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Calculus of Variations

Cone Sobolev inequality and Dirichlet problem for nonlinear elliptic equations on a manifold with conical singularities

Hua Chen · Xiaochun Liu · Yawei Wei

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Abstract In present work, we first establish the corresponding Sobolev inequality and Poincaré inequality on the cone Sobolev spaces, and then, as an application of such inequalities, we prove the existence of non-trivial weak solution for Dirichlet boundary value problem for a class of non-linear elliptic equation on manifolds with conical singularities.

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1 Introduction and preliminaries

Let X be a closed, compact, C^∞ manifold, and set

$$X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$$

this local model interpreted as a cone with the base X. Since the analysis is formulated off the singularity it makes sense to pass to

$$X^\wedge = \mathbb{R}_+ \times X$$

the open stretched cone with the base X.

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H. Chen · X. Liu
School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
e-mail: chenhua@whu.edu.cn

X. Liu
e-mail: xcliu@whu.edu.cn

Y. Wei (✉)
School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China
e-mail: weiyawei@nankai.edu.cn

A finite dimensional manifold B with conical singularities is a topological space with a finite subset $B_0 = \{b_1, \ldots, b_M\} \subset B$ of conical singularities, with the following two properties:

1. $B \setminus B_0$ is a C^∞ manifold.
2. Every $b \in B_0$ has an open neighhourhood U in B, such that there is a homeomorphism $\varphi : U \to X^\Delta$ for some closed compact C^∞ manifold $X = X(b)$, and φ restricts to a diffeomorphism $\varphi' : U \setminus \{b\} \to X^\wedge$.

Example 1.1 Let $\tilde{X}$ be an arbitrary closed compact C^∞ manifold, then there is an integer N and a C^∞ submanifold X of $S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$ which is diffeomorphism to $\tilde{X}$. The set $B := \{x \in \mathbb{R}^N \setminus \{0\} : \frac{x}{|x|} \in X\} \cup \{0\}$ is an infinite cone with the base X and the conical point $\{0\}$.

From now on, we assume that the manifold B is paracompact and of dimension $n + 1$. By this assumption we can define the stretched manifold $\mathbb{B}$, associated with B, as a C^∞ manifold with compact C^∞ boundary $\partial\mathbb{B} \cong \bigcup_{b \in B_0} X(b)$, such that there is a diffeomophism $B\backslash B_0 \cong \mathbb{B}\backslash\partial\mathbb{B} := \text{int}\mathbb{B}$, the restriction of which to $U_1\backslash B_0 \cong V_1\backslash\partial\mathbb{B}$ for an open neighbourhood $U_1 \subset B$ near the points of B_0 and a collar neighbourhood $V_1 \subset \mathbb{B}$ with $V_1 \cong \bigcup_{b \in B_0}\{[0, 1) \times X_b\}$.

The typical differential operators on a manifold with conical singularities, are called Fuchs type, if the operators in a neighbourhood of $t = 0$ are of the following form

$$A = t^{-m} \sum_{k=0}^{m} a_k(t) \left(-t\frac{\partial}{\partial t}\right)^k \tag{1.1}$$

with $(t, x) \in \mathbb{R}_+ \times X = X^\wedge$, $a_k(t) \in C^\infty(\overline{\mathbb{R}}_+, \text{Diff}^{m-k}(X))$. Examples of that kind of operators are as follows.

Example 1.2 Let $g_X(t)$ be an t-dependent family of Riemannian metrics on a closed compact C^∞ manifold X, which is infinitely differentiable in $t \in \overline{\mathbb{R}}_+$. Then

$$g := dt^2 + t^2 g_X(t)$$

is a Riemannian metric on $X^\wedge$. The Laplace–Beltrami operators corresponding to the metric g are then of the form

$$\Delta = t^{-2} \sum_{k=0}^{2} a_k(t) \left(-t\frac{\partial}{\partial t}\right)^k$$

with $a_k(t) \in C^\infty(\overline{\mathbb{R}}_+, \text{Diff}^{2-k}(X))$.

Example 1.3 Let $g_X(t, y)$ be an (t, y)-dependent family of Riemannian metrics on a closed compact C^∞ manifold X, which is infinitely differentiable in $(t, y) \in \overline{\mathbb{R}}_+ \times \Omega$. Then

$$g := dt^2 + t^2 g_X(t, y) + dy^2$$

is a Riemannian metric on $X^\wedge \times \Omega$. The Laplace–Beltrami operators related to the metric g are then of the form

$$\Delta = t^{-2} \sum_{k=0}^{2} a_k(t, y)(-t\frac{\partial}{\partial t})^k + \sum_{j=1}^{q} \frac{\partial^2}{\partial y_j^2}$$

with $a_k(t, y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \text{Diff}^{2-k}(X))$ and $q = \dim\Omega$.

The differentiation $t\partial_t$ in Fuchs type operators motivates us to employ the Mellin transform.

Definition 1.1 Let $u(t) \in C_0^\infty(\mathbb{R}_+)$, $z \in \mathbb{C}$, then the Mellin transform is defined by the formula

$$Mu(z) = \int_0^{+\infty} t^z u(t) \frac{dt}{t},$$

and

$$M : C_0^\infty(\mathbb{R}_+) \to \mathcal{A}(\mathbb{C}),$$

where $\mathcal{A}(\mathbb{C})$ denotes the space of entire functions.

Proposition 1.1 (cf. [12]) *The Mellin transform satisfies the following identities,*

(1) $M((-t\partial_t)u)(z) = zM(z)$,
(2) $M(t^{-p}u)(z) = (Mu)(z-p)$,
(3) $M((\log t)u)(z) = (\partial_z Mu)(z)$,
(4) $M(u(t^\beta))(z) = \beta^{-1}(Mu)(\beta^{-1}z)$,

for $t \in \mathbb{R}_+$, $z, p \in \mathbb{C}$, $\beta \in \mathbb{R} \setminus \{0\}$, *and* $u \in C_0^\infty(\mathbb{R}_+)$.

To extend M to more general distribution spaces on $\mathbb{R}_+$, we introduce the weighted Mellin transform. The so-called weight line Γ_β is defined as $\Gamma_\beta = \{z \in \mathbb{C} : \operatorname{Re} z = \beta\}$. Then we define the weighted Mellin transform with weight data γ as follows

$$M_\gamma u := Mu|_{\Gamma_{\frac{1}{2}-\gamma}} = \int_0^{+\infty} t^{1/2-\gamma+i\tau} u(t) \frac{dt}{t},$$

and the inverse weighted Mellin transform is defined as

$$(M_\gamma^{-1} g)(t) = \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{2}-\gamma}} t^{-z} g(z) dz.$$

For $u(t) \in C_0^\infty(\mathbb{R}_+)$, set $S_\gamma u(r) = e^{-(\frac{1}{2}-\gamma)r} u(e^{-r})$, then we have

$$(M_\gamma u)\left(\frac{1}{2} - \gamma + i\tau\right) = (\mathcal{F} S_\gamma u)(\tau), \tag{1.2}$$

where $\mathcal{F}$ is the 1-dimensional Fourier transform corresponding to t. In fact, by changing variables $t = e^{-r}$ and set $z = \frac{1}{2} - \gamma + i\tau \in \mathbb{C}$, it is easy to see

$$\left(\mathcal{F} S_\gamma u\right)(\tau) = \int_{-\infty}^{+\infty} e^{-ir\tau} e^{-\left(\frac{1}{2}-\gamma\right)r} u(e^{-r}) dr = \int_{-\infty}^{+\infty} e^{-\left(\frac{1}{2}-\gamma+i\tau\right)r} u(e^{-r}) dr \tag{1.3}$$

$$= \int_0^{+\infty} t^z u(t) \frac{dt}{t} = \left(M_\gamma u\right)\left(\frac{1}{2} - \gamma + i\tau\right). \tag{1.4}$$

Accordingly, we have the following result.

Proposition 1.2 (cf. [12]) *The operator* $M_\gamma : C_0^\infty(\mathbb{R}_+) \to \mathcal{S}(\Gamma_{\frac{1}{2}-\gamma})$ *extends by continuity to an isomorphism*

$$M_\gamma : L_2^\gamma(\mathbb{R}_+) \to L^2\left(\Gamma_{\frac{1}{2}-\gamma}\right)$$

for all $\gamma \in \mathbb{R}$ *and* $L_2^\gamma(\mathbb{R}_+) = t^\gamma L^2(\mathbb{R}_+)$, *where*

$$\| u \|_{L_2^\gamma(\mathbb{R}_+)} = (2\pi)^{-\frac{1}{2}} \| M_\gamma u \|_{L^2\left(\Gamma_{\frac{1}{2}-\gamma}\right)} \tag{1.5}$$

The so-called weighted Mellin Sobolev spaces can be defined by using the property (1.5) as follows.

Definition 1.2 For $s, \gamma \in \mathbb{R}$, we denote by $\mathcal{H}_2^{s,\gamma}(\mathbb{R}_+^{n+1})$ the space of all $u \in \mathcal{D}'(\mathbb{R}_+^{n+1})$ such that

$$\frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{n+1}{2}-\gamma}} \int\limits_{\mathbb{R}^n} \left(1 + |z|^2 + |\xi|^2\right)^s | \left(M_{\gamma-\frac{n+1}{2}, t\to z} \mathcal{F}_{x\to\xi} u\right)(z, \xi)|^2 dz d\xi < +\infty,$$

where $M_{\gamma-\frac{n+1}{2}}$ is the weighted Mellin transform and $\mathcal{F}_{x\to\xi}$ the n-dimensional Fourier transform. Naturally, the space $\mathcal{H}_2^{s,\gamma}(\mathbb{R}_+^{n+1})$ admits a norm

$$\|u\|_{\mathcal{H}_2^{s,\gamma}\left(\mathbb{R}_+^{n+1}\right)} = \left\{ \frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{n+1}{2}-\gamma}} \int\limits_{\mathbb{R}^n} \left(1+|z|^2+|\xi|^2\right)^s | \left(M_{\gamma-\frac{n+1}{2}, t\to z} \mathcal{F}_{x\to\xi} u\right)(z, \xi)|^2 dz d\xi \right\}^{1/2}.$$

Now we turn to natural scales of the weighted Mellin Sobolev space of integer smoothness.

Definition 1.3 Let $L^2(\mathbb{R}_+^{n+1})$ be the space of square integrable functions on $\mathbb{R}_+^{n+1}$, with respect to $dtdx$, and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$. For $m \in \mathbb{N}$, and $\gamma \in \mathbb{R}$, we define

$$\mathcal{H}_2^{m,\gamma}\left(\mathbb{R}_+^{n+1}\right) = \left\{ u \in \mathcal{D}'\left(\mathbb{R}_+^{n+1}\right) : (t\partial_t)^\alpha \partial_x^\beta u \in t^{\gamma-\frac{n}{2}} L^2\left(\mathbb{R}_+^{n+1}, dtdx\right) \right\}, \tag{1.6}$$

for arbitrary $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, and $|\alpha| + |\beta| \le m$. Then $\mathcal{H}_2^{m,\gamma}(\mathbb{R}_+^{n+1})$ is a Hilbert space with the norm

$$\|u\|_{\mathcal{H}_2^{m,\gamma}\left(\mathbb{R}_+^{n+1}\right)} = \sum_{|\alpha|+|\beta|\le m} \left\{ \int\limits_{\mathbb{R}_+\times\mathbb{R}^n} |t^{\frac{n}{2}-\gamma} (t\partial_t)^\alpha \partial_x^\beta u(t, x)|^2 dtdx \right\}^{1/2}.$$

If we denote by $L_2(\mathbb{R}_+^{n+1})$ the space of square integrable functions with respect to the measure $\frac{dt}{t}dx$, we can modify (1.6) as follows:

$$\mathcal{H}_2^{m,\gamma}\left(\mathbb{R}_+^{n+1}\right) = \left\{ u \in \mathcal{D}'\left(\mathbb{R}_+^{n+1}\right) : t^{\frac{n+1}{2}-\gamma} (t\partial_t)^\alpha \partial_x^\beta u \in L_2\left(\mathbb{R}_+^{n+1}, \frac{dt}{t}dx\right) \right\}, \tag{1.7}$$

for arbitrary $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, and $|\alpha| + |\beta| \le m$. Here $m \in \mathbb{N}$ is called the smoothness of Sobolev spaces, and $\gamma \in \mathbb{R}$ the flatness of t-variable.

Next, we introduce a map

$$\left(S_{\frac{n+1}{2},\gamma} u\right)(r, x) = e^{-\left(\frac{n+1}{2}-\gamma\right)r} u\left(e^{-r}, x\right) \tag{1.8}$$

for $u(t,x) \in C_0^\infty(\mathbb{R}_+^{n+1})$, which is a continuous map $S_{\frac{n+1}{2},\gamma} : C_0^\infty(\mathbb{R}_+^{n+1}) \to C_0^\infty(\mathbb{R}^{n+1})$. Analogous to (1.2), we can extend (1.8) to an isomorphism

$$S_{\frac{n+1}{2},\gamma} : \mathcal{H}_2^{m,\gamma}\left(\mathbb{R}_+^{n+1}\right) \to H_2^m\left(\mathbb{R}^{n+1}\right). \tag{1.9}$$

In other words, we have

$$\|u\|_{\mathcal{H}_2^{m,\gamma}\left(\mathbb{R}_+^{n+1}\right)} \approx \|S_{\frac{n+1}{2},\gamma} u\|_{H_2^m(\mathbb{R}^{n+1})}$$

in the sense of norm equivalence, where $H_2^m(\mathbb{R}^{n+1})$ denotes the distribution space for $(r,x) \in \mathbb{R}^{n+1}$ such that

$$H_2^m\left(\mathbb{R}^{n+1}\right) = \left\{v(r,x) \in \mathcal{D}'\left(\mathbb{R}^{n+1}\right) | \partial_r^\alpha \partial_x^\beta v(r,x) \in L^2\left(\mathbb{R}^{n+1}, drdx\right)\right\}$$

for $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^n$ and $|\alpha|+|\beta| \le m$. One can obtain more details and information on Fuchs type operators and the weighted Mellin Sobolev spaces in [5], and [12].

Next, we generalize spaces $\mathcal{H}_2^{m,\gamma}(\mathbb{R}_+^{n+1})$ to $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^{n+1})$ for $1 \le p < +\infty$, and later on to $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ (the cone Sobolev spaces) on manifolds with conical singularities. Since the Eq. 1.5 is only valid for $p=2$, we introduce here the definition of $\mathcal{H}_p^{s,\gamma}(\mathbb{R}_+^{n+1})$ for s integers. We first modify the spaces $L^p(\mathbb{R}_+^{n+1}, dtdx)$ to $L_p(\mathbb{R}_+^{n+1}, \frac{dt}{t}dx)$.

Definition 1.4 For $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, we say that $u(t,x) \in L_p(\mathbb{R}_+^{n+1}, \frac{dt}{t}dx)$ if

$$\| u \|_{L_p} = \left(\int\int_{\mathbb{R}_+\mathbb{R}^n} t^{n+1}|u(t,x)|^p \frac{dt}{t} dx\right)^{\frac{1}{p}} < +\infty.$$

Moreover, the weighted L_p-spaces with weight data $\gamma \in \mathbb{R}$ is denoted by $L_p^\gamma(\mathbb{R}_+^{n+1}, \frac{dt}{t}dx)$, namely, if $u(t,x) \in L_p^\gamma(\mathbb{R}_+^{n+1}, \frac{dt}{t}dx)$, then $t^{-\gamma}u(t,x) \in L_p(\mathbb{R}_+^{n+1}, \frac{dt}{t}dx)$, and

$$\|u\|_{L_p^\gamma} = \left(\int\int_{\mathbb{R}_+\mathbb{R}^n} t^{n+1}|t^{-\gamma}u(t,x)|^p \frac{dt}{t} dx\right)^{\frac{1}{p}} < +\infty.$$

Now we can define the weighted Sobolev space for all $1 \le p < +\infty$.

Definition 1.5 For $m \in \mathbb{N}$, and $\gamma \in \mathbb{R}$, the spaces

$$\mathcal{H}_p^{m,\gamma}\left(\mathbb{R}_+^{n+1}\right) := \left\{u \in \mathcal{D}'\left(\mathbb{R}_+^{n+1}\right) : t^{\frac{n+1}{p}-\gamma}(t\partial_t)^\alpha \partial_x^\beta u \in L_p\left(\mathbb{R}_+^{n+1}, \frac{dt}{t}dx\right)\right\},$$

for arbitrary $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, and $|\alpha|+|\beta| \le m$. In other words, if $u(t,x) \in \mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^{n+1})$, then $(t\partial_t)^\alpha \partial_x^\beta u \in L_p^\gamma(\mathbb{R}_+^{n+1}, \frac{dt}{t}dx)$.

It is easy to see that $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^{n+1})$ is a Banach space with norm

$$\|u\|_{\mathcal{H}_p^{m,\gamma}\left(\mathbb{R}_+^{n+1}\right)} = \sum_{|\alpha|+|\beta|\le m} \left(\int\int_{\mathbb{R}_+^{n+1}} t^{n+1}|t^{-\gamma}(t\partial_t)^\alpha \partial_x^\beta u(t,x)|^p \frac{dt}{t} dx\right)^{\frac{1}{p}}.$$

Similarly (cf. [4]) we can define the weighted Sobolev spaces $\mathcal{H}_p^{m,\gamma}(X^\wedge)$ with $1\leq p<\infty$ on manifolds with conical singularities. Let X be a closed compact C^∞ manifold, and $\mathcal{U} = \{U_1, \ldots, U_N\}$ an open covering of X by coordinate neighborhoods. If we fix a subordinate partition of unity $\{\varphi_1, \ldots, \varphi_N\}$ and charts $\chi_j : U_j \to \mathbb{R}^n$, $j = 1, \ldots, N$, then $u \in \mathcal{H}_p^{m,\gamma}(X^\wedge)$ if and only if $u \in \mathcal{D}'(X^\wedge)$ with the norm

$$\|u\|_{\mathcal{H}_p^{m,\gamma}(X^\wedge)} = \left\{ \sum_{j=1}^{N} \| \left(1 \times \chi_j^*\right)^{-1} \varphi_j u \|^p_{\mathcal{H}_p^{m,\gamma}\left(\mathbb{R}_+^{n+1}\right)} \right\}^{\frac{1}{p}} < +\infty.$$

Here $1 \times \chi_j^* : C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n) \to C_0^\infty(\mathbb{R}_+ \times U_j)$ is the pull-back function with respect to $1 \times \chi_j : \mathbb{R}_+ \times U_j \to \mathbb{R}_+ \times \mathbb{R}^n$. Denote $\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge)$ as the subspace of $\mathcal{H}_p^{m,\gamma}(X^\wedge)$ which is defined as the closure of $C_0^\infty(X^\wedge)$ with respect to the norm $\|\cdot\|_{\mathcal{H}_p^{m,\gamma}(X^\wedge)}$.

Proposition 1.3 (cf. [11]) *We have* $\mathcal{H}_p^{m,\gamma}(X^\wedge) \subset W_{\text{loc}}^{m,p}(X^\wedge)$ *for all* $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$, *where* $W_{\text{loc}}^{m,p}(X^\wedge)$ *denotes the subspace of all* $u \in \mathcal{D}'(X^\wedge)$ *such that* $\varphi u \in W^{m,p}(X^\wedge)$ *for every* $\varphi \in C_0^\infty(X^\wedge)$.

Let $\mathbb{B}$ be the stretched manifold of B, we will always denote $\omega(t) \in C_0^\infty(\mathbb{B})$ as a real-valued cut-off function which equals 1 near $\{0\} \times \partial\mathbb{B}$.

Definition 1.6 Let $\mathbb{B}$ be the stretched manifold to a manifold B with conical singularities. Then $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ for $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$ denotes the subspace of all $u \in W_{\text{loc}}^{m,p}(\text{int}\mathbb{B})$, such that

$$\mathcal{H}_p^{m,\gamma}(\mathbb{B}) = \left\{u \in W_{\text{loc}}^{m,p}(\text{int}\mathbb{B}) \mid \omega u \in \mathcal{H}_p^{m,\gamma}\left(X^\wedge\right)\right\}$$

for any cut-off function ω, supported by a collar neighbourhood of $[0, 1) \times \partial\mathbb{B}$. Moreover, the subspace $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B})$ of $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is defined as follows:

$$\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{B}) := [\omega]\mathcal{H}_{p,0}^{m,\gamma}\left(X^\wedge\right) + [1-\omega]W_0^{m,p}(\text{int}\mathbb{B}),$$

where $W_0^{m,p}(\text{int}\mathbb{B})$ denotes the closure of $C_0^\infty(\text{int}\mathbb{B})$ in Sobolev spaces $W^{m,p}(\tilde{X})$ when $\tilde{X}$ is a closed compact C^∞ manifold of dimension $n+1$ that contains $\mathbb{B}$ as a submanifold with boundary.

Remark 1.1 (cf. [11]) We have the following properties:

(1) $\mathcal{H}_p^{m,\gamma}(\mathbb{B})$ is Banach space for $1 \leq p < \infty$, and is Hilbert space for $p = 2$.
(2) $L_p^\gamma(\mathbb{B}) := \mathcal{H}_p^{0,\gamma}(\mathbb{B})$.
(3) $L_p(\mathbb{B}) := \mathcal{H}_p^{0,0}(\mathbb{B})$.
(4) $t^{\gamma_1}\mathcal{H}_p^{m,\gamma_2}(\mathbb{B}) = \mathcal{H}_p^{m,\gamma_1+\gamma_2}(\mathbb{B})$.
(5) The embedding $\mathcal{H}_p^{m,\gamma}(\mathbb{B}) \hookrightarrow \mathcal{H}_p^{m',\gamma'}(\mathbb{B})$ is continuous if $m \geq m'$, $\gamma \geq \gamma'$; and is compact embedding if $m > m'$, $\gamma > \gamma'$.

2 Cone Sobolev inequality

In this section we will prove the so-called cone Sobolev inequality on $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^{n+1})$. The discussion will be separated into two parts. One is for $\frac{1}{p} > \frac{m}{n+1}$, and the other is for $\frac{1}{p} < \frac{m}{n+1}$. First let us recall two well-known results.

Proposition 2.1 (Gagliardo–Nirenberg–Sobolev inequality) *Assume $1 \le p < n$. There exists a constant c, depending only on p and n, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le c\|\nabla u\|_{L^p(\mathbb{R}^n)} \tag{2.1}$$

for all $u \in C_0^1(\mathbb{R}^n)$, with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Here the constant $c = \frac{(n-1)p}{n(n-p)}$ is the optimal constant.

Proposition 2.2 *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose ∂U is C^1. Assume $1 \le p < n$, and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$ and we have the following Sobolev inequality:*

$$\|u\|_{L^{p^*}(U)} \le C\|u\|_{W^{1,p}(U)},$$

the constant C depending only on p, n, and U.

One can obtain all the details in [6]. Here we will generalize the Sobolev inequality on the cone Sobolev spaces $\mathcal{H}_p^{s,\gamma}$, namely,

Theorem 2.1 (Cone Sobolev Inequality) *Assume that $1 \le p < n+1$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n+1}$, and $\gamma \in \mathbb{R}$. Let $\mathbb{R}_+^{n+1} := \mathbb{R}_+ \times \mathbb{R}^n$, $t \in \mathbb{R}_+$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The following estimate*

$$\|u\|_{L_{p^*}^{\gamma^*}\left(\mathbb{R}_+^{n+1}\right)} \le c_1\|(t\partial_t)u\|_{L_p^{\gamma}\left(\mathbb{R}_+^{n+1}\right)} + (c_1+c_2)\sum_{i=1}^{n}\|\partial_{x_i}u\|_{L_p^{\gamma}\left(\mathbb{R}_+^{n+1}\right)} + \frac{c_2}{\alpha}\|u\|_{L_p^{\gamma}\left(\mathbb{R}_+^{n+1}\right)} \tag{2.2}$$

holds for all $u \in C_0^\infty(\mathbb{R}_+^{n+1})$, where $\gamma^* = \gamma - 1$, $c_1 = \frac{np}{(n+1)(n+1-p)}$, and $c_2 = \frac{np|n-\frac{(\gamma-1)np}{n+1-p}|^{\frac{1}{n+1}}}{(n+1-p)(n+1)}$, $\alpha = \frac{np}{n+1-p}$. Moreover, if $u \in \mathcal{H}_{p,0}^{1,\gamma}(\mathbb{R}_+^{n+1})$, we have

$$\|u\|_{L_{p^*}^{\gamma^*}\left(\mathbb{R}_+^{n+1}\right)} \le c\|u\|_{\mathcal{H}_p^{1,\gamma}\left(\mathbb{R}_+^{n+1}\right)}, \tag{2.3}$$

where the constant $c = c_1 + c_2$, and c_1, c_2 are given in (2.2).

Proof First, we consider $p = 1$, and then $p^* = \frac{n+1}{n}$. For arbitrary $\gamma' \in \mathbb{R}$, since $u(t,x) \in C_0^\infty(\mathbb{R}_+^{n+1})$, where $(t,x) = (t, x_1, x_2, \ldots, x_n) \in \mathbb{R}_+ \times \mathbb{R}^n$, we have

$$\begin{aligned}|t^{n-\gamma'}u| &= |\int_0^t (r\partial_r)\left(r^{n-\gamma'}u\right)\frac{dr}{r}| \\ &\le \int_0^{+\infty} |r^{n-\gamma'}(r\partial_r)u|\frac{dr}{r} + |n-\gamma'|\int_0^{+\infty} |r^{n-\gamma'}u|\frac{dr}{r}.\end{aligned}$$

Analogously, we get

$$|t^{n-\gamma'}u| \le \int_{-\infty}^{+\infty} t^{n-\gamma'}|\partial_{y_i}u|dy_i$$

for $i = 1, 2, \dots, n$. Multiplying the above $n+1$ inequalities, one has

$$\begin{aligned}|t^{n-\gamma'}u|^{n+1} \le & \int_0^{+\infty} |r^{n-\gamma'}(r\partial_r)u|\frac{dr}{r}\prod_{i=1}^{n}\int_{-\infty}^{+\infty} t^{n-\gamma'}|\partial_{y_i}u|dy_i \\ & + |n-\gamma'|\int_0^{+\infty} |r^{n-\gamma'}u|\frac{dr}{r}\prod_{i=1}^{n}\int_{-\infty}^{+\infty} t^{n-\gamma'}|\partial_{y_i}u|dy_i.\end{aligned}$$

Then, we can obtain

$$\begin{aligned}|t^{n-\gamma'}u|^{\frac{n+1}{n}} \le & \left[\int_0^{+\infty} |r^{n-\gamma'}(r\partial_r)u|\frac{dr}{r}\prod_{i=1}^{n}\int_{-\infty}^{+\infty} t^{n-\gamma'}|\partial_{y_i}u|dy_i\right. \\ & \left. + |n-\gamma'|\int_0^{+\infty} |r^{n-\gamma'}u|\frac{dr}{r}\prod_{i=1}^{n}\int_{-\infty}^{+\infty} t^{n-\gamma'}|\partial_{y_i}u|dy_i\right]^{\frac{1}{n}} \\ \le & \left(\int_0^{+\infty} |r^{n-\gamma'}(r\partial_r)u|\frac{dr}{r}\right)^{\frac{1}{n}}\prod_{i=1}^{n}\left(\int_{-\infty}^{+\infty} t^{n-\gamma'}|\partial_{y_i}u|dy_i\right)^{\frac{1}{n}} \\ & + |n-\gamma'|^{\frac{1}{n}}\left(\int_0^{+\infty} |r^{n-\gamma'}u|\frac{dr}{r}\right)^{\frac{1}{n}}\prod_{i=1}^{n}\left(\int_{-\infty}^{+\infty} t^{n-\gamma'}|\partial_{y_i}u|dy_i\right)^{\frac{1}{n}}.\end{aligned}$$

Integrating both sides of the last inequality with respect to $\frac{dt}{t}$, we have

$$\begin{aligned}\int_0^{+\infty} |t^{n-\gamma'}u|^{\frac{n+1}{n}}\frac{dt}{t} \le & \left(\int_0^{+\infty} |r^{n-\gamma'}(r\partial_r)u|\frac{dr}{r}\right)^{\frac{1}{n}}\int_0^{+\infty}\prod_{i=1}^{n}\left(\int_{-\infty}^{+\infty} t^{n-\gamma'}|\partial_{y_i}u|dy_i\right)^{\frac{1}{n}}\frac{dt}{t} \\ & + |n-\gamma'|^{\frac{1}{n}}\left(\int_0^{+\infty} |r^{n-\gamma'}u|\frac{dr}{r}\right)^{\frac{1}{n}}\int_0^{+\infty}\prod_{i=1}^{n}\left(\int_{-\infty}^{+\infty} t^{n-\gamma'}|\partial_{y_i}u|dy_i\right)^{\frac{1}{n}}\frac{dt}{t}.\end{aligned}$$

By Hölder inequality, it yields

$$\begin{aligned}\int_0^{+\infty} |t^{n-\gamma'}u|^{\frac{n+1}{n}}\frac{dt}{t} \le & \left(\int_0^{+\infty} |r^{n-\gamma'}(r\partial_r)u|\frac{dr}{r}\right)^{\frac{1}{n}}\left(\prod_{i=1}^{n}\int_{-\infty}^{+\infty}\int_0^{+\infty} t^{n-\gamma'}|\partial_{y_i}u|\frac{dt}{t}dy_i\right)^{\frac{1}{n}} \\ & + |n-\gamma'|^{\frac{1}{n}}\left(\int_0^{+\infty} |r^{n-\gamma'}u|\frac{dr}{r}\right)^{\frac{1}{n}}\left(\prod_{i=1}^{n}\int_{-\infty}^{+\infty}\int_0^{+\infty} t^{n-\gamma'}|\partial_{y_i}u|\frac{dt}{t}dy_i\right)^{\frac{1}{n}}. \qquad (2.4)\end{aligned}$$

Now integrating (2.4) with respect to dy_1, and applying Hölder inequality again, we get

$$\iint |t^{n-\gamma'}u|^{\frac{n+1}{n}}\frac{dt}{t}dy_1 \le \left(\iint t^{n-\gamma'}|\partial_{y_1}u|\frac{dt}{t}dy_1\right)^{\frac{1}{n}}\left(\iint |r^{n-\gamma'}(r\partial_r)u|\frac{dr}{r}dy_1\right)^{\frac{1}{n}}$$
$$\times\prod_{i=2}^{n}\left(\iiint t^{n-\gamma'}|\partial_{y_i}u|\frac{dt}{t}dy_1dy_i\right)^{\frac{1}{n}}$$
$$+|n-\gamma'|^{\frac{1}{n}}\left(\iint t^{n-\gamma'}|\partial_{y_1}u|\frac{dt}{t}dy_1\right)^{\frac{1}{n}}\left(\iint |r^{n-\gamma'}u|\frac{dr}{r}dy_1\right)^{\frac{1}{n}}$$
$$\times\prod_{i=2}^{n}\left(\iiint t^{n-\gamma'}|\partial_{y_i}u|\frac{dt}{t}dy_1dy_i\right)^{\frac{1}{n}}.$$

Integrating with respect to $dy_2,\dots,dy_n$, and using Hölder inequality, we obtain

$$\int\cdots\int |t^{n-\gamma'}u|^{\frac{n+1}{n}}\frac{dt}{t}dy_1\dots dy_n$$
$$\le\left(\int\cdots\int r^{n-\gamma'}|(r\partial_r)u|\frac{dr}{r}dy_1\dots dy_n\right)^{\frac{1}{n}}\prod_{i=1}^{n}\left(\int\cdots\int t^{n-\gamma'}|\partial_{y_i}u|\frac{dt}{t}dy_1\dots dy_n\right)^{\frac{1}{n}}$$
$$+|n-\gamma'|^{\frac{1}{n}}\left(\int\cdots\int r^{n-\gamma'}|u|\frac{dr}{r}dy_1\dots dy_n\right)^{\frac{1}{n}}$$
$$\times\prod_{i=1}^{n}\left(\int\cdots\int t^{n-\gamma'}|\partial_{y_i}u|\frac{dt}{t}dy_1\dots dy_n\right)^{\frac{1}{n}},$$

which means that

$$\iint |t^{n-\gamma'}u|^{\frac{n+1}{n}}\frac{dt}{t}dx \le \left(\iint t^{n-\gamma'}|(t\partial_t)u|\frac{dt}{t}dx\right)^{\frac{1}{n}}\prod_{i=1}^{n}\left(\iint t^{n-\gamma'}|\partial_{x_i}u|\frac{dt}{t}dx\right)^{\frac{1}{n}}$$
$$+|n-\gamma'|^{\frac{1}{n}}\left(\iint t^{n-\gamma'}|u|\frac{dt}{t}dx\right)^{\frac{1}{n}}\prod_{i=1}^{n}\left(\iint t^{n-\gamma'}|\partial_{x_i}u|\frac{dt}{t}dx\right)^{\frac{1}{n}}.$$

Here we used the notation of integral "$\iint$" to stand for "$\int_0^{+\infty}\int_{\mathbb{R}^n}$".

Then, by the inequality $a+b<(a^\alpha+b^\alpha)^{1/\alpha}$ for $0<\alpha<1$, we have

$$\left(\iint |t^{n-\gamma'}u|^{\frac{n+1}{n}}\frac{dt}{t}dx\right)^{\frac{n}{n+1}}$$
$$\le\left[\left(\iint t^{n-\gamma'}|(t\partial_t)u|\frac{dt}{t}dx\right)^{\frac{1}{n}}\prod_{i=1}^{n}\left(\iint t^{n-\gamma'}|\partial_{x_i}u|\frac{dt}{t}dx\right)^{\frac{1}{n}}\right.$$
$$\left.+|n-\gamma'|^{\frac{1}{n}}\left(\iint t^{n-\gamma'}|u|\frac{dt}{t}dx\right)^{\frac{1}{n}}\prod_{i=1}^{n}\left(\iint t^{n-\gamma'}|\partial_{x_i}u|\frac{dt}{t}dx\right)^{\frac{1}{n}}\right]^{\frac{n}{n+1}}$$
$$\le\left(\iint t^{n-\gamma'}|(t\partial_t)u|\frac{dt}{t}dx\right)^{\frac{1}{n+1}}\prod_{i=1}^{n}\left(\iint t^{n-\gamma'}|\partial_{x_i}u|\frac{dt}{t}dx\right)^{\frac{1}{n+1}}$$

$$+|n-\gamma'|^{\frac{1}{n+1}}\left(\iint t^{n-\gamma'}|u|\frac{dt}{t}dx\right)^{\frac{1}{n+1}}\prod_{i=1}^{n}\left(\iint t^{n-\gamma'}|\partial_{x_i}u|\frac{dt}{t}dx\right)^{\frac{1}{n+1}}$$

$$\leq\frac{1}{n+1}\iint t^{n-\gamma'}|\,(t\partial_t)\,u|\frac{dt}{t}dx+\frac{1}{n+1}\sum_{i=1}^{n}\iint t^{n-\gamma'}|\partial_{x_i}u|\frac{dt}{t}dx$$

$$+\frac{|n-\gamma'|^{\frac{1}{n+1}}}{n+1}\left(\iint t^{n-\gamma'}|u|\frac{dt}{t}dx\right)+\frac{|n-\gamma'|^{\frac{1}{n+1}}}{n+1}\sum_{i=1}^{n}\iint t^{n-\gamma'}|\partial_{x_i}u|\frac{dt}{t}dx$$

Set $\gamma=\gamma'+1$, it yields

$$\left(\iint|t^{n-\gamma'}u|^{\frac{n+1}{n}}\frac{dt}{t}dx\right)^{\frac{n}{n+1}}\leq\frac{1}{n+1}\iint t^{n+1-\gamma}|\,(t\partial_t)\,u|\frac{dt}{t}dx$$
$$+\left(\frac{1}{n+1}+\frac{|n+1-\gamma|^{\frac{1}{n+1}}}{n+1}\right)\sum_{i=1}^{n}\iint t^{n+1-\gamma}|\partial_{x_i}u|\frac{dt}{t}dx$$
$$+\frac{|n+1-\gamma|^{\frac{1}{n+1}}}{n+1}\left(\iint t^{n+1-\gamma}|u|\frac{dt}{t}dx\right). \tag{2.5}$$

This is the estimate (2.2) for $p=1$.

Now we consider the case for $1<p<n+1$. Applying (2.5) for $v=|u|^{\alpha}$ with $\alpha>1$, and write $\gamma=\gamma'+1$, we have

$$\left(\iint|t^{n-\gamma'}\left(|u|^{\alpha}\right)|^{\frac{n+1}{n}}\frac{dt}{t}dx\right)^{\frac{n}{n+1}}\leq\frac{1}{n+1}\iint t^{n-\gamma'}|\,(t\partial_t)\left(|u|^{\alpha}\right)|\frac{dt}{t}dx$$
$$+\left(\frac{1}{n+1}+\frac{|n-\gamma'|^{\frac{1}{n+1}}}{n+1}\right)\sum_{i=1}^{n}\iint t^{n-\gamma'}|\partial_{x_i}\left(|u|^{\alpha}\right)|\frac{dt}{t}dx$$
$$+\frac{|n-\gamma'|^{\frac{1}{n+1}}}{n+1}\iint t^{n-\gamma'}|u|^{\alpha}\frac{dt}{t}dx. \tag{2.6}$$

Since $|\partial_{x_i}(|u|^{\alpha})|=|\partial_{x_i}(u^{\frac{\alpha}{2}}\cdot\bar{u}^{\frac{\alpha}{2}})|\leq\alpha|u|^{\alpha-1}|\partial_{x_i}u|$, we can rewrite (2.6) to get

$$\left(\iint t^{\frac{(n-\gamma')(n+1)}{n}}|u|^{\frac{\alpha(n+1)}{n}}\frac{dt}{t}dx\right)^{\frac{n}{n+1}}\leq\frac{\alpha}{n+1}\iint t^{n-\gamma'}|u|^{\alpha-1}|\,(t\partial_t)\,u|\frac{dt}{t}dx$$
$$+\alpha\left(\frac{1}{n+1}+\frac{|n-\gamma'|^{\frac{1}{n+1}}}{n+1}\right)\sum_{i=1}^{n}\iint t^{n-\gamma'}|u|^{\alpha-1}|\partial_{x_i}u|\frac{dt}{t}dx$$
$$+\frac{|n-\gamma'|^{\frac{1}{n+1}}}{n+1}\left(\iint t^{n-\gamma'}|u|^{\alpha}\frac{dt}{t}dx\right).$$

Set $\psi + \varphi = n - \gamma'$, and using Hölder inequality again, we obtain

$$\left(\iint t^{\frac{(n-\gamma')(n+1)}{n}} |u|^{\frac{\alpha(n+1)}{n}} \frac{dt}{t} dx\right)^{\frac{n}{n+1}}$$
$$\leq \frac{\alpha}{n+1}\left(\iint \left(t^{\varphi}|(t\partial_t)u|\right)^p \frac{dt}{t}dx\right)^{\frac{1}{p}}\left(\iint \left(t^{\psi}|u|^{\alpha-1}\right)^{\frac{p}{p-1}} dtdx\right)^{\frac{p-1}{p}}$$
$$+\alpha\left(\frac{1}{n+1}+\frac{|n-\gamma'|^{\frac{1}{n+1}}}{n+1}\right)\sum_{i=1}^{n}\left(\iint \left(t^{\varphi}|\partial_{x_i}u|\right)^p \frac{dt}{t}dx\right)^{\frac{1}{p}}$$
$$\times\left(\iint \left(t^{\psi}|u|^{\alpha-1}\right)^{\frac{p}{p-1}} dtdx\right)^{\frac{p-1}{p}}$$
$$+\frac{|n-\gamma'|^{\frac{1}{n+1}}}{n+1}\left(\iint \left(t^{\varphi}|u|\right)^p \frac{dt}{t}dx\right)^{\frac{1}{p}}\left(\iint \left(t^{\psi}|u|^{\alpha-1}\right)^{\frac{p}{p-1}} dtdx\right)^{\frac{p-1}{p}}. \quad (2.7)$$

Choose φ and ψ such that $\varphi p = n + 1 - p\gamma$, $\psi \cdot \frac{p}{p-1} = \frac{(n+1)(n-\gamma')}{n}$, $p^*\gamma^* = n + 1 - \frac{(n+1)(n-\gamma')}{n}$, and $(\alpha - 1)\frac{p}{p-1} = p^*$. Thus from $\psi + \varphi = n - \gamma'$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n+1}$, we can get $\psi = \frac{(n+1)(n-\gamma')(p-1)}{np}$, $\varphi = \frac{(n-\gamma')(n+1-p)}{np}$, $\gamma^* = \frac{(n+1-p)\gamma'}{np}$, $\gamma = 1 + \frac{(n+1-p)\gamma'}{np}$ and $\alpha = \frac{np}{n+1-p}$ for $1 < p < n + 1$. It is easy to see that $\frac{n}{n+1} - \frac{p-1}{p} = \frac{1}{p^*}$ and $\alpha > 1$. Consequently, (2.7) becomes the following estimate

$$\left(\iint t^{n+1-p^*\gamma^*}|u|^{p^*}\frac{dt}{t}dx\right)^{\frac{1}{p^*}}$$
$$\leq \frac{\alpha}{n+1}\left(\iint t^{n+1-p\gamma}|\,(t\partial_t)\,u|^p\frac{dt}{t}dx\right)^{\frac{1}{p}}$$
$$+\alpha\left(\frac{1}{n+1}+\frac{|n-\gamma'|^{\frac{1}{n+1}}}{n+1}\right)\sum_{i=1}^{n}\left(\iint t^{n+1-p\gamma}|\left(\partial_{x_i}\right)u|^p\frac{dt}{t}dx\right)^{\frac{1}{p}}$$
$$+\frac{|n-\gamma'|^{\frac{1}{n+1}}}{n+1}\left(\iint t^{n+1-p\gamma}|u|^p\frac{dt}{t}dx\right)^{\frac{1}{p}},$$

which means that

$$\|u\|_{L^{\gamma^*}_{p^*}} \leq c_1\|\,(t\partial_t)\,u\|_{L^{\gamma}_p} + (c_1+c_2)\sum_{i=1}^{n}\|\partial_{x_i}u\|_{L^{\gamma}_p} + \frac{c_2}{\alpha}\|u\|_{L^{\gamma}_p}, \quad (2.8)$$

where $c_1 = \frac{\alpha}{n+1}$ $c_2 = \alpha \cdot \frac{|n-\frac{(\gamma-1)np}{n+1-p}|^{\frac{1}{n+1}}}{n+1}$, with $\alpha = \frac{np}{n+1-p}$, and $\gamma = \gamma^* + 1$. □

Remark 2.1 If we take $\tau = \ln t$, then $\tau \to -\infty$ as $t \to 0+$, and $t\partial_t = \partial_\tau$, $\frac{dt}{t} = d\tau$. Thus in case of $\gamma = \frac{n+1}{p}$ in (2.2), then $c_2 = 0$, $c_1 = \frac{np}{(n+1)(n+1-p)}$, $L^{\gamma}_p(\mathbb{R}^{n+1}_+) = L^p(\mathbb{R}^{n+1})$ and $L^{\gamma^*}_{p^*}(\mathbb{R}^{n+1}_+) = L^{p^*}(\mathbb{R}^{n+1})$. Thus the cone Sobolev inequality becomes the standard Sobolev inequality (2.1), that means the standard Sobolev inequality is the special case of the cone Sobolev inequality here and the constants, appeared in (2.2), would be the best possible.

Remark 2.2 There rises a natural question on whether the spaces $\mathcal{H}_{p,0}^{1,\frac{n+1}{p}}(\mathbb{R}_+^{n+1})$ have certain inclusion relation with $W_0^{1,p}(\mathbb{R}_+^{n+1})$. Indeed, it is not the case. By definition, if $u(t,x) \in \mathcal{H}_{p,0}^{1,\frac{n+1}{p}}(\mathbb{R}_+^{n+1})$, then

$$t^{\frac{n+1}{p}-\frac{n+1}{p}}(t\partial_t)^\alpha\,\partial_x^\beta u \in L_p\left(\mathbb{R}_+^{n+1}, \frac{dt}{t}dx\right),$$

for $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, and $|\alpha|+|\beta| \le 1$. On the other hand, if $v(t,x) \in W_0^{1,p}(\mathbb{R}_+^{n+1})$, then

$$\partial_t^\alpha\partial_x^\beta v \in L^p\left(\mathbb{R}_+^{n+1}, dtdx\right),$$

for $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, and $|\alpha|+|\beta| \le 1$. So the spaces $\mathcal{H}_{p,0}^{1,\frac{n+1}{p}}(\mathbb{R}_+^{n+1})$ and $W_0^{1,p}(\mathbb{R}_+^{n+1})$ have no inclusion relation, since the two conditions of $|\alpha|=1, |\beta|=0$ and $|\alpha|=0, |\beta|=1$ lead to different results. However, if we set $\gamma = \frac{n}{p}$, then we have the equivalent relation, i.e. $L_p^\gamma(\mathbb{R}_+^{n+1}, \frac{dt}{t}dx) = L^p(\mathbb{R}_+^{n+1}, dtdx)$.

Theorem 2.2 *Suppose* $\frac{1}{p}-\frac{\mu}{n+1} > 0$, $m, \mu \in \mathbb{N}, \mu < m$, *and* $\gamma \in \mathbb{R}$, *then there are continuous embedding*

$$\mathcal{H}_{p,0}^{m,\gamma}\left(\mathbb{R}_+^{n+1}\right) \hookrightarrow \mathcal{H}_{p',0}^{m',\gamma'}\left(\mathbb{R}_+^{n+1}\right)$$

with $\frac{1}{p'} = \frac{1}{p}-\frac{\mu}{n+1}$, $m' = m-\mu$, *and* $\gamma' = \gamma-\mu$. *More precisely, for* $u \in \mathcal{H}_{p,0}^{m,\gamma}(\mathbb{R}_+^{n+1})$, *we have*

$$\|u\|_{\mathcal{H}_{p'}^{m',\gamma'}} \le c^\mu \|u\|_{\mathcal{H}_p^{m,\gamma}}, \tag{2.9}$$

where $c = c_1 + c_2$ *is defined as that in* (2.8). *In particular, if* $\frac{1}{p}-\frac{m}{n+1} > 0$, *then there are continuous embedding*

$$\mathcal{H}_{p,0}^{m,\gamma}\left(\mathbb{R}_+^{n+1}\right) \hookrightarrow L_{p^*}^{\gamma^*}\left(\mathbb{R}_+^{n+1}\right)$$

with $\frac{1}{p^*} = \frac{1}{p}-\frac{m}{n+1}$ *and* $\gamma^* = \gamma - m$, *and for* $u \in \mathcal{H}_{p,0}^{m,\gamma}(\mathbb{R}_+^{n+1})$, *we have*

$$\|u\|_{L_{p^*}^{\gamma^*}} \le c^m \|u\|_{\mathcal{H}_p^{m,\gamma}}. \tag{2.10}$$

Proof Since $C_0^\infty(\mathbb{R}_+^{n+1})$ is dense in $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{R}_+^{n+1})$, Theorem 2.1 is valid for $u(t,x) \in \mathcal{H}_{p,0}^{1,\gamma}$. Note that if $u(t,x) \in \mathcal{H}_{p,0}^{m,\gamma}(\mathbb{R}_+^{n+1})$, then $D^\alpha u = (t\partial_t)^{\alpha_1}\partial_x^{\alpha'}u \in \mathcal{H}_{p,0}^{1,\gamma}(\mathbb{R}_+^{n+1})$ for $|\alpha| = \alpha_1 + |\alpha'| \le m-1$. We can apply Theorem 2.1 iteratively to obtain the estimate of continuous embedding (2.9) and (2.10). □

Theorem 2.3 *Let* $m \in \mathbb{N}$, $0 < \alpha < 1$, $\gamma \ge 1 + \frac{n}{p} + \frac{m-1}{n+1}$ *and* $m - \frac{n+1}{p} = \alpha$. *Then there exists a constant* c, *depending only on* p, n *and* γ, *such that for any* $u(y) \in \mathcal{H}_{p,0}^{m,\gamma}(\mathbb{R}_+^{n+1})$, *and* $y_1, y_2 \in \mathbb{R}_+^{n+1}$, $|y_1 - y_2| \approx \rho$ *for some* $\rho > 0$ *small enough, we have*

$$|u(y_1)-u(y_2)| \le c\|u\|_{\mathcal{H}_p^{m,\gamma}}|y_1-y_2|^\alpha.$$

Proof Since $u \in \mathcal{H}^{m,\gamma}_{p,0}(\mathbb{R}^{n+1}_+)$, then $(t\partial_t)u, \partial_{x_i}u \in \mathcal{H}^{m-1,\gamma}_{p,0}(\mathbb{R}^{n+1}_+)$, for $i = 1, \ldots, n$. In view of $(m-1) - \frac{n+1}{p} = \alpha - 1 < 0$, we have $\frac{1}{p} - \frac{m-1}{n+1} = \frac{1-\alpha}{n+1} > 0$. Apply Theorem 2.1, we obtain $(t\partial_t)u, \partial_{x_i}u \in L^{\gamma^*}_{p^*}(\mathbb{R}^{n+1}_+)$, for $\frac{1}{p^*} = \frac{1}{p} - \frac{m-1}{n+1}$, and $\gamma^* = \gamma - (m-1)$. Since $\gamma \geq 1 + \frac{n}{p} + \frac{m-1}{n+1}$, then $\gamma^* > 0$. Furthermore, we have following estimate

$$\|\,(t\partial_t)\,u\|_{L^{\gamma^*}_{p^*}} \leq c_0\|u\|_{\mathcal{H}^{m,\gamma}_p} \tag{2.11}$$

and

$$\|\partial_{x_i}u\|_{L^{\gamma^*}_{p^*}} \leq c_0\|u\|_{\mathcal{H}^{m,\gamma}_p}, \tag{2.12}$$

with $c_0 = (c_1 + c_2)^{m-1}$, where c_1 and c_2 are defined in (2.8).

Let $\Omega_\rho \subseteq \mathbb{R}^{n+1}_+$ be a $(n+1)$-dimensional polyhedron, with center $y_0 = (t_0, x_0)$, side length ρ and the side line being parallel with the coordinate axis, such that $|t - t_0| \leq \sqrt{n+1}\rho$, $|x - x_0| \leq \sqrt{n+1}\rho$ for $y = (t, x) \in \Omega_\rho$. Here we first suppose $u(y) \in C^\infty_0(\mathbb{R}^{n+1}_+)$, and get

$$\begin{aligned} |u(y) - u(y_0)| \leq \int_0^1 |\frac{d}{d\theta}u\,(t_0 + \theta t, x_0 + \theta x)\,|d\theta \leq \sqrt{n+1}\rho \\ \times \int_0^1 \left(|\partial_t u\,(t_0 + \theta t, x_0 + \theta x)\,| + \sum_{i=1}^n |\partial_{x_i}u\,(t_0 + \theta t, x_0 + \theta x)\,|\right)d\theta. \end{aligned} \tag{2.13}$$

Integrating (2.13) on Ω_ρ, we obtain

$$\begin{aligned} &\frac{1}{\rho^{n+1}}\int_{\Omega_\rho} u(t,x)dtdx - u\,(t_0, x_0) \\ &\leq \frac{1}{\rho^{n+1}}\int_{\Omega_\rho} |u(t,x) - u\,(t_0,x_0)\,|dtdx \\ &\leq \frac{\sqrt{n+1}}{\rho^n}\int_{\Omega_\rho}\int_0^1 \left(|\partial_t u\,(t_0 + \theta t, x_0 + \theta x)\,| + \sum_{i=1}^n |\partial_{x_i}u\,(t_0 + \theta t, x_0 + \theta x)\,|\right)d\theta dtdx. \end{aligned}$$

Change the parameters $t_0 + \theta t = t'$, $x_0 + \theta x = x'$, and $dt = \frac{1}{\theta}dt'$, $dx = \frac{1}{\theta^n}dx'$ and write t, x in the place of t', x' for convenience, we get for ρ small enough,

$$\begin{aligned} |u(y) - u(y_0)| &\leq \frac{\sqrt{n+1}}{\rho^n}\int_{\Omega_{\theta\rho}} \frac{1}{\theta^{n+1}}\int_0^1 \left(|\partial_t u(t,x)| + \sum_{i=1}^n |\partial_{x_i}u(t,x)|\right)d\theta dtdx \\ &:= I_1 + I_2, \end{aligned} \tag{2.14}$$

where

$$I_1 = \frac{\sqrt{n+1}}{\rho^n}\int_0^1 \frac{1}{\theta^{n+1}}\int_{\Omega_{\theta\rho}} |\partial_t u(t,x)|dtdxd\theta,$$

and

$$I_2 = \frac{\sqrt{n+1}}{\rho^n} \sum_{i=1}^{n} \int_0^1 \frac{1}{\theta^{n+1}} \int_{\Omega_{\theta\rho}} |\partial_{x_i} u(t,x)| dt dx d\theta.$$

Consider $|\partial_t u| = |t^{\varphi} \partial_t u| t^{-\varphi}$ and set $\varphi = \frac{n}{p^*} - \gamma^* + 1$. By using Hölder inequality for $\frac{1}{p^*} + \frac{1}{q} = 1$ to estimate I_1, we obtain

$$I_1 \leq \frac{\sqrt{n+1}}{\rho^n} \int_0^1 \frac{1}{\theta^{n+1}} \left(\int_{\Omega_{\theta\rho}} t^{-\varphi q} dt dx \right)^{\frac{1}{q}} \left(\int_{\Omega_{\theta\rho}} t^{1+(\varphi-1)p^*} |(t\partial_t) u|^{p^*} \frac{dt}{t} dx \right)^{\frac{1}{p^*}} d\theta.$$

Since $\varphi = \frac{n}{p^*} - \gamma^* + 1$, we have $1 + (\varphi - 1)p^* = n + 1 - p^* \gamma^*$, and

$$\left(\int_{\Omega_{\theta\rho}} t^{-\varphi q} dt dx \right)^{\frac{1}{q}} = \left(\int_0^{\theta\rho} t^{-\varphi q} dt \int_{\omega_{\theta\rho}} 1 dx \right)^{\frac{1}{q}} = \left(\frac{1}{1-\varphi q} \right)^{\frac{1}{q}} \theta^{\frac{n+1-\varphi q}{q}} \rho^{\frac{n+1}{q} - \varphi},$$

where $\Omega_{\theta\rho} = (0, \theta\rho) \times \omega_{\theta\rho}$ with $\dim \omega_{\theta\rho} = n$. Then, we can get

$$I_1 \leq \frac{\sqrt{n+1}}{\rho^n} \int_0^1 \frac{1}{\theta^{n+1}} \left(\frac{1}{1-\varphi q} \right)^{\frac{1}{q}} \theta^{\frac{n+1-\varphi q}{q}} \rho^{\frac{n+1}{q} - \varphi} d\theta \left(\iint_{\mathbb{R}_+^{n+1}} t^{n+1-p^*\gamma^*} |(t\partial_t) u|^{p^*} \frac{dt}{t} dx \right)^{\frac{1}{p^*}}$$

$$\leq \sqrt{n+1} \left(\frac{1}{1-\varphi q} \right)^{\frac{1}{q}} \rho^{\frac{n+1}{q} - \varphi - n} \int_0^1 \theta^{-(n+1) + \frac{n+1-\varphi q}{q}} d\theta \cdot \| (t\partial_t) u \|_{L_{p^*}^{\gamma^*}}.$$

Since $\gamma \geq 1 + \frac{n}{p} + \frac{m-1}{n+1}$, then $\frac{n+1}{q} - \varphi - n = \alpha + \varepsilon_1$, with $\varepsilon_1 = \gamma - (1 + \frac{n}{p} + \frac{m-1}{n+1}) \geq 0$. Note that $\rho^{\varepsilon_1} \leq 1$ and $\int_0^1 \theta^{-(n+1)+\frac{n+1-\varphi q}{q}} d\theta$ is bounded, then we have

$$I_1 \leq c\rho^{\alpha} \| (t\partial_t) u \|_{L_{p^*}^{\gamma^*}} \tag{2.15}$$

for some constant c depending on p, n and γ. Analogously, we can obtain the following estimate

$$I_2 \leq c' \rho^{\alpha} \| \left(\partial_{x_i} \right) u \|_{L_{p^*}^{\gamma^*}} \tag{2.16}$$

with constant c' depending on p, n and γ.

Thus for any fixed $y = (t, x)$, $\tilde{y} = (\tilde{t}, \tilde{x}) \in \mathbb{R}_+^{n+1}$, and $|y - \tilde{y}| \approx \rho$, we have proved that

$$|u(y) - u(\tilde{y})| \leq c \|u\|_{\mathcal{H}_p^{m,\gamma}} |y - \tilde{y}|^{\alpha},$$

where the constant c depends on p, n and γ.

Finally, since $C_0^{\infty}(\mathbb{R}_+^{n+1})$ is dense in $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{R}_+^{n+1})$, the estimate above holds in case of $u \in \mathcal{H}_{p,0}^{m,\gamma}(\mathbb{R}_+^{n+1})$. □

Theorem 2.3 can be extended to the following result:

Theorem 2.4 *Assume* $m - \frac{n+1}{p} = k + \alpha, k \in \mathbb{N}_+, 0 < \alpha < 1, \gamma \geq 1 + \frac{n}{p} + \frac{m-k-1}{n+1}$ *and multi-index* $\beta = (\beta_1, \beta'), |\beta| = k$. *If we denote that* $D^\beta = (t\partial_t)^{\beta_1}\partial_x^{\beta'}$, *then we have*

$$|D^\beta u(y_1) - D^\beta u(y_2)| \leq c\|u\|_{\mathcal{H}_p^{m,\gamma}}|y_1 - y_2|^\alpha,$$

where the constant c depends on p, n *and* $\gamma, u(y) = u(t,x) \in \mathcal{H}_{p,0}^{m,\gamma}(\mathbb{R}_+^{n+1}), y_1, y_2 \in \mathbb{R}_+^{n+1}$ *and* $|y_1 - y_2| \approx \rho$ *for some* $\rho > 0$ *small enough.*

We can also prove the corresponding Poincaré inequality, which will be useful in the next section.

Theorem 2.5 (Poincaré inequality) *Let* $\mathbb{B} = (0,1) \times X$ *be a bounded subspace in* $\mathbb{R}_+^{n+1}$ *with* $X \subset \mathbb{R}^n$, *and* $1 < p < +\infty, \gamma \in \mathbb{R}$. *If* $u(t,x) \in \mathcal{H}_{p,0}^{1,\gamma}(\mathbb{B})$, *then*

$$\|u(t,x)\|_{L_p^\gamma(\mathbb{B})} \leq c\|\nabla_{\mathbb{B}}u(t,x)\|_{L_p^\gamma(\mathbb{B})}, \tag{2.17}$$

where $\nabla_{\mathbb{B}} = (t\partial_t, x_1, \ldots, x_n)$ *is the gradient operator on* $\mathbb{B}$, *and the constant c depends only on* $\mathbb{B}$ *and* p.

Proof Set $Q = \{(t,x) \in \mathbb{R}_+^{n+1} | 0 < t < d, a_i < x_i < a_i + d, i = 1, \ldots, n\}$, where $d \in \mathbb{R}$ should be chosen large enough such that $\mathbb{B} \subset Q$. Now suppose $u(t,x) \in C_0^\infty(\mathbb{B})$. For $(t,x) \in \mathbb{B} \subset Q$, we have

$$u(t,x) = \int_{a_1}^{x_1} \partial_{x_1}u\,(t,s,x_2,\ldots,x_n)\,ds,$$

and then

$$|u(t,x)|^p \leq \left(\int_{a_1}^{x_1} |\partial_{x_1}u\,(t,s,x_2,\ldots,x_n)|ds\right)^p.$$

By Hölder inequality, we can choose $q > 1$ such that $\frac{1}{q} + \frac{1}{p} = 1$, and then we can get

$$\begin{aligned} |u(t,x)|^p &\leq \left(\int_{a_1}^{x_1} 1^q ds\right)^{\frac{p}{q}} \int_{a_1}^{x_1} |\partial_{x_1}u\,(t,s,x_2,\ldots,x_n)|^p ds \\ &\leq d^{p-1} \int_{a_1}^{a_1+d} |\partial_{x_1}u\,(t,s,x_2,\ldots,x_n)|^p ds. \end{aligned}$$

By the mean value theorem, we have

$$\int_{a_1}^{a_1+d} |\partial_{x_1}u\,(t,s,x_2,\ldots,x_n)|^p ds = d \cdot |\partial_{x_1}u\left(t,x_1',x_2,\ldots,x_n\right)|^p$$

with $a_1 < x_1' < a_1 + d$, and then

$$t^{n+1-\gamma p}|u(t,x)|^p \leq d^p \cdot t^{n+1-\gamma p}|\partial_{x_1}u\left(t,x_1',x_2,\ldots,x_n\right)|^p. \tag{2.18}$$

Integrating both sides of (2.18) with respect to $\frac{dt}{t}dx$ on Q, we have

$$\int_Q t^{n+1-\gamma p}|u(t,x)|^p \frac{dt}{t}dx \leq d^p \cdot \int_Q t^{n+1-\gamma p}|\partial_{x_1} u\left(t, x_1', x_2, \ldots, x_n\right)|^p \frac{dt}{t}dx.$$

By the definition of Q and $u(t,x) \in C_0^\infty(\mathbb{B})$, we can also obtain

$$\int_{\mathbb{B}} t^{n+1-\gamma p}|u(t,x)|^p \frac{dt}{t}dx \leq d^p \cdot \int_{\mathbb{B}} t^{n+1-\gamma p}|\partial_{x_1} u\left(t, x_1', x_2, \ldots, x_n\right)|^p \frac{dt}{t}dx$$
$$\leq d^p \cdot \|\partial_{x_1} u(t,x)\|_{L_p^\gamma(\mathbb{B})}.$$

Since $C_0^\infty(\mathbb{B})$ is dense in $\mathcal{H}_{p,0}^{1,\gamma}(\mathbb{B})$, the estimate above implies that,

$$\|u(t,x)\|_{L_p^\gamma(\mathbb{B})} \leq c\|\partial_{x_1} u(t,x)\|_{L_p^\gamma(\mathbb{B})} \leq c\|\nabla_{\mathbb{B}} u(t,x)\|_{L_p^\gamma(\mathbb{B})},$$

for $u(t,x) \in \mathcal{H}_{p,0}^{1,\gamma}(\mathbb{B})$, where the constant c depends only on $\mathbb{B}$ and p. Theorem 2.5 is proved. □

3 Nonlinear Dirichlet boundary value problems on manifolds with conical singularities

Let B be a n-dimensional compact manifold with conical singularity at the point $b \in \partial B$, and $\mathbb{B}$ be the stretched manifold of B, i.e. without loss of generality, we suppose $\mathbb{B} = [0,1) \times X$, X is a closed compact manifold of dimension $n-1$, $\partial\mathbb{B} = \{0\} \times X$. Let $\Delta_{\mathbb{B}} = (x_1\partial_{x_1})^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2$, be Fuchs Laplacian operator (cf. [8]) and defined in int$\mathbb{B}$. Then we consider the following Dirichlet problem:

$$\begin{cases} -\Delta_{\mathbb{B}} u \;=\; u|u|^{p-1}, & \text{for } \; 1 < p < \frac{n+2}{n-2}, \quad x \in \text{int}(\mathbb{B}) \\ u = 0 & \text{on } \; \partial\mathbb{B}. \end{cases} \tag{3.1}$$

We say that $u(x) \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ is a weak solution of (3.1) if for any $\psi \in C_0^\infty(\mathbb{B})$, we have

$$\int_{\mathbb{B}} \nabla_{\mathbb{B}} u \cdot \overline{\nabla_{\mathbb{B}}\psi} \frac{dx_1}{x_1}dx' - \int_{\mathbb{B}} u|u|^{p-1}\overline{\psi}\frac{dx_1}{x_1}dx' = 0, \tag{3.2}$$

where $\nabla_{\mathbb{B}} = (x_1\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$, and $\Delta_{\mathbb{B}} = |\nabla_{\mathbb{B}}|^2$.

We can prove the following result:

Theorem 3.1 *The Dirichlet boundary value problem* (3.1) *has a non-trivial weak solution in the weighted Mellin Sobolev space* $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$.

Remark 3.1 As an important application, the Sobolev inequality is used to prove the existence theorem for nonlinear elliptic partial differential equations (cf. [7,9]), in particular in case of nonlinear term with critical Sobolev exponent. In this aspect, the typical example can be found in [2]. In a forthcoming paper [3], we will apply our cone Sobolev inequality and Poincaré inequality to prove the existence of solutions for Dirichlet problem of nonlinear elliptic equations, defined on the conical manifold, with critical Sobolev exponent $p = \frac{n+2}{n-2}$.

The following lemma will be used for solving our problem.

Lemma 3.1 *If $p > 1$ and $p+1 < 2^* = \frac{2n}{n-2}$, then the embedding $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) \hookrightarrow \mathcal{H}_{p+1,0}^{0,\frac{n}{p+1}}(\mathbb{B})$ is compact.*

Proof According to Definition 1.6, we can write $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ and $\mathcal{H}_{p+1,0}^{0,\frac{n}{p+1}}(\mathbb{B})$ as follows,

$$\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}) := [\omega]\mathcal{H}_{2,0}^{1,\frac{n}{2}}\left(X^\wedge\right) + [1-\omega]H_0^1(\text{int}\mathbb{B}),$$
$$\mathcal{H}_{p+1,0}^{0,\frac{n}{p+1}}(\mathbb{B}) := [\omega]\mathcal{H}_{p+1,0}^{0,\frac{n}{p+1}}\left(X^\wedge\right) + [1-\omega]L^{p+1}(\text{int}\mathbb{B}).$$

By using the Sobolev inequality, we know that the embedding $[1-\omega]H_0^1(\text{int}\mathbb{B}) \hookrightarrow [1-\omega]L^{p+1}(\text{int}\mathbb{B})$ is compact for $p > 1$, and $p+1 < 2^*$. It is sufficient to show that the embedding $[\omega]\mathcal{H}_{2,0}^{1,\frac{n}{2}}(X^\wedge) \hookrightarrow [\omega]\mathcal{H}_{p+1,0}^{0,\frac{n}{p+1}}(X^\wedge)$ is compact. Similarly to (1.8) and (1.9), for $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$, and $1 < q < \infty$, we define

$$\left(S_{\frac{n}{q},\gamma} v\right)\left(r, x'\right) = e^{-r\left(\frac{n}{q}-\gamma\right)} v(e^{-r}, x'),$$

for $v(x_1, x') \in \mathcal{H}_q^{m,\gamma}(X^\wedge)$. Then $S_{\frac{n}{q},\gamma}$ induces an isomorphism as follows,

$$S_{\frac{n}{q},\gamma} : [\omega]\mathcal{H}_q^{m,\gamma}\left(X^\wedge\right) \to [\tilde{\omega}]W^{m,q}(\mathbb{R}\times X),$$

with a cut-off function $\tilde{\omega}(r) = \omega(e^{-r}) \in C_0^\infty(\mathbb{R}_+)$. In the present case, we take $\gamma = \frac{n}{p+1}$, and $q = p+1$ for $u(x_1, x') \in \mathcal{H}_{p+1}^{0,\frac{n}{p+1}}(X^\wedge)$. Thus we have the following isomorphism mapping

$$S_{\frac{n}{p+1},\frac{n}{p+1}} : [\omega]\mathcal{H}_{p+1}^{0,\frac{n}{p+1}}(X^\wedge) \to [\tilde{\omega}]L^{p+1}(\mathbb{R}\times X), \tag{3.3}$$

i.e.

$$S_{\frac{n}{p+1},\frac{n}{p+1}}\left(\omega\left(x_1\right)u\left(x_1,x'\right)\right) = \omega\left(e^{-r}\right)e^{-r\left(\frac{n}{p+1}-\frac{n}{p+1}\right)}u\left(e^{-r},x'\right) = \omega\left(e^{-r}\right)u\left(e^{-r},x'\right).$$

Analogously if we take $\gamma = \frac{n}{2}$, $q = 2$ and $v(x_1, x') \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(X^\wedge)$, then we have an isomorphism

$$S_{\frac{n}{2},\frac{n}{2}} : [\omega]\mathcal{H}_{2,0}^{1,\frac{n}{2}}\left(X^\wedge\right) \to [\tilde{\omega}]H_0^1(\mathbb{R}\times X),$$

i.e.

$$S_{\frac{n}{2},\frac{n}{2}}\left(\omega(x_1)v\left(x_1,x'\right)\right) = \omega\left(e^{-r}\right)e^{-r\left(\frac{n}{2}-\frac{n}{2}\right)}v\left(e^{-r},x'\right) = \omega\left(e^{-r}\right)v\left(e^{-r},x'\right).$$

Moreover, $S_{\frac{n}{p+1},\frac{n}{p+1}}$ induces another isomorphism. In fact, for every $v(x_1, x') \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(X^\wedge)$, one has

$$\begin{aligned}
S_{\frac{n}{p+1},\frac{n}{p+1}}\left(\omega\left(x_1\right)v\left(x_1,x'\right)\right) &= \omega\left(e^{-r}\right)e^{-r\left(\frac{n}{p+1}-\frac{n}{p+1}\right)}v\left(e^{-r},x'\right)\\
&= \omega\left(e^{-r}\right)e^{-r\left(\frac{n}{2}-\frac{n}{2}\right)}e^{r\left(\frac{n}{2}-\frac{n}{2}\right)}e^{-r\left(\frac{n}{p+1}-\frac{n}{p+1}\right)}v\left(e^{-r},x'\right)\\
&= \omega\left(e^{-r}\right)e^{-r\left(\frac{n}{2}-\frac{n}{2}\right)}v\left(e^{-r},x'\right)\\
&= S_{\frac{n}{2},\frac{n}{2}}\left(\omega\left(x_1\right)v\left(x_1,x'\right)\right),
\end{aligned}$$

and then the mapping

$$S_{\frac{n}{p+1},\frac{n}{p+1}}:[\omega]\mathcal{H}^{1,\frac{n}{2}}_{2,0}\left(X^\wedge\right)\to[\tilde{\omega}]H^1_0\left(\mathbb{R}\times X\right) \tag{3.4}$$

is also an isomorphism. Since the embedding $[\tilde{\omega}]H^1_0(\mathbb{R}\times X)\hookrightarrow[\tilde{\omega}]L^{p+1}(\mathbb{R}\times X)$ is compact, then the embedding $[\omega]\mathcal{H}^{1,\frac{n}{2}}_{2,0}(X^\wedge)\hookrightarrow[\omega]\mathcal{H}^{0,\frac{n}{p+1}}_{p+1}(X^\wedge)$ is compact by the isomorphisms (3.3) and (3.4). □

Corresponding to the problem (3.1), we introduce the following functional form, which is defined on $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$,

$$J(v)=\frac{1}{2}\int\limits_{\mathbb{B}}|\nabla_{\mathbb{B}}v|^2\frac{dx_1}{x_1}dx'-\frac{1}{p+1}\int\limits_{\mathbb{B}}|v|^{p+1}\frac{dx_1}{x_1}dx'. \tag{3.5}$$

Then $J(v)\in C^1\left(\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B}),\mathbb{R}\right)$, and the critical point of $J(v)$ in $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$ is just the weak solution of (3.1), i.e., if $u\in\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$ is a weak solution of (3.1), then we have

$$\langle J'(u),v\rangle=\int\limits_{\mathbb{B}}\left(\nabla_{\mathbb{B}}u\cdot\overline{\nabla_{\mathbb{B}}v}-u|u|^{p-1}\overline{v}\right)\frac{dx_1}{x_1}dx'=0 \tag{3.6}$$

for any $v\in\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$. Here $J'(\cdot)$ denotes the Fréchet differentiation. According to (3.5), for $u\in\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$, we can rewrite J as

$$J(u)=\frac{1}{2}\parallel\nabla_{\mathbb{B}}u\parallel^2_{L_2^{\frac{n}{2}}(\mathbb{B})}-\frac{1}{p+1}\parallel u\parallel^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}. \tag{3.7}$$

By Lemma 3.1 and Poincaré inequality (2.17), there exist two constants $c,\tilde{c}>0$, such that the following estimates

$$\|u\|_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}=\|u\|_{\mathcal{H}^{0,\frac{n}{p+1}}_{p+1,0}(\mathbb{B})}\le c\|u\|_{\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})}\le\tilde{c}\parallel\nabla_{\mathbb{B}}u\parallel_{L_2^{\frac{n}{2}}(\mathbb{B})}$$

hold. Moreover, by (3.7) we have

$$J(u)\ge\left(\frac{1}{2}-\frac{\tilde{c}^{p+1}}{p+1}\parallel\nabla_{\mathbb{B}}u\parallel^{p-1}_{L_2^{\frac{n}{2}}(\mathbb{B})}\right)\parallel\nabla_{\mathbb{B}}u\parallel^2_{L_2^{\frac{n}{2}}(\mathbb{B})}. \tag{3.8}$$

That means $J(u)$ has lower bound in $\mathcal{H}^{1,\frac{n}{2}}_{2,0}(\mathbb{B})$.

For proving Theorem 3.1, we will use the well-known Mountain Pass Lemma.

Proposition 3.1 (cf. [1,10]) *Let E be a Banach space and $I\in C^1(E,\mathbb{R})$. Suppose $I(0)=0$ and it satisfies*

(1) *there exists $R>0$, $a>0$ such that if $\|u\|_E=R$, then $I(u)\ge a$;*
(2) *there exists $e\in E$ such that $\|e\|>R$ and $I(e)<a$.*

If I satisfies $(PS)_c$ condition with

$$c=\inf_{h\in\Gamma}\max_{t\in[0,1]}I(h(t)),$$

where

$$\Gamma=\{h\in C([0,1];E);\quad h(0)=0\quad\text{and}\quad h(1)=e\},$$

then c is a critical value of I and $c\ge a$.

Here $(PS)_c$ condition (also called Palais–Smale condition) means the following:

Definition 3.1 We say that I satisfies the $(\mathrm{PS})_c$ condition, if for any sequence $\{u_k\} \subset E$ with the properties:

$$I(u_k) \to c \quad \text{and} \quad \| I'(u_k) \|_{E'} \to 0,$$

there exists a subsequence which is convergent, where $I'(\cdot)$ is the Fréchet differentiation of I and E' is the dual space of E. If it is hold for any $c \in \mathbb{R}$, we say that I satisfies (PS) condition.

Lemma 3.2 *The functional form $J(u)$, defined by (3.7), satisfies (PS) condition on $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$.*

Proof Let $\{u_k\} \subset \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ satisfying

$$J(u_k) \to c \quad \text{and} \quad \| J'(u_k) \|_{\mathcal{H}_{2,0}^{-1,-\frac{n}{2}}(\mathbb{B})} \to 0,$$

where $J'(\cdot)$ is the Fréchet differentiation, and $\mathcal{H}_{2,0}^{-1,-\frac{n}{2}}(\mathbb{B})$ the dual space of $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, and

$$\| J'(u) \|_{\mathcal{H}_{2,0}^{-1,-\frac{n}{2}}(\mathbb{B})} = \sup_{\varphi \in C_0^\infty(\mathbb{B})} \frac{|\langle J'(u), \varphi \rangle|}{\| \varphi \|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}}.$$

First we can deduce that $\{u_k\}$ is bounded in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$. In fact, if we assume that $\{u_k\}$ is non-bounded in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, and then by Poincaré inequality (2.17), the norm of $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ is equivalent to the norm $\| \nabla_{\mathbb{B}} u_k \|_{L_2^{\frac{n}{2}}(\mathbb{B})}$, thus we have

$$\| \nabla_{\mathbb{B}} u_k \|_{L_2^{\frac{n}{2}}(\mathbb{B})} \to +\infty \quad \text{as} \quad k \to +\infty.$$

Let $v_k = \frac{u_k}{\|\nabla_{\mathbb{B}} u_k\|_{L_2^{\frac{n}{2}}(\mathbb{B})}}$, then $\| \nabla_{\mathbb{B}} v_k \|_{L_2^{\frac{n}{2}}(\mathbb{B})} = 1$, which means $\{v_k\}$ being bounded in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, i.e., there exists $v \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, and a subsequence $\{v_{k_j}\}$, such that

$$v_{k_j} \rightharpoonup v \quad \text{in} \quad \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B}).$$

Here, for simplicity, we still denote v_{k_j} by v_k, then from $\| J'(u_k) \|_{\mathcal{H}_{2,0}^{-1,-\frac{n}{2}}(\mathbb{B})} \to 0$ we know that for any $\varphi \in C_0^\infty(\mathbb{B})$,

$$\int_{\mathbb{B}} \left(\nabla_{\mathbb{B}} u_k \nabla_{\mathbb{B}} \bar{\varphi} - u_k |u_k|^{p-1} \bar{\varphi} \right) \frac{dx_1}{x_1} dx' = o(1) \| \varphi \|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}, \tag{3.9}$$

and then

$$\int_{\mathbb{B}} \left(\nabla_{\mathbb{B}} v_k \nabla_{\mathbb{B}} \bar{\varphi} - v_k |u_k|^{p-1} \bar{\varphi} \right) \frac{dx_1}{x_1} dx' = \frac{o(1) \| \varphi \|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})}}{\| \nabla_{\mathbb{B}} u_k \|_{L_2^{\frac{n}{2}}(\mathbb{B})}}, \tag{3.10}$$

i.e.,

$$\int_{\mathbb{B}} \left(\nabla_{\mathbb{B}} v_k \nabla_{\mathbb{B}} \bar{\varphi} - v_k |u_k|^{p-1} \bar{\varphi} \right) \frac{dx_1}{x_1} dx' = o(1). \tag{3.11}$$

From $J(u_k) \to c$, we know

$$\frac{1}{2}\int\limits_{\mathbb{B}}\left(|\nabla_{\mathbb{B}}u_k|^2-\frac{1}{p+1}|u_k|^{p+1}\right)\frac{dx_1}{x_1}dx'=o(1),$$

that means

$$\frac{1}{2}\int\limits_{\mathbb{B}}|\nabla_{\mathbb{B}}v_k|^2\frac{dx_1}{x_1}dx'=\frac{1}{p+1}\int\limits_{\mathbb{B}}|u_k|^{p-1}|v_k|^2\frac{dx_1}{x_1}dx'+o(1). \tag{3.12}$$

Take $\varphi = v_k$ in (3.11), then we have

$$\int\limits_{\mathbb{B}}|\nabla_{\mathbb{B}}v_k|^2\frac{dx_1}{x_1}dx'=\int\limits_{\mathbb{B}}|u_k|^{p-1}|v_k|^2\frac{dx_1}{x_1}dx'+o(1).$$

Comparing with (3.12), we can deduce that $p+1=2$, which contradicts with the assumption $p>1$. That means $\{u_k\}$ would be bounded in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$. Thus there exists $u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ and a subsequence, which still denotes by u_k, such that $u_k \rightharpoonup u$ in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$.

Now we want to prove that $u_k \to u$ in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$. In fact, by Poincaré inequality (2.17), we have

$$\| u_k-u \|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} \approx \| \nabla_{\mathbb{B}}u_k-\nabla_{\mathbb{B}}u \|_{L_2^{\frac{n}{2}}(\mathbb{B})}.$$

Then, by (3.6), we can obtain

$$\begin{aligned}\| u_k-u \|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} &\approx \| \nabla_{\mathbb{B}}u_k-\nabla_{\mathbb{B}}u \|_{L_2^{\frac{n}{2}}(\mathbb{B})} = \langle J'(u_k)-J'(u),u_k-u\rangle\\ &\quad+\int\limits_{\mathbb{B}}\left(|u_k|^{p-1}u_k-|u|^{p-1}u\right)\overline{(u_k-u)}\frac{dx_1}{x_1}dx'.\end{aligned}$$

Since $u_k \rightharpoonup u$ in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, then from Lemma 3.1, we have $u_k \to u$ in $L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})$, that is, $|u_k|^{p-1}u_k \to |u|^{p-1}u$ in $L_{p+1}^{\frac{np}{p+1}}(\mathbb{B})$. So from (3.9), one has $\langle J'(u),\varphi\rangle = 0$ for any $\varphi \in C_0^{\infty}(\mathbb{B})$. Since $C_0^{\infty}(\mathbb{B})$ is dense in $\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ and $J'(u_k) \to 0$ in $\mathcal{H}_{2,0}^{-1,-\frac{n}{2}}(\mathbb{B})$, we can deduce that

$$\langle J'(u_k)-J'(u),u_k-u\rangle \to 0 \quad \text{as } k \to +\infty.$$

Moreover, by Hölder inequality we obtain

$$\begin{aligned}&\int\limits_{\mathbb{B}}\left(|u_k|^{p-1}u_k-|u|^{p-1}u\right)\overline{(u_k-u)}\frac{dx_1}{x_1}dx'\\ &\quad\le \int\limits_{\mathbb{B}} |\, |u_k|^{p-1}u_k-|u|^{p-1}u \,| \cdot |\overline{u_k-u}|\frac{dx_1}{x_1}dx'\\ &\quad\le \| |u_k|^{p-1}u_k-|u|^{p-1}u \|_{L_{\frac{p+1}{p}}^{\frac{np}{p+1}}(\mathbb{B})} \| u_k-u \|_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})}.\end{aligned}$$

Thus $u_k \to u$ in $L_{p+1}^{\frac{n}{p+1}}(\mathbb{B})$ implies that

$$\| u_k - u \|_{\mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})} \approx \| \nabla_{\mathbb{B}} u_k - \nabla_{\mathbb{B}} u \|_{L_2^{\frac{n}{2}}(\mathbb{B})} \to 0, \quad \text{as } k \to +\infty,$$

as required. □

Proof of Theorem 3.1 We choose $E = \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, $I(u) = J(u)$ as defined by (3.7). Then it is obvious that $I(0) = J(0) = 0$. From (3.8), we have

$$J(u) \geq \left(\frac{1}{2} - \frac{\tilde{c}^{p+1}}{p+1} \| \nabla_{\mathbb{B}} u \|^{p-1}_{L_2^{\frac{n}{2}}(\mathbb{B})} \right) \| \nabla_{\mathbb{B}} u \|^2_{L_2^{\frac{n}{2}}(\mathbb{B})}.$$

Choose R satisfying $0 < R < (\frac{p+1}{2\tilde{c}^{p+1}})^{\frac{1}{p-1}}$, and let $\| \nabla_{\mathbb{B}} u \|_{L_2^{\frac{n}{2}}(\mathbb{B})} = R > 0$. Then there exists $a(R)$, satisfying $0 < a(R) \leq (\frac{1}{2} - \frac{\tilde{c}^{p+1}}{p+1} R^{p-1}) R^2$, such that

$$J(u) \geq \inf_{\| \nabla_{\mathbb{B}} u \|_{L_2^{\frac{n}{2}}(\mathbb{B})} = R} J(u) = a(R).$$

That means the condition (1) of Proposition 3.1 holds.

For $u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$ with $\| \nabla_{\mathbb{B}} u \|_{L_2^{\frac{n}{2}}(\mathbb{B})} = R > 0$, we take $\theta > 0$, and then obtain

$$J(\theta u) = \frac{\theta^2}{2} \int_{\mathbb{B}} |\nabla_{\mathbb{B}} u|^2 \frac{dx_1}{x_1} dx' - \frac{\theta^{p+1}}{p+1} \int_{\mathbb{B}} |u|^{p+1} \frac{dx_1}{x_1} dx'.$$

Since $p + 1 > 2$, we can get $\lim_{\theta \to +\infty} J(\theta u) = -\infty$. Therefore, we can find a positive constant θ_1 large enough, such that for $e = \theta_1 u \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, we have $\| \nabla_{\mathbb{B}} e \|_{L_2^{\frac{n}{2}}(\mathbb{B})} > R$ and $J(e) < 0 < a(R)$. Set $\Gamma = \{h \in C([0,1]; \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})) \mid h(0) = 0 \text{ and } h(1) = e\}$, and from continuity, we then get

$$c = \inf_{h \in \Gamma} \max_{t \in [0,1]} J(h(t)) \geq a(R) > 0. \tag{3.13}$$

Moreover, Lemma 3.2 tells us that the functional J satisfies $(PS)_c$ condition, and then the condition (2) of Proposition 3.1 holds. Consequently, from Proposition 3.1, $c \geq a(R) > 0$ in (3.13) is a critical value of $J(u)$ with non-trivial critical point $\tilde{u} \in \mathcal{H}_{2,0}^{1,\frac{n}{2}}(\mathbb{B})$, which is the non-trivial weak solution of the Dirichlet problem (3.1). The proof of Theorem 3.1 is now completed. □

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