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Calculus of Variations

Inequalities for eigenvalues of a clamped plate problem

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Abstract In this paper we study eigenvalues of a clamped plate problem on compact domains in complete manifolds. For complete manifolds admitting special functions, we prove universal inequalities for eigenvalues of clamped plate problem independent of the domains of Payne–Pólya–Weinberger–Yang type. These manifolds include Hadamard manifolds with Ricci curvature bounded below, a class of warped product manifolds, the product of Euclidean spaces with any complete manifolds and manifolds admitting eigenmaps to a sphere. In the case of warped product manifolds, our result implies a universal inequality on hyperbolic space proved by Cheng–Yang. We also strengthen an inequality for eigenvalues of clamped plate problem on submanifolds in a Euclidean space obtained recently by Cheng, Ichikawa and Mametsuka.

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1 Introduction

Let M be an n-dimensional complete Riemannian manifold and let Ω be a bounded connected domain in M. Let Δ be the Laplace operator acting on functions on M. The so called *Dirichlet eigenvalue problem* or the *fixed membrane problem* is stated as:

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$$\begin{cases} \Delta u = -\lambda u \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.1}$$

Let

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots,$$

denote the successive eigenvalues of (1.1). Here each eigenvalue is repeated according to its multiplicity. The study of the spectrum of Δ is an important topic and many works have been done in this area during the past years (see, e.g., [1,7,31] and the references therein). When M is a Euclidean space $\mathbf{R}^n$, namely, when Ω is a bounded domain in $\mathbf{R}^n$, Payne et al. [29] proved that

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{k=1}^{k} \lambda_i, \quad k = 1, 2, \ldots. \tag{1.2}$$

In 1980, Hile and Protter [25] generalized (1.2) to

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{kn}{4}, \quad \text{for } k = 1, 2, \ldots. \tag{1.3}$$

In 1991, Yang [36] proved the following much stronger inequality (cf. [10]):

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_{k+1} - \left(1 + \frac{4}{n}\right) \lambda_i \right) \leq 0, \quad \text{for } k = 1, 2, \ldots. \tag{1.4}$$

which is called Yang's first inequality (see [1,2]). According to the inequality, one has

$$\lambda_{k+1} \leq \frac{1}{k} \left(1 + \frac{4}{n}\right) \sum_{i=1}^{k} \lambda_i, \tag{1.5}$$

which is called Yang's second inequality.

For the Dirichlet eigenvalue problem on a complete Riemannian manifold isometrically immersed in a Euclidean space, Chen and Cheng [8] and El Soufi et al. [17] have proved, independently,

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_i + \frac{n^2}{4} H_0^2 \right), \tag{1.6}$$

where $H_0 = \max_{x \in \Omega} |\mathbf{H}|$ and $\mathbf{H}$ is the mean curvature vector of M. When M is a unit n-sphere, the above inequality has also been obtained in [10]. When M is an n-dimensional hypersurface in $\mathbf{R}^{n+1}$, Harrell [20] has also proved the above inequality.

The inequalities on the higher eigenvalues of the Laplacian on a connected bounded domain in $\mathbf{R}^n$ obtained by Payne–Pólya–Weinberger, Hile–Protter, Yang have also been extended to some other eigenvalue problems (cf. [1–5,8–17,19–23,26,27,32–35,37], etc). Here let us consider the case of the eigenvalue problem for the *Dirichlet biharmonic operator* or the *clamped plate problem* which describes the characteristic vibrations of a clamped plate. This problem is given by

$$\begin{cases} \Delta^2 u = \Gamma u \quad \text{in } \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial n}\big|_{\partial\Omega} = 0. \end{cases} \tag{1.7}$$

Here Ω is a bounded domain in a complete manifold and Δ^2 denotes the biharmonic operator on M.

When M is a Euclidaen space $\mathbf{R}^n$, Payne et al. [29] proved that the eigenvalues $\{\Gamma_i\}_{i=1}^{\infty}$ of the problem (1.7) satisfy

$$\Gamma_{k+1} - \Gamma_k \leq \frac{8(n+2)}{n^2} \frac{1}{k} \sum_{i=1}^{k} \Gamma_i. \tag{1.8}$$

Chen and Qian [9], Hook [26] proved, independently, the following inequality

$$\frac{n^2 k^2}{8(n+2)} \leq \left(\sum_{i=1}^{k} \Gamma_i^{1/2} \right) \left(\sum_{i=1}^{k} \frac{\Gamma_i^{1/2}}{\Gamma_{k+1} - \Gamma_i} \right). \tag{1.9}$$

In a survey paper on recent developments on eigenvalue problems, Ashbaugh [1] asked if one can obtain universal inequality for the eigenvalues of the clamped plate problem (1.6) which is similar to Yang's universal inequality (1.4) for the eigenvalues of the fixed membrane problem (1.1). This problem has been solved by Cheng and Yang in [13]. Namely, they proved that

$$\Gamma_{k+1} - \frac{1}{k} \sum_{i=1}^{k} \Gamma_i \leq \left(\frac{8(n+2)}{n^2} \right)^{1/2} \frac{1}{k} \sum_{i=1}^{k} \left(\Gamma_i (\Gamma_{k+1} - \Gamma_i) \right)^{1/2}. \tag{1.10}$$

When M is an n-dimensional unit sphere or an n-dimensional minimal submanifold in a Euclidean space, universal inequalities for eigenvalues of the clamped plate problems have been proved in [34].

When M is a hyperbolic space $\mathbf{H}^n(-1)$, Cheng and Yang [14] have proved the following remarkable inequality

$$\sum_{i,j=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \leq 24 \sum_{i,j=1}^{k} (\Gamma_{k+1} - \Gamma_i) \left\{ \Gamma_i^{1/2} - \frac{(n-1)^2}{4} \right\} \left\{ \Gamma_j^{1/2} - \frac{(n-1)^2}{6} \right\}. \tag{1.11}$$

More recently, Cheng et al. [16] proved that if M is an n-dimensional submanifold isometrically immersed in a Euclidean space with mean curvature vector $\mathbf{H}$, then

$$\sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i)^2 \leq \frac{1}{n^2} \sum_{i=1}^{k} (\Gamma_{k+1} - \Gamma_i) \left(n^2 H_0^2 + (2n+4)\Gamma_i^{1/2} \right) \left(n^2 H_0^2 + 4\Gamma_i^{1/2} \right), \tag{1.12}$$

where $H_0 = \sup_\Omega |\mathbf{H}|$. Since any complete Riemannian manifold can be isometrically immersed in some Euclidean space (cf. [28]), the inequality (1.12) shows that for any bounded connected domain Ω in an n-dimensional complete Riemannian manifold M, there is an upper bound for the $(k+1)$-th eigenvalue of the clamped problem in terms of the first k eigenvalues and a constant which depends on n, M and Ω.

In this paper we study eigenvalues of a clamped plate problem on compact domains in complete Riemannian manifolds. When the manifolds are isometrically immersed in a Euclidean space, we will strengthen the inequality (1.12) by Cheng, Ichikawa and Mametsuka. For complete manifolds admitting special functions, we prove universal inequalities independent of the domains of Payne–Pólya–Weinberger–Yang type. These manifolds include the

Hadamard manifolds with Ricci curvature bounded below, a class of warped product manifolds containing the hyperbolic space, the product of Euclidean spaces with any complete manifolds and manifolds admitting eigenmaps to a sphere. In the case of the warped product manifolds, our inequality generalizes Cheng–Yang's inequality (1.11). We can state the main results in this paper as follows:

Theorem 1.1 *Let M be an n-dimensional complete Riemannian manifold and let Ω be a bounded domain with smooth boundary $\partial\Omega$ in M. Denote by ν the outward unit normal of $\partial\Omega$. Let Δ be the Laplacian of M and denote by Γ_i the i-th eigenvalue of the problem:*

$$\begin{cases} \Delta^2 u = \Gamma u \quad \text{in } \ \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}\big|_{\partial\Omega} = 0. \end{cases} \tag{1.13}$$

(i) *If M is isometrically immersed in $\mathbf{R}^m$ with mean curvature vector $\mathbf{H}$, then*

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le \frac{1}{n}\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\left(n^2H_0^2+(2n+4)\Gamma_i^{1/2}\right)\right\}^{1/2} \times\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(n^2H_0^2+4\Gamma_i^{1/2}\right)\right\}^{1/2}, \tag{1.14}$$

where $H_0=\sup_\Omega|\mathbf{H}|$.

(ii) *If there exists a function $\phi:\Omega\to\mathbf{R}$ and a constant A_0 such that*

$$|\nabla\phi|=1,\ |\Delta\phi|\le A_0, \quad \text{on } \Omega, \tag{1.15}$$

then

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le \left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\left(A_0^2+4A_0\Gamma_i^{1/4}+6\Gamma_i^{1/2}\right)\right\}^{1/2} \times\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(2\Gamma_i^{1/4}+A_0\right)^2\right\}^{1/2}. \tag{1.16}$$

(iii) *If there exists a function $\psi:\Omega\to\mathbf{R}$ and a constant B_0 such that*

$$|\nabla\psi|=1,\ \Delta\psi=B_0, \quad \text{on } \Omega, \tag{1.17}$$

then

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le \left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\left(6\Gamma_i^{1/2}-B_0^2\right)\right\}^{1/2} \times\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(4\Gamma_i^{1/2}-B_0^2\right)\right\}^{1/2}. \tag{1.18}$$

(iv) *If there exist l functions $\phi_p:\Omega\to\mathbf{R}$ such that*

$$\langle\nabla\phi_p,\nabla\phi_q\rangle=\delta_{pq},\ \Delta\phi_p=0, \quad \text{on } \Omega, \quad p,\ q=1,\ldots,l, \tag{1.19}$$

then

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le \frac{2(2(l+2))^{1/2}}{l}\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\Gamma_i^{1/2}\right\}^{1/2} \times\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Gamma_i^{1/2}\right\}^{1/2}. \tag{1.20}$$

(v) *If Ω admits an eigenmap $f=(f_1,f_2,\ldots,f_{m+1}):\Omega\to\mathbf{S}^m$ corresponding to an eigenvalue μ, that is,*

$$\Delta f_\alpha=-\mu f_\alpha,\ \alpha=1,\ldots,m+1,\ \sum_{\alpha=1}^{m+1}f_\alpha^2=1,$$

then

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\left(6\Gamma_i^{1/2}+\mu\right)\right\}^{1/2}\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(4\Gamma_i^{1/2}+\mu\right)\right\}^{1/2}, \tag{1.21}$$

where $\mathbf{S}^m$ is the unit m-sphere.

Remark 1.1 The functions in items (ii)–(v) of Theorem 1.1 are only required to be defined on Ω and the corresponding inequalities for the eigenvalues depend on Ω. In the next section, we will give examples of complete manifolds on which there exist globally defined functions of similar nature. Thus for those manifolds, the inequalities for the eigenvalues of the clamped plate problem as stated in items (ii)–(v) of Theorem 1.1 are independent of the domains and therefore are universal.

2 Proof of the main results

The following lemma which can be also deduced from (2.4) of [34] is necessary for proving our result. For the sake of completeness, we will give its proof.

Lemma 2.1 *Let $\Gamma_i, i=1,\ldots,$ be the i-th eigenvalue of the problem (1.13) and u_i the orthonormal eigenfunction corresponding to Γ_i, that is,*

$$\Delta^2 u_i=\Gamma_i u_i\quad\text{in }\Omega,\quad u_i|_{\partial\Omega}=\left.\frac{\partial u_i}{\partial\nu}\right|_{\partial\Omega}=0,\quad\int_M u_iu_j=\delta_{ij},\quad\forall\ i,\ j=1,2,\ldots. \tag{2.1}$$

Then for any smooth function $h : \Omega \to \mathbf{R}$, *we have*

$$\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\int_\Omega u_i^2|\nabla h|^2$$
$$\leq \delta\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\int_\Omega (u_i^2(\Delta h)^2+4(\langle\nabla h,\nabla u_i\rangle^2+u_i\Delta h\langle\nabla h,\nabla u_i\rangle)-2u_i|\nabla h|^2\Delta u_i)$$
$$+\sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\int_\Omega\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right)^2, \tag{2.2}$$

where δ *is any positive constant.*

Proof of Lemma 2.1. For $i = 1, \ldots, k$, consider the functions $\phi_i : \Omega \to \mathbf{R}$ given by

$$\phi_i = hu_i - \sum_{j=1}^{k} r_{ij}u_j,$$

where

$$r_{ij} = \int_\Omega hu_iu_j.$$

Since $\phi_i|_{\partial\Omega} = \frac{\partial\phi_i}{\partial\nu}\Big|_{\partial\Omega} = 0$ and

$$\int_\Omega u_j\phi_i = 0, \quad \forall\, i, j = 1, \ldots, k,$$

it follows from the Rayleigh-Ritz inequality that

$$\Gamma_{k+1} \leq \frac{\int_\Omega \phi_i\Delta^2\phi_i}{\int_\Omega \phi_i^2}. \tag{2.3}$$

We have

$$\int_\Omega \phi_i\Delta^2\phi_i$$
$$= \int_\Omega \phi_i\left(\Delta^2(hu_i) - \sum_{j=1}^{k} r_{ij}\Gamma_j u_j\right)$$
$$= \int_\Omega \phi_i\Delta^2(hu_i)$$
$$= \int_\Omega \phi_i(\Delta(u_i\Delta h) + 2\Delta\langle\nabla h, \nabla u_i\rangle + 2\langle\nabla h, \nabla(\Delta u_i)\rangle + \Delta h\Delta u_i + \Gamma_i hu_i)$$
$$= \Gamma_i||\phi_i||^2 + \int_\Omega \phi_i(\Delta(u_i\Delta h) + 2\Delta\langle\nabla h, \nabla u_i\rangle + 2\langle\nabla h, \nabla(\Delta u_i)\rangle + \Delta h\Delta u_i)$$

$$= \Gamma_i ||\phi_i||^2 + \int_\Omega hu_i(\Delta(u_i\Delta h) + 2\Delta\langle\nabla h, \nabla u_i\rangle + 2\langle\nabla h, \nabla(\Delta u_i)\rangle + \Delta h\Delta u_i)$$
$$- \sum_{j=1}^{k} r_{ij}s_{ij}, \tag{2.4}$$

where $||\phi_i||^2 = \int_\Omega \phi_i^2$ and

$$s_{ij} = \int_\Omega u_j(\Delta(u_i\Delta h) + 2\Delta\langle\nabla h, \nabla u_i\rangle + 2\langle\nabla h, \nabla(\Delta u_i)\rangle + \Delta h\Delta u_i). \tag{2.5}$$

Multiplying the equation $\Delta^2 u_i = \Gamma_i u_i$ by hu_j, we have

$$hu_j\Delta^2 u_i = \Gamma_i hu_i u_j. \tag{2.6}$$

Changing the roles of i and j, one gets

$$hu_i\Delta^2 u_j = \Gamma_j hu_i u_j. \tag{2.7}$$

Subtracting (2.6) from (2.7) and integrating the resulted equation on Ω, we get by using the divergence theorem that

$$\begin{aligned}
(\Gamma_j - \Gamma_i)r_{ij} &= \int_\Omega (hu_i\Delta^2 u_j - hu_j\Delta^2 u_i)\\
&= \int_\Omega (\Delta(hu_i)\Delta u_j - \Delta(hu_j)\Delta u_i)\\
&= \int_\Omega ((u_i\Delta h + 2\langle\nabla h, \nabla u_i\rangle)\Delta u_j - ((u_j\Delta h + 2\langle\nabla h, \nabla u_j\rangle)\Delta u_i)\\
&= \int_\Omega (u_j(\Delta(u_i\Delta h) + 2\Delta\langle\nabla h, \nabla u_i\rangle) - u_j\Delta h\Delta u_i + 2u_j \mathrm{div}(\Delta u_i\nabla h))\\
&= \int_\Omega u_j(\Delta(u_i\Delta h) + 2\Delta\langle\nabla h, \nabla u_i\rangle + \Delta h\Delta u_i + 2\langle\nabla\Delta u_i, \nabla h\rangle)\\
&= s_{ij},
\end{aligned} \tag{2.8}$$

where for a vector field Z on Ω, div$Z$ denotes the divergence of Z. Observe that

$$\begin{aligned}
&\int_\Omega hu_i(\Delta(u_i\Delta h) + 2\Delta\langle\nabla h, \nabla u_i\rangle + 2\langle\nabla h, \nabla(\Delta u_i)\rangle + \Delta h\Delta u_i)\\
&= \int_\Omega (\Delta(hu_i)u_i\Delta h + 2\Delta(hu_i)\langle\nabla h, \nabla u_i\rangle - 2\Delta u_i \mathrm{div}(hu_i\nabla h) + hu_i\Delta h\Delta u_i)\\
&= \int_\Omega (u_i^2(\Delta h)^2 + 4(\langle\nabla h, \nabla u_i\rangle^2 + u_i\Delta h\langle\nabla h, \nabla u_i\rangle) - 2u_i|\nabla h|^2\Delta u_i).
\end{aligned} \tag{2.9}$$

It follows from (2.3), (2.4), (2.8) and (2.9) that

$$\begin{aligned}(\Gamma_{k+1}-\Gamma_i)\|\phi_i\|^2 &\le \int_\Omega \left(u_i^2(\Delta h)^2 + 4\left(\langle\nabla h, \nabla u_i\rangle^2 + u_i\Delta h\langle\nabla h, \nabla u_i\rangle\right) - 2u_i|\nabla h|^2\Delta u_i\right)\\ &\quad + \sum_{j=1}^{k}(\Gamma_i-\Gamma_j)r_{ij}^2. \end{aligned} \tag{2.10}$$

Set

$$t_{ij} = \int_\Omega u_j\left(\langle\nabla h, \nabla u_i\rangle + \frac{u_i\Delta h}{2}\right); \tag{2.11}$$

then $t_{ij}+t_{ji}=0$ and

$$\begin{aligned}\int_\Omega (-2)\phi_i\left(\langle\nabla h, \nabla u_i\rangle + \frac{u_i\Delta h}{2}\right) &= \int_\Omega (-2hu_i\langle\nabla h, \nabla u_i\rangle - u_i^2 h\Delta h) + 2\sum_{j=1}^{k} r_{ij}t_{ij}\\ &= \int_\Omega u_i^2|\nabla h|^2 + 2\sum_{j=1}^{k} r_{ij}t_{ij}. \end{aligned} \tag{2.12}$$

Multiplying (2.12) by $(\Gamma_{k+1}-\Gamma_i)^2$ and using the Schwarz inequality and (2.10), we get

$$\begin{aligned}&(\Gamma_{k+1}-\Gamma_i)^2\left(\int_\Omega u_i^2|\nabla h|^2 + 2\sum_{j=1}^{k} r_{ij}t_{ij}\right)\\ &= (\Gamma_{k+1}-\Gamma_i)^2\int_M (-2)\phi_i\left(\langle\nabla h, \nabla u_i\rangle + \frac{u_i\Delta h}{2}\right)\\ &= (\Gamma_{k+1}-\Gamma_i)^2\int_M (-2)\phi_i\left(\left(\langle\nabla h, \nabla u_i\rangle + \frac{u_i\Delta h}{2}\right) - \sum_{j=1}^{k} t_{ij}u_j\right)\\ &\le \delta(\Gamma_{k+1}-\Gamma_i)^3\|\phi_i\|^2 + \frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\int_M \left|\langle\nabla h, \nabla u_i\rangle + \frac{u_i\Delta h}{2} - \sum_{j=1}^{k} t_{ij}u_j\right|^2\\ &= \delta(\Gamma_{k+1}-\Gamma_i)^3\|\phi_i\|^2 + \frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\left(\int_\Omega \left(\langle\nabla h, \nabla u_i\rangle + \frac{u_i\Delta h}{2}\right)^2 - \sum_{j=1}^{k} t_{ij}^2\right)\\ &\le \delta(\Gamma_{k+1}-\Gamma_i)^2\left(\int_\Omega \left(u_i^2(\Delta h)^2 + 4\left(\langle\nabla h, \nabla u_i\rangle^2 + u_i\Delta h\langle\nabla h, \nabla u_i\rangle\right) - 2u_i|\nabla h|^2\Delta u_i\right)\right.\\ &\quad \left.+ \sum_{j=1}^{k}(\Gamma_i-\Gamma_j)r_{ij}^2\right) + \frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\left(\int_\Omega \left(\langle\nabla h, \nabla u_i\rangle + \frac{u_i\Delta h}{2}\right)^2 - \sum_{j=1}^{k} t_{ij}^2\right). \end{aligned} \tag{2.13}$$

Summing over i from 1 to k for (2.13) and noticing $r_{ij} = r_{ji},\ t_{ij} = -t_{ji}$, we get

$$\begin{aligned}
&\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\int_\Omega u_i^2|\nabla h|^2 - 2\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)(\Gamma_i-\Gamma_j)r_{ij}t_{ij}\\
&\le \sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\delta\int_\Omega \left(u_i^2(\Delta h)^2+4(\langle\nabla h,\nabla u_i\rangle^2+u_i\Delta h\langle\nabla h,\nabla u_i\rangle)-2u_i|\nabla h|^2\Delta u_i\right)\\
&\quad+\sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\int_\Omega\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\Delta h}{2}\right)^2\\
&\quad-\sum_{i,j=1}^{k}(\Gamma_{k+1}-\Gamma_i)\delta(\Gamma_i-\Gamma_j)^2r_{ij}^2-\sum_{i,j=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}t_{ij}^2,
\end{aligned}\tag{2.14}$$

which implies (2.2). □

Proof of Theorem 1.1. Let $\{u_i\}_{i=1}^{\infty}$ be the orthonormal eigenfunctions corresponding to the eigenvalues $\{\Gamma_i\}_{i=1}^{\infty}$ of the problem (1.13).

(i) Let $x_\alpha, \alpha = 1, \dots, m$, be the standard coordinate functions of $\mathbf{R}^m$. Taking $h = x_\alpha$ in (2.2) and summing over α, we have

$$\begin{aligned}
&\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\sum_{\alpha=1}^{m}\int_\Omega u_i^2|\nabla x_\alpha|^2\\
&\le \delta\sum_{i=1}^{k+1}(\Gamma_{k+1}-\Gamma_i)^2\sum_{\alpha=1}^{m}\int_\Omega \Big(u_i^2\,(\Delta x_\alpha)^2+4\left(\langle\nabla x_\alpha,\nabla u_i\rangle^2\right.\\
&\quad\left.+\,u_i\Delta x_\alpha\langle\nabla x_\alpha,\nabla u_i\rangle\right)-2u_i|\nabla x_\alpha|^2\Delta u_i\Big)\\
&\quad+\sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\sum_{\alpha=1}^{m}\int_\Omega\left(\langle\nabla x_\alpha,\nabla u_i\rangle+\frac{u_i\Delta x_\alpha}{2}\right)^2,
\end{aligned}\tag{2.15}$$

Since M is isometrically immersed in $\mathbf{R}^m$, we have

$$\sum_{\alpha=1}^{m}|\nabla x_\alpha|^2 = n$$

which implies that

$$\sum_{\alpha=1}^{m}\int_\Omega u_i^2|\nabla x_\alpha|^2 = n \tag{2.16}$$

Also, we have

$$\Delta(x_1,\dots,x_m)\equiv(\Delta x_1,\dots,\Delta x_m)=n\mathbf{H}, \tag{2.17}$$

$$\sum_{\alpha=1}^{m}\langle\nabla x_\alpha,\nabla u_i\rangle^2=\sum_{\alpha=1}^{m}(\nabla u_i(x_\alpha))^2=|\nabla u_i|^2 \tag{2.18}$$

and

$$\sum_{\alpha=1}^{m} \Delta x_\alpha \langle \nabla x_\alpha, \nabla u_i \rangle = \sum_{\alpha=1}^{m} \Delta x_\alpha \nabla u_i(x_\alpha) = \langle n\mathbf{H}, \nabla u_i \rangle = 0. \tag{2.19}$$

Substituting (2.16)–(2.19) into (2.15), we get

$$\begin{aligned} n\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le \delta \sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 &\int_\Omega (n^2u_i^2|\mathbf{H}|^2 + 4|\nabla u_i|^2 - 2nu_i\Delta u_i) \\ &+ \sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\int_\Omega \left(|\nabla u_i|^2 + \frac{n^2u_i^2|\mathbf{H}|^2}{4}\right) \\ &\le \delta\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2(n^2H_0^2 + (2n+4)\Gamma_i^{1/2}) \\ &+ \sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\left(\Gamma_i^{1/2} + \frac{n^2H_0^2}{4}\right). \end{aligned} \tag{2.20}$$

Here in the last inequality, we have used the fact that $|\mathbf{H}| \le H_0$ and

$$\int_\Omega |\nabla u_i|^2 = -\int_\Omega u_i\Delta u_i \le \left(\int_\Omega u_i^2\right)^{1/2}\left(\int_\Omega (\Delta u_i)^2\right)^{1/2} = \Gamma_i^{1/2}. \tag{2.21}$$

Taking

$$\delta = \left\{\frac{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(\Gamma_i^{1/2} + \frac{n^2H_0^2}{4}\right)}{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2(n^2H_0^2 + (2n+4)\Gamma_i^{1/2})}\right\}^{1/2}$$

in (2.20), one gets (1.14).

(ii) Substituting $h = \phi$ into (2.2) and using (1.15) and the Schwarz inequality, we get

$$\begin{aligned} \sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 &\le \delta\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \\ &\times \int_\Omega \left(u_i^2(\Delta\phi)^2 + 4\left(\langle\nabla\phi, \nabla u_i\rangle^2 + u_i\Delta\phi\langle\nabla\phi,\nabla u_i\rangle\right) - 2u_i\Delta u_i\right) \\ &+ \sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\int_\Omega\left(\langle\nabla\phi,\nabla u_i\rangle + \frac{u_i\Delta\phi}{2}\right)^2 \\ &\le \delta\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\int_\Omega \left(A_0^2u_i^2 + 4\left(|\nabla u_i|^2 + A_0|u_i||\nabla u_i|\right) - 2u_i\Delta u_i\right) \\ &+ \sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\int_\Omega\left(|\nabla u_i|^2 + A_0|u_i||\nabla u_i| + \frac{A_0^2u_i^2}{4}\right). \end{aligned} \tag{2.22}$$

Substituting (2.21) and

$$\int_\Omega |u_i||\nabla u_i| \leq \left(\int_\Omega u_i^2\right)^{1/2} \left(\int_\Omega |\nabla u_i|^2\right)^{1/2} \leq \Gamma_i^{1/4}$$

into (2.22), we get

$$\begin{aligned}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \leq\ & \delta \sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \left(A_0^2 + 4A_0\Gamma_i^{1/4} + 6\Gamma_i^{1/2}\right) \\ & + \sum_{i=1}^{k} \frac{(\Gamma_{k+1}-\Gamma_i)}{\delta} \left(\Gamma_i^{1/4} + \frac{A_0}{2}\right)^2.\end{aligned}$$

Taking

$$\delta = \left\{ \frac{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(\Gamma_i^{1/4}+\frac{A_0}{2}\right)^2}{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\left(A_0^2+4A_0\Gamma_i^{1/4}+6\Gamma_i^{1/2}\right)} \right\}^{1/2},$$

we obtain (2.16).

(iii) Introducing $h = \psi$ into (2.2) and using (1.17), we have

$$\begin{aligned}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \leq\ & \delta \sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \\ & \times \int_\Omega \left(B_0^2 u_i^2 + 4(|\nabla u_i|^2 + B_0 u_i \langle \nabla\psi, \nabla u_i\rangle) - 2u_i\Delta u_i\right) \\ & + \sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\int_\Omega \left(|\nabla u_i|^2 + B_0 u_i\langle\nabla\psi,\nabla u_i\rangle + \frac{B_0^2 u_i^2}{4}\right) \\ \leq\ & \delta\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\left(6\Gamma_i^{1/2} - B_0^2\right) \\ & + \sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\left(\Gamma_i^{1/2} - \frac{B_0^2}{4}\right),\end{aligned} \tag{2.23}$$

where in the last inequality, we have used the fact that

$$\int_\Omega u_i\langle\nabla\psi,\nabla u_i\rangle = -\frac{1}{2}\int_\Omega u_i^2\Delta\psi = -\frac{B_0^2}{2}.$$

Taking

$$\delta = \left\{ \frac{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(\Gamma_i^{1/2}-\frac{B_0^2}{4}\right)}{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\left(6\Gamma_i^{1/2}-B_0^2\right)} \right\}^{1/2},$$

we obtain (1.18).

(iv) Substituting $h = \phi_p$ into (2.2), summing over p and using (1.19), we get

$$l\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le \delta\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\int_{\Omega}\left(4\sum_{p=1}^{l}\langle\nabla\phi_p,\nabla u_i\rangle^2 - 2lu_i\Delta u_i\right) + \sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\int_{\Omega}\left(\sum_{p=1}^{l}\langle\nabla\phi_p,\nabla u_i\rangle^2\right). \tag{2.24}$$

Since $\{\nabla\phi_p\}_{p=1}^{l}$ is a set of orthonormal vector fields, we have

$$\sum_{p=1}^{l}\langle\nabla\phi_p,\nabla u_i\rangle^2 \le |\nabla u_i|^2. \tag{2.25}$$

Thus, we have

$$\begin{aligned} l\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 &\le \delta\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\int_{\Omega}(4+2l)|\nabla u_i|^2 \\ &\quad + \sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\int_{\Omega}|\nabla u_i|^2 \\ &\le \delta\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2(2l+4)\Gamma_i^{1/2} \\ &\quad + \frac{1}{\delta}\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Gamma_i^{1/2}. \end{aligned} \tag{2.26}$$

Taking

$$\delta = \left\{\frac{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\Gamma_i^{1/2}}{(2l+4)\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\Gamma_i^{1/2}}\right\}^{1/2}$$

in (2.26), one gets (1.21).

(v) Taking the Laplacian of the equation

$$\sum_{\alpha=1}^{m+1} f_\alpha^2 = 1$$

and using the fact that

$$\Delta f_\alpha = -\mu f_\alpha,\ \alpha = 1,\ldots,m+1,$$

we have

$$\sum_{\alpha=1}^{m+1}|\nabla f_\alpha|^2 = \mu. \tag{2.27}$$

It then follows by taking $h = f_\alpha$ in (2.2) and summing over α that

$$\mu\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2 \le \delta\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\int_\Omega\left(\mu^2u_i^2+4\sum_{\alpha=1}^{m+1}\langle\nabla f_\alpha,\nabla u_i\rangle^2-\mu^2u_i\Delta u_i\right)$$
$$+\sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\int_\Omega\left(\sum_{\alpha=1}^{m+1}\langle\nabla f_\alpha,\nabla u_i\rangle^2+\frac{\mu^2u_i^2}{4}\right)$$
$$\le\delta\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\left(\mu^2+6\mu\Gamma_i^{1/2}\right)$$
$$+\sum_{i=1}^{k}\frac{(\Gamma_{k+1}-\Gamma_i)}{\delta}\left(\mu\Gamma_i^{1/2}+\frac{\mu^2}{4}\right). \tag{2.28}$$

We get (1.22) by taking

$$\delta=\left\{\frac{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(\Gamma_i^{1/2}+\frac{\mu}{4}\right)}{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)^2\left(6\Gamma_i^{1/2}+\mu\right)}\right\}^{1/2}.$$

This completes the proof of Theorem 1.1.

□

Remark 2.1 We can obtain (1.12) by using (1.14). In order to see this, let us list the following algebraic lemma which can be proved by using induction.

Lemma 2.2 *Let $\{a_i\}_{i=1}^m$, $\{b_i\}_{i=1}^m$ and $\{c_i\}_{i=1}^m$ be three sequences of non-negative real numbers with $\{a_i\}$ decreasing and $\{b_i\}$ and $\{c_i\}_{i=1}^m$ increasing. Then we have*

$$\left(\sum_{i=1}^{m}a_i^2b_i\right)\left(\sum_{i=1}^{m}a_ic_i\right)\le\left(\sum_{i=1}^{m}a_i^2\right)\left(\sum_{i=1}^{m}a_ib_ic_i\right). \tag{2.29}$$

Since $\{(\Gamma_{k+1}-\Gamma_i)\}_{i=1}^k$ is decreasing and $\{(n^2H_0^2+(2n+4)\Gamma_i^{1/2})\}_{i=1}^k$ and $\{(n^2H_0^2+4\Gamma_i^{1/2})\}_{i=1}^k$ are increasing, we conclude from Lemma 2.2 that

$$\left\{\sum_{i=1}^{k+1}(\Gamma_{k+1}-\Gamma_i)^2\left(n^2H_0^2+(2n+4)\Gamma_i^{1/2}\right)\right\}\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(n^2H_0^2+4\Gamma_i^{1/2}\right)\right\}$$
$$\le\left\{\sum_{i=1}^{k+1}(\Gamma_{k+1}-\Gamma_i)^2\right\}\left\{\sum_{i=1}^{k}(\Gamma_{k+1}-\Gamma_i)\left(n^2H_0^2+(2n+4)\Gamma_i^{1/2}\right)\left(n^2H_0^2+4\Gamma_i^{1/2}\right)\right\}. \tag{2.30}$$

Substituting (2.30) into (1.14), one gets (1.12).

Now we give some examples of manifolds admitting the functions on the whole manifolds as stated in items (ii)–(v) of Theorem 1.1.

Example 2.1 Let M be an n-dimensional Hadamard manifold with Ricci curvature satisfying $\mathrm{Ric}_M\ge-(n-1)c^2$, $c\ge 0$ and let $\gamma:[0,+\infty)\to M$ be a geodesic ray, namely a unit speed geodesic with $d(\gamma(s), \gamma(t)) = t - s$ for any $t > s > 0$. The Busemann function b_γ corresponding to γ defined by

$$b_\gamma(x) = \lim_{t\to+\infty}(d(x, \gamma(t)) - t)$$

satisfies $|\nabla b_\gamma| \equiv 1$ (cf. [6,24]). Also, it follows from Theorem 3.5 in [30] that $|\Delta b_\gamma| \leq (n-1)c$ on M. Thus any Hadamard manifold with Ricci curvature bounded below admits functions satisfying (1.15).

Example 2.2 Let (N, ds_N^2) be a complete Riemannian manifold and define a Riemannian metric on $M = \mathbf{R} \times N$ by

$$ds_M^2 = dt^2 + \eta^2(t)ds_N^2, \tag{2.31}$$

where η is a positive smooth function defined on $\mathbf{R}$ with $\eta(0) = 1$. The manifold (M, ds_M^2) is called a warped product and denoted by $M = \mathbf{R} \times_\eta N$. It is known that M is a complete Riemannian manifold.

Set $\eta = e^{-t}$ and consider the warped product $M = \mathbf{R} \times_{e^{-t}} N$. Define $\psi : M \to \mathbf{R}$ by $\psi(x, t) = t$. Let us calculate $|\nabla\psi|$ and $\Delta\psi$. Assume that $\{\overline{\omega}_2, \ldots, \overline{\omega}_n\}$ be an orthonormal coframe on N with respect to ds_N^2. If we define $\omega_1 = dt$ and $\omega_\alpha = e^{-t}\overline{\omega}_\alpha$ for $2 \leq \alpha \leq n$, then the set $\{\omega_i\}_{i=1}^n$ forms an orthonormal coframe of M with respect to ds_M^2. The connection 1-forms ω_{ij} are defined by

$$d\omega_i = \sum_{j=1}^n \omega_{ij} \wedge \omega_j, \ \ \omega_{ij} + \omega_{ji} = 0.$$

Direct exterior differentiation yields

$$d\omega_1 = 0$$

and

$$\begin{aligned} d\omega_\alpha &= -e^{-t}dt \wedge \overline{\omega}_\alpha + e^{-t}\sum_{\beta=2}^n \overline{\omega}_{\alpha\beta} \wedge \overline{\omega}_\beta \\ &= \omega_\alpha \wedge \omega_1 + \sum_{\beta=2}^n \overline{\omega}_{\alpha\beta} \wedge \omega_\beta, \end{aligned} \tag{2.32}$$

where $\overline{\omega}_{\alpha\beta}$ are the connection 1-forms on N. Hence we conclude that the connection 1-forms of M are given by

$$\omega_{1\alpha} = -\omega_{\alpha 1} = -\omega_\alpha, \ \ \omega_{\alpha\beta} = \overline{\omega}_{\alpha\beta}.$$

For any C^2 function f on M, its gradient and hessian can be calculated by the following formulas

$$df = \sum_{i=1}^n f_i\omega_i, \quad df_i + \sum_{j=1}^n f_j\omega_{ji} = \sum_{j=1}^n f_{ij}\omega_j. \tag{2.33}$$

From $d\psi = \omega_1$, we know that

$$\psi_i = \delta_{i1} \tag{2.34}$$

and so

$$d\psi_i = 0. \tag{2.35}$$

Taking $f = \psi$ in (2.33) and using (2.32), (2.34) and (2.35), we get

$$\psi_{1j} = 0, \ \psi_{\alpha\beta} = -\delta_{\alpha\beta}. \tag{2.36}$$

Hence, we have

$$|\nabla\psi| = 1, \quad \Delta\psi = 1 - n. \tag{2.37}$$

That is, a warped product manifold $M = \mathbf{R} \times_{e^{-t}} N$ admits functions satisfying (1.17).

Let $\mathbf{H}^n$ be the n-dimensional hyperbolic space with constant curvature -1. Using the upper half-space model, $\mathbf{H}^n$ is given by

$$\mathbf{R}^n_+ = \{(x_1, x_2, \dots, x_n)|x_n > 0\} \tag{2.38}$$

with metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2} \tag{2.39}$$

One can check that the map $\Phi : \mathbf{R} \times_{e^{-t}} \mathbf{R}^{n-1}$ given by

$$\Phi(t, x) = (x, e^t)$$

is an isometry. Therefore, $\mathbf{H}^n$ admits a warped product model, $\mathbf{H}^n = \mathbf{R} \times_{e^{-t}} \mathbf{R}^{n-1}$.

Remark 2.2 We can show that the inequality (1.18) implies Cheng–Yang's inequality (1.11) by using the following *Reverse Chebyshev Inequality* (cf. [18]):

Lemma 2.3 (Reverse Chebyshev Inequality). *Suppose $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$ are two real sequences with $\{a_i\}$ increasing and $\{b_i\}$ decreasing. Then we have*

$$\sum_{i=1}^k a_i b_i \le \frac{1}{k}\left(\sum_{i=1}^k a_i\right)\left(\sum_{i=1}^k b_i\right). \tag{2.40}$$

In fact, consider the warped product manifold $M = \mathbf{R} \times_{e^{-t}} N$. Note that when $N = \mathbf{R}^{n-1}$, M is the hyperbolic space $\mathbf{H}^n(-1)$. Since (2.37) holds, we have by taking $B_0 = 1 - n$ in (1.18) that

$$\sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \le \left\{\sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 (6\Gamma_i^{1/2} - (n-1)^2)\right\}^{1/2}$$
$$\times \left\{\sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)\left(4\Gamma_i^{1/2} - (n-1)^2\right)\right\}^{1/2}. \tag{2.41}$$

Since $\{(\Gamma_{k+1} - \Gamma_i)\}_{i=1}^k$ is decreasing and $\{(6\Gamma_i^{1/2} - (n-1)^2)\}_{i=1}^k$ is increasing, it follows from (2.40) that

$$\sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \left(6\Gamma_i^{1/2} - (n-1)^2\right) \le \frac{1}{k}\left\{\sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2\right\}$$
$$\times \left\{\sum_{j=1}^k \left(6\Gamma_j^{1/2} - (n-1)^2\right)\right\}. \tag{2.42}$$

Substituting (2.42) into (2.41) and simplifying, we get (1.11).

Example 2.3 Let N be any complete Riemannian manifold. Let

$$M = \mathbf{R}^l \times N = \{(x_1, x_2, \dots x_l, z) | (x_1, x_2, \dots, x_l) \in \mathbf{R}^l, z \in N\}$$

be the product of $\mathbf{R}^l$ and N endowed with the product metric. Consider the functions $\phi_p : M \to \mathbf{R},\ p = 1, \dots, l$, defined by

$$\phi_p(x_1, x_2, \dots x_l, z) = x_p. \tag{2.43}$$

One can check that the functions $\phi_p,\ p = 1, \dots, l$, satisfy (1.19).

Example 2.4 Any compact homogeneous Riemannian manifold admits eigenmaps to some unit sphere for the first positive eigenvalue of the Laplacian (cf. [27]).

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