

Parabolic stable surfaces with constant mean curvature

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Abstract We prove that if u is a bounded smooth function in the kernel of a nonnegative Schrödinger operator $-L = -(\Delta + q)$ on a parabolic Riemannian manifold M , then u is either identically zero or it has no zeros on M , and the linear space of such functions is 1-dimensional. We obtain consequences for orientable, complete stable surfaces with constant mean curvature $H \in \mathbb{R}$ in homogeneous spaces $\mathbb{E}(\kappa, \tau)$ with four dimensional isometry group. For instance, if M is an orientable, parabolic, complete immersed surface with constant mean curvature H in $\mathbb{H}^2 \times \mathbb{R}$, then $|H| \leq \frac{1}{2}$ and if equality holds, then M is either an entire graph or a vertical horocylinder.

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1 Introduction

Constant mean curvature $H \in \mathbb{R}$ surfaces (briefly, H -surfaces) in a Riemannian three-manifold N constitute a natural object of study since they are critical points of the functional

$$F = \text{Area} - 2H \cdot \text{Volume}$$

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for compactly supported normal variations (along the paper, we will restrict ourselves to 2-sided surfaces in order to insure that the variational vector field of such a variation is essentially given by a function instead of a section of the normal bundle of the surface).

Among complete H -surfaces in N , the subclass of *stable* H -surfaces is perhaps the first one to be understood and completely described; one of the main reasons for this is that, under some natural conditions, limits of complete H -surfaces produce complete stable H -surfaces (see for instance the recent Stable Limit Leaf Theorem by Meeks et al. [21]). Here, stability for an H -surface M means that the second derivative of F is nonnegative for all compactly supported normal variations of M in N (we remark that this is a stronger notion than the usual stability associated to isoperimetry, where only normal variations which preserve volume up to first order are considered). It turns out that such a second derivative is explicitly given by an expression of the type

$$Q(f, f) = - \int_M f Lf,$$

where f is the normal part of the variational vector field of the variation and L is a Schrödinger type operator on M , namely $L = \Delta + q$ where Δ is the Laplacian with respect to the induced metric on M and q is certain smooth function on M . This extrinsic formulation justifies the interest of the study of purely intrinsic operators $L = \Delta + q$ on an abstract Riemannian surface (M, ds^2) where $q \in C^\infty(M)$, under the condition that $-\int_M f Lf \geq 0$ for all $f \in C_0^\infty(M)$ (we will brief this condition by writing $-L \geq 0$).

We will start by imposing a global condition on the metric ds^2 of conformal nature, namely to be *parabolic*,¹ to conclude that some properties of operators $L = \Delta + q$ with $-L \geq 0$ on a parabolic Riemannian manifold mimic the ones of the first eigenfunctions of a classical Schrödinger operator defined on a compact subdomain. For instance, if there exists a non identically zero *bounded* solution of $Lu = 0$ on M , then u vanishes nowhere and the linear space of such functions is 1-dimensional (see Theorem 1 and Corollary 1 for slightly more general statements of this property).

Next we will apply the above intrinsic result to obtain conditions under which a complete stable H -surface in a simply-connected homogeneous 3-manifold $\mathbb{E}(\kappa, \tau)$ with four dimensional isometry group is a (vertical) multigraph or even an entire graph. The spaces $\mathbb{E}(\kappa, \tau)$ appear naturally in the classification of the Thurston geometries, together with the space forms $\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3$ (with 6-dimensional isometry group) and the solvable Lie group Sol_3 (3-dimensional isometry group). $\mathbb{E}(\kappa, \tau)$ gives a common framework to the product spaces $\mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$ and the nontrivial Riemannian fibrations given by the Heisenberg group Nil_3 (over \mathbb{R}^2), the Berger spheres (over \mathbb{S}^2) and the universal cover $\widetilde{\text{PSL}}_2(\mathbb{R})$ of the unit tangent bundle of the hyperbolic plane (over \mathbb{H}^2). The geometry of H -surfaces in these spaces $\mathbb{E}(\kappa, \tau)$ is a field of intensive research nowadays, especially since Abresch and Rosenberg [1] described a Hopf-type holomorphic quadratic differential on any such surface. As an application of the results obtained in the abstract setting, we prove the following results:

- (1) There are no orientable, complete, immersed, parabolic, stable H -surfaces M in $\mathbb{H}^2 \times \mathbb{R}$ for any value of $H > \frac{1}{2}$, and if $H = \frac{1}{2}$ then either M is a vertical horocylinder or an entire vertical graph (Theorem 2 and Corollary 2). If we drop the hypothesis of parabolicity, we obtain nonexistence of orientable, complete, immersed stable H -surfaces M

¹ A Riemannian manifold M is *parabolic* if every positive superharmonic function on M must be constant. We remark that all manifolds are assumed to have empty boundary.

in $\mathbb{H}^2 \times \mathbb{R}$ for values $H > \frac{1}{\sqrt{3}} - \varepsilon$ for some $\varepsilon > 0$ (the case $H > \frac{1}{\sqrt{3}}$ had been proven by Nelli and Rosenberg [23], see also Remark 9.11 in [20] where this nonexistence property is stated for $H > \frac{1}{\sqrt{3}} - \varepsilon$; we also remark that Salavessa obtained a related inequality for $|H|$ valid for certain global graphs with parallel mean curvature in product manifolds and in calibrated manifolds [19,27]).

- (2) There are no orientable, complete, immersed, parabolic, stable H -surfaces M in Nil_3 for any $H > 0$, and if $H = 0$, then either M is a vertical plane or an entire vertical graph (Theorem 2 and Corollary 2). This result is sharp, since every solution of the Bernstein problem in Nil_3 being conformally \mathbb{C} provides an example of the last ones (Fernández and Mira [12] constructed a 2-parameter family of such solutions for every nonzero quadratic differential on \mathbb{C}). By the Daniel sister correspondence [7], the result in the first sentence of item 1 above is also sharp. The second and third sentences of item 1 can be also adapted to Nil_3 , see Corollary 3.
- (3) There are no orientable, complete, immersed, parabolic, stable H -surfaces M in any Berger sphere (it is expected that this result holds without assuming parabolicity, see Meeks et al. [20] where it is proved assuming either compactness of M or nonnegative scalar curvature of the ambient Berger sphere).
- (4) We start the study of the Bernstein problem for *horizontal* minimal graphs in Nil_3 . A horizontal graph over a domain Ω of the (y, z) -plane is the set $\{(0, y, z) * (u(y, z), 0, 0) \mid (y, z) \in \Omega\}$ where u is a C^2 -function defined in Ω (here we are using the natural Lie group multiplication $*$ of Nil_3 , see Sect. 4). Recall that a usual *vertical* graph over a domain Ω of the (x, y) -plane in Nil_3 is the set $\{(x, y, 0) * (0, 0, u(x, y)) \mid (x, y) \in \Omega\}$, where $u(x, y)$ is a C^2 -function defined on Ω ; a direct comparison justifies the use of the word *horizontal* for the graphs in our study. In this setting, we prove that a parabolic *complete* horizontal minimal graph M in Nil_3 must be a vertical plane (see Theorem 3 for a more general version of this result for horizontal multigraphs). We remark that every *entire* horizontal graph (i.e. Ω equals the (y, z) -plane) is proper, hence complete.

2 The operator $\Delta + q$

Along this section, (M, ds^2) will denote a connected (not necessarily complete) Riemannian n -manifold without boundary. Consider the operator in M given by

$$L = \Delta + q,$$

where Δ stands for the Laplacian with respect to the metric ds^2 acting on functions and $q \in C^\infty(M)$.

We next state the main result of this section.

Theorem 1 *Let (M^n, ds^2) be a Riemannian parabolic manifold. Consider an operator $L = \Delta + q$, where $q \in C^\infty(M)$. Let $u, v \in C^\infty(M)$ such that u is bounded, $v > 0$ and $uvLu \geq u^2Lv$ on M . Then, u/v is constant.*

Before proving Theorem 1, we will make some comments.

- (1) Since $L = \Delta + q$, the hypothesis $uvLu \geq u^2Lv$ is obviously equivalent to $uv\Delta u \geq u^2\Delta v$. In other words, we can restrict to the particular case $q = 0$.
- (2) We have stated the theorem under the hypothesis (M, ds^2) is parabolic, although the equivalent condition (see e.g. Proposition 4.1 in [30]) used in its proof will be

- (★) There exists a sequence of cut-off functions $\{\varphi_j\}_j \subset C_0^\infty(M)$ such that $0 \leq \varphi_j \leq 1$ in M , the compact sets $\varphi_j^{-1}(1)$ form an increasing exhaustion of M , and the sequence of energies $\{\int_M |\nabla \varphi_j|^2\}_j$ tends to zero as $j \rightarrow \infty$.

Proof The argument is inspired in ideas of Berestycki et al. [4, Sect. 4] (also see Ambrosio and Cabré [2, Proposition 2.1]). As said before this proof, we can assume $q = 0$. First note that u/v is smooth on M and its gradient is $\nabla(u/v) = v^{-2}(v\nabla u - u\nabla v)$. Hence, the divergence of the smooth vector field $v^2\nabla(u/v)$ is $\operatorname{div}(v^2\nabla(u/v)) = v\Delta u - u\Delta v$, from where we obtain

$$u \operatorname{div}(v^2\nabla(u/v)) = uvLu - u^2Lv \geq 0. \tag{1}$$

Consider a sequence $\{\varphi_j\}_j \subset C_0^\infty(M)$ satisfying the property (★). Given $j \in \mathbb{N}$, the standard argument in the sense of distributions applied to the function $\varphi_j^2 u/v \in H_0^1(M)$ (as usual, $H_0^1(M)$ denotes the closure with respect to the standard Sobolev norm of the linear space $C_0^\infty(M)$ of compactly supported smooth functions on M) and to the smooth vector field $v^2\nabla(u/v)$ yields

$$\begin{aligned} 0 &= \int_M \langle \nabla(\varphi_j^2 u/v), v^2\nabla(u/v) \rangle + \int_M \frac{\varphi_j^2 u}{v} \operatorname{div}(v^2\nabla(u/v)) \\ &= 2 \int_M \varphi_j uv \langle \nabla \varphi_j, \nabla(u/v) \rangle + \int_M \varphi_j^2 v^2 |\nabla(u/v)|^2 + \int_M \frac{\varphi_j^2 u}{v} \operatorname{div}(v^2\nabla(u/v)) \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ stands for the Riemannian metric on M . We define $\Omega_j = \varphi_j^{-1}(1)$. The last equality together with (1) lead to

$$\begin{aligned} \int_M \varphi_j^2 v^2 |\nabla(u/v)|^2 &\leq -2 \int_M \varphi_j uv \langle \nabla \varphi_j, \nabla(u/v) \rangle \\ &= -2 \int_{\operatorname{supp}(\varphi_j) - \Omega_j} \varphi_j uv \langle \nabla \varphi_j, \nabla(u/v) \rangle \\ &\leq 2 \left| \int_{\operatorname{supp}(\varphi_j) - \Omega_j} \varphi_j uv \langle \nabla \varphi_j, \nabla(u/v) \rangle \right| \\ &\leq 2 \left(\int_{\operatorname{supp}(\varphi_j) - \Omega_j} \varphi_j^2 v^2 |\nabla(u/v)|^2 \right)^{1/2} \left(\int_{\operatorname{supp}(\varphi_j) - \Omega_j} u^2 |\nabla \varphi_j|^2 \right)^{1/2}, \tag{2} \end{aligned}$$

where we have used the Schwarz inequality. Since u is bounded and the functions φ_j satisfy (★), we have

$$\int_{\text{supp}(\varphi_j) - \Omega_j} u^2 |\nabla \varphi_j|^2 \leq C E(\varphi_j) \tag{3}$$

where $C > 0$ is independent of j and the energy $E(\varphi_j) = \int_M |\nabla \varphi_j|^2$ tends to zero as $j \rightarrow \infty$. Plugging this information in (2) and splitting its left-hand-side in two integrals over Ω_j and $\text{supp}(\varphi_j) - \Omega_j$, we have

$$\begin{aligned} & \int_{\Omega_j} v^2 |\nabla(u/v)|^2 + \int_{\text{supp}(\varphi_j) - \Omega_j} \varphi_j^2 v^2 |\nabla(u/v)|^2 \\ & \leq 2\sqrt{C E(\varphi_j)} \left(\int_{\text{supp}(\varphi_j) - \Omega_j} \varphi_j^2 v^2 |\nabla(u/v)|^2 \right)^{1/2}. \end{aligned} \tag{4}$$

Estimating by below the first summand in the last left-hand-side by zero and simplifying we have

$$\int_{\text{supp}(\varphi_j) - \Omega_j} \varphi_j^2 v^2 |\nabla(u/v)|^2 \leq 4C E(\varphi_j). \tag{5}$$

Estimating by below the second summand in (4) by zero and using (5),

$$\int_{\Omega_j} v^2 |\nabla(u/v)|^2 \leq 2\sqrt{C E(\varphi_j)} \left(\int_{\text{supp}(\varphi_j) - \Omega_j} \varphi_j^2 v^2 |\nabla(u/v)|^2 \right)^{1/2} \leq 4C E(\varphi_j),$$

for all $j \in \mathbb{N}$. Taking $j \rightarrow \infty$ we infer that $\int_M v^2 |\nabla(u/v)|^2 = 0$. Since $v > 0$ in M , we conclude that u/v is constant, as desired. \square

We continue with an operator $L = \Delta + q$ on (M^n, ds^2) , $q \in C^\infty(M)$. As usual, we associate to L the quadratic form

$$\mathcal{Q}(f, f) = - \int_M f Lf = \int_M (|\nabla f|^2 - qf^2), \quad f \in C_0^\infty(M), \tag{6}$$

which can be continuously extended to the Sobolev space $H_0^1(M)$. We say that $-L$ is *nonnegative* on M if $\mathcal{Q}(f, f) \geq 0$ for all $f \in C_0^\infty(M)$, and we write $-L \geq 0$ in this case. Functions in the nullity of \mathcal{Q} are called *Jacobi functions*. By standard elliptic theory, Jacobi functions are always smooth and lie in the kernel of L . Fischer-Colbrie [14] proved the following useful characterization of nonnegativity of $-L$ in terms of existence of positive Jacobi functions (see also Lemma 2.1 in Meeks et al. [20]); note that both [14] and [20] are stated and proved for surfaces, but the proof can be directly extended to the n -dimensional case):

Lemma 1 *In the above situation, the following statements are equivalent:*

1. $-L \geq 0$ on M .
2. There exists a positive Jacobi function on M .
3. There exists a positive function $u \in C^\infty(M)$ such that $Lu \leq 0$.

Suppose (M^n, ds^2) is complete. Given $p_0 \in M$, we denote by $d(\cdot, p_0)$ the Riemannian distance on M to p_0 and by $B(p_0, r) = \{p \in M \mid d(p, p_0) < r\}$ the metric ball in M

of radius r centered at p_0 . We say that M has *at most quadratic volume growth* when the function $r > 0 \mapsto r^{-2}\text{Vol}(B(p_0, r))$ is bounded. This condition is independent of the point $p_0 \in M$. It is well-known that if M has at most quadratic volume growth, then it is parabolic (Cheng–Yau [5], see also Corollary 7.4 of [15]). In Theorem 2.11 of [20], Meeks et al. proved that if M is a simply-connected surface with quadratic area growth (although only parabolicity is required in their argument) and $-L \geq 0$, then a bounded solution of $Lu = 0$ does not change sign on M . The following corollary is a slight generalization of that result.

Corollary 1 *Let (M^n, ds^2) be a parabolic Riemannian manifold. Suppose $L = \Delta + q$, $q \in C^\infty(M)$, satisfies $-L \geq 0$ on M . If $u \in C^\infty(M)$ is a bounded function such that $uLu \geq 0$, then $Lu = 0$ and either $u = 0$ or u never vanishes.*

Proof As $-L \geq 0$ on M , Lemma 1 insures that there exists $v \in C^\infty(M)$ such that $v > 0$ and $Lv = 0$ on M . By Theorem 1, u must be a multiple of v from where the corollary follows directly. □

Next we show a direct application of Corollary 1. Consider functions $f \in C^\infty(\mathbb{R})$ and $u \in C^\infty(M)$ such that $Lu = 0$, and define $v = f(u) \in C^\infty(M)$. Note that

$$Lv = f''(u)|\nabla u|^2 + q(f(u) - f'(u)u). \tag{7}$$

Lemma 2 *Suppose that the hypotheses of Corollary 1 hold.*

1. *If $q \geq 0$, then $q = 0$. In particular, any Jacobi function bounded above or below on M must be constant.*
2. *If $q \leq 0$, then any bounded Jacobi function u on M is constant, and either $u = 0$ or $q = 0$ in M .*

Proof We first prove 1. By Lemma 1, there exists a positive solution w of $Lw = 0$. Thus $v = e^{-w}$ is smooth, positive and bounded on M . Furthermore, (7) implies that it satisfies $Lv = v(|\nabla w|^2 + q(1 + w))$. Since $q \geq 0$ on M and w is positive, then $Lv \geq 0$ on M . Therefore, Corollary 1 implies $Lv = 0$, and so $|\nabla w| = 0$ and $q = 0$. The second sentence in item 1 follows directly from the parabolicity of M , since $L = \Delta$.

To show 2, we apply (7) to the bounded smooth function $v = u^2$ to have $Lv = 2|\nabla u|^2 - qu^2 \geq 0$. From Corollary 1 we get $Lv = 0$, hence $|\nabla u| = 0$ and $qu^2 = 0$ and we are done. □

3 Stable H -surfaces in 3-manifolds

Let M be a two-sided immersed surface of constant mean curvature $H \in \mathbb{R}$ (briefly, an H -surface) in a Riemannian 3-manifold N . From a variational viewpoint, M is a critical point of the functional $\text{Area} - 2H \cdot \text{Volume}$. More precisely, denote by $x : M \rightarrow N$ an isometric immersion with constant mean curvature H and globally defined unit normal vector field η . Given a function $f \in C_0^\infty(M)$ and a compact smooth domain $\Omega \subset M$ which contains the support of f , we consider any variation of x given by a differentiable map $X : (-\varepsilon, \varepsilon) \times M \rightarrow N$, $\varepsilon > 0$, such that $X(0, \cdot) = x$ on M , $X(t, p) = x(p)$ for all $(t, p) \in (-\varepsilon, \varepsilon) \times [M - \Omega]$, and $\frac{\partial X}{\partial t} \Big|_0 = f\eta$. We associate to X the area function $\text{Area}(t) = \text{Area}(X(t, \cdot))$ (note that for small t , the map $X(t, \cdot) : M \rightarrow N$ is an immersion) and the signed volume function $\text{Vol}(t) = \int_{[0,t] \times \Omega} \text{Jac}(X) dV$. Then, the first derivative at $t = 0$ of

$\text{Area}(t) - 2H\text{Vol}(t)$ vanishes. It is also well-known that the second variation formula of $\text{Area} - 2H \cdot \text{Volume}$ at $t = 0$ is given by (see e.g. Barbosa et al. [3])

$$Q(f, f) = \left. \frac{d^2}{dt^2} \right|_{t=0} [\text{Area}(t) - 2H\text{Vol}(t)] = - \int_M f Lf \, dA = \int_M (|\nabla f|^2 - qf^2) \, dA, \tag{8}$$

where $L = \Delta + q$ is a Schrödinger type operator acting on smooth functions called the *stability operator* of M , Δ stands for the Laplacian with respect to the induced metric on M and q is the smooth function given by

$$q = |A|^2 + \text{Ric}(\eta)$$

(here A denotes the shape operator of M). The surface M is said to be *stable* when $-L$ is a nonnegative operator. We also say that $u \in C^\infty(M)$ is a *Jacobi function* when $Lu = 0$.

Note that if X is a Killing vector field on N , then $\langle X, \eta \rangle$ is a Jacobi function on M . Since every stable H -surface admits a positive Jacobi function by Lemma 1, then Theorem 1 has the following direct consequences:

- If M is a parabolic stable H -surface and N admits a nonzero Killing vector field X which is bounded when restricted to M , then either M is invariant under the 1-parameter group of isometries generated by X , or the linear space of bounded Jacobi functions and the cone of positive Jacobi functions on M are both generated by the same function. From here we deduce that if M is a parabolic stable H -surface in a product manifold $\Sigma \times \mathbb{R}$, then either M entirely vertical or it is transverse to the unit vertical Killing vector field $\frac{\partial}{\partial t}$.
- If M is a parabolic stable H -surface and N admits two linearly independent Killing vector fields X_1, X_2 which are bounded when restricted to M , then M is invariant under the 1-parameter group of isometries generated by some linear combination of X_1, X_2 .
- If M is a parabolic stable H -surface and N admits three linearly independent Killing vector fields which are bounded when restricted to M , then M has constant Gaussian curvature.

In the sequel, we will study the special case in which N is a simply-connected homogeneous 3-manifold whose isometry group has dimension 4. These homogeneous spaces are classified in terms of two real numbers κ, τ with $\kappa \neq 4\tau^2$, and are usually denoted by $\mathbb{E}(\kappa, \tau)$. The space $\mathbb{E}(\kappa, \tau)$ admits a fibration π over the complete simply-connected surface $\mathbb{M}^2(\kappa)$ of constant curvature κ (the sphere $\mathbb{S}^2(\kappa)$ when $\kappa > 0$, the Euclidean plane \mathbb{R}^2 when $\kappa = 0$ and the hyperbolic plane $\mathbb{H}^2(\kappa)$ when $\kappa < 0$); here τ is the bundle curvature. The fibers of $\pi: \mathbb{E}(\kappa, \tau) \rightarrow \mathbb{M}^2(\kappa)$ are geodesics, and translations along these fibers generate a unit Killing vector field E_3 , called the *vertical vector field*.

If $\tau = 0$, then $\mathbb{E}(\kappa, \tau)$ is the product space $\mathbb{M}^2(\kappa) \times \mathbb{R}$. In the case $\tau \neq 0$, we get three types of manifolds depending on the sign of κ : if $\kappa > 0$ we have the *Berger spheres*, if $\kappa = 0$ we obtain the *Heisenberg space* Nil_3 , and the case $\kappa < 0$ corresponds to the universal covering of $\text{PSL}_2(\mathbb{R})$, the unit tangent bundle of \mathbb{H}^2 . The above description could be extended for $\kappa = 4\tau^2$, obtaining the Euclidean space \mathbb{R}^3 (when $\kappa = \tau = 0$) or a round 3-sphere \mathbb{S}^3 (when $\kappa = 4\tau^2 \neq 0$), although these space forms have isometry group of dimension 6 and we will not consider them.

A consequence of an intrinsic estimate of the distance to the boundary of an immersed stable H -surface due to Rosenberg [26] is the nonexistence of immersed, stable H -surfaces in $\mathbb{E}(\kappa, \tau)$ provided that $H^2 > \frac{\tau^2 - \kappa}{3}$, other than $\mathbb{S}^2(\kappa) \times \{0\}$ in $\mathbb{S}^2(\kappa) \times \mathbb{R}$. It is expected that the right condition for such nonexistence result is $H^2 > \frac{\kappa}{4}$. Our next result achieves

such bound under the additional assumption of parabolicity. Also see Corollary 3 below for a slight improvement of the inequality $H^2 > \frac{\tau^2 - \kappa}{3}$ when the hypothesis of parabolicity is removed.

Given an immersed H -surface $M \looparrowright \mathbb{E}(\kappa, \tau)$ with unit normal vector field η , the bounded Jacobi function $\eta_3 = \langle \eta, E_3 \rangle$ is called the *angle function* of M . The surface M is called a *vertical multigraph* if η_3 does not vanish (i.e. it is transverse to the fibration $\pi : \mathbb{E}(\kappa, \tau) \rightarrow \mathbb{M}^2(\kappa)$), and is said to be a *cylinder* over a curve $\gamma \subset \mathbb{M}^2(\kappa)$ if $\eta_3 = 0$ (i.e. $M = \pi^{-1}(\gamma)$).

Theorem 2 *Let M be an orientable, parabolic, complete, immersed stable H -surface in $\mathbb{E}(\kappa, \tau)$. Then, one of the following statements hold:*

1. $\mathbb{E}(\kappa, \tau) = \mathbb{S}^2(\kappa) \times \mathbb{R}$, $H = 0$ and M is a slice $\mathbb{S}^2(\kappa) \times \{t\}$ for some $t \in \mathbb{R}$.
2. $H^2 \leq \frac{-\kappa}{4}$ and M is either a vertical multigraph or a vertical cylinder over a complete curve of geodesic curvature $2H$ in $\mathbb{M}^2(\kappa)$.

Proof Since η_3 is a bounded Jacobi function and M is parabolic, then Corollary 1 assures that either η_3 is identically zero or it never vanishes. In the first case, M is a vertical cylinder over a complete curve of geodesic curvature $2H$ by Lemma 3 in the appendix. Since M is stable, we conclude from Proposition 1 in the same appendix that $\kappa \leq -4H^2$, which finishes this case.

Assume from now on that η_3 has no zeros on M . Then M is a vertical multigraph and we can suppose $\eta_3 > 0$. Assume that $H^2 > \frac{-\kappa}{4}$ and we will prove that item 1 holds.²

If there exists a constant $c > 0$ such that $\eta_3 \geq c$, then the vertical projection π restricts to M as a covering map over the simply-connected surface $\mathbb{M}^2(\kappa)$. Hence M is an entire vertical graph (i.e. M intersects exactly once every fiber of π or equivalently, M is the image of a global section of the Riemmanian fibration $\pi : \mathbb{E}(\kappa, \tau) \rightarrow \mathbb{M}^2(\kappa)$). Now assume $\mathbb{E}(\kappa, \tau)$ is not a Berger sphere and consider a sphere $S_H \subset \mathbb{E}(\kappa, \tau)$ of constant mean curvature H , which exists since $\frac{-\kappa}{4} < H^2$ (see Remark 1 below). By applying the maximum principle to M and S_H we conclude that $M = S_H$, which can be easily seen to be possible only if item 1 of the theorem holds. In the case $\mathbb{E}(\kappa, \tau)$ is a Berger sphere the above argument might fail, since we need to start with a sphere S_H disjoint from M , which does not necessarily exist. In this case, we simply observe that M is compact since it is a global graph over $\mathbb{S}^2(\kappa)$, which contradicts item 1 of Corollary 9.6 in [20].

Therefore, there exists a sequence of points $\{p_n\}_n$ in M such that $\eta_3(p_n) \rightarrow 0$ as $n \rightarrow +\infty$. Consider the isometry ϕ_n of $\mathbb{E}(\kappa, \tau)$ obtained as composition of the vertical translation T_n which maps p_n to $\pi(p_n)$ (here we identify $\pi(p_n)$ with its related point in $\mathbb{E}(\kappa, \tau)$ at height zero with respect to the fibration π) with the horizontal translation which maps $T_n(p_n)$ to a previously chosen point $\mathbf{0}$ of $\mathbb{E}(\kappa, \tau)$, to which we will call the origin (here, the word “horizontal” refers to the legendrian lift of the isometry of $\mathbb{M}^2(\kappa)$ through π , a lift which preserves the natural distribution orthogonal to the vertical fibers; for instance, in Sect. 4 below, this distribution is generated by the vector fields E_1, E_2). Then $\phi_n(p_n) = \mathbf{0}$ and the surface $M_n = \phi_n(M)$ is a complete stable H -surface which contains $\mathbf{0}$. Let $\eta_{n,3}$ be the angle function of M_n (which is nothing but $\eta_3 \circ \phi_n^{-1}$). Then $\eta_{n,3} > 0$ on M_n and $\eta_{n,3}(\mathbf{0}) \rightarrow 0$ as $n \rightarrow +\infty$.

Since $\{M_n\}_n$ is a sequence of H -surfaces with uniformly bounded second fundamental form (this follows from curvature estimates for stable constant mean curvature surfaces, see

² It should be mentioned that Espinar and Rosenberg [10] proved that a complete H -surface in $\mathbb{E}(-1, 0) = \mathbb{H}^2 \times \mathbb{R}$ with $H > 1/2$ and whose angle function does not change sign is a vertical cylinder over a complete curve in \mathbb{H}^2 of constant geodesic curvature $2H$.

Schoen [28]) and an accumulation point at the origin, standard convergence theorems (see for instance Theorem 4.2.2 in Pérez and Ros [25] for the minimal case, which can be extended to the case of fixed constant mean curvature with minor changes) give that after extracting a subsequence, there exist neighborhoods U_n of $\mathbf{0}$ in M_n which converge uniformly in the C^k -topology for every k to a (not necessarily complete) H -surface U_∞ which contains the origin $\mathbf{0}$. Clearly, U_∞ is stable and its angle function $\eta_{3,\infty} = \lim_n \eta_{3,n}|_{U_n}$ satisfies $\eta_{3,\infty} \geq 0$ in U_∞ and $\eta_{3,\infty}(\mathbf{0}) = 0$. Applying the maximum principle to $\eta_{3,\infty}$ (see e.g. Assertion 2.2 in [20]) we conclude that $\eta_{\infty,3}$ is identically zero in U_∞ . Therefore, U_∞ is contained in a vertical cylinder C_γ over a curve $\gamma \subset \mathbb{M}^2(\kappa)$ with constant geodesic curvature $2H$. Since the surfaces M_n are complete without boundary and have uniformly bounded fundamental form, an analytic prolongation argument insures that the maximal sheet which contains U_∞ in the accumulation set of the $\{M_n\}_n$ is the whole cylinder C_γ (after extracting a subsequence). In particular C_γ is stable, which is a contradiction since as we said above, there are no stable cylinders of constant mean curvature H with $H^2 > \frac{-\kappa}{4}$. Now the proof is complete. \square

Remark 1 In the above proof, we have used the existence of (immersed) H -spheres in $\mathbb{E}(\kappa, \tau)$ for all values of H with $H^2 > \frac{-\kappa}{4}$. We now give a brief explanation of this well-known result. In the product space $\mathbb{M}^2(\tilde{\kappa}) \times \mathbb{R}$, there are (embedded) rotational spheres of constant mean curvature \tilde{H} whenever $4\tilde{H}^2 + \tilde{\kappa} > 0$ (see Hsiang and Hsiang [18], Pedrosa and Ritoré [24], and Abresch and Rosenberg [1]). Let $\tilde{\kappa} = \kappa - 4\tau^2$. By the Daniel correspondence [7], there are (possibly nonembedded, see Torralbo [29]) spheres of constant mean curvature H in $\mathbb{E}(\kappa, \tau)$ whenever $H^2 + \tau^2 = \tilde{H}^2 > \frac{-\tilde{\kappa}}{4} = \frac{-\kappa}{4} + \tau^2$.

Hauswirth et al. [17, Theorem 1.2] used a half-space type theorem for properly embedded $\frac{1}{2}$ -surfaces in $\mathbb{H}^2 \times \mathbb{R}$ to conclude that a complete, immersed $\frac{1}{2}$ -surface in $\mathbb{H}^2 \times \mathbb{R}$ which is transverse to E_3 must be an entire vertical graph. A bit later and using a different approach to prove a related half-space type theorem for properly immersed minimal surfaces in the Heisenberg space Nil_3 , Hauswirth and Daniel [8, Theorem 3.1] demonstrated the corresponding result, namely that a complete immersed minimal surface in Nil_3 which is transverse to E_3 must be an entire vertical graph. Using results from [12] and the sister correspondence in [7], these results can be extended to the case of a complete vertical multigraph M in $\mathbb{E}(\kappa, \tau)$ with mean curvature H satisfying $H^2 = \frac{-\kappa}{4}$, concluding that M is an entire vertical graph (see also Daniel et al. [9] and Fernández and Mira [11]). As a direct consequence, we can improve the statement of item 2 in Theorem 2:

Corollary 2 *Under the same hypotheses of Theorem 2, if $H^2 = \frac{-\kappa}{4}$ then M is either an entire vertical graph or a vertical cylinder over a complete curve of geodesic curvature $2H$ in $\mathbb{M}^2(\kappa)$.*

Remark 2 Entire vertical minimal graphs in Nil_3 have been analytically classified by Fernández and Mira [12] in terms of their conformal type (\mathbb{C} or $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$) and a holomorphic quadratic differential (called the *Abresch-Rosenberg quadratic differential* of the graph [1]), whose only restriction is to be nonzero when the conformal structure is \mathbb{C} . In spite of this analytic description, the geometry of the entire minimal graphs in Nil_3 is not well understood yet. All simple examples of such entire minimal graphs have area growth strictly greater than quadratic. We conjecture that no entire minimal graph in Nil_3 has quadratic area growth, which would imply that the only complete, immersed, stable H -surfaces in Nil_3 with quadratic area growth are vertical planes.

Using the Daniel sister surface correspondence [7, Corollary 3.3] and the above paragraph, we conclude that there exist entire vertical graphs of constant mean curvature $\frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$

which are parabolic. It is natural to expect nonexistence of entire vertical graphs of constant mean curvature $\frac{1}{2}$ and quadratic area growth in $\mathbb{H}^2 \times \mathbb{R}$. Regarding the case $H \in [0, \frac{1}{2})$ in $\mathbb{H}^2 \times \mathbb{R}$, for each such a value of H there exist complete stable H -surfaces of revolution in $\mathbb{H}^2 \times \mathbb{R}$ which are entire graphs (see Nelli and Rosenberg [23]); even more, there exists an entire minimal graph which is conformally \mathbb{C} (Collin and Rosenberg [6]). Note that the Daniel sister correspondence relates these H -surfaces in $\mathbb{H}^2 \times \mathbb{R}$, $0 < H < \frac{1}{2}$, with \widehat{H} -surfaces in $\widehat{\text{PSL}}_2(\mathbb{R}) = \mathbb{E}(\kappa, \tau)$ where $-1 < \kappa < 0$, $\tau^2 = \frac{1}{4}(\kappa + 1)$ and $\tau^2 + \widehat{H}^2 = H^2 < \frac{1}{4}$.

We have already mentioned that if M is an orientable, complete, immersed, stable \widehat{H} -surface in $\mathbb{E}(\kappa, \tau)$, then $H^2 \leq \frac{\tau^2 - \kappa}{3}$ (except the special case of $M = \mathbb{S}^2(\kappa) \times \{t\}$ in $\mathbb{S}^2(\kappa) \times \mathbb{R}$) by a result due to Rosenberg [26]. Next we will improve slightly this inequality. We point out that Corollary 3 below was stated in Remark 9.11 of [20] in the particular case of $\mathbb{E}(\kappa, \tau) = \mathbb{H}^2 \times \mathbb{R}$; although this particular case can be seen to imply the whole statement by an application of the Daniel sister correspondence, we will supply an independent proof. We will need the following expression for the stability operator of M (see e.g. [26] or formula (28) in [20]):

$$L = \Delta - K + \tilde{q}, \quad \text{where } \tilde{q} = 3H^2 + \kappa - \tau^2 + (H^2 - \det(A)), \tag{9}$$

and K denotes the Gauss curvature of M .

Orientable, complete, immersed, stable H -surfaces M in $\mathbb{E}(\kappa, \tau)$, for $\kappa \geq \tau^2$, are classified (this inequality leads to $\mathbb{E}(\kappa, \tau)$ being $\mathbb{S}^2(\kappa) \times \mathbb{R}$ or a Berger sphere with nonnegative scalar curvature; in the first case M must be a horizontal slice, while in the second case there are no such complete stable H -surfaces for any value of H , see Rosenberg [26] and Meeks et al. [20]). Thus we can assume $\kappa < \tau^2$.

Corollary 3 *Let M be an orientable, complete, immersed, stable H -surface in $\mathbb{E}(\kappa, \tau)$, with $\kappa < \tau^2$. Then there exists $\varepsilon > 0$ such that $H^2 < \frac{\tau^2 - \kappa}{3} - \varepsilon$.*

Proof We will first prove that the case $H^2 = \frac{\tau^2 - \kappa}{3}$ cannot occur (recall that the case $H^2 > \frac{\tau^2 - \kappa}{3}$ does not occur by [26]). Since $H^2 - \det(A) = \frac{1}{4}(k_1 - k_2)^2 \geq 0$ where k_1, k_2 are the principal curvatures of M with respect to any unit normal vector field, then the function \tilde{q} in (9) satisfies $\tilde{q} \geq 0$. By [20, Theorem 2.9]), M has quadratic area growth, in particular it is parabolic. In this setting, Theorem 2 implies that $H^2 \leq \frac{-\kappa}{4}$. Plugging the value of H^2 in this inequality we obtain $4\tau^2 \leq \kappa$, which contradicts our hypothesis. Therefore, $H^2 < \frac{\tau^2 - \kappa}{3}$.

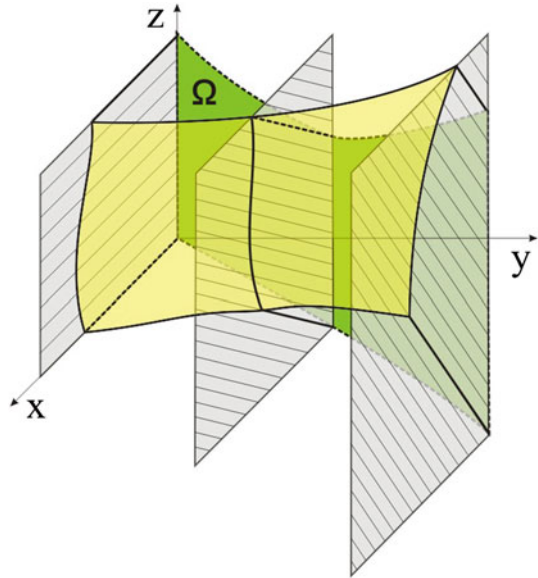
Next suppose that for any $n \in \mathbb{N}$, there exists an orientable, complete, stable, H_n -surface M_n in the same ambient space $\mathbb{E}(\kappa, \tau)$, with $H_n^2 \in [0, \frac{\tau^2 - \kappa}{3})$ converging to $H_\infty := \frac{\tau^2 - \kappa}{3}$ as $n \rightarrow \infty$. Similarly as in the proof of Theorem 2, we can assume after an ambient isometry that M_n contains the origin $\mathbf{0} \in \mathbb{E}(\kappa, \tau)$. Since the sequence $\{M_n\}_n$ has uniformly bounded second fundamental form (by stability), an accumulation point at the origin and their mean curvatures H_n satisfy $H_n \rightarrow H_\infty$, then after extracting a subsequence the M_n converge to an orientable, complete, stable H_∞ -surface $M_\infty \subset \mathbb{E}(\kappa, \tau)$ with $\mathbf{0} \in M_\infty$. This contradicts the arguments in the previous paragraph, and finishes the proof. \square

4 A Bernstein-type theorem for horizontal minimal graphs in Nil₃

Consider the model of the Heisenberg space $\text{Nil}_3 = \mathbb{E}(0, \frac{1}{2})$ given by (\mathbb{R}^3, ds^2) with the Riemannian metric

$$ds^2 = dx^2 + dy^2 + \left(dz + \frac{1}{2}(y dx - x dy) \right)^2. \tag{10}$$

Fig. 1 A horizontal graph in the direction of X , over a domain Ω of the (y, z) -plane. The *parallel lines in vertical planes* are integral curves of X



In this model, the Riemannian fibration is the vertical projection $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2, \pi(x, y, z) = (x, y)$ and the vector fields

$$E_1 = \partial_x - \frac{y}{2}\partial_z, \quad E_2 = \partial_y + \frac{x}{2}\partial_z, \quad E_3 = \partial_z \tag{11}$$

define a global orthonormal basis of left invariant vector fields. E_3 is a (unit) Killing vector field of Nil_3 , although this property no longer holds for E_1, E_2 . Instead,

$$X := E_1 + yE_3$$

is Killing (not bounded), and its associated 1-parameter group of isometries is given by

$$\phi_t(x, y, z) = \left(x + t, y, z + \frac{ty}{2} \right), \quad t \in \mathbb{R}.$$

Given a domain $\Omega \subset \mathbb{R}^2 \equiv \{(0, y, z) \mid y, z \in \mathbb{R}\}$ and a C^2 -function $u : \Omega \rightarrow \mathbb{R}$, the *horizontal graph* Σ_u defined by u (in the direction of X) is the surface of Nil_3 parameterized by

$$F(y, z) = \left(u(y, z), y, z + \frac{y}{2}u(y, z) \right), \quad (y, z) \in \Omega, \tag{12}$$

see Fig. 1. Note that since the rotations around the x_3 axis are isometries of Nil_3 , this notion of horizontal graph does not really give priority to the special horizontal direction defined by X . A straightforward computation gives that a horizontal graph Σ_u is minimal if and only if u satisfies the following PDE:

$$\begin{aligned} & [1 + 2yu_z + (1 + y^2)u_z^2] u_{yy} - 2u_y [y + (1 + y^2)u_z] u_{yz} \\ & + \left[1 + (1 + y^2)u_y^2 \right] u_{zz} - u_y u_z (1 + yu_z) = 0. \end{aligned}$$

A horizontal graph Σ_u is said to be *entire* when $\Omega = \mathbb{R}^2$, i.e. Σ_u intersects exactly once every integral curve of X . As simple examples of entire horizontal minimal graphs, we have the following ones:

- (1) For any $a, b \in \mathbb{R}$ the vertical plane $\Pi_{a,b} = \{x = ay + b\}$ is the entire horizontal minimal graph Σ_u associated to the function $u(y, z) = ay + b$. Moreover, the induced metric on $\Pi_{a,b}$ is flat.
- (2) Given $c, d \in \mathbb{R}, c \neq 0$, the function $u(y, z) = cz + d$ describes an entire horizontal minimal graph Σ_u , which fails to be parabolic by Theorem 3 below. Σ_u can also be seen as an entire vertical minimal graph parameterized by

$$\tilde{F}(x, y) = \left(x, y, \frac{xy}{2} + \frac{x-d}{c} \right), \quad (x, y) \in \mathbb{R}^2 \equiv \{(x, y, 0) \mid x, y \in \mathbb{R}\}.$$

If $M \subset \text{Nil}_3$ is a horizontal minimal graph and η denotes a unit normal vector field to M , then $\langle X, \eta \rangle$ is a nonvanishing Jacobi function on M . Hence, Lemma 1 insures that M is stable. In fact, M is area-minimizing by the standard calibration argument: Suppose that $u : \Omega \rightarrow \mathbb{R}$ is a function defining a horizontal minimal graph over a domain $\Omega \subset \{(0, y, z) \mid (y, z) \in \mathbb{R}^2\}$. Consider the differential 2-form ω defined on the ‘‘horizontal solid cylinder’’ $\cup_{t \in \mathbb{R}} \phi_t(\Omega)$ by

$$\omega_{\phi_t(p)}(v_1, v_2) = dV_{F(p)} \left((d\phi_{u(p)-t})_{\phi_t(p)}(v_1), (d\phi_{u(p)-t})_{\phi_t(p)}(v_2), \eta_{F(p)} \right),$$

where $p \in \Omega, t \in \mathbb{R}, v_1, v_2 \in T_{\phi_t(p)}\text{Nil}_3, dV$ is the volume form for the metric ds^2 on Nil_3, F is given by (12) and $\{\phi_t\}_{t \in \mathbb{R}}$ is the one-parameter group generated by X (which consists of isometries of Nil_3). Then ω is closed (since M is minimal), has unit comass and M is a calibrated surface for ω , i.e. $\omega|_M$ equals the area element of M . By the Fundamental Theorem of Calibrations (see e.g. Morgan [22] Theorem 6.4, also see Harvey and Lawson [16]), each compact subdomain of the graph of u is area minimizing for its boundary.

A natural version of the Bernstein problem in Nil_3 is to consider entire horizontal minimal graphs (or more generally, with constant mean curvature $H \in \mathbb{R}$) in the sense above. As we said in Remark 2, the Bernstein problem for the minimal case in the vertical direction (i.e. when the minimal surface is an entire vertical graph) has been solved by Fernández and Mira [12] and the solutions of this problem are essentially described in terms of holomorphic quadratic differentials on \mathbb{C} (not identically zero) or on \mathbb{D} . Regarding the case of nonzero constant mean curvature, Figueroa et al. [13, Theorem 4] proved that there are no entire vertical graphs of nonzero constant mean curvature in Nil_3 . Their argument, based on the maximum principle applied to spheres with the appropriate value of the mean curvature (see the third paragraph in the proof of Theorem 2 for a similar argument), works without changes when exchanging vertical graphs by horizontal ones. Hence, we only have to concentrate on the minimal case of the horizontal Bernstein problem. Theorem 3 below characterizes the examples $\Pi_{a,b}$ above among solutions of the horizontal Bernstein problem with parabolic conformal structure.

An orientable surface M of Nil_3 with unit normal vector field η is said to be a *horizontal multigraph* when $\langle X, \eta \rangle$ does not vanish. By Lemma 1, every horizontal multigraph is stable.

Theorem 3 *Let $M \subset \text{Nil}_3$ be a complete minimal horizontal multigraph. If M is parabolic, then it is a vertical plane.*

Proof Suppose M is not a vertical plane. By Corollary 2, M is an entire vertical graph. Consider the Jacobi functions $u = \langle \eta, E_3 \rangle, v = \langle \eta, X \rangle$, where η is a unit normal vector field on M . Since M is a horizontal multigraph, up to a change of orientation we can assume $v > 0$. Hence Theorem 1 gives that $u = lv$ for some $l \in \mathbb{R} - \{0\}$; this is, $\langle \eta, lX - E_3 \rangle = 0$ on M . Then $lX - E_3$ restricts to a tangent vector field along M , which has no zeros since $l \neq 0$ and E_1, E_3 are linearly independent. As $lX - E_3$ is Killing, then M is invariant under

the one-parameter group of isometries generated by $IX - E_3$. By the results in Figueroa et al. [13], M must be ambiently isometric to an entire minimal vertical graph M_θ defined by

$$z = z(x, y) = \frac{xy}{2} + \frac{\sinh(2\theta)}{2} \left[y\sqrt{1+y^2} + \ln \left(y + \sqrt{1+y^2} \right) \right], \quad (x, y) \in \mathbb{R}^2, \quad (13)$$

for some $\theta \in \mathbb{R}$. Since all isometries of Nil_3 preserve both the vertical direction and the horizontal distribution generated by E_1, E_2 (see Eq. 11), then it suffices to show that for all $\theta \in \mathbb{R}$, the surface M_θ is not a horizontal multigraph in any horizontal direction, i.e. the function $\langle X_\alpha, \eta \rangle$ has a zero on M_θ for every $\alpha \in [0, 2\pi)$, where X_α is the horizontal Killing vector field of Nil_3 given by

$$X_\alpha = \cos \alpha (E_1 + yE_3) + \sin \alpha (E_2 - xE_3).$$

The vector fields

$$T_1 = E_1 + yE_3, \quad T_2 = E_2 + \sinh(2\theta)\sqrt{1+y^2}E_3$$

restrict to M_θ as a global basis of the tangent bundle of M_θ , and

$$\det(T_1, T_2, X_\alpha) = -\sin \alpha \left(x + \sinh(2\theta)\sqrt{1+y^2} \right).$$

Therefore, X_α is tangent to M_θ (hence $\langle X_\alpha, \eta \rangle$ vanishes) along the curve with $x = -\sinh(2\theta)\sqrt{1+y^2}$, for every $\alpha \in [0, 2\pi)$. □

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Appendix: Vertical cylinders in $\mathbb{E}(\kappa, \tau)$

Consider the Riemannian submersion $\pi : \mathbb{E}(\kappa, \tau) \rightarrow \mathbb{M}^2(\kappa)$. The vertical fibers of this submersion are geodesics, and they are also integral curves of the unit Killing vector field E_3 , which generates the kernel of $d\pi$. $\mathbb{E}(\kappa, \tau)$ is parallelizable: there exists a global orthonormal frame (E_1, E_2, E_3) of the tangent bundle, and

$$[E_1, E_2] = 2\tau E_3, \quad [E_2, E_3] = \sigma E_1, \quad [E_3, E_1] = \sigma E_2$$

where

$$\sigma = \begin{cases} 0 & \text{if } \tau = 0, \\ \frac{\kappa}{2\tau} & \text{if } \tau \neq 0. \end{cases}$$

In particular, the 2-dimensional horizontal distribution $\text{Span}(E_1, E_2) = \ker(d\pi)^\perp$ is integrable if $\tau = 0$ (in this case $\mathbb{E}(\kappa, \tau) = \mathbb{M}^2(\kappa) \times \mathbb{R}$) and it is completely nonintegrable otherwise. Denote by $\bar{\nabla}$ the Riemannian connection on $\mathbb{E}(\kappa, \tau)$. The nonvanishing Christoffel symbols $\bar{\Gamma}_{ij}^k = \langle \bar{\nabla}_{E_i} E_j, E_k \rangle$ are the following:

$$\bar{\Gamma}_{12}^3 = \bar{\Gamma}_{23}^1 = -\bar{\Gamma}_{21}^3 = -\bar{\Gamma}_{13}^2 = \tau, \quad \bar{\Gamma}_{32}^1 = -\bar{\Gamma}_{31}^2 = \tau - \sigma$$

and

$$\left. \begin{aligned} \bar{\nabla}_{E_i} E_i &= 0, \quad i = 1, 2, 3, \\ \bar{\nabla}_{E_1} E_2 &= -\bar{\nabla}_{E_2} E_1 = \tau E_3, \\ \bar{\nabla}_X E_3 &= \tau X \times E_3, \end{aligned} \right\} \quad (14)$$

for any vector field X on $\mathbb{E}(\kappa, \tau)$, where \times denotes the vector product in $\mathbb{E}(\kappa, \tau)$ with respect to the direct orthonormal frame (E_1, E_2, E_3) .

Given a curve $\gamma \subset \mathbb{M}^2(\kappa)$, we define the vertical cylinder over γ as $C_\gamma = \pi^{-1}(\gamma)$. If we parameterize γ by its arc-length, then a global orthonormal frame of the tangent bundle of C_γ is given by $\{\gamma', E_3\}$ (here we are identifying γ' with its isometric preimage under $d\pi$). Let η be the unit normal vector field along C_γ such that $\{\gamma', E_3, \eta\}$ is positively oriented.

Lemma 3 *The mean curvature with respect to η , the Gauss curvature and the extrinsic curvature of C_γ are respectively given by*

$$H = \frac{k_\gamma}{2}, \quad K = 0, \quad K_{ext} = -\tau^2,$$

where k_γ is the geodesic curvature of γ in $\mathbb{M}^2(\kappa)$.

Proof A straightforward computation using (14) gives that the matrix of the second fundamental form of C_γ with respect to the orthonormal basis $\{\gamma', E_3\}$ of the tangent bundle to C_γ is given by

$$\Pi = \begin{pmatrix} \langle \bar{\nabla}_{\gamma'} \gamma', \eta \rangle & \langle \bar{\nabla}_{\gamma'} E_3, \eta \rangle \\ \langle \bar{\nabla}_{E_3} \gamma', \eta \rangle & \langle \bar{\nabla}_{E_3} E_3, \eta \rangle \end{pmatrix} = \begin{pmatrix} k_\gamma & \tau \\ \tau & 0 \end{pmatrix} \tag{15}$$

where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric on $\mathbb{E}(\kappa, \tau)$. In order to compute K , we use the Gauss equation:

$$K = \bar{K} + \det(\Pi) = \bar{K} - \tau^2,$$

where \bar{K} is the sectional curvature of the tangent plane to C_γ , i.e.

$$\bar{K} = \langle \bar{\nabla}_{\gamma'} \bar{\nabla}_{E_3} E_3, \gamma' \rangle - \langle \bar{\nabla}_{E_3} \bar{\nabla}_{\gamma'} E_3, \gamma' \rangle - \langle \bar{\nabla}_{[\gamma', E_3]} E_3, \gamma' \rangle.$$

Equations 14 and 15 imply that the first term in the last expression vanishes and the second one equals τ^2 . It remains to check that the third one is zero and we will have that $K = 0$. Now, $[\gamma', E_3] = \bar{\nabla}_{\gamma'} E_3 - \bar{\nabla}_{E_3} \gamma'$. Note that $\bar{\nabla}_{\gamma'} E_3 = \tau \eta$ is normal to C_γ and that $\langle \bar{\nabla}_{E_3} \gamma', \gamma' \rangle = \langle \bar{\nabla}_{E_3} \gamma', E_3 \rangle = 0$. Hence, $[\gamma', E_3]$ is normal to C_γ . Since $[\gamma', E_3]$ is clearly tangent to C_γ , it must be zero. \square

Proposition 1 *Let $\gamma \subset \mathbb{M}^2(\kappa)$ be a complete curve with constant geodesic curvature $k_\gamma \in \mathbb{R}$. Then the cylinder C_γ is stable if and only if $\kappa \leq -k_\gamma^2$.*

Proof The stability operator of C_γ is given by $L = \Delta + |A|^2 + \text{Ric}(\eta)$, where A is the shape operator of C_γ . By Eq. 15 we have $|A|^2 = k_\gamma^2 + 2\tau^2$, while a direct calculation using (14) gives $\text{Ric}(\eta) = \kappa - 2\tau^2$. Therefore, L is just a translation (with constant potential) of the Laplacian on C_γ , namely

$$L = \Delta + k_\gamma^2 + \kappa.$$

Hence, the Dirichlet problem with zero boundary values for this operator has first eigenvalue $l_1(\Omega) = l_1^\Delta(\Omega) - (k_\gamma^2 + \kappa)$ on every relatively compact domain $\Omega \subset C_\gamma$, where $l_1^\Delta(\Omega)$ is the first eigenvalue of the Dirichlet problem with zero boundary values for the Laplacian on C_γ . Since the induced metric on C_γ is flat, then $\inf_\Omega l_1(\Omega) = -(k_\gamma^2 + \kappa)$, from where the proposition follows. \square

Remark 3 From Proposition 1, all constant mean curvature cylinders $C_\gamma \subset \mathbb{E}(\kappa, \tau)$ are unstable when $\kappa > 0$, and the only stable constant mean curvature cylinders in $\mathbb{E}(0, \tau)$ are vertical planes (which are minimal).

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