma(L^\infty, L^1)$ then

$$v = u' \quad \text{a.e. on } [R_1, R_2]. \tag{7}$$

Indeed, as

$$\int_{R_1}^{R_2} u'_k(r) f(r)\, \mathrm{d}r \to \int_{R_1}^{R_2} v(r) f(r)\, \mathrm{d}r \quad \text{for all} \quad f \in L^1,$$

taking $f \equiv \chi_{r_1,r_2}$, the characteristic function of the interval having the endpoints $r_1, r_2 \in [R_1, R_2]$, it follows

$$\int_{r_1}^{r_2} u'_k(r)\, \mathrm{d}r \to \int_{r_1}^{r_2} v(r)\, \mathrm{d}r \quad (r_1, r_2 \in [R_1, R_2]).$$

Then, letting $k \to \infty$ in

$$u_k(r_2) - u_k(r_1) = \int_{r_1}^{r_2} u'_k(r)\, \mathrm{d}r$$

we obtain

$$u(r_2) - u(r_1) = \int_{r_1}^{r_2} v(r)\,\mathrm{d}r \quad (r_1, r_2 \in [R_1, R_2])$$

which, clearly implies (7).

Now, to prove (ii) it suffices to show that if $\{u'_j\}$ is an arbitrary subsequence of $\{u'_n\}$, then it contains itself a subsequence $\{u'_k\}$ such that $u'_k \to u'$ in the w^*–topology $\sigma(L^\infty, L^1)$. Since L^1 is separable and $\{u'_j\}$ is bounded in $L^\infty = (L^1)^*$, we know that it has a subsequence $\{u'_k\}$ convergent to some $v \in L^\infty$ in the w^*—topology $\sigma(L^\infty, L^1)$. Then, as shown before (see (7)), we have $v = u'$. □

3 A minimization problem

Let $h \in C$ and $\mathcal{F} : K \to \mathbb{R}$ be given by

$$\mathcal{F}(v) = \int_{R_1}^{R_2} \left\{\Phi[v'(r)] + G(r, v(r)) + h(r)v(r)\right\} r^{N-1}\mathrm{d}r \quad (v \in K).$$

On account of hypotheses (H_Φ) and (H_g) the functional $\mathcal{F}$ is well defined.

Proposition 1 *If* $\overline{h} = 0$ *then* $\mathcal{F}$ *has a minimum over* K.

Proof Step I. We prove that if $\{u_n\} \subset K$ is a sequence which converges uniformly on $[R_1, R_2]$ to some $u \in K$, then

$$\liminf_{n\to\infty} \int_{R_1}^{R_2} \Phi[u'_n(r)]\, r^{N-1}\mathrm{d}r \geq \int_{R_1}^{R_2} \Phi[u'(r)]\, r^{N-1}\mathrm{d}r. \tag{8}$$

By virtue of (H_Φ) the function Φ is convex, hence for all $y \in [-a, a]$ and $z \in (-a, a)$ one has

$$\Phi(y) - \Phi(z) \geq \phi(z)(y - z). \tag{9}$$

This implies that for any $\lambda \in [0, 1)$ it holds

$$\int_{R_1}^{R_2} \Phi[u'_n(r)]\, r^{N-1}\mathrm{d}r \geq \int_{R_1}^{R_2} \Phi[\lambda u'(r)]\, r^{N-1}\mathrm{d}r + \int_{R_1}^{R_2} \phi[\lambda u'(r)][u'_n(r) - \lambda u'(r)]\, r^{N-1}\mathrm{d}r. \tag{10}$$

From Lemma 1 we have that ${u_n}' \to u'$ in the w^*–topology $\sigma(L^\infty, L^1)$. Since the map $r \mapsto r^{N-1}\phi[\lambda u'(r)]$ belongs to $L^\infty \subset L^1$, using (10) we infer that

$$\liminf_{n\to\infty} \int_{R_1}^{R_2} \Phi[{u'}_n(r)]\, r^{N-1}\mathrm{d}r \geq \int_{R_1}^{R_2} \Phi[\lambda u'(r)]\, r^{N-1}\mathrm{d}r + (1-\lambda)\int_{R_1}^{R_2} \phi[\lambda u'(r)]u'(r)\, r^{N-1}\mathrm{d}r.$$

As $\phi(t)t \geq 0$, for all $t \in (-a, a)$, we get

$$\liminf_{n\to\infty} \int_{R_1}^{R_2} \Phi[{u'}_n(r)]\, r^{N-1}\mathrm{d}r \geq \int_{R_1}^{R_2} \Phi[\lambda u'(r)]\, r^{N-1}\mathrm{d}r,$$

which, using Lebesgue's dominated convergence theorem, gives (8) by letting $\lambda \to 1$.

Step II. Due to the 2π–periodicity of $G(r, \cdot)$ (see (H_g)) and because of $\overline{h} = 0$, we have

$$\mathcal{F}(v + 2\pi) = \mathcal{F}(v), \quad \forall v \in K.$$

Therefore, if u minimizes $\mathcal{F}$ over K, then the same is true for $u + 2k\pi$ for any $k \in \mathbb{Z}$. This means that we can search, without loss of generality, a minimizer $u \in K$ with $\overline{u} \in [0, 2\pi]$. Thus, the problem reduces to minimize $\mathcal{F}$ over

$$\hat{K} = \{v \in K \ : \ \overline{v} \in [0, 2\pi]\}.$$

If $v \in \hat{K}$ then, using (6) we obtain

$$|v(r)| \leq |\overline{v}| + |\tilde{v}(r)| \leq 2\pi + (R_2 - R_1)a.$$

This, together with $\|v'\|_\infty \leq a$ shows that $\hat{K}$ is bounded in $W^{1,\infty}$ and, by the compactness of the embedding $W^{1,\infty} \subset C$, the set $\hat{K}$ is relatively compact in C. Let $\{u_n\} \subset \hat{K}$ be a minimizing sequence for $\mathcal{F}$. Passing to a subsequence if necessary and using Lemma 1, we may assume that $\{u_n\}$ converges uniformly to some $u \in K$. It is easily seen that actually $u \in \hat{K}$. By *Step I* we obtain

$$\inf_{\hat{K}} \mathcal{F} = \lim_{n\to\infty} \mathcal{F}(u_n) \geq \mathcal{F}(u),$$

showing that u minimizes $\mathcal{F}$ over $\hat{K}$. □

Remark 1 If $\{u_n\} \subset K$ and $u \in C$ are such that $u_n(r) \to u(r)$ for all $r \in [R_1, R_2]$, then by Lemma 1 and the reasoning in *Step I* of the above proof we have that $u \in K$ and (8) still holds true.

Lemma 2 *If u minimizes $\mathcal{F}$ over K then u satisfies the variational inequality*

$$\int_{R_1}^{R_2} \left(\Phi[v'(r)] - \Phi[u'(r)] + \{g[r, u(r)] + h(r)\}[v(r) - u(r)]\right) r^{N-1}\mathrm{d}r \geq 0$$

for all $v \in K$.

Proof The argument is standard. See for example Lemma 2 in [10]. □

4 An existence result

We show that the minimizers of $\mathcal{F}$ provide classical solutions for the Neumann boundary value problem

$$[r^{N-1}\phi(u')]' = r^{N-1}[g(r,u) + h(r)], \quad u'(R_1) = 0 = u'(R_2), \tag{11}$$

under the basic assumptions (H_Φ) and (H_g). Recall that by a *solution* of (11) we mean a function $u \in C^1[R_1, R_2]$, such that $\|u'\|_\infty < a$, $\phi(u')$ is differentiable and (11) is satisfied.

Let us begin with the simpler problem

$$[r^{N-1}\phi(u')]' = r^{N-1}[u + f(r)], \quad u'(R_1) = 0 = u'(R_2). \tag{12}$$

Proposition 2 *For any $f \in C$, problem (12) has a unique solution $\widehat{u}_f$ and $\widehat{u}_f$ satisfies the variational inequality*

$$\int_{R_1}^{R_2} \left(\Phi[v'(r)] - \Phi[\widehat{u}'_f(r)] + \{\widehat{u}_f(r) + f(r)\}[v(r) - \widehat{u}_f(r)]\right) r^{N-1}\mathrm{d}r \geq 0 \tag{13}$$

for all $v \in K$.

Proof The existence part follows from Corollary 2.4 in [5]. If u and v are two solutions of (12), then

$$\int_{R_1}^{R_2} \{r^{N-1}[\phi(u'(r)) - \phi(v'(r))]\}'[u(r) - v(r)]\,\mathrm{d}r = \int_{R_1}^{R_2} [u(r) - v(r)]^2\, r^{N-1}\mathrm{d}r$$

and hence, integrating the first term by parts and using the boundary conditions we obtain

$$\int_{R_1}^{R_2} \{[\phi(u'(r)) - \phi(v'(r))][u'(r) - v'(r)] + [u(r) - v(r)]^2\}\, r^{N-1}\mathrm{d}r = 0.$$

The monotonicity of ϕ yields $u = v$.

From (9) we have

$$\begin{aligned}
&\int_{R_1}^{R_2} \{\Phi[v'(r)] - \Phi[\widehat{u}'_f(r)]\}\, r^{N-1}\mathrm{d}r \\
&\quad\geq \int_{R_1}^{R_2} \phi[\widehat{u}'_f(r)][v'(r) - \widehat{u}'_f(r)]\, r^{N-1}\mathrm{d}r \\
&\quad= -\int_{R_1}^{R_2} \{r^{N-1}\phi[\widehat{u}'_f(r)]\}'[v(r) - \widehat{u}_f(r)]\,\mathrm{d}r \\
&\quad= -\int_{R_1}^{R_2} [\widehat{u}_f(r) + f(r)][v(r) - \widehat{u}_f(r)]\, r^{N-1}\mathrm{d}r,
\end{aligned}$$

showing that (13) holds for all $v \in K$. □

Theorem 1 *If hypotheses* (H_Φ) *and* (H_g) *hold true, then, for any* $h \in C$ *with* $\overline{h} = 0$, *problem* (11) *has at least one solution which minimizes* $\mathcal{F}$ *over* K.

Proof For any $w \in K$ we set

$$f_w := g(\cdot, w) + h - w \in C.$$

By Proposition 2, the unique solution $\widehat{u}_{f_w}$ of problem (12) with $f = f_w$ satisfies the variational inequality

$$\int_{R_1}^{R_2} \{\Phi[v'(r)] - \Phi[\widehat{u}_{f_w}{}'(r)] + [\widehat{u}_{f_w}(r) + f_w(r)][v(r) - \widehat{u}_{f_w}(r)]\} r^{N-1} \mathrm{d}r \geq 0 \tag{14}$$

for all $v \in K$. Let $u \in K$ be a minimizer of $\mathcal{F}$ over K; we know that it exists by Proposition 1. From Lemma 2, u satisfies the variational inequality

$$\int_{R_1}^{R_2} \{\Phi[v'(r)] - \Phi[u'(r)] + [u(r) + f_u(r)][v(r) - u(r)]\} r^{N-1} \mathrm{d}r \geq 0 \tag{15}$$

for all $v \in K$. Taking $v = \widehat{u}_{f_u}$ in (15) and $w = v = u$ in (14) and adding the resulting inequalities, we get

$$\int_{R_1}^{R_2} [u(r) - \widehat{u}_{f_u}(r)]^2 r^{N-1} \mathrm{d}r \leq 0.$$

It follows that $u = \widehat{u}_{f_u}$. Consequently, the minimizer u solves (11). □

Corollary 1 *For any* $\mu \in \mathbb{R}$ *and* $h \in C$ *with* $\overline{h} = 0$ *the problem*

$$\left(r^{N-1} \frac{u'}{\sqrt{1 - u'^2}} \right)' + r^{N-1} \mu \sin u = r^{N-1} h(r), \quad u'(R_1) = 0 = u'(R_2)$$

has at least one solution.

Corollary 2 *For any* $\mu \in \mathbb{R}$ *and* $h \in C$ *such that*

$$\int_{\mathcal{A}} h(|x|)\, \mathrm{d}x = 0,$$

the problem

$$\nabla \cdot \left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) + \mu \sin v = h(|x|) \quad in \quad \mathcal{A}, \quad \partial_\nu v = 0 \quad on \quad \partial\mathcal{A}$$

has at least one classical radial solution.

Proof Indeed, going to spherical coordinates, we have

$$\int_{\mathcal{A}} h(|x|)\, \mathrm{d}x = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_{R_1}^{R_2} h(r)\, r^{N-1} \mathrm{d}r.$$

□

Remark 2 If $\mathcal{D}$ is a bounded domain with sufficiently smooth boundary, a necessary condition for the existence of at least one solution to the Neumann problem

$$\nabla \cdot \left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) + \mu \sin v = h(x) \quad \text{in} \quad \mathcal{D}, \quad \partial_\nu v = 0 \quad \text{on} \quad \partial \mathcal{D} \tag{16}$$

for any $\mu > 0$ is that condition

$$\int_{\mathcal{D}} h(x) \, \mathrm{d}x = 0 \tag{17}$$

holds, as it is easily seen by integrating both members of (16) over $\mathcal{D}$ and using divergence theorem and the boundary conditions. It is an open problem to know if condition (17) is sufficient. A proof of the existence of a minimum for the functional

$$\mathcal{G}(u) = \int_{\mathcal{D}} \left[-\sqrt{1 - |\nabla v(x)|^2} + \mu \cos v(x) + h(x) v(x) \right] \mathrm{d}x$$

on the closed convex set

$$K := \{ v \in W^{1,\infty}(\mathcal{D}) : |\nabla v(x)| \leq 1 \text{ a.e. on } \mathcal{D} \}$$

can be done following the lines of the proof of Proposition 1, but our way to go from the variational inequality to the differential equation seems to be specific to a one-dimensional situation, i.e., to the radial case.

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