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Calculus of Variations

On the entire self-shrinking solutions to Lagrangian mean curvature flow

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Abstract The authors prove that the logarithmic Monge–Ampère flow with uniformly bound and convex initial data satisfies uniform decay estimates away from time $t = 0$. Then applying the decay estimates, we conclude that every entire classical strictly convex solution of the equation

$$\det D^2 u = \exp\left\{ n\left(-u + \frac{1}{2}\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} \right) \right\},$$

should be a quadratic polynomial if the inferior limit of the smallest eigenvalue of the function $|x|^2 D^2 u$ at infinity has an uniform positive lower bound larger than $2(1 - 1/n)$. Using a similar method, we can prove that every classical convex or concave solution of the equation

$$\sum_{i=1}^{n} \arctan \lambda_i = -u + \frac{1}{2}\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}$$

must be a quadratic polynomial, where λ_i are the eigenvalues of the Hessian $D^2 u$.

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1 Introduction

In 1915, Bernstein [17] proved his celebrated theorem that the only entire minimal graphs in three-dimensional Euclidean space are planes. In 1954, Jörgens [14] proved that every classical strictly convex solution of the equation

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$$\det D^2u = 1, \quad x \in \mathbb{R}^2 \tag{1.1}$$

must be a quadratic polynomial, and Bernstein's theorem can be proved using this result. Meanwhile, Calabi ($n \le 5$) [4] and Pogorelov ($n \ge 2$) [18] extended Jörgens' theorem to $\mathbb{R}^n$. Later Jost and Xin had an alternative proof for this result [15]. In 2003, Cafarelli and Li [2] gave an extension to the theorem of Jörgens, Calabi and Pogorelov. In that paper, they presented another proof of the theorem of Jörgens, Calabi and Pogorelov and did research on the asymptotic behavior of convex solutions. They also used their results to reprove the Bernstein theorems. Recently, Li and Xu [16] showed that every smooth strictly convex solution on $\mathbb{R}^n$ of the Monge–Ampère type equation

$$\det D^2u = \exp\left\{-\sum_{i=1}^{n} d_i \frac{\partial u}{\partial x_i} - d_0\right\}, \quad x \in \mathbb{R}^n \tag{1.2}$$

must be a quadratic polynomial where $d_0, d_1, \ldots, d_n$ are constants.

From [11], we know that the gradient graph $(x, \nabla u)$ determines a volume minimizing surface in $\mathbb{C}^n$ if and only if u satisfies the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = \Theta.$$

Here, λ_i are the eigenvalues of the Hessian D^2u and Θ is a constant. It belongs to an important class of fully nonlinear elliptic equations which has been studied by various authors (cf. Bao et al. [1] and Yuan [21]). A Bernstein type theorem has been proved in [21]. It tells us, that if u is a smooth convex function and satisfies the special Lagrangian equation in $\mathbb{R}^n$ then u must be a quadratic polynomial.

Lagrangian self-similar solution being part of a minimal cone was investigated in [10] with additional conditions on Maslov class and the Lagrangian angle. In this paper we mainly do research on a Bernstein type problem for self-shrinking equations of the Lagrangian mean curvature flow in Euclidean and pseudo-Euclidean space.

Consider the logarithmic Monge–Ampère flow (cf. [20])

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{n} \ln \det D^2u = 0, & t > 0, \quad x \in \mathbb{R}^n, \\ u = u_0(x), & t = 0, \quad x \in \mathbb{R}^n. \end{cases} \tag{1.3}$$

By Proposition 2.1 in [12], there exists a family of diffeomorphisms

$$r_t : \mathbb{R}^n \to \mathbb{R}^n,$$

such that the maps

$$F(x,t) = (r_t(x), Du(r_t(x), t)) \subset \mathbb{R}^{2n}_n,$$
$$F_0(x) = (x, Du_0(x)),$$

satisfy the mean curvature flow in pseudo-Euclidean space:

$$\begin{cases} \dfrac{dF}{dt} = \overrightarrow{H}, \\ F(x,0) = F_0(x), \end{cases} \tag{1.4}$$

where $\overrightarrow{H}$ is the mean curvature vector of the submanifold defined by F.

Definition 1.1 Assume that function $u_0(x) \in C^2(\mathbb{R}^n)$. We call $u_0(x)$ satisfying Condition A, if

$$\Lambda I \geq D^2 u_0(x) \geq \lambda I, \qquad x \in \mathbb{R}^n.$$

Here Λ, λ are two positive constants and I is the identity matrix.

For the logarithmic Monge–Ampère flow, the first author has obtained the long time existence and the global estimates of derivatives of solutions (cf. [12, Theorem 1.2]).

Proposition 1.2 *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function which satisfies Condition A. Then there exists a unique strictly convex solution of* (1.3) *such that*

$$u(x,t) \in C^\infty(\mathbb{R}^n \times (0,+\infty)) \cap C(\mathbb{R}^n \times [0,+\infty)) \tag{1.5}$$

where $u(\cdot,t)$ satisfies Condition A. More generally, for $l \in \{3,4,5\ldots\}$ and $\varepsilon_0 > 0$, there holds

$$\sup_{x\in\mathbb{R}^n} |D^l u(x,t)|^2 \leq C, \qquad \forall t \in (\varepsilon_0, +\infty), \tag{1.6}$$

where C depends only on $n, \lambda, \Lambda, \frac{1}{\varepsilon_0}$.

In fact, in this paper we will prove the following stronger result:

Theorem 1.3 *Assume that $u(x,t)$ is a strictly convex solution of* (1.3)*, and $u(\cdot,t)$ satisfies Condition A. Then there exists a positive constant C depending only on $n, \lambda, \Lambda, \frac{1}{\varepsilon_0}$, such that*

$$\sup_{x\in\mathbb{R}^n} |D^3 u(x,t)|^2 \leq \frac{C}{t}, \qquad \forall t \geq \varepsilon_0. \tag{1.7}$$

More generally, for all $l \in \{3,4,5\ldots\}$ there holds

$$\sup_{x\in\mathbb{R}^n} |D^l u(x,t)|^2 \leq \frac{C}{t^{l-2}}, \qquad \forall t \geq \varepsilon_0. \tag{1.8}$$

Remark 1.4 For the special Lagrangian evolution equation (1.13), there are similar results in the paper [5].

Next we consider the following Monge–Ampère type equation

$$\det D^2 u = \exp\left\{ n\left(-u + \frac{1}{2}\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} \right) \right\}. \tag{1.9}$$

According to definitions in [7], we can show that an entire solution to (1.9) is a self-shrinking solution to Lagrangian mean curvature flow in pseudo-Euclidean space. As an application of Proposition 1.2 and Theorem 1.3, we can prove that

Theorem 1.5 *Assume that $u : \mathbb{R}^n \to \mathbb{R}$ is a C^2 strictly convex solution of* (1.9) *which satisfies*

$$\liminf_{x\to\infty} |x|^2 \mu(x) > \frac{2(n-1)}{n}, \tag{1.10}$$

where $\mu(x)$ is the smallest eigenvalue of D^2u. Then u must be a quadratic polynomial. Furthermore, there exists a symmetric real matrix A such that

$$u = u(0) + \frac{1}{2} <x, Ax>, \tag{1.11}$$

where $\det A = e^{-nu(0)}$.

Remark 1.6 In dimension 1, assume that u is a smooth solution of (1.9) with $u'(0) = 0$, then

$$u = u(0) + \frac{1}{2} e^{-nu(0)} x^2.$$

This follows from the existence and the uniqueness results of ordinary differential equations.

By Theorem 1.5 and by (1.11) we know that condition (1.10) implies

$$\nabla u(0) = 0. \tag{1.12}$$

Then, a natural question is presented. If we weaken condition (1.10) to (1.12), does the same result in Theorem 1.5 still hold?

In [19], the special Lagrangian evolution equation can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{\sqrt{-1}} \ln \dfrac{\det(I + \sqrt{-1} D^2 u)}{\sqrt{\det(I + (D^2u)^2)}} = 0, & t > 0, \quad x \in \mathbb{R}^n, \\ u = u_0(x), & t = 0, \quad x \in \mathbb{R}^n. \end{cases} \tag{1.13}$$

It is well-known that there exists a family of diffeomorphisms

$$r_t : \mathbb{R}^n \to \mathbb{R}^n,$$

such that

$$F(x,t) = (r_t(x), Du(r_t(x), t)) \subset \mathbb{R}^{2n},$$
$$F_0(x) = (x, Du_0(x)),$$

satisfies the mean curvature flow in Euclidean space:

$$\begin{cases} \dfrac{dF}{dt} = \overrightarrow{H}, \\ F(x,0) = F_0(x), \end{cases} \tag{1.14}$$

where $\overrightarrow{H}$ is the mean curvature vector of the submanifold defined by F.

Consider the entire self-shrinking solutions to Lagrangian mean curvature flow in Euclidean space. When the Hessian of the potential function u has eigenvalues strictly uniformly between -1 and 1, Chau et al. [6] showed that all self-shrinking solutions must be quadratic polynomials. The next two theorems generalize their results.

Theorem 1.7 *Let u be a C^2 self-shrinking solution to Lagrangian mean curvature flow in Euclidean space:*

$$\sum_{i=1}^{n} \arctan \lambda_i = -u + \frac{1}{2} \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}, \tag{1.15}$$

where λ_i $(i = 1, 2, \ldots, n)$ are the eigenvalues of Hessian D^2u. Suppose that

$$-1 \le \lambda_i \le 1, \tag{1.16}$$

then u must be a quadratic polynomial.

Theorem 1.8 *Let u be a C^2 convex or concave solution to (1.15). Then u must be a quadratic polynomial.*

Here we use some techniques in [21] and some ideas developed in the proof of Lemma 3.2. We only use the elliptic equation (1.15), but do not need the parabolic equation (1.13).

This paper is organized as follows. In Sect. 2, we obtain the differential inequality (2.1), which plays an important role in the third-order decay estimates (see Lemma 2.2). Then we complete the proof of Theorem 1.3 by the blow-up argument. In Sect. 3, we give the proof of Theorem 1.5 by the second-order derivative estimates for the equations of Monge–Ampère type (1.9). In Sect. 4, we prove Theorems 1.7 and 1.8.

2 The decay estimates of the logarithmic Monge–Ampère flow

Throughout the following Einstein's convention of summation over repeated indices will be adopted. Denote

$$u_i = \frac{\partial u}{\partial x_i}, \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad u_{ijk} = \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}, \ldots, \text{and } [u^{ij}] = [u_{ij}]^{-1}.$$

We introduce the comparison principle for solutions of Cauchy problems which belongs to Giga et al. [8] (cf. a special version of Theorem 4.1 in [8]).

Lemma 2.1 *Suppose that the functions $\sigma_*, \sigma^* \in C^{2,1}(\mathbb{R}^n \times (0, +\infty)) \cap C(\mathbb{R}^n \times [0, +\infty))$ and u satisfies Condition A. If there exists a positive constant C, such that*

$$\sigma_* \le C, \quad \sigma^* \le C,$$

and σ_* , σ^* satisfy

$$\partial_t \sigma_* - \frac{1}{n} u^{ij} \sigma_{*ij} + \frac{1}{2n^2} \sigma_*^2 \le 0, \quad \forall t > 0, \quad x \in \mathbb{R}^n;$$

$$\partial_t \sigma^* - \frac{1}{n} u^{ij} \sigma^*_{ij} + \frac{1}{2n^2} \sigma^{*2} \ge 0, \quad \forall t > 0, \quad x \in \mathbb{R}^n;$$

$$\sigma_* \le \sigma^*, \quad t = 0, \quad \forall x \in \mathbb{R}^n.$$

Then there holds

$$\sigma_* \le \sigma^*, \quad \forall t > 0, \quad x \in \mathbb{R}^n.$$

We are now in a position to describe Calabi's computation. It is used by Pogorelov [18] and Caffarelli et al. [3], to estimate the third derivatives of Monge–Ampère equation. Here we use his methods to carry out the third derivatives of Monge–Ampère equation of parabolic type.

Let

$$\sigma = u^{kl} u^{pq} u^{rs} u_{kpr} u_{lqs}.$$

Then the expression measures the square of the third derivatives in terms of the Riemannian metric $ds^2 = u_{ij} dx^i dx^i$. We establish the following lemma which is a parabolic version of [3, Lemma 3.1].

Lemma 2.2 *Let u be a solution of* (1.3). *If* $u(\cdot, t)$ *satisfies* (1.5) *and Condition A. Then* σ *satisfies a parabolic inequality:*

$$\partial_t \sigma - \frac{1}{n} u^{ij} \sigma_{ij} + \frac{1}{2n^2} \sigma^2 \le 0, \quad \forall t > 0, \quad x \in \mathbb{R}^n. \tag{2.1}$$

Proof Note that

$$\partial u^{ab} = -u^{ac} \partial u_{cd} u^{db},$$
$$\partial_t \sigma = 2u^{kl} u^{pq} u^{rs} \partial_t u_{kpr} u_{lqs} - 3u^{ka} \partial_t u_{ab} u^{bl} u^{pq} u^{rs} u_{kpr} u_{lqs}.$$

By Eq. (1.3), we have

$$\begin{aligned}
\partial_t u_a &= \frac{1}{n} u^{ij} u_{aij}, \\
\partial_t u_{ab} &= \frac{1}{n} u^{ij} u_{abij} - \frac{1}{n} u^{ic} u^{jd} u_{aij} u_{bcd}, \\
\partial_t u_{kpr} &= \frac{1}{n} u^{ij} u_{kprij} - \frac{1}{n} u^{ia} u^{jb} u_{rab} u_{kpij} \\
&\quad - \frac{1}{n} u^{ic} u^{jd} u_{pcd} u_{krij} - \frac{1}{n} u^{ic} u^{jd} u_{kij} u_{prcd} \\
&\quad + \frac{1}{n} u^{ia} u^{cb} u_{rab} u^{jd} u_{kij} u_{pcd} + \frac{1}{n} u^{ic} u^{ja} u^{db} u_{rab} u_{kij} u_{pcd}.
\end{aligned}$$

Then

$$\begin{aligned}
n \partial_t \sigma &= 2u^{kl} u^{pq} u^{rs} u^{ij} u_{lqs} u_{kprij} - 6u^{kl} u^{pq} u^{rs} u^{ia} u^{jb} u_{lqs} u_{rab} u_{kpij} \\
&\quad + 4u^{kl} u^{pq} u^{rs} u^{ia} u^{cb} u^{jd} u_{lqs} u_{rab} u_{kij} u_{pcd} \\
&\quad - 3u^{ka} u^{bl} u^{pq} u^{rs} u^{ij} u_{kpr} u_{lqs} u_{abij} + 3u^{ka} u^{bl} u^{pq} u^{rs} u^{ic} u^{jd} u_{kpr} u_{lqs} u_{aij} u_{bcd}.
\end{aligned} \tag{2.2}$$

By the computation in [3], we have

$$\begin{aligned}
u^{ij} \sigma_{ij} &= 2u^{kl} u^{pq} u^{rs} u^{ij} u_{lqs} u_{kprij} + 2u^{kl} u^{pq} u^{rs} u^{ij} u_{kpri} u_{lqsj} \\
&\quad - 12u^{ka} u^{bl} u^{pq} u^{rs} u^{ij} u_{abi} u_{lqs} u_{kprj} \\
&\quad + 6u^{ka} u^{bl} u^{pc} u^{dq} u^{rs} u^{ij} u_{kpr} u_{lqs} u_{abi} u_{cdj} \\
&\quad - 3u^{ka} u^{bl} u^{pq} u^{rs} u^{ij} u_{kpr} u_{lqs} u_{abij} \\
&\quad + 3u^{kc} u^{ad} u^{bl} u^{pq} u^{rs} u^{ij} u_{kpr} u_{lqs} u_{abi} u_{cdj} \\
&\quad + 3u^{ka} u^{bc} u^{dl} u^{pq} u^{rs} u^{ij} u_{kpr} u_{lqs} u_{abi} u_{cdj}.
\end{aligned} \tag{2.3}$$

At any point x, we may assume that u_{ij} is diagonal after a suitable rotation. So the simplified versions of (2.2), (2.3) are

$$\begin{aligned}
n \partial_t \sigma &= 2u^{kk} u^{pp} u^{rr} u^{ii} u_{kpr} u_{kprii} - 6u^{kk} u^{pp} u^{rr} u^{ii} u^{jj} u_{kpr} u_{rij} u_{kpij} \\
&\quad + 4u^{kk} u^{pp} u^{rr} u^{ii} u^{cc} u^{jj} u_{kpr} u_{ric} u_{kij} u_{pcj} \\
&\quad - 3u^{kk} u^{bb} u^{pp} u^{rr} u^{ii} u_{kpr} u_{bpr} u_{kbii} + 3u^{kk} u^{bb} u^{pp} u^{rr} u^{ii} u^{jj} u_{kpr} u_{bpr} u_{kij} u_{bij}, \\
u^{ij} \sigma_{ij} &= 2u^{kk} u^{pp} u^{rr} u^{ii} u_{kpr} u_{kprii} + 2u^{kk} u^{pp} u^{rr} u^{ii} u_{kpri} u_{kpri} \\
&\quad - 12u^{kk} u^{bb} u^{pp} u^{rr} u^{ii} u_{kbi} u_{bpr} u_{kpri} + 6u^{kk} u^{bb} u^{pp} u^{dd} u^{rr} u^{ii} u_{kpr} u_{bdr} u_{kbi} u_{pdi} \\
&\quad - 3u^{kk} u^{bb} u^{pp} u^{rr} u^{ii} u_{kpr} u_{bpr} u_{kbii} + 3u^{kk} u^{aa} u^{bb} u^{pp} u^{rr} u^{ii} u_{kpr} u_{bpr} u_{abi} u_{kai} \\
&\quad + 3u^{kk} u^{bb} u^{dd} u^{pp} u^{rr} u^{ii} u_{kpr} u_{dpr} u_{kbi} u_{bdi}.
\end{aligned}$$

Let

$$A = u^{kk}u^{pp}u^{rr}u^{ll}u^{qq}u^{ii}u_{kpr}u_{lqr}u_{kli}u_{pqi},$$
$$B = u^{kk}u^{pp}u^{rr}u^{ll}u^{qq}u^{ii}u_{kpr}u_{lpr}u_{kqi}u_{lqi}.$$

Then, we get

$$\begin{aligned} n\partial_t\sigma &= 2u^{kk}u^{pp}u^{rr}u^{ii}u_{kpr}u_{kprii} - 6u^{kk}u^{pp}u^{rr}u^{ii}u^{jj}u_{kpr}u_{rij}u_{kpij} \\ &\quad -3u^{kk}u^{bb}u^{pp}u^{rr}u^{ii}u_{kpr}u_{bpr}u_{kbii} + 4A + 3B, \\ u^{ij}\sigma_{ij} &= 2u^{kk}u^{pp}u^{rr}u^{ii}u_{kpr}u_{kprii} + 2u^{kk}u^{pp}u^{rr}u^{ii}u_{kpri}u_{kpri} \\ &\quad -12u^{kk}u^{bb}u^{pp}u^{rr}u^{ii}u_{kbi}u_{bpr}u_{kpri} \\ &\quad -3u^{kk}u^{bb}u^{pp}u^{rr}u^{ii}u_{kpr}u_{bpr}u_{kbii} \\ &\quad +6A + 3B + 3B. \end{aligned}$$

It is easy to verify that

$$u^{kk}u^{bb}u^{pp}u^{rr}u^{ii}u_{kbi}u_{bpr}u_{kpri} = u^{kk}u^{pp}u^{rr}u^{ii}u^{jj}u_{kpr}u_{rij}u_{kpij}.$$

So we obtain

$$\begin{aligned} u^{ij}\sigma_{ij} - n\partial_t\sigma &= 2u^{kk}u^{pp}u^{rr}u^{ii}u_{kpri}u_{kpri} - 6u^{kk}u^{pp}u^{rr}u^{ii}u^{jj}u_{kpr}u_{rij}u_{kpij} \\ &\quad +3B + 2A. \end{aligned} \tag{2.4}$$

Thus

$$\begin{aligned} &2u^{kk}u^{pp}u^{rr}u^{ii}u_{kpri}u_{kpri} - 6u^{kk}u^{pp}u^{rr}u^{ii}u^{jj}u_{kpr}u_{rij}u_{kpij} \\ &= 2u^{kk}u^{pp}u^{rr}u^{ii}\left[u_{kpri} - \frac{1}{2}u^{ll}(u_{kli}u_{plr} + u_{pli}u_{klr} + u_{rli}u_{kpl})\right]^2 \\ &\quad -\frac{1}{2}u^{kk}u^{pp}u^{rr}u^{ii} \mid u^{ll}(u_{kli}u_{plr} + u_{pli}u_{klr} + u_{rli}u_{kpl}) \mid^2 \\ &= 2u^{kk}u^{pp}u^{rr}u^{ii}\left[u_{kpri} - \frac{1}{2}u^{ll}(u_{kli}u_{plr} + u_{pli}u_{klr} + u_{rli}u_{kpl})\right]^2 \\ &\quad -\frac{3}{2}B - \frac{6}{2}A \\ &\geq -\frac{3}{2}B - 3A. \end{aligned}$$

By $B \geq A$ and $B \geq \frac{1}{n}\sigma^2$ (cf. [3]), (2.4) tells us that

$$\begin{aligned} u^{ij}\sigma_{ij} - n\partial_t\sigma &\geq \frac{1}{2}B + B - A \\ &\geq \frac{1}{2n}\sigma^2. \end{aligned}$$

□

Corollary 2.3 *Assume $u_0(x)$ be a smooth function satisfying Condition A and*

$$\sup_{x\in\mathbb{R}^n} |D^3 u_0| < +\infty. \tag{2.5}$$

Set $\sigma_0 = \sigma\mid_{t=0}$. *Then*

$$\sup_{x\in\mathbb{R}^n}\sigma \leq \frac{\sup_{x\in\mathbb{R}^n}\sigma_0}{1+\frac{1}{2n^2}\sup_{x\in\mathbb{R}^n}\sigma_0 t}, \quad \forall t>0, \tag{2.6}$$

i.e,

$$\sup_{x\in\mathbb{R}^n}|D^3u|^2 \leq \frac{C\sup_{x\in\mathbb{R}^n}|D^3u_0|^2}{1+\sup_{x\in\mathbb{R}^n}|D^3u_0|^2 t}, \quad \forall t>0, \tag{2.7}$$

where C *is positive constant depending only on* n, λ, Λ.

Proof By Schauder estimates, as in the proof of Proposition 1.2 (cf. [12]), we have

$$\sup_{x\in\mathbb{R}^n}\sigma \leq C.$$

Here, C is a positive constant depending only on n, λ, Λ and $\sup_{x\in\mathbb{R}^n}|D^3u_0|$. Set $\sigma_* = \sigma$ and

$$\sigma^* = \frac{\sup_{x\in\mathbb{R}^n}\sigma_0}{1+\frac{1}{2n^2}\sup_{x\in\mathbb{R}^n}\sigma_0 t}.$$

In this case, one can verify that

$$\frac{d}{dt}\sigma^* + \frac{1}{2n^2}\sigma^{*2} = 0,$$

with

$$\sigma^*|_{t=0} = \sup_{x\in\mathbb{R}^n}\sigma_0.$$

Then by Lemma 2.1 we obtain (2.6) and (2.7). □

By now we have proved (1.7) with an additional condition (2.5). Using Krylov–Safonov theory and interior Schauder estimates of parabolic equations, we need not that u_0 satisfies (2.5) for our theorem.

Proof of Theorem 1.3 By Proposition 1.2, we have

$$\sup_{x\in\mathbb{R}^n}|D^3u|_{t=\varepsilon_0} \leq C, \tag{2.8}$$

where C is a positive constant depending only on n, λ, Λ and $\frac{1}{\varepsilon_0}$. Using Corollary 2.3, it follows from (2.8) that we obtain (1.7).

We will derive high order estimates (1.8) via the blow up argument. To do so, by Chau et al. [5], we employ a parabolic scaling now. The remaining proof is routine. Define

$$y = \mu(x-x_0), \qquad s = \mu^2(t-t_0),$$
$$u_\mu(y,s) = \mu^2[u(x,t) - u(x_0,t_0) - Du(x_0,t_0)\cdot(x-x_0)].$$

It is easy to see that

$$D_y^2 u_\mu = D_x^2 u, \qquad \frac{\partial}{\partial s}u_\mu = \frac{\partial}{\partial t}u$$

and

$$D_y^l u_\mu = \mu^{2-l} D_x^l u$$

for all nonnegative integers l. By computing, $u_\mu(y, s)$ satisfies

$$\begin{cases} \dfrac{\partial u_\mu}{\partial s} - \dfrac{1}{n} \ln\det D^2 u_\mu = 0, & s > 0, \quad y \in \mathbb{R}^n, \\ u_\mu = u_\mu(y, s)|_{t=t_0}, & s = 0, \quad y \in \mathbb{R}^n, \end{cases}$$

with

$$u_\mu(0, 0) = Du_\mu(0, 0) = 0. \tag{2.9}$$

Without loss of generality, we prove (1.8) for $l = 4$ only since the statement follows in a similar way for all l by induction on l.

Note that

$$\sup_{x\in\mathbb{R}^n} |D^4 u| < +\infty, \quad t \geq \varepsilon_0.$$

Suppose that $|D^4 u|^2 t^2$ were not bounded on $\mathbb{R}^n \times [\varepsilon_0, +\infty)$. By [13, Lemma 3.5], there would be a sequence $t_k \to +\infty$, such that

$$2\rho_k := \sup_{x\in\mathbb{R}^n} |D^4 u(x, t_k)|^2 t_k^2 \to +\infty \tag{2.10}$$

and

$$\sup_{x\in\mathbb{R}^n, t\leq t_k} |D^4 u(x, t)|^2 t^2 \leq 2\rho_k. \tag{2.11}$$

Then there exists x_k such that

$$|D^4 u(x_k, t_k)|^2 t_k^2 \geq \rho_k \to +\infty \quad \text{as} \quad t_k \to +\infty. \tag{2.12}$$

Let $(y, Du_{\mu_k}(y, s))$ be a parabolic scaling of $(x, Du(x, t))$ by $\mu_k = \left(\frac{\rho_k}{t_k^2}\right)^{\frac{1}{4}}$ at (x_k, t_k) for each k. Thus $u_{\mu_k}(y, s)$ is a solution of a fully nonlinear parabolic equation

$$\frac{\partial u_{\mu_k}}{\partial s} - \frac{1}{n} \ln\det D^2 u_{\mu_k} = 0, \quad -\mu_k^2 t_k < s \leq 0, \quad y \in \mathbb{R}^n. \tag{2.13}$$

Combining (2.10), (2.11) with (2.12), there holds

$$|D_y^2 u_{\mu_k}| = |D_x^2 u| \leq n\Lambda, \qquad (y, s) \in \mathbb{R}^n \times (-\mu_k^2 t_k, 0]; \tag{2.14}$$

$$\begin{aligned} \forall y \in \mathbb{R}^n, \quad |D_y^3 u_{\mu_k}|^2 &= \mu_k^{-2} |D_x^3 u|^2 \\ &\leq \mu_k^{-2} t_k^{-1} C \\ &= \rho_k^{-\frac{1}{2}} C \to 0 \end{aligned}$$

and

$$\begin{aligned} \forall y \in \mathbb{R}^n, \quad |D_y^4 u_{\mu_k}|^2 &= \mu_k^{-4} |D_x^4 u|^2 \leq 2; \\ |D_y^4 u_{\mu_k}(0, 0)| &\geq 1. \end{aligned} \tag{2.15}$$

Using (2.13), by Schauder estimates, there exists a constant C depending only on $n, \lambda, \Lambda, \dfrac{1}{\varepsilon_0}$, such that for $l \geq 4$, we derive

$$\forall (y, s) \in \mathbb{R}^n \times (-\mu_k^2 t_k, 0], \quad |D_y^l u_{\mu_k}|^2 \leq C. \tag{2.16}$$

Combining (2.9), (2.14)–(2.16) together, a diagonal sequence argument shows that u_{μ_k} converges subsequently and uniformly on compact subsets in $\mathbb{R}^n \times (-\infty, 0]$ to a smooth function u_∞ with

$$\forall (y, s) \in \mathbb{R}^n \times (-\infty, 0], \quad |D_y^3 u_\infty| = 0$$

and

$$|D_y^4 u_\infty(0, 0)| \geq 1.$$

It is a contradiction. □

3 Self-shrinking solutions to Lagrangian mean curvature flow in pseudo-Euclidean space

We now describe the relationship between Monge–Ampère type equations (1.9) and the logarithmic Monge–Ampère flow.

A solution $F(\cdot, t)$ of (1.4) is called self-shrinking if it has the form

$$M_t = \sqrt{-t} M_{-1} \quad \text{for all } t < 0, \tag{3.1}$$

where $M_t = F(\cdot, t)$.

Assume that $F(x, t)$ is a self-shrinking solution of (1.4). Following [12, Proposition 2.1], $u(x, t)$ satisfies

$$\frac{\partial u}{\partial t} - \frac{1}{n} \ln \det D^2 u = 0, \quad t < 0, \qquad x \in \mathbb{R}^n. \tag{3.2}$$

Hence,

$$D\left(u(x, t) + tu\left(\frac{x}{\sqrt{-t}}, -1\right)\right) = 0,$$

i.e.,

$$u(x, t) = -tu\left(\frac{x}{\sqrt{-t}}, -1\right), \quad t < 0. \tag{3.3}$$

Thus combining (3.2), (3.3) and letting $t = -1$, we can verify that $u(x, -1)$ satisfies (1.9).

Conversely, if $u(x)$ solves (1.9) , then using (3.3), we can obtain a solution $F(x, t)$ to (1.4) which is shrinking. Suppose that $u(x)$ solves (1.9). Define

$$u(x, t) = -tu\left(\frac{x}{\sqrt{-t}}\right).$$

One can easily check the family $M_t = \{(x, Du(x, t)) | x \in \mathbb{R}^n\}$ satisfying (3.1) and we also have

$$\frac{\partial u}{\partial t}(x, t) = -u\left(\frac{x}{\sqrt{-t}}\right) + \frac{1}{2} < \nabla u, \frac{x}{\sqrt{-t}} >= \frac{1}{n} \ln \det D^2 u.$$

In other words, $u(x, t)$ solves the logarithmic gradient flow. By the above discussion, there exists a family r_t, such that $F(x, t) = (r_t(x), Du(r_t(x), t))$ is a self-shrinking solution of (1.4).

Based on Theorem 1.3 according to the parabolic equation (1.3), we will prove the following lemma by the same methods in [6].

Lemma 3.1 *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth solution of* (1.9) *which satisfies condition A. Then u must be a quadratic polynomial.*

Proof If u is a smooth solution to (1.9), then

$$v(x,t) = (1-t)u\left(\frac{x}{\sqrt{1-t}}\right)$$

is a solution to (1.3) for $t \in (0,1)$ with initial data $u(x)$. Hence applying Proposition 1.2 to $v(x,t)$ we show that this solution is unique. By Theorem 1.3, there is some constant C, such that $|D^3 v(x,t)| \le C$ for $t \ge \varepsilon_0$ and any $x \in \mathbb{R}^n$. But one checks directly that

$$D^3 v(x,t) = \frac{1}{\sqrt{1-t}} D^3 u\left(\frac{x}{\sqrt{1-t}}\right).$$

This implies

$$|D^3 u(x)| = \left|D^3 u\left(\frac{x\sqrt{1-t}}{\sqrt{1-t}}\right)\right| = \sqrt{1-t}|D^3 v(x\sqrt{1-t}, t)| \le C\sqrt{1-t}$$

for any x. It follows that $D^3 u(x) \equiv 0$ by letting $t \to 1$. Then u must be a quadratic polynomial. Lemma 3.1 is established. □

In fact, using the interior estimated skills (cf. [9]), we can get the upper bound for the second derivatives of solutions of (1.9) under the condition (1.10).

Lemma 3.2 *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth strictly convex solution to* (1.9) *and suppose $\mu(x)$ satisfies* (1.10)*. Then there exists a positive constant R_0 depending only on $\mu(x)$, such that*

$$D^2 u(x) \le CI, \qquad x \in \mathbb{R}^n, \tag{3.4}$$

where C is a positive constant depending only on $\mu(x)$ and $\|u\|_{C^2(\bar{B}_{R_0+1})}$. B_{R_0} is a ball centered at 0 *with radius R_0 in $\mathbb{R}^n$.*

Proof Denote

$$u_i = \frac{\partial u}{\partial x_i}, \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad u_{ijk} = \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}, \ldots$$

and

$$[u^{ij}] = [u_{ij}]^{-1}, \quad L = u^{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

Let γ denotes a vector field. Set

$$u_\gamma = D_\gamma u, \ \ u_{\gamma\gamma} = D^2_{\gamma\gamma} u.$$

We will prove that

$$\sup_{x\in\mathbb{R}^n,\, \gamma\in\mathbb{S}^{n-1}} u_{\gamma\gamma} \le C.$$

By (1.10), there is some constant $\lambda > \dfrac{2(n-1)}{n}$ and some constant R_0, such that

$$|x|^2\mu(x) \ge \lambda,$$

for $|x| > R_0 + 1$. One can define a family of smooth function by

$$f_k(t) = \begin{cases} 1, & 0 \le t \le R_0, \\ \varphi & R_0 \le t \le R_0 + 1, \\ -k[t^2 - (R_0+1)^2] + \frac{3}{4}, & t \ge R_0 + 1, \end{cases}$$

where $0 < k \le 1$, and $(t, \varphi(t))$ is a smooth curve connecting two points $(R_0, 1)$, $(R_0 + 1, \dfrac{3}{4})$ satisfying $\dfrac{3}{4} \le \varphi \le 1$.

We view $u_{\gamma\gamma}$ as a function on $\mathbb{R}^n \times \mathbb{S}^{n-1}$. It is easy to see that $f_k(|x|)u_{\gamma\gamma}$ always attains its maximum at

$$(p, \xi) \in \{(x, \gamma) \in \mathbb{R}^n \times \mathbb{S}^{n-1} | f_k(|x|) > 0\}.$$

By (1.10), we have $u_{\gamma\gamma} > 0$. Let

$$\eta_k(x) = f_k(|x|), \quad w = \eta_k(x)u_{\xi\xi}.$$

Then at p,

$$0 \ge Lw = u^{ij}(\eta_k u_{\xi\xi})_{ij} = u^{ij}(\eta_k)_{ij}u_{\xi\xi} + 2u^{ij}(\eta_k)_i(u_{\xi\xi})_j + \eta_k u^{ij}(u_{\xi\xi})_{ij}. \tag{3.5}$$

We assume that

$$p \in \{x \in \mathbb{R}^n | |x| > R_0 + 1\}.$$

Then at p, the derivative $u_{\xi\xi}$ will be the maximum eigenvalue of the Hessian D^2u. By a rotation, we can assume that D^2u is diagonal with ξ as the x_1 direction. In this case, $u_{\xi\xi} = u_{11}$. Then at p, there holds

$$(\eta_k u_{11})_j = 0, \quad j = 1, 2, \ldots, n.$$

Hence

$$(u_{11})_j = -u_{11}\frac{(\eta_k)_j}{\eta_k}, \quad (\eta_k)_j = -\eta_k \frac{(u_{11})_j}{u_{11}}, \quad j = 1, 2, \ldots, n. \tag{3.6}$$

Clearly, by (3.6),

$$\begin{aligned} 2u^{ij}(\eta_k)_i(u_{11})_j &= u^{11}(\eta_k)_1 u_{111} + u^{11}(\eta_k)_1 u_{111} + 2\sum_{i\ne 1} \frac{(\eta_k)_i u_{11i}}{u_{ii}} \\ &= -u^{11}\frac{(\eta_k)_1(\eta_k)_1}{\eta_k}u_{11} - u^{11}\eta_k \frac{u_{111}^2}{u_{11}} - 2\sum_{i\ne 1} \eta_k \frac{u_{11i}^2}{u_{ii}u_{11}}. \end{aligned} \tag{3.7}$$

Let $< \cdot, \cdot >$ be the inner product in $\mathbb{R}^n$. Differentiating Eq. (1.9), we have

$$\begin{aligned} \frac{1}{n}u^{ij}u_{ij1} &= -\frac{1}{2}u_1 + \frac{1}{2} <x, Du_1>, \\ \frac{1}{n}u^{ij}u_{11ij} &= \frac{1}{n}\sum_{i,j=1}^{n} \frac{u_{ij1}^2}{u_{ii}u_{jj}} + \frac{1}{2} <x, Du_{11}>. \end{aligned} \tag{3.8}$$

Substituting (3.7), (3.8) into (3.5) and using

$$(\eta_k)_i = -2kx_i, \quad (\eta_k)_{ij} = -2k\delta_{ij},$$

we have, at p,

$$0 \geq -\frac{1}{n}2k\sum_{i=1}^{n} u^{ii}u_{11} - \frac{1}{n}\frac{(\eta_k)_1^2}{\eta_k} - \frac{1}{n}\eta_k\frac{u_{111}^2}{u_{11}^2} - \frac{1}{n}2\eta_k\sum_{i\neq 1}\frac{u_{11i}^2}{u_{ii}u_{11}}$$

$$+\frac{1}{n}\eta_k\sum_{i,j=1}^{n}\frac{u_{ij1}^2}{u_{ii}u_{jj}} + \frac{\eta_k}{2} <x, Du_{11}>.$$

Note that

$$\eta_k\sum_{i,j=1}^{n}\frac{u_{ij1}^2}{u_{ii}u_{jj}} \geq \eta_k\frac{u_{111}^2}{u_{11}^2} + 2\eta_k\sum_{i\neq 1}\frac{u_{11i}^2}{u_{ii}u_{11}}.$$

Combining the above two inequalities, at p, we get

$$0 \geq -\frac{1}{n}2k\sum_{i=1}^{n} u^{ii}u_{11} - \frac{1}{n}\frac{(\eta_k)_1^2}{\eta_k} + \frac{\eta_k}{2} <x, Du_{11}>.$$

In view of (3.6),

$$\frac{\eta_k}{2} <x, Du_{11}> = -\frac{u_{11}}{2} <x, D\eta_k>.$$

Then at p,

$$\frac{(\eta_k)_1^2}{\eta_k} \geq -2k\sum_{i=1}^{n} u^{ii}u_{11} - n\frac{u_{11}}{2} <x, D\eta_k>.$$

Using $u_{ii} \geq \dfrac{\lambda}{|x|^2}$ for $i \geq 2$, we deduce from the above that

$$\frac{4k^2x_1^2}{\eta_k} \geq -2k - \frac{2k(n-1)}{\lambda}|x|^2u_{11} + nk|x|^2u_{11},$$

i.e., at p,

$$\frac{4kx_1^2 + 2\eta_k}{n|x|^2 - \frac{2(n-1)}{\lambda}|x|^2} \geq \eta_k u_{11}.$$

Thus if $p \in \{x \in \mathbb{R}^n \,||x| > R_0 + 1\}$, then there holds

$$\max_{x\in\mathbb{R}^n,\gamma\in\mathbb{S}^{n-1}} \eta_k u_{\gamma\gamma} \leq \frac{4k\lambda x_1^2 + 2\lambda\eta_k}{\left(\lambda - \frac{2(n-1)}{n}\right)n|x|^2} \leq \frac{6\lambda}{\left(\lambda - \frac{2(n-1)}{n}\right)n}. \tag{3.9}$$

And if $p \in \{x \in \mathbb{R}^n \,||x| \leq R_0 + 1\}$, then

$$\max_{x\in\mathbb{R}^n,\gamma\in\mathbb{S}^{n-1}} \eta_k u_{\gamma\gamma} \leq \|u\|_{C^2(\bar{B}_{R_0+1})}. \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$\max_{x\in\mathbb{R}^n,\gamma\in\mathbb{S}^{n-1}} \eta_k u_{\gamma\gamma} \leq \frac{6\lambda}{\left(\lambda - \frac{2(n-1)}{n}\right)n} + \|u\|_{C^2(\bar{B}_{R_0+1})}. \tag{3.11}$$

For any fixed $x \in \mathbb{R}^n$ and $\gamma \in \mathbb{S}^{n-1}$, let k converges to 0, then

$$\frac{3}{4}u_{\gamma\gamma} \le \frac{6\lambda}{(\lambda - \frac{2(n-1)}{n})n} + \|u\|_{C^2(\bar{B}_{R_0+1})}.$$

So we obtain

$$u_{\gamma\gamma} \le \frac{24\lambda}{(3\lambda - \frac{6(n-1)}{n})n} + \frac{4}{3}\|u\|_{C^2(\bar{B}_{R_0+1})}$$

and Lemma 3.2 is established. □

Proof of Theorem 1.5 Introduce the Legendre transformation of u,

$$y_i = \frac{\partial u}{\partial x_i}, \;\; i = 1, 2, \dots, n, \quad u^*(y_1, \dots, y_n) := \sum_{i=1}^n x_i \frac{\partial u}{\partial x_i} - u(x).$$

In terms of $y_1, \dots, y_n, u^*(y_1, \dots, y_n)$, one can easily check that

$$\frac{\partial^2 u^*}{\partial y_i \partial y_j} = \left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]^{-1}.$$

Thus, in view of (3.4),

$$D^2 u^* \ge \frac{1}{C} I.$$

And the PDE (1.9) can be rewritten as

$$\det D^2 u^* = \exp\left\{n\left(-u^* + \frac{1}{2}\sum_{i=1}^n y_i \frac{\partial u^*}{\partial y_i}\right)\right\}.$$

Using Lemma 3.2, we have

$$D^2 u^* \le CI.$$

So

$$\frac{1}{C} I \le D^2 u \le CI.$$

An application of Lemma 3.1 yields the desired result. □

4 Self-shrinking solutions to Lagrangian mean curvature flow in Euclidean space

In this section, first we present the proof of Theorems 1.7. Then, by the Lewy rotation, we obtain Theorem 1.8.

Proof of Theorem 1.7 For $x \in \mathbb{R}^n$, let

$$\eta_k(x) = \begin{cases} 1 & |x| \leqq R_0 \\ -k(|x|^2 - R_0^2) + 1 & |x| \geqq R_0 \end{cases}.$$

Here R_0 and $k < 1$ be two positive constants which we will determine later. Similar to [21], we denote

$$g_{ij} = \delta_{ij} + \sum_{k=1}^n u_{ik} u_{kj}.$$

By (1.16), we have

$$-I \leq D^2u \leq I.$$

We set

$$\phi(x) = \eta_k e^{\alpha \ln \det g},$$

where α is a positive constant which will be determined later. Assume that at the point p, ϕ attains its maximum value. Obviously, at p, $\eta_k(p) > 0$. If

$$p \in \{X \in \mathbb{R}^n \| X| > R_0\},$$

then

$$(\eta_k)_{ij} = -2k\delta_{ij}, \quad (\eta_k)_i = -2kx_i. \tag{4.1}$$

At p, we get

$$D\eta_k + \eta_k \alpha D \ln \det g = 0, \tag{4.2}$$

and

$$\begin{aligned}
0 &\geq g^{ij}\phi_{ij} \\
&= g^{ij}(\eta_k)_{ij}e^{\alpha \ln \det g} + 2g^{ij}(\eta_k)_i(e^{\alpha \ln \det g})_j + g^{ij}\eta_k(e^{\alpha \ln \det g})_{ij} \\
&= e^{\alpha \ln \det g}[g^{ij}(\eta_k)_{ij} + 2g^{ij}(\eta_k)_i(\alpha \ln \det g)_j + \eta_k g^{ij}(\alpha \ln \det g)_{ij} \\
&\quad + \eta_k g^{ij}(\alpha \ln \det g)_i(\alpha \ln \det g)_j].
\end{aligned}$$

We pick a coordinate system satisfying $u_{ij} = u_{ii}\delta_{ij}$ at p. Then, inserting (4.1) and (4.2) to the above inequality, at p, we get

$$0 \geq -2k \sum_i \frac{1}{1+u_{ii}^2} + \eta_k g^{ij}(\alpha \ln \det g)_{ij} - \eta_k g^{ij}(\alpha \ln \det g)_i(\alpha \ln \det g)_j. \tag{4.3}$$

Differentiating (1.15) twice, we have

$$\begin{aligned}
g^{lk}u_{lki} &= -\frac{u_i}{2} + \frac{1}{2} < x, Du_i >, \\
g^{lk}u_{lkij} &= g^{lm}g^{nk}u_{lki}\sum_{s=1}^{n}(u_{msj}u_{sn} + u_{ms}u_{snj}) + \frac{1}{2} < x, Du_{ij} >.
\end{aligned}$$

Similar to Lemma 2.1 in [21], we arrive at, at p,

$$\begin{aligned}
&g^{ij}(\ln \det g)_{ij} \\
&= g^{ij}(g^{ab})_i(g_{ab})_j + g^{ij}g^{ab}(g_{ab})_{ij} \\
&= \sum_{i,a,b=1}^{n} -g^{ii}g^{aa}g^{bb}u_{abi}^2(u_{aa}+u_{bb})^2 + \sum_{a,b=1}^{n} g^{ij}g^{ab}u_{abij}(u_{aa}+u_{bb}) \\
&\quad + \sum_{i,k,a=1}^{n} 2g^{aa}g^{ii}u_{aki}^2
\end{aligned}$$

$$= 2\sum_{a,b,c=1}^{n} g^{aa}g^{bb}g^{cc}u_{abc}^2(1+u_{aa}u_{bb}) + \sum_{a,b,c=1}^{n} g^{aa}u_{aa} <x, Du_{aa}>$$

$$= 2\sum_{a,b,c=1}^{n} g^{aa}g^{bb}g^{cc}u_{abc}^2(1+u_{aa}u_{bb}) + \frac{1}{2} <x, D\ln\det g> . \tag{4.4}$$

Inserting the above equality into (4.3) and combining (4.1) with (4.2), we obtain

$$0 \geq -2kn + \eta_k\alpha\frac{1}{2} <x, D\ln\det g> +2\eta_k\alpha \sum_{a,b,c=1}^{n} g^{aa}g^{bb}g^{cc}u_{abc}^2(1+u_{aa}u_{bb})$$
$$-4\eta_k\alpha^2\sum_{i=1}^{n} g^{ii}\left(\sum_{a=1}^{n} g^{aa}u_{aa}u_{aai}\right)^2$$
$$\geq -\frac{1}{2} <x, D\eta_k> -2kn + 2\eta_k\alpha \sum_{a,b,c=1}^{n} g^{aa}g^{bb}g^{cc}u_{abc}^2(1+u_{aa}u_{bb})$$
$$-4n^2\eta_k\alpha^2 \sum_{a,b=1}^{n} g^{bb}g^{aa}g^{aa}u_{aa}^2u_{aab}^2$$
$$\geq k(|x|^2 - 2n) + 2\eta_k\alpha(1-2n^2\alpha)\sum_{a,b=1}^{n} g^{bb}g^{aa}g^{aa}u_{aa}^2u_{aab}^2.$$

If we take

$$R_0 > \sqrt{2n}, \text{ and } \alpha < \frac{1}{2n^2}, \tag{4.5}$$

we have a contradiction.

Assume the function $\ln\det g$ is not constant in $\mathbb{R}^n$. Then there is a ball B_{R_0} centered at 0 with radius R_0 satisfying (4.5), such that the function $\ln\det g$ is not a constant in B_{R_0}. Suppose that $\ln\det g$ attains its maximum value in B_{R_0}. Applying strong maximum principle to (4.4), we obtain $\ln\det g$ is a constant. This is a contradiction. Hence $\ln\det g$ attains its maximum value only on the boundary ∂B_{R_0}. Similarly, in $B_{\sqrt{R_0^2+1}}$, $\ln\det g$ also attains its maximum value only on the boundary $\partial B_{\sqrt{R_0^2+1}}$. We assume that the points p_1 and p_2 be maximum value points with respect to ∂B_{R_0} and $\partial B_{\sqrt{R_0^2+1}}$, namely,

$$\max_{\overline{B}_{R_0}} \ln\det g = \ln\det g(p_1),$$
$$\max_{\overline{B}_{(R_0^2+1)^{1/2}}} \ln\det g = \ln\det g(p_2).$$

Then

$$\ln\det g(p_1) \leq \ln\det g(p_2).$$

But the equality can not hold. In fact, if the equality holds, then the function $\ln\det g$ achieves its maximum value in the interior of the domain $B_{\sqrt{R_0^2+1}}$. This is a contradiction. So we can choose k sufficiently small such that

$$\phi(P_1) = (\det g)^\alpha(p_1) < (1-k)(\det g)^\alpha(p_2) = \phi(p_2).$$

This means that, for fixed u, we can choose suitable k such that the maximum value of ϕ only occurs in the set

$$\{X \in \mathbb{R}^n \,||X| > R_0\}.$$

But we have proved that it is impossible. Thus the discussion implies the function $\ln \det g$ is a constant. So by (4.4), we have

$$g^{aa}g^{bb}g^{cc}u_{abc}^2(1+u_{aa}u_{bb}) = 0.$$

Now we can use the same argument of Proposition 2.1 in [21]. We obtain

$$u_{abc}^2(1+u_{aa}u_{bb}) = u_{abc}^2(1+u_{bb}u_{cc}) = u_{abc}^2(1+u_{cc}u_{aa}).$$

Observe that one of $u_{aa}u_{bb}$, $u_{bb}u_{cc}$ and $u_{cc}u_{aa}$ must be nonnegative, we get, at every point,

$$u_{abc} = 0.$$

Consequently, u is a quadric polynomial. □

Proposition 4.1 *Assume that u be a smooth solution to* (1.15) *and* D^2u *satisfies*

$$D^2u \geq 0. \tag{4.6}$$

Set Lewy rotation [21],

$$\begin{cases} \bar{x} = \dfrac{x + Du(x)}{\sqrt{2}} \\ D\bar{u}(\bar{x}) = \dfrac{-x + Du(x)}{\sqrt{2}} \end{cases}. \tag{4.7}$$

Then $\bar{u}$ *is a smooth solution to* (1.15) *and* $D^2\bar{u}$ *satisfies* (1.16).

Proof Suppose that

$$F = \arctan(\lambda_1) + \cdots + \arctan(\lambda_n), \quad G = -u + \frac{1}{2} <x, Du>,$$
$$x = (x_1, \ldots, x_n), \quad \frac{\partial F}{\partial x} = \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right).$$

Then

$$\frac{\partial F}{\partial \bar{x}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \bar{x}}. \tag{4.8}$$

By (4.7), we get

$$\begin{cases} x = \dfrac{\bar{x} - D\bar{u}(\bar{x})}{\sqrt{2}} \\ Du(x) = \dfrac{\bar{x} + D\bar{u}(\bar{x})}{\sqrt{2}} \end{cases}. \tag{4.9}$$

So

$$\frac{\partial x}{\partial \bar{x}} = \frac{I - D^2\bar{u}(\bar{x})}{\sqrt{2}}. \tag{4.10}$$

By (1.15) and (4.9), we have

$$\frac{\partial F}{\partial x} = \frac{\partial G}{\partial x} = -\frac{1}{2}Du + \frac{1}{2}xD^2u$$
$$= -\frac{\bar{x} + D\bar{u}(\bar{x})}{2\sqrt{2}} + \frac{1}{4}(\bar{x} - D\bar{u}(\bar{x}))(I + D^2\bar{u}(\bar{x}))\frac{\partial \bar{x}}{\partial x}. \tag{4.11}$$

Using (4.8), (4.10), (4.11) and

$$\frac{\partial \bar{x}}{\partial x}\frac{\partial x}{\partial \bar{x}} = I,$$

we obtain

$$\frac{\partial F}{\partial \bar{x}} = -\frac{1}{4}(\bar{x} + D\bar{u}(\bar{x}))(I - D^2\bar{u}(\bar{x})) + \frac{1}{4}(\bar{x} - D\bar{u}(\bar{x}))(I + D^2\bar{u}(\bar{x}))$$
$$= -\frac{1}{2}D\bar{u} + \frac{1}{2}\bar{x}D^2\bar{u}. \tag{4.12}$$

From (4.7), we see that

$$D^2\bar{u} = (I + D^2u)^{-1}(-I + D^2u). \tag{4.13}$$

Hence,

$$\arctan(\lambda_1) + \cdots + \arctan(\lambda_n) = \frac{n\pi}{4} + (\arctan(\bar{\lambda}_1) + \cdots + \arctan(\bar{\lambda}_n)). \tag{4.14}$$

Set

$$\bar{F} = \arctan(\bar{\lambda}_1) + \cdots + \arctan(\bar{\lambda}_n), \quad \bar{G} = -\bar{u} + \frac{1}{2} < \bar{x}, D\bar{u} >.$$

Combining (4.12) with (4.14), we obtain

$$\frac{\partial \bar{F}}{\partial \bar{x}} = -\frac{1}{2}D\bar{u} + \frac{1}{2}\bar{x}D^2\bar{u} = \frac{\partial \bar{G}}{\partial \bar{x}}.$$

By (4.13),

$$-I \leq D^2\bar{u} \leq I.$$

This completes the proof of Proposition 4.1. □

Proof of Theorem 1.8 Case 1. Assume that u be a smooth convex solution to (1.15). By Lewy rotation (4.7) in Proposition 4.1, $\bar{u}$ is a smooth solution to (1.15) and $D^2\bar{u}$ satisfies (1.16). Using Theorem 1.7, $D^2\bar{u}$ must be a constant matrix. From (4.13), we deduce that u is a quadric polynomial.

Case 2. Assume that u is a smooth concave solution to (1.15). Set $u^* = -u$, then u^* must be a quadric polynomial by case 1.

So we have the desired results. □

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