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Calculus of Variations

Quasilinear asymptotically periodic Schrödinger equations with critical growth

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Abstract It is established the existence of solutions for a class of asymptotically periodic quasilinear elliptic equations in $\mathbb{R}^N$ with critical growth. Applying a change of variable, the quasilinear equations are reduced to semilinear equations, whose respective associated functionals are well defined in $H^1(\mathbb{R}^N)$ and satisfy the geometric hypotheses of the Mountain Pass Theorem. The Concentration–Compactness Principle and a comparison argument allow to verify that the problem possesses a nontrivial solution.

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1 Introduction

This article is concerned with the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$i\varepsilon\partial_t z = -\varepsilon^2\Delta z + W(x)z - l(x,|z|^2)z - \kappa\varepsilon^2\Delta[\rho(|z|^2)]\rho'(|z|^2)z, \tag{1.1}$$

where $z:\mathbb{R}^N\times\mathbb{R}\to\mathbb{C}$, $W:\mathbb{R}^N\to\mathbb{R}$ is a given potential, $\varepsilon>0$, κ is a real constant and l, ρ are suitable real functions. The semilinear case corresponding to $\kappa=0$ has been studied extensively in recent years (see [1,3,22] and references therein). Quasilinear equations of the form (1.1) have been established in several areas of physics corresponding to various types

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of ρ. We refer the reader to the introduction in [17] and its references for a discussion on the subject.

Here we consider the existence of standing wave solutions for quasilinear Schrödinger equations of form (1.1) with $\rho(s) = s$ and $\kappa = \varepsilon = 1$. Seeking solutions of the type stationary waves, namely, the solutions of the form

$$z(x,t) = \exp(-iFt)u(x), \quad F \in \mathbb{R},$$

we get an equation of elliptic type which has the formal structure:

$$-\Delta u + V(x)u - u\Delta(u^2) = K(x)|u|^{22^*-2}u + g(x,u), \quad u > 0,\ x \in \mathbb{R}^N, \tag{1.2}$$

where $V(x) = W(x) - F$ is the new potential function and $K(x)|u|^{22^*-2}u + g(x,u) = l(x,u^2)u$ is the new nonlinearity.

Our main goal is to establish, under an asymptotic periodicity condition at infinity, the existence of a solution for the critical problem

$$\begin{cases} -\Delta u - u\Delta(u^2) + V(x)u \;=\; K(x)|u|^{22^*-2}u + g(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N),\ u > 0, \end{cases} \tag{1.3}$$

where $V, K : \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Note that $22^* = 4N/(N-2)$ corresponds to the critical exponent for Problem 1.3.

Considering $\mathcal{F}$ the class of functions $h \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$ such that, for every $\varepsilon > 0$, the set $\{x \in \mathbb{R}^N : |h(x)| \geq \varepsilon\}$ has finite Lebesgue measure, we assume V and K satisfy

(V) there exist a constant $a_0 > 0$ and a function $V_0 \in C(\mathbb{R}^N, \mathbb{R})$, 1- periodic in x_i, $1 \leq i \leq N$, such that $V_0 - V \in \mathcal{F}$ and

$$V_0(x) \geq V(x) \geq a_0 > 0, \quad \text{for all } \ x \in \mathbb{R}^N;$$

(K) there exist a function $K_0 \in C(\mathbb{R}^N, \mathbb{R})$, 1-periodic in x_i, $1 \leq i \leq N$, and a point $x_0 \in \mathbb{R}^N$, such that $K - K_0 \in \mathcal{F}$ and

(i) $K(x) \geq K_0(x) > 0, \quad \text{for all } \ x \in \mathbb{R}^N$,
(ii) $K(x) = \|K\|_\infty + O(|x - x_0|^{N-2}), \quad \text{as } \ x \to x_0$.

Considering $G(x,s) = \int_0^s g(x,t)\,dt$, the primitive of g, we also suppose the following hypotheses:

(g_1) $g(x,s) = o(|s|)$, as $s \to 0^+$, uniformly in $\mathbb{R}^N$;
(g_2) there exist constants $a_1, a_2 > 0$ and $4 \leq q_1 < 22^*$ such that

$$|g(x,s)| \leq a_1 + a_2|s|^{q_1-1}, \quad \text{for all } \ (x,s) \in \mathbb{R}^N \times [0,+\infty);$$

(g_3) there exist a constant $2 \leq q_2 < 22^*$ and functions $h_1 \in L^1(\mathbb{R}^N)$, $h_2 \in \mathcal{F}$ such that

$$\frac{1}{4}g(x,s)s - G(x,s) \geq -h_1(x) - h_2(x)s^{q_2}, \quad \text{for all } \ (x,s) \in \mathbb{R}^N \times [0,+\infty).$$

We observe that the conditions (g_1) and (g_2) allow us to employ variational methods to study Problem 1.3 and to verify that the associated functional has a local minimum at the origin. As a matter of fact, we study the functional associated with the modified problem. The condition (g_2) imposes a subcritical growth on g. Under the above hypotheses the associated functional does not satisfy a compactness condition of Palais-Smale type since the term $K(x)|u|^{22^*-2}u$ is critical and the domain is all $\mathbb{R}^N$.

The asymptotic periodicity of g at infinity is given by the following condition.

(g_4) there exist a constant $2 \leq q_3 < 22^*$ and functions $h_3 \in \mathcal{F}$, $g_0 \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^+)$, 1-periodic in x_i, $1 \leq i \leq N$, such that

(i) $G(x, s) \geq G_0(x, s) = \int_0^s g_0(x, t)\,dt, \quad$ for all $(x, s) \in \mathbb{R}^N \times [0, +\infty)$,

(ii) $|g(x, s) - g_0(x, s)| \leq h_3(x)|s|^{q_3-1}, \quad$ for all $(x, s) \in \mathbb{R}^N \times [0, +\infty)$,

(iii) the function $s \to g_0(x, s)/s^3$ is nondecreasing in the variable $s > 0$.

Finally, we also suppose that g satisfies:

(g_5) there exists an open bounded set $\Omega \subset \mathbb{R}^N$, containing x_0 given by $(K) - (ii)$, such that

(i) $\dfrac{G(x, s)}{s^{22^*-1}} \to \infty, \quad$ as $s \to \infty$, uniformly in Ω, if $3 \leq N < 10$,

(ii) $\dfrac{G(x, s)}{s^4} \to \infty, \quad$ as $s \to \infty$, uniformly in Ω, if $N \geq 10$.

We observe that the conditions (g_2) and (g_5) imply that $22^* - 1 \leq q_1 < 22^*$ if $3 \leq N < 10$. Now we may state our main result.

Theorem 1 *Suppose that* (V), (K) *and* $(g_1)-(g_5)$ *are satisfied. Then Problem 1.3 possesses a solution.*

We observe that in the particular case: $V = V_0, K = K_0, g = g_0$, Theorem 1 clearly gives us a solution for the periodic problem. Actually, the condition $(g_4)(iii)$ is not necessary when we look for the existence of a solution for the periodic problem. More specifically, considering the problem

$$\begin{cases} -\Delta u - u\Delta(u^2) + V_0(x)u = K_0(x)|u|^{22^*-2}u + g_0(x, u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \; u > 0, \end{cases} \tag{1.4}$$

under the hypotheses:

(V_0) the function $V_0 \in C(\mathbb{R}^N, \mathbb{R})$ is 1- periodic in x_i, $1 \leq i \leq N$, and there exists a constant $a_0 > 0$ such that

$$V_0(x) \geq a_0 > 0, \quad \text{for all } x \in \mathbb{R}^N;$$

(K_0) the function $K_0 \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x_i, $1 \leq i \leq N$, and there is a point $x_0 \in \mathbb{R}^N$ such that

(i) $K_0(x) > 0, \quad$ for all $x \in \mathbb{R}^N$,

(ii) $K_0(x) = \|K_0\|_\infty + O(|x - x_0|^{N-2}), \quad$ as $x \to x_0$;

and the function g_0 satisfying $(g_1) - (g_3)$ and (g_5), we may state:

Theorem 2 *Suppose that* (V_0) *and* $(g_1) - (g_3)$ *and* (g_5) *are satisfied. Then Problem 1.4 possesses a solution.*

Recent mathematical studies [16–18,21] have focused on the existence of solutions for (1.2) in the subcritical case ($K \equiv 0$) and $g(x, s) = |s|^{p-1}s$, $4 \leq p + 1 < 22^*$, $N \geq 3$. In [17], by a change of variable, the quasilinear problem was reduced to a semilinear one and an Orlicz space framework was used to prove the existence of a positive solution for every positive μ (in front of the nonlinear term) via Mountain Pass Theorem. In [7], Colin and Jeanjean also made use of a change of variable in order to reduce Eq. 1.2, with $K \equiv 0$, to a semilinear equation. By using the Sobolev space $H^1(\mathbb{R}^N)$, they proved the existence of solutions based on classical results given by Berestycki and Lions [3] when $N = 1$ or $N \geq 3$, and Berestycki et al. [4] when $N = 2$. In a recent article [23], the authors generalized the earlier results for the subcritical case by supposing (V), (g_1), (g_2) and $(g_5)(ii)$ (for $N \geq 3$), and a version of (g_3). We should also mention the article [18] where the authors used the Nehari method and considered a more general quasilinear problem including the ones to which the change of variables does not apply.

We notice that the main results in this article provide the existence of solutions for the critical exponent case, mentioned as an open problem in [17]. A problem of type (1.2) was studied by Moameni [20], for $N = 2$, with V and g being two continuous 1-periodic functions and g with critical exponential growth. For $N \geq 3$, Moameni [19] established the existence of a nonnegative solution for the critical exponent case, when the potential function V is radially symmetrical and satisfies some geometric condition other than periodic one. Furthermore, an Orlicz space framework was used.

Let us also mention that during the last stage of preparation of this article we became of aware of [9] which contains a result similar to Theorem 2 under a more restricted hypotheses for the periodic potential V. Moreover in [9] the nonlinear term g is supposed to be a pure power.

The underling idea for proving our main results is motivated by the argument used in [7,17]. We also use a change of variable to reformulate the problem obtaining a semilinear problem which has an associated functional well-defined in the Sobolev space $H^1(\mathbb{R}^N)$ and satisfies the geometric hypotheses of the Mountain Pass Theorem (see [2]). After that, we adapt the argument employed in [15], supposing by contradiction that the only possible solution for the Problem 1.3 is the trivial solution. Considering the functional associated with the modified problem, we use a version of the Mountain Pass Theorem without compactness condition [12] to get a Cerami sequence associated with the minimax level. Next, we use this sequence and a technical result due to Lions (see [8]) to get a nontrivial critical point of the functional associated with the periodic problem. Furthermore, we are able to prove that the value of the functional associated with the Problem 1.3 at this point is less than or equal to the mountain pass minimax level and that this level is attained. Finally, we employ a local version of the Mountain Pass Theorem (Theorem 4) to obtain a nontrivial solution for the Problem 1.3.

The outline of the article is as follows: in Sect. 2 we present the versions of the Mountain Pass Theorem which will be employed and introduce the variational framework associated with the Problem 1.3. In Sect. 3 we present estimates on the minimax level of the functional associated with the modified problem, which ensure that this minimax level is below of a certain critical level. In Sect. 4 we prove Theorems 1 and 2.

Notation. In this paper we make use of the following notations:

$M, C, C_0, C_1, \ldots$ denote (possibly different) positive constants. $\int_{\mathbb{R}^N} f(x)\,dx$ will be represented by $\int_{\mathbb{R}^N} f$. $B_R(p)$ denotes the open ball with the radius R centered at the point $p \in \mathbb{R}^N$; the symbol $\partial B_R(p)$ denotes the boundary of this ball. $C_0^\infty(\mathbb{R}^N)$ denotes the functions infinitely differentiable with compact support in $\mathbb{R}^N$. The symbol $\|u\|_p$ is used for the norm of the space $L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$. $D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$ endowed with the norm $\|\nabla u\|_2$. $|A|$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}^N$.

2 Preliminary results

In this section we state two versions of the Mountain Pass Theorem [2] which are used to in our proofs of Theorems 1 and 2. We also introduce the variational framework associated with the Problem 1.3. Furthermore, we verify the geometric conditions of the Mountain Pass Theorem and we present results concerning the behaviour of the Cerami sequences of the associated functional: we show the boundedness for the Cerami sequences and a proposition which will be essential to guarantee that the solutions that we provide in our proofs of Theorems 1 and 2 are not trivial.

2.1 Versions of the Mountain Pass Theorem

Let E be a real Banach space and $I : E \to \mathbb{R}$ a functional of class C^1. Let K be the set of critical points of I. Given $b \in \mathbb{R}$, we define $I^b = \{u \in E : I(u) \leq b\}$ and $K_b = \{u \in E : u \in K, I(u) = b\}$.

As we observed in the introduction, the functional associated with the Problem 1.3 does not satisfy a condition Palais-Smale type. To overcome this difficulty, we shall use two versions of the Mountain Pass Theorem. Next, we state the first version of this theorem (see also [11,15,25]).

We recall that $I \in C^1(E, \mathbb{R})$ satisfies the Cerami condition on level b, denoted by $(Ce)_b$, if any sequence $(u_n) \subset E$ for which (i) $I(u_n) \to b$ and (ii) $\|I'(u_n)\|_{E'} \cdot (\|u_n\|_E + 1) \to 0$, as $n \to \infty$, possesses a convergent subsequence. I satisfies the Cerami condition, denoted by (Ce), if it satisfies $(Ce)_b$ for every $b \in \mathbb{R}$. We say that $(u_n) \subset E$ is a $(Ce)_b$ sequence if it satisfies $(i) - -(ii)$. We also say that $(u_n) \subset E$ is a (Ce) sequence if it is a $(Ce)_b$ sequence for some $b \in \mathbb{R}$.

Theorem 3 *Let E be a real Banach space and let $I \in C^1(E, \mathbb{R})$. Let S be a closed subset of E which disconnects (archwise) E in distinct connected components E_1 and E_2. Suppose further that $I(0) = 0$ and*

(I_1) $0 \in E_1$ *and there is $\alpha > 0$ such that $I|_S \geq \alpha > 0$,*
(I_2) *there is $e \in E_2$ such that $I(e) \leq 0$.*

Then I possesses a $(Ce)_c$ sequence with $c \geq \alpha > 0$ given by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \tag{2.1}$$

where

$$\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) \in I^0 \cap E_2\}. \tag{2.2}$$

We will also need to establish a local version of Theorem 3, which has been proved in [15] (see also [14]).

Theorem 4 *Let E be a real Banach space. Suppose that $I \in C^1(E, \mathbb{R})$ satisfies $I(0) = 0$, (I_1) and (I_2). If there exists $\gamma_0 \in \Gamma$, Γ defined by (2.2), such that*

$$c = \max_{t \in [0,1]} I(\gamma_0(t)) > 0, \tag{2.3}$$

then I possesses a nontrivial critical point $u \in K_c \cap \gamma_0([0, 1])$.

2.2 The variational framework

We observe that formally the Problem 1.3 is the Euler-Lagrange equation associated with the energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (1 + 2u^2)|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 - \frac{1}{22^*}\int_{\mathbb{R}^N} K(x)|u|^{22^*} - \int_{\mathbb{R}^N} G(x, u).$$

From the variational point of view, the first difficulty associated with the Problem 1.3 is to find an appropriate function space where the functional J is well defined. In order to avoid such difficult, we make use of the change of variable introduced by [17], that is, we consider $v = f^{-1}(u)$, where f is defined by

$$\begin{aligned} f'(t) &= \frac{1}{(1+2f^2(t))^{1/2}} \quad &\text{on}[0,+\infty), \\ f(t) &= -f(-t) \quad &\text{on}(-\infty,0], \end{aligned} \tag{2.4}$$

having the following properties, which have been proved in [7] and [11].

Lemma 1 *The function f satisfies the following properties:*

(1) *f is uniquely defined, C^∞ and invertible;*
(2) *$|f'(t)| \le 1$ for all $t \in \mathbb{R}$;*
(3) *$|f(t)| \le |t|$ for all $t \in \mathbb{R}$;*
(4) *$f(t)/t \to 1$ as $t \to 0$;*
(5) *$f(t)/\sqrt{t} \to 2^{1/4}$ as $t \to +\infty$;*
(6) *$f(t)/2 \le tf'(t) \le f(t)$ for all $t \ge 0$;*
(7) *$|f(t)| \le 2^{1/4}|t|^{1/2}$ for all $t \in \mathbb{R}$;*
(8) *$f^2(t)/2 \le f(t)f'(t)t \le f^2(t)$ for all $t \in \mathbb{R}$;*
(9) *There exists a positive constant C such that*

$$|f(t)| \ge \begin{cases} C|t|, & |t| \le 1, \\ C|t|^{1/2}, & |t| \ge 1; \end{cases}$$

(10) *$|f(t)f'(t)| \le 1/\sqrt{2}$ for all $t \in \mathbb{R}$.*

Remark 1 Property (5) can also be obtained from the properties (3) and (7) (see Remark 2 in Sect. 3).

As a consequence of Lemma 1, we have

Corollary 1 (i) *The function $f'(t)f(t)t^{-1}$ is decreasing for all $t > 0$.*
(ii) *The function $f^3(t)f'(t)t^{-1}$ is increasing for all $t > 0$.*
(iii) *The function $f^{22^*-1}(t)f'(t)t^{-1}$ is increasing for all $t > 0$.*

Proof The items (i), (ii) have been proved in [10]. The proof of the item (iii) is similar to the proof of (ii). □

After the change of variables $v = f^{-1}(u)$, from J, we obtain the following functional

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(v) - \frac{1}{22^*}\int_{\mathbb{R}^N} K(x)|f(v)|^{22^*} - \int_{\mathbb{R}^N} G(x, f(v)), \tag{2.5}$$

which is well defined in $H^1(\mathbb{R}^N)$ and belongs to C^1 under the hypotheses (V), (K), (g_1) and (g_2). Moreover, the critical points of I are the weak solutions of the problem

$$\begin{cases} -\Delta v + V(x)f'(v)f(v) = K(x)|f(v)|^{22^*-2}f(v)f'(v) + g(x, f(v))f'(v), \\ v \in H^1(\mathbb{R}^N), \ v > 0, \end{cases} \tag{2.6}$$

that is,

$$\langle I'(v), w\rangle = \int_{\mathbb{R}^N} \nabla v \nabla w + \int_{\mathbb{R}^N} V(x) f'(v) f(v) w$$
$$- \int_{\mathbb{R}^N} K(x)|f(v)|^{22^*-2} f(v) f'(v) w - \int_{\mathbb{R}^N} g(x, f(v)) f'(v) w,$$

for all $v, w \in H^1(\mathbb{R}^N)$. We observe (see [7] or [11]) that if $v \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ is a critical point of the functional I, then the function $u = f(v)$ is a classical solution of (1.3). We also observe that, in order to obtain a nonnegative solution for (2.6), we set $g(x, s) = 0$ for all $x \in \mathbb{R}^N$, $s < 0$. Indeed, from (g_4), we have $I(|v|) \leq I(v)$. Then, by a result due to Brezis and Nirenberg (cf. [5, Theorem 10]), we conclude that $v \geq 0$. Consequently $u = f(v)$ is a nonnegative solution for the Problem 1.3.

In a similar fashion, associated with the periodic problem, we have the functional $I_0 \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$, defined by

$$I_0(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_0(x) f^2(v) - \frac{1}{22^*}\int_{\mathbb{R}^N} K_0(x)|f(v)|^{22^*} - \int_{\mathbb{R}^N} G_0(x, f(v)), \tag{2.7}$$

with $g_0(x, s) = 0$ for all $x \in \mathbb{R}^N$, $s < 0$.

Here, we consider the space $H^1(\mathbb{R}^N)$ endowed with one of the following norms

$$\|v\| = \left(\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x) v^2)\right)^{1/2},$$
$$\|v\|_0 = \left(\int_{\mathbb{R}^N} (|\nabla v|^2 + V_0(x) v^2)\right)^{1/2}.$$

Note that, in view of (V), the above norms are both equivalent to the standard norm on $H^1(\mathbb{R}^N)$. We also observe that, given $\delta > 0$, in view of (g_1) and (g_2) there is a constant $C_\delta > 0$ such that

$$|g(x, s)| \leq \delta|s| + C_\delta |s|^{q_1 - 1}, \quad \text{for all } (x, s) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.8}$$
$$|G(x, s)| \leq \frac{\delta}{2}|s|^2 + \frac{C_\delta}{q_1}|s|^{q_1}, \quad \text{for all } (x, s) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.9}$$

2.3 Mountain pass geometry

Next lemma shows that the (modified) functional associated with the Problem 1.3 satisfies the geometric properties of the Mountain Pass Theorem.

Lemma 2 *Suppose that (V), (K), (g_1) and (g_2) are satisfied. Then the functional I, defined by (2.5), satisfies the conditions $I(0) = 0$, (I_1) and (I_2) of Theorem 3.*

Proof First note that $I(0)=0$. Now, for every $\rho>0$, define

$$S_\rho := \left\{ v \in H^1(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} |\nabla v|^2 + \int\limits_{\mathbb{R}^N} V(x) f^2(v) = \rho^2 \right\}.$$

Since $\Psi : H^1(\mathbb{R}^N) \to \mathbb{R}$, given by

$$\Psi(v) = \int\limits_{\mathbb{R}^N} |\nabla v|^2 + \int\limits_{\mathbb{R}^N} V(x) f^2(v),$$

is continuous, S_ρ is a closed subset which disconnects the space $H^1(\mathbb{R}^N)$. Taking $0<\lambda<1$ such that $q_1/2 = \lambda + (1-\lambda)2^*$, by condition (V), relation (2.9), Hölder's inequality, Lemma 1-(7) and the Sobolev Embedding Theorem, we have

$$\begin{aligned}
\int\limits_{\mathbb{R}^N} G(x, f(v)) &\le \frac{\delta}{2} \int\limits_{\mathbb{R}^N} f^2(v) + \frac{C_\delta}{q_1} \int\limits_{\mathbb{R}^N} |f^2(v)|^{q_1/2} \\
&\le \frac{\delta}{2a_0}\rho^2 + \frac{C_\delta}{q_1} \left(\int\limits_{\mathbb{R}^N} f^2(v) \right)^{\lambda} \left(\int\limits_{\mathbb{R}^N} (f^2(v))^{2^*} \right)^{1-\lambda} \\
&\le \frac{\delta}{2a_0}\rho^2 + 2^{2^*(1-\lambda)/2} \frac{C_\delta}{q_1} \left(\int\limits_{\mathbb{R}^N} f^2(v) \right)^{\lambda} \left(\int\limits_{\mathbb{R}^N} |v|^{2^*} \right)^{1-\lambda} \\
&\le \frac{\delta}{2a_0}\rho^2 + \frac{2^{2^*(1-\lambda)/2} C_0 C_\delta}{q_1 a_0^\lambda} \rho^{2\lambda} \left(\int\limits_{\mathbb{R}^N} |\nabla v|^2 \right)^{2^*(1-\lambda)/2} \\
&\le \frac{\delta}{2a_0}\rho^2 + \frac{2^{2^*(1-\lambda)/2} C_0 C_\delta}{q_1 a_0^\lambda} \rho^{\lambda 2 + (1-\lambda)2^*},
\end{aligned}$$

for every $v \in S_\rho$ and some constant $C_0>0$. Furthermore, by Lemma 1-(7) and the Sobolev Embedding Theorem,

$$\begin{aligned}
\int\limits_{\mathbb{R}^N} K(x)|f(v)|^{22^*} &\le 2^{2^*/2}\|K\|_\infty \int\limits_{\mathbb{R}^N} |v|^{2^*} \le 2^{2^*/2} C_0 \|K\|_\infty \left(\int\limits_{\mathbb{R}^N} |\nabla v|^2 \right)^{2^*/2} \\
&\le 2^{2^*/2} C_0 \|K\|_\infty \rho^{2^*}.
\end{aligned}$$

Hence, we have

$$I(v) \ge \left(\frac{1}{2} - \frac{\delta}{2a_0} \right) \rho^2 - \frac{2^{2^*/2} C_0 \|K\|_\infty}{22^*} \rho^{2^*} - \frac{2^{2^*(1-\lambda)/2} C_0 C_\delta}{q_1 a_0^\lambda} \rho^{\lambda 2 + (1-\lambda)2^*},$$

for every $v \in S_\rho$. Since $\lambda 2 + (1-\lambda)2^* > 2$, choosing $0<\delta<a_0$, we conclude, for ρ sufficiently small, that

$$c_0 := \inf_{S_\rho} I \ge \alpha > 0.$$

The condition (I_1) is satisfied.

In order to verify the condition (I_2) it suffices to exhibit $\varphi \in H^1(\mathbb{R}^N)$ such that

$$I(t\varphi) \to -\infty, \quad \text{as} \quad t \to +\infty. \tag{2.10}$$

Indeed, consider $\varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^+)$, $\varphi \not\equiv 0$. From the properties (3) and (9) of Lemma 1 and $(g_4)(i)$, we get for every $t > 0$,

$$I(t\varphi) \le \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \varphi|^2 + \frac{t^2}{2} \int_{\mathbb{R}^N} V(x)\varphi^2 - \frac{1}{22^*} \inf_{\mathbb{R}^N} K \int_{\mathbb{R}^N} f^{22^*}(t\varphi)$$

$$\le \frac{t^2}{2}\|\varphi\|^2 - \frac{C^{22^*}}{22^*} t^{2^*} \inf_{\mathbb{R}^N} K \int_{\{t\varphi(x) \ge 1\}} \varphi^{2^*}.$$

(2.10) is proved and the condition (I_2) is satisfied. The proof of the lemma is complete. □

As a consequence of Theorem 3 and Lemma 2, we have

Corollary 2 *Suppose that* (V), (K), (g_1) *and* (g_2) *are satisfied. Then the functional* I *possesses a* $(Ce)_c$ *sequence, with* c *given by* (2.1).

2.4 Behaviour of the Cerami sequences

Here we verify the boundedness of the (Ce) sequences associated with the functional I. Before stating the next lemma, we establish a simple result that will be employed several times in our work. In the following lemma, given $h \in \mathcal{F}$, we set $D_\varepsilon = \{x \in \mathbb{R}^N : |h(x)| \ge \varepsilon\}$ and $D_\varepsilon(R) = \{x \in \mathbb{R}^N : |h(x)| \ge \varepsilon, |x| \ge R\}$.

Lemma 3 *Suppose* $h \in \mathcal{F}$. *Then* $|D_\varepsilon(R)| \to 0$ *as* $R \to \infty$.

Proof Since $h \in \mathcal{F}$, $|D_\varepsilon| < \infty$ for all $\varepsilon > 0$. Lemma is equivalent to the following claim:

$$\lim_{n\to\infty} |D_\varepsilon \cap (\mathbb{R}^N \backslash B_{R_n})| = 0,$$

for every sequence $(R_n) \subset \mathbb{R}$ such that $R_n \to \infty$. Consider the real function $\zeta : \mathbb{R}^N \to \mathbb{R}$ given by $\zeta(x) = \chi_{D_\varepsilon}(x)$, that is,

$$\zeta(x) = \begin{cases} 1 & \text{for } x \in D_\varepsilon \\ 0 & \text{for } x \notin D_\varepsilon \end{cases}$$

Then $\zeta \in L^1(\mathbb{R}^N)$ and $\|\zeta\|_1 = \int_{\mathbb{R}^N} |\zeta| = |D_\varepsilon|$. Moreover, defining the sequence of functions $\zeta_n : \mathbb{R}^N \to \mathbb{R}$ by $\zeta_n(x) = \chi_{D_\varepsilon \cap (\mathbb{R}^N \backslash B_{R_n})}(x)$, it follows that $|\zeta_n(x)| \le |\zeta(x)|$. Since $\zeta_n(x) \to 0$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$, our claim follows from Lebesgue Dominated Convergence Theorem. □

Lemma 4 *Suppose that* (V), (K), $(g_1) - (g_3)$ *are satisfied. Then every Cerami sequence* (v_n) *in* $H^1(\mathbb{R}^N)$ *associated with the functional* I *is bounded.*

Proof First of all we observe that if a sequence $(v_n) \subset H^1(\mathbb{R}^N)$ satisfies

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 + \int_{\mathbb{R}^N} V(x) f^2(v_n) \le M \tag{2.11}$$

for some constant $M > 0$, then it is bounded in $H^1(\mathbb{R}^N)$. For that, we just need to show that $\int_{\mathbb{R}^N} v_n^2$ is bounded. By condition (V), Lemma 1-(9), (2.11) and the Sobolev Imbedding Theorem, we find a constant $C > 0$ such that

$$\int_{\{|v_n(x)|\leq 1\}} v_n^2 \leq \frac{1}{C^2} \int_{\{|v_n(x)|\leq 1\}} f^2(v_n) \leq \frac{1}{C^2 a_0} \int_{\mathbb{R}^N} V(x) f^2(v_n) \leq \frac{M}{a_0 C^2}$$

and

$$\int_{\{|v_n(x)|>1\}} v_n^2 \leq \int_{\{|v_n(x)|>1\}} |v_n|^{2^*} \leq \left(\int_{\mathbb{R}^N} |\nabla v_n|^2 \right)^{2^*/2} \leq M^{2^*/2}.$$

Therefore,

$$\int_{\mathbb{R}^N} v_n^2 = \int_{\{|v_n(x)|\leq 1\}} v_n^2 + \int_{\{|v_n(x)|>1\}} v_n^2 \leq \frac{M}{a_0 C^2} + M^{2^*/2}.$$

The claim is proved.

Now let $(v_n) \subset H^1(\mathbb{R}^N)$ be an arbitrary Cerami sequence for I on level $c \in \mathbb{R}$, that is,

$$I(v_n) = c + o_n(1) \quad \text{and} \quad \|I'(v_n)\|(1 + \|v_n\|) = o_n(1). \tag{2.12}$$

By the first condition in (2.12), (2.9) and the conditions (V) and (K), we have

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(v_n) \\
&= \frac{1}{22^*} \int_{\mathbb{R}^N} K(x) |f(v_n)|^{22^*} + \int_{\mathbb{R}^N} G(x, f(v_n)) + c + o_n(1) \\
&\leq \frac{\|K\|_\infty}{22^*} \int_{\mathbb{R}^N} |f(v_n)|^{22^*} + \frac{\delta}{2a_0} \int_{\mathbb{R}^N} V(x) f^2(v_n) + \frac{C_\delta}{q_1} \int_{\mathbb{R}^N} |f(v_n)|^{q_1} + c + o_n(1).
\end{aligned}$$

Given $0 < \varepsilon \leq 1$ to be chosen later, there exists $0 < \delta_1 < 1$ such that $|s|^{q_1} \leq \varepsilon |s|^2$ for all $|s| \leq \delta_1$. Then, from Lemma 1-(3), it follows that

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 + \left(\frac{1}{2} - \frac{\delta}{2a_0} \right) \int_{\mathbb{R}^N} V(x) f^2(v_n) \\
&\leq \frac{C_\delta}{q_1} \int_{\{|v_n(x)|\leq \delta_1\}} |f(v_n)|^{q_1} + \frac{C_\delta}{q_1} \int_{\{|v_n(x)|>\delta_1\}} |f(v_n)|^{q_1} + \frac{\|K\|_\infty}{22^*} \int_{\mathbb{R}^N} |f(v_n)|^{22^*} + c + o_n(1) \\
&\leq \frac{\varepsilon C_\delta}{q_1} \int_{\{|v_n(x)|\leq \delta_1\}} f^2(v_n) + \frac{C_\delta}{q_1} \int_{\{|v_n(x)|>\delta_1\}} |f(v_n)|^{q_1} + \frac{\|K\|_\infty}{22^*} \int_{\mathbb{R}^N} |f(v_n)|^{22^*} + c + o_n(1),
\end{aligned}$$

which yields

$$\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2+\left(\frac{1}{2}-\frac{\delta}{2a_0}-\frac{\varepsilon C_\delta}{q_1 a_0}\right)\int_{\mathbb{R}^N}V(x)f^2(v_n)$$
$$\leq \frac{C_\delta}{q_1}\int_{\{|v_n(x)|>\delta_1\}}|f(v_n)|^{q_1}+\frac{\|K\|_\infty}{22^*}\int_{\mathbb{R}^N}|f(v_n)|^{22^*}+c+o_n(1).$$

Note that if $|s|>\delta_1$, there exists a constant $C_1>0$ such that $|s|^{q_1}\leq C_1|s|^{22^*}$. Thus, we get

$$\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2+\left(\frac{1}{2}-\frac{\delta}{2a_0}-\frac{\varepsilon C_\delta}{q_1 a_0}\right)\int_{\mathbb{R}^N}V(x)f^2(v_n)\leq C_2\int_{\mathbb{R}^N}|f(v_n)|^{22^*}+c+o_n(1), \tag{2.13}$$

where $C_2=C_\delta C_1/q_1+\|K\|_\infty/22^*$. Take δ and ε sufficiently small so that $\frac{1}{2}-\frac{\delta}{2a_0}-\frac{\varepsilon C_\delta}{q_1 a_0}>0$. Hence, in order to conclude the proof of lemma, it suffices to show that the right hand side in (2.13) is bounded.

By Lemma 1-(8), we have

$$\langle I'(v_n),v_n\rangle\leq\int_{\mathbb{R}^N}|\nabla v_n|^2+\int_{\mathbb{R}^N}V(x)f^2(v_n)-\frac{1}{2}\int_{\mathbb{R}^N}K(x)|f(v_n)|^{22^*}$$
$$-\int_{\mathbb{R}^N}g(x,f(v_n))f'(v_n)v_n.$$

Using Lemma 1-(6) jointly with $(g_4)(ii)$, it follows, for $s\geq 0$,

$$\begin{aligned}g(x,f(s))f'(s)s&\geq g_0(x,f(s))f'(s)s-h_3(x)|f(s)|^{q_3-1}f'(s)s\\&\geq\frac{1}{2}g_0(x,f(s))f(s)-h_3(x)|f(s)|^{q_3}\\&\geq\frac{1}{2}g_(x,f(s))f(s)-\frac{3}{2}h_3(x)|f(s)|^{q_3}.\end{aligned}$$

Hence, noting that $g(x,s)=0$ if $s<0$,

$$\langle I'(v_n),v_n\rangle\leq\int_{\mathbb{R}^N}|\nabla v_n|^2+\int_{\mathbb{R}^N}V(x)f^2(v_n)-\frac{1}{2}\int_{\mathbb{R}^N}K(x)|f(v_n)|^{22^*}$$
$$-\int_{\mathbb{R}^N}\frac{1}{2}g(x,f(v_n))f(v_n)+\frac{3}{2}\int_{\mathbb{R}^N}h_3|f(v_n)|^{q_3}.$$

Consequently, by (g_3),

$$\begin{aligned}&I(v_n)-\frac{1}{2}\langle I'(v_n),v_n\rangle\\&\geq\frac{1}{2N}\int_{\mathbb{R}^N}K(x)|f(v_n)|^{22^*}-\frac{3}{4}\int_{\mathbb{R}^N}h_3|f(v_n)|^{q_3}+\int_{\mathbb{R}^N}\left[\frac{1}{4}g(x,f(v_n))f(v_n)-G(x,f(v_n))\right]\\&\geq\frac{1}{2N}\int_{\mathbb{R}^N}K(x)|f(v_n)|^{22^*}-\frac{3}{4}\int_{\mathbb{R}^N}h_3|f(v_n)|^{q_3}-\int_{\mathbb{R}^N}h_1-\int_{\mathbb{R}^N}h_2|f(v_n)|^{q_2}.\end{aligned}$$

Then, by (2.12) and the fact that $h_1 \in L^1(\mathbb{R}^N)$, we find a constant $C_3 > 0$ such that

$$\frac{1}{2N}\int_{\mathbb{R}^N} K(x)|f(v_n)|^{22^*} \leq \int_{\mathbb{R}^N} h_2|f(v_n)|^{q_2} + \frac{3}{4}\int_{\mathbb{R}^N} h_3|f(v_n)|^{q_3} + C_3. \tag{2.14}$$

Given $\epsilon > 0$, we set $D_\epsilon(R) = \{x \in \mathbb{R}^N : |h_2(x)| \geq \epsilon, |x| \geq R\}$ for all $R > 0$. Then, since $h_2 \in \mathcal{F}$, applying Lemma 3, we find $R = R_\epsilon > 0$ such that $|D_\epsilon(R)| < \epsilon$. By Hölder's inequality,

$$\begin{aligned}\int_{D_\epsilon(R)} h_2|f(v_n)|^{q_2} &\leq \|h_2\|_\infty \left(\int_{D_\epsilon(R)} 1^{\frac{22^*}{22^*-q_2}}\right)^{\frac{22^*-q_2}{22^*}} \left(\int_{D_\epsilon(R)} |f(v_n)|^{22^*}\right)^{\frac{q_2}{22^*}} \\ &\leq \|h_2\|_\infty \epsilon^{\frac{22^*-q_2}{22^*}} \left(\int_{\mathbb{R}^N} |f(v_n)|^{22^*}\right)^{\frac{q_2}{22^*}}.\end{aligned} \tag{2.15}$$

On the other hand,

$$\int_{\mathbb{R}^N\setminus D_\epsilon(R)} h_2|f(v_n)|^{q_2} \leq \|h_2\|_\infty \left(\frac{\omega_N R^N}{N}\right)^{\frac{22^*-q_2}{22^*}} \left(\int_{\mathbb{R}^N} |f(v_n)|^{22^*}\right)^{\frac{q_2}{22^*}} + \epsilon\int_{\mathbb{R}^N} |f(v_n)|^{q_2}. \tag{2.16}$$

Furthermore, considering $0 < \lambda \leq 1$ such that $q_2 = 2\lambda + (1-\lambda)22^*$, we apply Hölder's inequality, condition (V) and (2.13) to find $C_4 > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} |f(v_n)|^{q_2} &\leq \left(\int_{\mathbb{R}^N} f^2(v_n)\right)^{\lambda} \left(\int_{\mathbb{R}^N} |f(v_n)|^{22^*}\right)^{1-\lambda} \\ &\leq \left(\frac{1}{a_0}\int_{\mathbb{R}^N} V(x)f^2(v_n)\right)^{\lambda} \left(\int_{\mathbb{R}^N} |f(v_n)|^{22^*}\right)^{1-\lambda} \\ &\leq C_4\left(\int_{\mathbb{R}^N} |f(v_n)|^{22^*}\right)^{\lambda} \left(\int_{\mathbb{R}^N} |f(v_n)|^{22^*}\right)^{1-\lambda} \\ &= C_4\int_{\mathbb{R}^N} |f(v_n)|^{22^*}.\end{aligned} \tag{2.17}$$

Thus, from (2.15)–(2.17), we obtain $C_5 > 0$ such that

$$\int_{\mathbb{R}^N} h_2|f(v_n)|^{q_2} \leq C_5\left(\int_{\mathbb{R}^N} |f(v_n)|^{22^*}\right)^{\frac{q_2}{22^*}} + \epsilon C_4\int_{\mathbb{R}^N} |f(v_n)|^{22^*}.$$

Observing that an analogous estimate holds for $\int_{\mathbb{R}^N} h_3|f(v_n)|^{q_3}$, we find $C_6 > 0$ such that

$$\begin{aligned}
&\frac{1}{2N}\inf_{\mathbb{R}^N} K \int_{\mathbb{R}^N} |f(v_n)|^{22^*} \\
&\leq C_5\left(\int_{\mathbb{R}^N} |f(v_n)|^{22^*}\right)^{\frac{q_2}{22^*}} + \epsilon C_4 \int_{\mathbb{R}^N} |f(v_n)|^{22^*} \\
&\quad + \frac{3}{4}C_6\left(\int_{\mathbb{R}^N} |f(v_n)|^{22^*}\right)^{\frac{q_3}{22^*}} + \frac{3}{4}\epsilon C_4 \int_{\mathbb{R}^N} |f(v_n)|^{22^*} + C_3,
\end{aligned}$$

which yields

$$\begin{aligned}
&\left(\frac{1}{2N}\inf_{\mathbb{R}^N} K - \frac{7}{4}\epsilon C_4\right)\int_{\mathbb{R}^N} |f(v_n)|^{22^*} \\
&\leq C_5\left(\int_{\mathbb{R}^N} |f(v_n)|^{22^*}\right)^{\frac{q_2}{22^*}} + \frac{3}{4}C_6\left(\int_{\mathbb{R}^N} |f(v_n)|^{22^*}\right)^{\frac{q_3}{22^*}} + C_3.
\end{aligned}$$

Since $q_2, q_3 < 22^*$, taking $\epsilon > 0$ sufficiently small, we obtain the desired result. □

Before stating the next result, we recall that the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$ is given by

$$S = \inf_{\substack{v \in D^{1,2}(\mathbb{R}^N)\\ v \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla v|^2}{\left(\int_{\mathbb{R}^N} |v|^{2^*}\right)^{2/2^*}}. \tag{2.18}$$

Proposition 1 *Suppose that* (V), (K), (g_1) *and* (g_2) *are satisfied. Let* $(v_n) \subset H^1(\mathbb{R}^N)$ *be a* $(Ce)_b$ *sequence with* $0 < b < \frac{1}{2N}\|K\|_\infty^{\frac{2-N}{2}} S^{\frac{N}{2}}$, *and* $v_n \rightharpoonup 0$ *weakly in* $H^1(\mathbb{R}^N)$. *Then there exist a sequence* $(y_n) \subset \mathbb{R}^N$ *and* $r, \eta > 0$ *such that* $|y_n| \to \infty$ *and*

$$\limsup_{n\to\infty} \int_{B_r(y_n)} |v_n|^2 \geq \eta > 0.$$

Proof Supposing that the result does not hold, we have (see [8] or [25]):

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^\sigma = 0, \quad \text{for every } \ \sigma \in (2, 2^*). \tag{2.19}$$

Since g is subcritical, as in [23], we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g(x, f(v_n)f'(v_n)v_n = \lim_{n\to\infty} \int_{\mathbb{R}^N} G(x, f(v_n)) = 0. \tag{2.20}$$

Thus, since $(v_n) \subset H^1(\mathbb{R}^N)$ is a $(Ce)_b$ sequence for the functional I, it follows that

$$\begin{aligned} b + o_n(1) &= I(v_n) - \frac{1}{2}\langle I'(v_n), v_n\rangle \\ &= \frac{1}{2}\int_{\mathbb{R}^N} V(x)[f^2(v_n) - f(v_n)f'(v_n)v_n] \\ &\quad + \int_{\mathbb{R}^N} K(x)\left[\frac{1}{2}|f(v_n)|^{22^*-2} f(v_n)f'(v_n)v_n - \frac{1}{22^*}|f(v_n)|^{22^*}\right]. \end{aligned} \tag{2.21}$$

We claim that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)[f^2(v_n) - f(v_n)f'(v_n)v_n] = 0; \tag{2.22}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)\left[2^{\frac{2^*}{2}}|v_n|^{2^*} - |f(v_n)|^{22^*}\right] = 0; \tag{2.23}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)\left[\frac{1}{2}|f(v_n)|^{22^*-2} f(v_n)f'(v_n)v_n - \frac{1}{2}2^{\frac{2^*-2}{2}}|v_n|^{2^*}\right] = 0. \tag{2.24}$$

Assuming that the claim is true, and using (2.21), we obtain

$$b + o_n(1) = \int_{\mathbb{R}^N} K(x)|v_n|^{2^*}\left[\frac{1}{2}2^{\frac{2^*-2}{2}} - \frac{2^{\frac{2^*-2}{2}}}{2^*}\right] = \frac{2^{\frac{2^*-2}{2}}}{N}\int_{\mathbb{R}^N} K(x)|v_n|^{2^*}.$$

Hence

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)|v_n|^{2^*} = \frac{Nb}{2^{\frac{2^*-2}{2}}} > 0. \tag{2.25}$$

Consequently, using (2.24), we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)|f(v_n)|^{22^*-2} f(v_n)f'(v_n)v_n = Nb. \tag{2.26}$$

On the other hand, taking the first limit in (2.20), the second one in (2.28) below and the fact that $\langle I'(v_n), v_n\rangle \to 0$, we get

$$\lim_{n\to\infty}\left[\int_{\mathbb{R}^N} K(x)|f(v_n)|^{22^*-2} f(v_n)f'(v_n)v_n - \|v_n\|^2\right] = 0.$$

Therefore, from (2.26), it follows that

$$\lim_{n\to\infty}\|v_n\|^2 = Nb. \tag{2.27}$$

We recall that, from the definition of S (2.18),

$$\frac{1}{\|K\|_\infty}\int_{\mathbb{R}^N} K(x)|v_n|^{2^*} \le \int_{\mathbb{R}^N} |v_n|^{2^*} \le \left(\frac{1}{S}\int_{\mathbb{R}^N} |\nabla v_n|^2\right)^{\frac{2^*}{2}} \le \left(\frac{\|v_n\|^2}{S}\right)^{\frac{2^*}{2}}.$$

Passing to the limit in the above inequality, in view of (2.25) and (2.27), we obtain

$$\frac{1}{\|K\|_\infty}\frac{Nb}{2^{\frac{2^*-2}{2}}} \le \left(\frac{Nb}{S}\right)^{\frac{2^*}{2}},$$

that is,

$$b \ge \frac{1}{2N}\|K\|_\infty^{\frac{2-N}{2}} S^{\frac{N}{2}},$$

contradicting the assumption that $b < \frac{1}{2N}\|K\|_\infty^{\frac{2-N}{2}} S^{\frac{N}{2}}$. That concludes the proof of Proposition 1, except for (2.22), (2.23) and (2.24).

For proving (2.22), we simply verify that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)[f^2(v_n)-v_n^2]=0 \text{ and } \lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)[v_n^2-f(v_n)f'(v_n)v_n]=0. \tag{2.28}$$

Indeed, for $\delta > 0$ to be chosen later, we may write

$$\int_{\mathbb{R}^N} V(x)[f^2(v_n)-v_n^2] = \int_{\{|v_n(x)|>\delta\}} V(x)[f^2(v_n)-v_n^2] + \int_{\{|v_n(x)|\le\delta\}} V(x)[f^2(v_n)-v_n^2].$$

Using (2.19), condition (V) and Lemma 1-(3), we obtain

$$\int_{\{|v_n(x)|>\delta\}} \left|V(x)[f^2(v_n)-v_n^2]\right| \le 2\|V\|_\infty \int_{\{|v_n(x)|>\delta\}} v_n^2 < \frac{2\|V\|_\infty}{\delta^{\sigma-2}}\int_{\mathbb{R}^N}|v_n|^\sigma \to 0, \tag{2.29}$$

as $n\to\infty$. On the other hand, by Lemma 1-(4), given $\varepsilon > 0$, we choose $\delta > 0$ so that $\left|(f(s)/s)^2-1\right| < \varepsilon$, if $|s|\le\delta$. Thus, by the condition (V), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{\{0<|v_n(x)|\le\delta\}}\left|V(x)v_n^2\left[\left(\frac{f(v_n)}{v_n}\right)^2-1\right]\right| \\
&\le \|V\|_\infty \limsup_{n\to\infty}\int_{\{0<|v_n(x)|\le\delta\}} v_n^2\left|\left(\frac{f(v_n)}{v_n}\right)^2-1\right| \\
&< \varepsilon\|V\|_\infty \limsup_{n\to\infty}\int_{\mathbb{R}^N} v_n^2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left|V(x)[f^2(v_n)-v_n^2]\right| \\
&\le \limsup_{n\to\infty}\int_{\{|v_n(x)|>\delta\}}\left|V(x)[f^2(v_n)-v_n^2]\right| + \varepsilon\|V\|_\infty\limsup_{n\to\infty}\int_{\mathbb{R}^N} v_n^2.
\end{aligned}$$

Since $\varepsilon > 0$ can be taken as small as we like and $(v_n)\subset H^1(\mathbb{R}^N)$ is bounded, using (2.29), we have the first limit in (2.28). By the properties (3), (4) and (8) of Lemma 1 and the fact that $f'(s)\to 1$ as $s\to 0$, the verification of the second limit in (2.28) is similar to the first one. Therefore (2.22) holds.

The verification of (2.23) is similar to the previous one. Indeed, for $R > 0$ to be chosen later, we write

$$\int_{\mathbb{R}^N} K(x)\left[2^{\frac{2^*}{2}}|v_n|^{2^*} - |f(v_n)|^{22^*}\right] = \int_{\{|v_n(x)|\le R\}} K(x)\left[2^{\frac{2^*}{2}}|v_n|^{2^*} - |f(v_n)|^{22^*}\right] + \int_{\{|v_n(x)|> R\}} K(x)\left[2^{\frac{2^*}{2}}|v_n|^{2^*} - |f(v_n)|^{22^*}\right].$$

By the condition (K), Lemma 1-(7) and (2.19),

$$\begin{aligned}\int_{\{|v_n(x)|\le R\}} &\left|K(x)\left[2^{\frac{2^*}{2}}|v_n|^{2^*} - |f(v_n)|^{22^*}\right]\right| \\ &\le 2^{\frac{2^*+2}{2}}\|K\|_\infty \int_{\{|v_n(x)|\le R\}} |v_n|^{2^*} \le 2^{\frac{2^*+2}{2}}\|K\|_\infty R^{2^*-\sigma}\int_{\mathbb{R}^N}|v_n|^\sigma \\ &\to 0, \end{aligned} \tag{2.30}$$

as $n\to\infty$. On the other hand, by Lemma 1-(5), given $\varepsilon > 0$, we may choose $R > 0$ sufficiently large so that $\left|1 - \left(f(|s|)/2^{1/4}|s|^{1/2}\right)^{22^*}\right| < \varepsilon$ for $|s| > R$. Thus, by the condition (K),

$$\begin{aligned}\limsup_{n\to\infty} \int_{\{|v_n(x)|> R\}} &\left|K(x)\left[2^{\frac{2^*}{2}}|v_n|^{2^*} - |f(v_n)|^{22^*}\right]\right| \\ &\le \|K\|_\infty \limsup_{n\to\infty}\int_{\{|v_n(x)|> R\}} 2^{\frac{2^*}{2}}|v_n|^{2^*}\left|1-\left(\frac{f(|v_n|)}{2^{1/4}|v_n|^{1/2}}\right)^{22^*}\right| \\ &< \varepsilon 2^{\frac{2^*}{2}}\|K\|_\infty \limsup_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^{2^*}.\end{aligned}$$

Hence

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathbb{R}^N} &\left|K(x)\left[2^{\frac{2^*}{2}}|v_n|^{2^*} - |f(v_n)|^{22^*}\right]\right| \\ &\le \limsup_{n\to\infty}\int_{\{|v_n(x)|\le R\}}\left|K(x)\left[2^{\frac{2^*}{2}}|v_n|^{2^*} - |f(v_n)|^{22^*}\right]\right| \\ &\quad + \varepsilon 2^{\frac{2^*}{2}}\|K\|_\infty \limsup_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^{2^*}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and $(v_n)\subset H^1(\mathbb{R}^N)$ is bounded, we use (2.30) to conclude (2.23).

Finally, using the following identity

$$\frac{|f(s)|^{22^*-2}f(s)f'(s)s}{2^{(2^*-2)/2}|s|^{2^*}} = \frac{1}{(1/2f^2(s)+1)^{1/2}}\left(\frac{f(|s|)}{2^{1/4}|s|^{1/2}}\right)^{22^*-2}, \quad \text{for all } s\in\mathbb{R}\backslash\{0\},$$

by Lemma 1-(5), (7), (8) and the condition (K), a similar argument to the one used above shows (2.24). The Proposition 1 is proved. □

3 Estimates

In this section we verify that the minimax level associated with the Mountain Pass Theorem (Theorem 1) is in the interval where the Proposition 1 can be applied. To show this result, we use appropriate test functions as the ones employed by Brézis and Nirenberg [6], and we verify some auxiliary results about these functions. Then we show the main result of the section, providing the estimate for the minimax level c given by (2.1).

3.1 Test functions

Without loss of generality, we assume that x_0, given by the condition (K), is the origin of $\mathbb{R}^N$ and that $B_2(0) \subset \Omega$, with Ω given by the condition (g_5).

Given $\varepsilon > 0$, we consider the function $w_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ defined by

$$w_\varepsilon(x) = C(N)\frac{\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}},$$

where

$$C(N) = [N(N-2)]^{(N-2)/4}.$$

We observe (see [25]) that $\{w_\varepsilon\}_{\varepsilon>0}$ is a family of functions on which the infimum, that defines the best constant, S, for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$, is attained. We also consider $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$, $\phi \equiv 1$ in $B_1(0)$, $\phi \equiv 0$ in $\mathbb{R}^N \backslash B_2(0)$ and define

$$u_\varepsilon = \phi w_\varepsilon, \qquad v_\varepsilon = \frac{u_\varepsilon}{(\int_{\mathbb{R}^N} K u_\varepsilon^{2^*})^{1/2^*}}.$$

The following lemmas were inspired by [6,15,25]. We just state them, since their proofs are standard.

Lemma 5 *Suppose that (K) is satisfied. Then, there exist positive constants k_1, k_2 and ε_0 such that*

$$\int_{\mathbb{R}^N\backslash B_1(0)} |\nabla u_\varepsilon|^2 = O(\varepsilon^{N-2}), \quad as \ \ \varepsilon \to 0^+, \tag{3.1}$$

$$k_1 < \int_{\mathbb{R}^N} K u_\varepsilon^{2^*} < k_2, \quad for\ all \ \ 0 < \varepsilon < \varepsilon_0, \tag{3.2}$$

$$\int_{|x|\le 1} |x|^{N-2} w_\varepsilon^{2^*} = O(\varepsilon^{N-2}), \quad as \ \ \varepsilon \to 0^+, \tag{3.3}$$

$$\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 \le \|K\|_\infty^{(2-N)/N} S + O(\varepsilon^{N-2}), \quad as \ \ \varepsilon \to 0^+. \tag{3.4}$$

Lemma 6 *Suppose that (K) is satisfied. Then, as $\varepsilon \to 0$, we have*

(i) $$\|v_\varepsilon\|_2^2 = \begin{cases} O(\varepsilon), & \text{if } \ N = 3, \\ O\left(\varepsilon^2|\log\varepsilon|\right), & \text{if } \ N = 4, \\ O\left(\varepsilon^2\right), & \text{if } \ N \ge 5; \end{cases}$$

(ii) $$\|v_\varepsilon\|_{2^*-\frac{1}{2}}^{2^*-\frac{1}{2}} = O\left(\varepsilon^{(N-2)/4}\right), \qquad \text{if } N \ge 3.$$

3.2 Estimates for the minimax level

Before stating the main result of this section, we present two useful lemmas.

Lemma 7 *Suppose that* (V), (K), (g_1) *and* (g_2) *are satisfied. Consider* $t_\varepsilon > 0$ *such that* $I(t_\varepsilon v_\varepsilon) = \max_{t\geq 0} I(tv_\varepsilon)$. *Then, there exist* $\varepsilon_0 > 0$ *and positive constants* T_1 *and* T_2 *such that* $T_1 \leq t_\varepsilon \leq T_2$ *for every* $0 < \varepsilon < \varepsilon_0$.

Proof First, we observe that, in view of Lemma 2, I satisfies $I(0) = 0$ and (I_1). Consequently, since $I \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and $\|v_\varepsilon\|$ is bounded, we conclude that there exists $T_1 > 0$ such that $t_\varepsilon \geq T_1 > 0$ for every $\varepsilon > 0$ sufficiently small. On the other hand, from $(g_4)(i)$ and Lemma 1-(3), we have

$$I(t_\varepsilon v_\varepsilon) \leq \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(t_\varepsilon v_\varepsilon) - \frac{1}{22^*}\int_{\mathbb{R}^N} K(x) f^{22^*}(t_\varepsilon v_\varepsilon)$$
$$\leq \frac{t_\varepsilon^2}{2}\|v_\varepsilon\|^2 - \frac{1}{22^*}\inf_{\mathbb{R}^N} K \int_{\mathbb{R}^N} f^{22^*}(t_\varepsilon v_\varepsilon).$$

We claim that there is a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} f^{22^*}(t_\varepsilon v_\varepsilon) \geq C_1 t_\varepsilon^{2^*}. \tag{3.5}$$

Indeed, by property (9) of Lemma 1, we have

$$\int_{\mathbb{R}^N} f^{22^*}(t_\varepsilon v_\varepsilon) \geq C^{22^*} t_\varepsilon^{2^*} \int_{\{t_\varepsilon v_\varepsilon(x)\geq 1\}} v_\varepsilon^{2^*}.$$

We consider $x \in \mathbb{R}^N$ such that $|x| \leq \varepsilon < 1$. Then, from (3.2), we get

$$v_\varepsilon(x) \geq \frac{1}{k_2} w_\varepsilon(x) \geq \frac{C(N)}{k_2}\frac{1}{2^{\frac{N}{2}-1}\varepsilon^{\frac{N}{2}-1}}.$$

Consequently, since $t_\varepsilon \geq T_1 > 0$, for every $\varepsilon > 0$ sufficiently small, there exists $\varepsilon_0 > 0$ such that $t_\varepsilon v_\varepsilon(x) \geq 1$ whenever $|x| \leq \varepsilon < \varepsilon_0$. Now, by straightforward computations we find a constant $C_2 > 0$ such that

$$\int_{\{t_\varepsilon v_\varepsilon(x)\geq 1\}} v_\varepsilon^{2^*} \geq \frac{1}{k_2}\int_{B_\varepsilon(0)} w_\varepsilon^{2^*} = \frac{C^{2^*}(N)}{k_2}\int_{B_\varepsilon(0)} \frac{\varepsilon^N}{\left(\varepsilon^2 + |x|^2\right)^N} \geq C^{2^*}(N)\omega_N C_2.$$

This concludes the proof of the claim.

Thus, from (3.5), we have

$$\alpha \leq I(t_\varepsilon v_\varepsilon) \leq \frac{t_\varepsilon^2}{2}\|v_\varepsilon\|^2 - \left(\frac{C_1}{22^*}\inf_{\mathbb{R}^N} K\right) t_\varepsilon^{2^*},$$

which yields

$$\left(\frac{C_1}{22^*}\inf_{\mathbb{R}^N} K\right) t_\varepsilon^{2^*} \leq \frac{t_\varepsilon^2}{2}\|v_\varepsilon\|^2 - \alpha.$$

Since $\|v_\varepsilon\|$ is bounded, we obtain the estimate $t_\varepsilon \leq T_2 < \infty$. The lemma is proved. □

Lemma 8 *There exist constants* C_0, $R>0$ *such that*

$$f^{22^*}(t) - 2^{2^*/2}t^{2^*} \geq -C_0 t^{2^*-1/2}, \quad \text{for all } \ t \geq R.$$

Proof Consider $t \geq 1$. By definition, we have

$$f(t) - f(1) = \int_1^t \frac{1}{\left(1+2f^2(s)\right)^{1/2}}\, ds.$$

It follows from Lemma 1-(7) that $2f^2(s) \leq 2^{3/2}s$ for all $s \geq 0$. Therefore,

$$\begin{aligned} f(t) - f(1) &\geq \int_1^t \frac{1}{\left(1+2^{3/2}s\right)^{1/2}}\, ds = \frac{2}{2^{3/2}}\left(1+2^{3/2}s\right)^{1/2}\Big|_1^t \\ &= \frac{1}{2^{1/2}}\left(1+2^{3/2}t\right)^{1/2} - \frac{1}{2^{1/2}}\left(1+2^{3/2}\right)^{1/2}, \quad \text{for all } \ t \geq 1, \end{aligned}$$

which yields

$$f(t) \geq f(1) - \frac{1}{2^{1/2}}\left(1+2^{3/2}\right)^{1/2} + 2^{1/4}t^{1/2}, \quad \text{for all } \ t \geq 1.$$

Now, observing that by Lemma 1-(3), $f(1) \leq 1$, we have $-d = f(1) - \frac{1}{2^{1/2}}\left(1+2^{3/2}\right)^{1/2} < 0$. Hence, we obtain

$$f(t) \geq -d + 2^{1/4}t^{1/2}, \quad \text{for all } \ t \geq 1, \tag{3.6}$$

which amounts to

$$f^{22^*}(t) - 2^{2^*/2}t^{2^*} \geq \left(2^{1/4}t^{1/2} - d\right)^{22^*} - \left(2^{1/4}t^{1/2}\right)^{22^*}, \quad \text{for all } \ t \geq 1. \tag{3.7}$$

On the other hand, by Mean Value Theorem, there exists $0 < \theta < 1$ such that

$$\begin{aligned} x^{22^*} - (x-d)^{22^*} &= 22^*(x-\theta d)^{22^*-1}d \\ &\leq 22^* d\, x^{22^*-1}, \quad \text{for all } \ x \geq d > 0. \end{aligned}$$

Taking $x = 2^{1/4}t^{1/2} \geq d$, we obtain

$$\left(2^{1/4}t^{1/2}\right)^{22^*} - \left(2^{1/4}t^{1/2} - d\right)^{22^*} \leq 22^* d 2^{\frac{22^*-1}{4}} t^{2^*-1/2}, \quad \text{for all } \ t \geq 2^{-1/2}d^2. \tag{3.8}$$

(3.7) and (3.8) prove lemma with $C_0 = 2^* 2^{\frac{22^*+3}{4}} d$ and $R \geq \max\{1, 2^{-1/2}d^2\}$. □

Remark 2 From the proof of above Lemma 8, using the properties (3) and (7) of Lemma 1, we obtain the property

$$\lim_{t\to+\infty} \frac{f(t)}{t^{1/2}} = 2^{1/4}. \tag{5}$$

Indeed, by Lemma 1-(3), (7), we get (3.6). Consequently

$$\liminf_{t\to+\infty} \frac{f(t)}{t^{1/2}} \geq 2^{1/4}.$$

Using the property (7) of Lemma 1 one more time, it follows

$$\limsup_{t\to+\infty} \frac{f(t)}{t^{1/2}} \le 2^{1/4}.$$

Therefore, (5) holds.

Next we state the result which provides an appropriate estimate on the minimax level.

Proposition 2 *Suppose that* (V), (K), (g_1), (g_2) *and* (g_5) *are satisfied. Then there exists* $v \in H^1(\mathbb{R}^N)\backslash\{0\}$ *such that*

$$\max_{t\ge0} I(tv) < \frac{1}{2N}\|K\|_\infty^{\frac{2-N}{2}} S^{\frac{N}{2}}.$$

Proof Consider t_ε as defined by Lemma 7. Invoking Lemma 8 and observing that $\int_{\mathbb{R}^N} K(x)v_\varepsilon^{2^*} = 1$, we may write

$$\begin{aligned}
&\frac{1}{22^*}\int_{\mathbb{R}^N} K(x) f^{22^*}(t_\varepsilon v_\varepsilon)\\
&= \frac{1}{22^*}\int_{\{t_\varepsilon v_\varepsilon(x)<R\}} K(x)\left(f^{22^*}(t_\varepsilon v_\varepsilon) - 2^{\frac{2^*}{2}}(t_\varepsilon v_\varepsilon)^{2^*}\right)\\
&\quad + \frac{1}{22^*}\int_{\{t_\varepsilon v_\varepsilon(x)\ge R\}} K(x)\left(f^{22^*}(t_\varepsilon v_\varepsilon) - 2^{\frac{2^*}{2}}(t_\varepsilon v_\varepsilon)^{2^*}\right) + \frac{1}{22^*}\int_{\mathbb{R}^N} K(x)2^{\frac{2^*}{2}}(t_\varepsilon v_\varepsilon)^{2^*}\\
&\ge -\frac{2^{\frac{2^*}{2}-1}}{2^*}\|K\|_\infty \int_{\{t_\varepsilon v_\varepsilon(x)<R\}} (t_\varepsilon v_\varepsilon)^{2^*} - \frac{C_0\|K\|_\infty}{22^*}\int_{\mathbb{R}^N} (t_\varepsilon v_\varepsilon)^{2^*-\frac{1}{2}} + \frac{2^{\frac{2^*}{2}-1}}{2^*} t_\varepsilon^{2^*}.
\end{aligned}$$

Note that if $0 \le |s| < R$, there exists $C_1 > 0$ such that $|s|^{2^*} \le C_1|s|^{2^*-\frac{1}{2}}$. Hence,

$$\int_{\{t_\varepsilon v_\varepsilon(x)<R\}} (t_\varepsilon v_\varepsilon)^{2^*} \le C_1 \int_{\{t_\varepsilon v_\varepsilon(x)<R\}} (t_\varepsilon v_\varepsilon)^{2^*-\frac{1}{2}} \le C_1 \int_{\mathbb{R}^N} (t_\varepsilon v_\varepsilon)^{2^*-\frac{1}{2}}.$$

Thus, we obtain a constant $C_2 > 0$ such that

$$\frac{1}{22^*}\int_{\mathbb{R}^N} K(x) f^{22^*}(t_\varepsilon v_\varepsilon) \ge \frac{2^{\frac{2^*}{2}-1}}{2^*} t_\varepsilon^{2^*} - C_2\int_{\mathbb{R}^N} (t_\varepsilon v_\varepsilon)^{2^*-\frac{1}{2}}. \tag{3.9}$$

Using Lemma 1-(3) and the relation (3.9), we have

$$
\begin{aligned}
I(t_\varepsilon v_\varepsilon) &= \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(t_\varepsilon v_\varepsilon) \\
&\quad - \frac{1}{22^*}\int_{\mathbb{R}^N} K(x) f^{22^*}(t_\varepsilon v_\varepsilon) - \int_{\mathbb{R}^N} G(x, f(t_\varepsilon v_\varepsilon)) \\
&\le \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} V(x) v_\varepsilon^2 - \frac{2^{\frac{2^*}{2}-1}}{2^*} t_\varepsilon^{2^*} \\
&\quad + C_2 \int_{\mathbb{R}^N} (t_\varepsilon v_\varepsilon)^{2^*-\frac{1}{2}} - \int_{\mathbb{R}^N} G(x, f(t_\varepsilon v_\varepsilon)).
\end{aligned}
$$

Letting X_ε be the integral $\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2$, we get

$$
\begin{aligned}
I(t_\varepsilon v_\varepsilon) &\le \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 + \frac{t_\varepsilon^2}{2}\|V\|_\infty \|v_\varepsilon\|_2^2 - \frac{2^{\frac{2^*}{2}-1}}{2^*} t_\varepsilon^{2^*} \\
&\quad + C_2 t_\varepsilon^{2^*-\frac{1}{2}} \|v_\varepsilon\|_{2^*-\frac{1}{2}}^{2^*-\frac{1}{2}} - \int_{\mathbb{R}^N} G(x, f(t_\varepsilon v_\varepsilon)) \\
&\le \frac{1}{2N} X_\varepsilon^{N/2} + C_3 \|v_\varepsilon\|_2^2 + C_4 \|v_\varepsilon\|_{2^*-\frac{1}{2}}^{2^*-\frac{1}{2}} - \int_{\mathbb{R}^N} G(x, f(t_\varepsilon v_\varepsilon)),
\end{aligned}
$$

for some constants $C_3, C_4 > 0$. Indeed, considering the function $h = h_\varepsilon : [0, \infty) \to I\!R$ given by $h(t) = \frac{1}{2} X_\varepsilon t^2 - \frac{2^{(2^*-2)/2}}{2^*} t^{2^*}$, we have that $t_0 = \frac{1}{\sqrt{2}} X_\varepsilon^{1/(2^*-2)}$ is a maximum point of h and $h(t_0) = \frac{1}{2N} X_\varepsilon^{N/2}$. It follows from (3.4) that

$$
I(t_\varepsilon v_\varepsilon) \le \frac{1}{2N}\left(\|K\|_\infty^{\frac{2-N}{N}} S + O(\varepsilon^{N-2})\right)^{\frac{N}{2}} + C_3 \|v_\varepsilon\|_2^2 + C_4 \|v_\varepsilon\|_{2^*-\frac{1}{2}}^{2^*-\frac{1}{2}} - \int_{\mathbb{R}^N} G(x, f(t_\varepsilon v_\varepsilon)).
$$

Applying the inequality

$$
(b+c)^\zeta \le b^\zeta + \zeta (b+c)^{\zeta-1} c, \quad b, c \ge 0, \ \zeta \ge 1,
$$

we get

$$
I(t_\varepsilon v_\varepsilon) \le \frac{1}{2N}\|K\|_\infty^{\frac{2-N}{2}} S^{\frac{N}{2}} + C_3 \|v_\varepsilon\|_2^2 + C_4 \|v_\varepsilon\|_{2^*-\frac{1}{2}}^{2^*-\frac{1}{2}} - \int_{\mathbb{R}^N} G(x, f(t_\varepsilon v_\varepsilon)) + O(\varepsilon^{N-2}). \tag{3.10}
$$

Now consider

$$
\gamma(\varepsilon) = \max \begin{cases} \{\varepsilon, \varepsilon^{\frac{N-2}{4}}\}, & \text{if } N = 3, \\ \{\varepsilon^2 |\log \varepsilon|, \varepsilon^{\frac{N-2}{4}}\}, & \text{if } N = 4, \\ \{\varepsilon^2, \varepsilon^{\frac{N-2}{4}}\}, & \text{if } N \ge 5, \end{cases}
$$

that is,

$$\gamma(\varepsilon) = \begin{cases} \varepsilon^{\frac{N-2}{4}}, & \text{if } 3 \leq N < 10, \\ \varepsilon^2, & \text{if } N \geq 10. \end{cases} \tag{3.11}$$

In view of Lemma 6, (3.10) and (3.11), we find a constant $C_5 > 0$ such that

$$I(t_\varepsilon v_\varepsilon) \leq \frac{1}{2N}\|K\|_\infty^{(2-N)/2} S^{N/2} + \gamma(\varepsilon)\left[C_5 - \frac{1}{\gamma(\varepsilon)}\int_{\mathbb{R}^N} G(x, f(t_\varepsilon v_\varepsilon))\right].$$

In order to prove Proposition 2, we just need to verify that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\gamma(\varepsilon)}\int_{\mathbb{R}^N} G(x, f(t_\varepsilon v_\varepsilon)) > C_5. \tag{3.12}$$

By $(g_4)(i)$, we have

$$G(x, s) + s^2 \geq 0, \quad \text{for every } \ x \in \Omega, \ s \geq 0. \tag{3.13}$$

Given $A_0 > 0$, we invoke (g_5) to obtain $R = R(A_0) > 0$ such that, for $x \in \Omega$, $s \geq R$,

$$G(x, s) \geq \begin{cases} A_0 s^{22^*-1}, & \text{if } 3 \leq N < 10, \\ A_0 s^4, & \text{if } N \geq 10. \end{cases} \tag{3.14}$$

Now consider the function $\eta_\varepsilon : [0, \infty) \to \mathbb{R}$ defined by

$$\eta_\varepsilon(r) = \frac{\varepsilon^{(N-2)/2}}{(\varepsilon^2 + r^2)^{(N-2)/2}}.$$

Since $\phi \equiv 1$ in $B_1(0)$, in view of (3.2), we find a constant $C_6 > 0$ such that $v_\varepsilon(x) \geq C_6\eta_\varepsilon(|x|)$ for $|x| < 1$. Furthermore, since η_ε is decreasing and f is increasing, there exists a positive constant $\tilde{\alpha}$ such that, for $|x| < \varepsilon$,

$$f(t_\varepsilon v_\varepsilon(x)) \geq f(T_1 C_6 \eta_\varepsilon(|x|)) \geq f(T_1 C_6 \eta_\varepsilon(\varepsilon)) \geq f(\tilde{\alpha}\varepsilon^{\frac{2-N}{2}}).$$

Recall that T_1 is given by Lemma 7. Then we may choose $\varepsilon_1 > 0$ such that

$$\begin{aligned} &\tilde{\alpha}\varepsilon^{\frac{2-N}{2}} \geq 1, \\ &f(t_\varepsilon v_\varepsilon(x)) \geq f(\tilde{\alpha}\varepsilon^{\frac{2-N}{2}}) \geq R = R(A_0), \end{aligned} \tag{3.15}$$

for $|x| < \varepsilon$, $0 < \varepsilon < \varepsilon_1$. Since $B_\varepsilon(0) \subset \Omega$, it follows from (3.14) and (3.15) that

$$G(x, f(t_\varepsilon v_\varepsilon)) \geq \begin{cases} A_0 f^{22^*-1}(t_\varepsilon v_\varepsilon), & \text{if } 3 \leq N < 10, \\ A_0 f^4(t_\varepsilon v_\varepsilon), & \text{if } N \geq 10, \end{cases} \tag{3.16}$$

for $|x| < \varepsilon$, $0 < \varepsilon < \varepsilon_1$.

Now, in order to verify (3.12), we should consider the two possible cases:
Case 1: $3 \leq N < 10$**.** By (3.15), (3.16) and Lemma 1-(9),

$$\begin{aligned} G(x, f(t_\varepsilon v_\varepsilon)) &\geq A_0 f^{22^*-1}(t_\varepsilon v_\varepsilon) \geq A_0 f^{22^*-1}(\tilde{\alpha}\varepsilon^{\frac{2-N}{2}}) \\ &\geq A_0 C^{22^*-1}\tilde{\alpha}^{\frac{22^*-1}{2}}\varepsilon^{\frac{(2-N)(22^*-1)}{4}}, \end{aligned}$$

for every $x \in B_\varepsilon(0)$ and $0 < \varepsilon < \varepsilon_1$. Hence, since $B_2(0) \subset \Omega$, we invoke (3.13) and Lemma 1-(3) to get

$$\begin{aligned}\int_{\mathbb{R}^N} G(x, f(t_\varepsilon v_\varepsilon)) &= \int_{B_\varepsilon(0)} G(x, f(t_\varepsilon v_\varepsilon)) + \int_{\Omega\setminus B_\varepsilon(0)} G(x, f(t_\varepsilon v_\varepsilon)) \\ &\geq A_0 C^{22^*-1}\tilde{\alpha}^{\frac{22^*-1}{2}}\varepsilon^{\frac{-3N-2}{4}}|B_\varepsilon(0)| - \int_{\Omega\setminus B_\varepsilon(0)} f^2(t_\varepsilon v_\varepsilon) \\ &\geq A_0 C^{22^*-1}\tilde{\alpha}^{\frac{22^*-1}{2}}\omega_N \varepsilon^{\frac{-3N-2}{4}}\varepsilon^N - T_2^2 \int_{\Omega\setminus B_\varepsilon(0)} v_\varepsilon^2 \\ &\geq A_0 C^{22^*-1}\tilde{\alpha}^{\frac{22^*-1}{2}}\omega_N \varepsilon^{\frac{N-2}{4}} - T_2^2\|v_\varepsilon\|_2^2,\end{aligned}$$

for $0 < \varepsilon < \varepsilon_1$, where T_2 is given by Lemma 7. Consequently, by Lemma 6, we obtain

$$\frac{1}{\varepsilon^{\frac{N-2}{4}}}\int_{\mathbb{R}^N} G(x, f(t_\varepsilon v_\varepsilon)) \geq A_0 C^{22^*-1}\tilde{\alpha}^{\frac{22^*-1}{2}}\omega_N - \Gamma(\varepsilon), \tag{3.17}$$

where

$$\Gamma(\varepsilon) = \begin{cases} O(\varepsilon^{3/4}), & \text{if } N = 3, \\ O(\varepsilon^{3/2}|\log\varepsilon|), & \text{if } N = 4, \\ O(\varepsilon^{\frac{10-N}{4}}), & \text{if } 5 \leq N < 10. \end{cases}$$

Choosing $A_0 > 0$ sufficiently large, the above relation and (3.17) establish (3.12).
Case 2: $N \geq 10$**.** Applying the relations (3.15), (3.16) and the Lemma 1-(9) one more time,

$$G(x, f(t_\varepsilon v_\varepsilon)) \geq A_0 f^4(t_\varepsilon v_\varepsilon) \geq A_0 f^4(\tilde{\alpha}\varepsilon^{\frac{2-N}{2}}) \geq A_0 C^4\tilde{\alpha}^2\varepsilon^{2-N},$$

for every $x \in B_\varepsilon(0)$ and $0 < \varepsilon < \varepsilon_1$. Hence, using that $B_2(0) \subset \Omega$, relation (3.13) and the Lemma 1-(3), we get

$$\begin{aligned}\int_{\mathbb{R}^N} G(x, f(t_\varepsilon v_\varepsilon)) &= \int_{B_\varepsilon(0)} G(x, f(t_\varepsilon v_\varepsilon)) + \int_{\Omega\setminus B_\varepsilon(0)} G(x, f(t_\varepsilon v_\varepsilon)) \\ &\geq A_0 C^4\tilde{\alpha}^2\varepsilon^{2-N}|B_\varepsilon(0)| - \int_{\Omega\setminus B_\varepsilon(0)} f^2(t_\varepsilon v_\varepsilon) \\ &\geq A_0 C^4\tilde{\alpha}^2\omega_N\varepsilon^{2-N}\varepsilon^N - T_2^2\int_{\Omega\setminus B_\varepsilon(0)} v_\varepsilon^2 \\ &\geq A_0 C^4\tilde{\alpha}^2\omega_N\varepsilon^2 - T_2^2\|v_\varepsilon\|_2^2,\end{aligned}$$

for $0 < \varepsilon < \varepsilon_1$, with T_2 given by Lemma 7. Consequently, by Lemma 6, there exists a constant $C_7 > 0$ such that

$$\frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} G(x, f(t_\varepsilon v_\varepsilon)) \geq A_0 C^4\tilde{\alpha}^2\omega_N - C_7.$$

Since $A_0 > 0$ can be chosen sufficiently large, the above relation establishes (3.12). The proof of Proposition 2 is complete. □

4 Proofs of Theorems 1 and 2

In this section we prove Theorems 1 and 2 by verifying that the functionals I and I_0, defined by (2.5) and (2.7), respectively, have nonzero critical points. But, before proving them, we introduce two technical results needed to the proof of the first one.

Lemma 9 *Suppose that* (V), (K) *and* (g_4) *are satisfied. Let* $(v_n) \subset H^1(\mathbb{R}^N)$ *be a bounded sequence and* $w_n(x) = w(x - y_n)$, *where* $w \in H^1(\mathbb{R}^N)$ *and* $(y_n) \subset \mathbb{R}^N$. *If* $|y_n| \to \infty$, *then we have*

$$\begin{aligned}&[V_0(x) - V(x)]f'(v_n)f(v_n)w_n \to 0,\\&[K(x) - K_0(x)]|f(v_n)|^{22^*-2}f(v_n)f'(v_n)w_n \to 0,\\&[g_0(x, f(v_n)) - g(x, f(v_n))]f'(v_n)w_n \to 0,\end{aligned}$$

strongly in $L^1(\mathbb{R}^N)$, *as* $n \to \infty$.

Proof Considering that the other limits have already been proven when we deal with the problem in the subcritical case (see [23]), we shall establish the second limit in Lemma 9. Given $\delta > 0$, since $w \in L^{2^*}(\mathbb{R}^N)$, we find $0 < \varepsilon < \delta$ such that, for every measurable set $A \subset \mathbb{R}^N$ satisfying $|A| < \varepsilon$,

$$\int_A |w|^{2^*} < \delta. \tag{4.1}$$

We fix $\varepsilon > 0$ and set $D_\varepsilon(R) = \{x \in \mathbb{R}^N : |K(x) - K_0(x)| \geq \varepsilon, |x| \geq R\}$. By the condition (K) and the Lemma 3, we may find $R > 0$ such that $|D_\varepsilon(R)| < \varepsilon$. Then, applying Lemma 1-(10), (7), Hölder's inequality, condition (K), (4.1) and the fact that $(v_n) \subset H^1(\mathbb{R}^N)$ is bounded, we get

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus B_R(0)} |K(x) - K_0(x)|\,|f(v_n)|^{22^*-2}|f(v_n)|\,|f'(v_n)|\,|w_n|\\&\leq \frac{\|K\|_\infty}{\sqrt{2}}\int_{D_\varepsilon(R)} |f(v_n)|^{22^*-2}|w_n| + \frac{\varepsilon}{\sqrt{2}}\int_{\mathbb{R}^N\setminus[B_R(0)\cup D_\varepsilon(R)]} |f(v_n)|^{22^*-2}|w_n|\\&< \frac{\|K\|_\infty}{\sqrt{2}}\left(\int_{\mathbb{R}^N} |f(v_n)|^{22^*}\right)^{(2^*-1)/2^*}\|w_n\|_{L^{2^*}(D_\varepsilon(R))} + \frac{\delta}{\sqrt{2}}\left(\int_{\mathbb{R}^N} |f(v_n)|^{22^*}\right)^{(2^*-1)/2^*}\|w\|_{2^*}\\&\leq 2^{\frac{2^*-2}{2}}\|K\|_\infty\left(\int_{\mathbb{R}^N} |v_n|^{2^*}\right)^{(2^*-1)/2^*}\|w_n\|_{L^{2^*}(D_\varepsilon(R))} + 2^{\frac{2^*-2}{2}}\delta\left(\int_{\mathbb{R}^N} |v_n|^{2^*}\right)^{(2^*-1)/2^*}\|w\|_{2^*}\\&< C_1(\delta^{1/2^*} + \delta),\end{aligned} \tag{4.2}$$

for some constant $C_1 > 0$. On the other hand, by Lemma 1-(10), (7), Hölder's inequality, the condition (K) and the boundedness of $(v_n) \subset H^1(\mathbb{R}^N)$, there is $C_2 > 0$ such that

$$\begin{aligned}
&\int_{B_R(0)} |K(x) - K_0(x)|\,|f(v_n)|^{22^*-2}|f(v_n)|\,|f'(v_n)|\,|w_n| \\
&\leq \frac{\|K\|_\infty}{\sqrt{2}} \left(\int_{\mathbb{R}^N} |f(v_n)|^{22^*} \right)^{(2^*-1)/2^*} \left(\int_{B_R(0)} |w(x-y_n)|^{2^*} \right)^{1/2^*} \\
&\leq 2^{\frac{2^*-2}{2}} \|K\|_\infty \left(\int_{\mathbb{R}^N} |v_n|^{2^*} \right)^{(2^*-1)/2^*} \left(\int_{B_R(0)} |w(x-y_n)|^{2^*} \right)^{1/2^*} \\
&\leq C_2 \left(\int_{B_R(-y_n)} |w(x)|^{2^*} \right)^{1/2^*}.
\end{aligned}$$

Hence, since $w \in L^{2^*}(\mathbb{R}^N)$ and $|y_n| \to \infty$, there exists $n_0 \in \mathbb{N}$ such that

$$\int_{B_R(0)} |K(x) - K_0(x)|\,|f(v_n)|^{22^*-2}|f(v_n)|\,|f'(v_n)|\,|w_n| \leq C_2\delta, \quad \text{for all} \;\; n \geq n_0. \tag{4.3}$$

The inequalities (4.2), (4.3) and the fact that $\delta > 0$ can be chosen arbitrarily small imply that $[K(x) - K_0(x)]|f(v_n)|^{22^*-2} f(v_n) f'(v_n) w_n \to 0$ strongly in $L^1(\mathbb{R}^N)$, as $n \to \infty$, as desired. The proof of Lemma 9 is complete. □

In order to prove Theorem 1 we also need of the following result which has been proved in [23]:

Lemma 10 *Suppose* $2 \leq q < 22^*$ *and* $h \in \mathcal{F}$. *Let* $(v_n) \subset H^1(\mathbb{R}^N)$ *be a sequence such that* $v_n \rightharpoonup v$ *weakly in* $H^1(\mathbb{R}^N)$. *Then*

$$h(x)|f(v_n)|^q \to h(x)|f(v)|^q \quad \textit{strongly in } L^1(\mathbb{R}^N), \quad \textit{as} \;\; n \to \infty.$$

Now we are ready to prove Theorems 1 and 2.

4.1 Proof of Theorem 1

We infer by Corollary 2 that there exists a Cerami sequence on level c, that is, $(v_n) \subset H^1(\mathbb{R}^N)$ such that

$$I(v_n) \to c \geq \alpha > 0 \quad \text{and} \quad \|I'(v_n)\|(1 + \|v_n\|) \to 0, \quad \text{as} \;\; n \to \infty, \tag{4.4}$$

with c given by Theorem 3. Applying Lemma 4, we may assume, without loss generality, that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$. From this and (2.8), we have that v is a critical point of I, that is, $I'(v) = 0$. Effectively, because $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, it suffices to show that $\langle I'(v), \varphi\rangle = 0$ for every $\varphi \in C_0^\infty(\mathbb{R}^N)$. Note that

$$\begin{aligned}
&\langle I'(v_n), \varphi\rangle - \langle I'(v), \varphi\rangle - \int_{\mathbb{R}^N} (\nabla v_n - \nabla v)\nabla\varphi \\
&\quad = \int_{\mathbb{R}^N} \left[f(v_n)f'(v_n) - f(v)f'(v)\right] V(x)\varphi \\
&\qquad + \int_{\mathbb{R}^N} \left[|f(v)|^{22^*-2} f(v)f'(v) - |f(v_n)|^{22^*-2} f(v_n)f'(v_n)\right] K(x)\varphi \\
&\qquad + \int_{\mathbb{R}^N} \left[g(x, f(v))f'(v) - g(x, f(v_n))f'(v_n)\right]\varphi. \qquad (4.5)
\end{aligned}$$

Since $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$, we have that $v_n \to v$ in $L^p_{loc}(\mathbb{R}^N)$, with $p \in [1, 2^*)$. Then, up to a subsequence,

$$\begin{aligned}
&v_n(x) \to v(x) \quad \text{a.e. on } \ \mathcal{K} := supp\,\varphi, \quad \text{as } \ n \to \infty, \\
&|v_n(x)| \le |w_p(x)| \quad \text{for every } \ n \in \mathbb{N} \ \text{ and a.e. on } \ \mathcal{K}, \quad \text{with } \ w_p \in L^p(\mathcal{K}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&f(v_n)f'(v_n) \to f(v)f'(v) \quad \text{a.e. on } \ \mathcal{K}, \quad \text{as } \ n \to \infty, \\
&|f(v_n)|^{22^*-2} f(v_n)f'(v_n) \to |f(v)|^{22^*-2} f(v)f'(v) \quad \text{a.e. on } \ \mathcal{K}, \quad \text{as } \ n \to \infty, \\
&g(x, f(v_n))f'(v_n) \to g(x, f(v))f'(v) \quad \text{a.e. on } \ \mathcal{K}, \quad \text{as } \ n \to \infty.
\end{aligned}$$

Furthermore, from the condition (V) and the Lemma 1-(2), (3),

$$|V(x)f(v_n)f'(v_n)\varphi| \le |V(x)f(v_n)\varphi| \le \|V\|_\infty |w_2|\,|\varphi|,$$

and, by Lema 1-(10), (7) and condition (K),

$$\left|K(x)|f(v_n)|^{22^*-2} f(v_n)f'(v_n)\varphi\right| \le \|K\|_\infty 2^{\frac{2^*-2}{2}} |w_{2^*-1}|^{2^*-1}|\varphi|.$$

We also claim that there is a function $\psi \in L^1(\mathcal{K})$ such that

$$|g(x, f(v_n))f'(v_n)\varphi| \le \psi \quad \text{in } \ \mathbb{R}^N. \qquad (4.6)$$

Indeed, by (2.8), Lemma 1-(2), (3) we have, for $|v_n(x)| \le 1$,

$$\begin{aligned}
\left|g(x, f(v_n))f'(v_n)\varphi\right| &\le \delta|f(v_n)|\,|\varphi| + C_\delta|f(v_n)|^{q_1-1}|\varphi| \\
&\le (\delta + C_\delta)|\varphi|.
\end{aligned}$$

Again by (2.8), Lemma 1-(2), (3), (6), (7), if $|v_n(x)| > 1$,

$$\begin{aligned}
|g(x, f(v_n))f'(v_n)\varphi| &\le \delta|v_n|\,|\varphi| + C_\delta|f(v_n)|^{q_1-1}|f'(v_n)|\,|\varphi| \\
&\le \delta|w_2|\,|\varphi| + C_\delta|f(v_n)|^{q_1-1}\frac{|f(v_n)|}{|v_n|}|\varphi| \\
&\le \delta|w_2|\,|\varphi| + 2^{q_1/4}C_\delta|w_{2^*-1}|^{2^*-1}|\varphi|.
\end{aligned}$$

The above estimates imply the relation (4.6) as we claimed. Using this claim, (4.5), the Lebesgue Dominated Convergence Theorem and the weak convergence $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$, we have

$$\langle I'(v_n), \varphi\rangle - \langle I'(v), \varphi\rangle \to 0.$$

Since $I'(v_n) \to 0$, we conclude that $I'(v) = 0$. Thus, in order to prove Theorem 1, it suffices to assume that $v = 0$.

In view of Proposition 2, it follows that $0 < \alpha \le c < \frac{1}{2N} \|K\|_\infty^{(2-N)/2} S^{N/2}$. Furthermore, by Proposition 1, there exist a sequence $(y_n) \subset \mathbb{R}^N$ and $r, \eta > 0$ such that $|y_n| \to \infty$, as $n \to \infty$, and

$$\limsup_{n\to\infty} \int_{B_r(y_n)} |v_n|^2 \ge \eta > 0 \quad \text{for all } n \in \mathbb{N}. \tag{4.7}$$

Without loss of generality we may assume that $(y_n) \subset \mathbb{Z}^N$. Then, defining $u_n(x) = v_n(x + y_n)$, $n \in \mathbb{N}$, we have $\|u_n\|_0 = \|v_n\|_0$ for all $n \in \mathbb{N}$. Thus, passing to a subsequence if necessary, there exists $u \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $u_n \to u$ strongly in $L^2_{loc}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ almost everywhere in $\mathbb{R}^N$. From (4.7), we have $u \ne 0$.

We claim that u is a critical point of I_0. Indeed, we first observe that

$$\langle I_0'(u_n), \varphi\rangle \to \langle I_0'(u), \varphi\rangle, \quad \text{as } n \to \infty, \quad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^N). \tag{4.8}$$

Effectively, writing

$$\begin{aligned} &\langle I_0'(u_n), \varphi\rangle - \langle I_0'(u), \varphi\rangle - \int_{\mathbb{R}^N} (\nabla u_n - \nabla u)\nabla\varphi \\ &= \int_{\mathbb{R}^N} \left[f(u_n)f'(u_n) - f(u)f'(u)\right] V_0(x)\varphi \\ &\quad + \int_{\mathbb{R}^N} \left[|f(u)|^{22^*-2} f(u)f'(u) - |f(u_n)|^{22^*-2} f(u_n)f'(u_n)\right] K_0(x)\varphi \\ &\quad + \int_{\mathbb{R}^N} [g_0(x, f(u)) - g_0(x, f(u_n))]\, f'(v_n)\varphi, \end{aligned} \tag{4.9}$$

from the arguments used above, we deduce

$$\int_{\mathbb{R}^N} (\nabla u_n - \nabla u)\nabla\varphi \to 0, \quad as \ \ n \to \infty,$$

$$\int_{\mathbb{R}^N} \left[f(u_n)f'(u_n) - f(u)f'(u)\right] V_0(x)\varphi \to 0, \quad as \ \ n \to \infty$$

and

$$\int_{\mathbb{R}^N} \left[|f(u)|^{22^*-2} f(u)f'(u) - |f(u_n)|^{22^*-2} f(u_n)f'(u_n)\right] K_0(x)\varphi \to 0, \quad as \ \ n \to \infty.$$

Hence, in order to prove (4.8), it remains to analyze the last integral in (4.9). Note that

$$\begin{aligned} &[g_0(x, f(u)) - g_0(x, f(u_n))]\, f'(v_n)\varphi \\ &\quad = [g_0(x, f(u)) - g(x, f(u_n))]\, f'(v_n)\varphi \\ &\qquad + [g(x, f(u_n)) - g_0(x, f(u_n))]\, f'(v_n)\varphi. \end{aligned} \tag{4.10}$$

Now, by the condition $(g_4)-(ii)$ and the arguments used in the proof of (4.6), we get $\psi \in L^1(\mathcal{K})$, $\mathcal{K} := supp\,\varphi$, such that

$$|[g(x, f(u_n)) - g_0(x, f(u_n))]f'(u_n)\varphi| \leq \psi. \tag{4.11}$$

Hence, applying the Lebesgue Dominated Convergence Theorem one more time, we obtain

$$[g(x, f(u_n)) - g_0(x, f(u_n))]f'(u_n)\varphi \to [g(x, f(u)) - g_0(x, f(u))]f'(u)\varphi \tag{4.12}$$

in $L^1(\mathcal{K})$. The claim (4.6) and the Lebesgue Dominated Convergence Theorem also provide

$$g(x, f(u_n))f'(u_n)\varphi \to g(x, f(u))f'(u)\varphi \quad \text{in} \quad L^1(\mathcal{K}).$$

Furthermore, from (4.6), (4.11) and the Lebesgue Dominated Convergence Theorem again,

$$g_0(x, f(u))f'(u_n)\varphi \to g_0(x, f(u))f'(u)\varphi \quad \text{in} \quad L^1(\mathcal{K}).$$

Consequently,

$$[g_0(x, f(u)) - g(x, f(u_n))]\, f'(u_n)\varphi \to [g_0(x, f(u)) - g(x, f(u))]\, f'(u)\varphi \tag{4.13}$$

in $L^1(\mathcal{K})$. The relations (4.10), (4.12) and (4.13) establish the verification of (4.8). On the other hand, considering $\varphi_n(x) = \varphi(x - y_n)$, for $n \in \mathbb{N}$, in view of the periodicities from V_0, K_0 and g_0, we get

$$\langle I_0'(u_n), \varphi\rangle = \langle I_0'(v_n), \varphi_n\rangle, \quad \text{for all} \;\; n \in \mathbb{N}. \tag{4.14}$$

Moreover applying Lemma 9, we have

$$|\langle I_0'(v_n), \varphi_n\rangle - \langle I'(v_n), \varphi_n\rangle| \to 0, \quad \text{as} \;\; n \to \infty. \tag{4.15}$$

By $\|\varphi_n\|_0 = \|\varphi\|_0$ for all $n \in \mathbb{N}$, and the fact that $(v_n) \subset H^1(\mathbb{R}^N)$ is a $(Ce)_c$ sequence, we have that $\langle I'(v_n), \varphi_n\rangle \to 0$. Hence, by (4.15), we obtain

$$\langle I_0'(v_n), \varphi_n\rangle \to 0, \quad \text{as} \;\; n \to \infty.$$

The above limit, (4.14) and (4.8) show that u is a critical point of I_0 as claimed.

Our next task is to verify that $I_0(u) \leq c$. In order to show this fact, we apply condition (K) and the definition of (u_n) to get

$$\begin{aligned}
&I(v_n) - \frac{1}{2}\langle I'(v_n), v_n\rangle \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)[f^2(u_n) - f'(u_n)f(u_n)u_n] \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^N} (V(x) - V_0(x))[f^2(v_n) - f'(v_n)f(v_n)v_n] \\
&\quad +\int_{\mathbb{R}^N} K_0(x)\left[\frac{1}{2}|f(u_n)|^{22^*-2}f(u_n)f'(u_n)u_n - \frac{1}{22^*}|f(u_n)|^{22^*}\right] \\
&\quad +\int_{\mathbb{R}^N} \left[\frac{1}{2}g(x, f(v_n))f'(v_n)v_n - G(x, f(v_n))\right].
\end{aligned} \tag{4.16}$$

Now, by property (8) of Lemma 1, conditions (V), (K) and the Fatou's Lemma, we obtain

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N} K_0(x)\left[\frac{1}{2}|f(u_n)|^{22^*-2}f(u_n)f'(u_n)u_n-\frac{1}{22^*}|f(u_n)|^{22^*}\right]$$

$$\geq \int_{\mathbb{R}^N} K_0(x)\left[\frac{1}{2}|f(u)|^{22^*-2}f(u)f'(u)u-\frac{1}{22^*}|f(u)|^{22^*}\right] \tag{4.17}$$

and

$$\liminf_{n\to\infty}\frac{1}{2}\int_{\mathbb{R}^N} V_0(x)[f^2(u_n)-f'(u_n)f(u_n)u_n]\geq \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)[f^2(u)-f'(u)f(u)u]. \tag{4.18}$$

We observe that, in view of $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$, by Lemma 10, with $h = V - V_0$, and Lemma 1-(8), we get

$$\liminf_{n\to\infty}\frac{1}{2}\int_{\mathbb{R}^N}(V(x)-V_0(x))[f^2(v_n)-f'(v_n)f(v_n)v_n]=0. \tag{4.19}$$

We also claim that

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}\left[\frac{1}{2}g(x,f(v_n))f'(v_n)v_n-G(x,f(v_n))\right]$$

$$\geq \int_{\mathbb{R}^N}\left[\frac{1}{2}g_0(x,f(u))f'(u)u-G_0(x,f(u))\right]. \tag{4.20}$$

Assuming that the claim is true, we use (4.4), (4.16)–(4.20) and the fact that u is a critical point of I_0 to get

$$\begin{aligned} c \geq\ & \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)[f^2(u)-f'(u)f(u)u] \\ & + \int_{\mathbb{R}^N} K_0(x)\left[\frac{1}{2}|f(u)|^{22^*-2}f(u)f'(u)u-\frac{1}{22^*}|f(u)|^{22^*}\right] \\ & + \int_{\mathbb{R}^N}\left[\frac{1}{2}g_0(x,f(u))f'(u)u-G_0(x,f(u))\right] \\ =\ & I_0(u)-\frac{1}{2}\langle I_0'(u),u\rangle = I_0(u), \end{aligned} \tag{4.21}$$

that is, $I_0(u) \leq c$.

We shall verify that $\max_{t\geq 0} I_0(tu) = I_0(u)$. For that, we define the function $\eta(t) := I_0(tu)$ for $t \geq 0$. Since u is a critical point of I_0, it follows that $u > 0$ (see the argument below).

Hence, we may write

$$\eta'(t) = t\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} V_0(x)f'(tu)f(tu)u$$

$$-\int_{\mathbb{R}^N} K_0(x)f^{22^*-1}(tu)f'(tu)u - \int_{\mathbb{R}^N} g_0(x, f(tu))f'(tu)u$$

$$= t\left\{\int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N}\left[\frac{K_0 f^{22^*-1}(t|u|)f'(t|u|)}{t|u|}\right.\right.$$

$$\left.\left. + \frac{g_0(x, f(t|u|))f'(t|u|)}{t|u|} - \frac{V_0(x)f'(t|u|)f(t|u|)}{t|u|}\right]u^2\right\}.$$

Note that, fixed $x \in \mathbb{R}^N$, the function $\zeta : (0, +\infty) \to \mathbb{R}$ defined by

$$\zeta(s) = \frac{K_0(x)f^{22^*-1}(s)f'(s)}{s} + \frac{g_0(x, f(s))f'(s)}{s} - \frac{V_0(x)f'(s)f(s)}{s}$$

is increasing. Effectively, this is a direct consequence of $(g_4)(iii)$ and Corollary 1 applied to

$$\zeta(s) = K_0(x)\frac{f^{22^*-1}(s)f'(s)}{s} + \frac{g_0(x, f(s))}{f^3(s)}\frac{f^3(s)f'(s)}{s} + V_0(x)\left(-\frac{f'(s)f(s)}{s}\right).$$

Now we observe that $\eta'(1) = 0$ since u is a critical point of I_0. Moreover, we have that $\eta'(t) > 0$ for $0 < t < 1$ and $\eta'(t) < 0$ for $t > 1$. Therefore, $I_0(u) = \eta(1) = \max_{t\geq 0}\eta(t) = \max_{t\geq 0} I_0(tu)$. Consequently, by (4.21), $(g_4)(i)$ and the definition of c,

$$c \leq \max_{t\geq 0} I(tu) \leq \max_{t\geq 0} I_0(tu) = I_0(u) \leq c.$$

This implies that there exists $\gamma \in \Gamma$ such that (2.3) holds. In view of Theorem 4, I possesses a critical point v on level c. From $c \geq \alpha > 0 = I(0)$, we have that v is a nonzero critical point of I. This concludes the proof of Theorem 1, except for the claim (4.20).

We shall now show that $v > 0$ in $\mathbb{R}^N$. Since $v \geq 0$ in $\mathbb{R}^N$ is a weak solution of the equation

$$-\Delta v = w \quad \text{in } \mathbb{R}^N,$$

where

$$w(x, v) := f'(v)\left[K(x)f^{22^*-1}(v) + g(x, f(v)) - V(x)f(v)\right],$$

by the conditions (V), (K), relation (2.8) and the Lemma 1-(7),(10), we get

$$\begin{aligned}|w| &\leq f'(v)\left[\|K\|_\infty |f^{2(2^*-1)}(v)||f(v)| + \delta|f(v)| + C_\delta|f(v)|^{q_1-1}\right]\\ &\leq C_1\left[|v|^{2^*-1} + |v|^{(q_1-2)/2} + 1\right]\\ &\leq 2C_1\left[|v|^{2^*-1} + 1\right],\end{aligned}$$

for some constant $C_1 > 0$, since $1 \leq (q_1-2)/2 \leq 2^*-1$. Using a result due to Brezis-Kato (cf. [24]), it follows that $w \in L^p(B_R)$ for every $p < \infty$, with $R > 0$ arbitrary. By elliptic regularity theory, we may conclude that $v \in W^{2,p}(B_R)$. Hence, $v \in C^{1,\beta}_{loc}(\mathbb{R}^N)$ for some $\beta \in (0, 1)$. Now, arguing by contradiction, we suppose that there is $x_0 \in \mathbb{R}^N$ such that $v(x_0) = 0$. Eq. 2.6 can be rewritten as

$$-\Delta v + c(x)v = V(x)f'(v)(v - f(v)) + K(x)f^{22^*-1}(v)f'(v) + g^+(x, f(v))f'(v) \geq 0,$$

where $c(x) = [V(x) - \frac{g^-(x, f(v(x)))}{v(x)}]f'(v(x)) > 0$, for $x \in \mathbb{R}^N$, with $\frac{g^-(x, f(v))}{v}$ defined to be 0 at $v = 0$. Note that, from Lemma 1-(3), we have $v - f(v) \geq 0$. In view of (g_1) and the Lemma 1-(4), we have that c is a continuous function in $\mathbb{R}^N$. Hence, applying the Strong Maximum Principle for weak solutions (cf. [13]) on an arbitrary ball centered in x_0, we obtain that $v \equiv 0$. This contradicts the fact that v is not trivial.

Finally, we conclude the proof of Theorem 1 by showing that (4.20) holds. First we observe that, in view of Lemma 10,

$$\int_{\mathbb{R}^N} h_3|f(v_n)|^{q_3} \to 0 \quad \text{as} \quad n \to \infty,$$

since $h_3 \in \mathcal{F}$, $2 \leq q_3 < 22^*$ and $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$. Invoking (g_4) and Lemma 1-(6), we have

$$\begin{aligned} |g(x, f(s))f'(s)s - g_0(x, f(s))f'(s)s| &= |[g(x, f(s)) - g_0(x, f(s))]f'(s)s| \\ &\leq |[g(x, f(s)) - g_0(x, f(s))]f(s)| \\ &\leq h_3(x)|f(s)|^{q_3}. \end{aligned}$$

Hence,

$$g(x, f(v_n))f'(v_n)v_n - g_0(x, f(v_n))f'(v_n)v_n \to 0$$

strongly in $L^1(\mathbb{R}^N)$ as $n \to \infty$. Similarly,

$$G(x, f(v_n)) - G_0(x, f(v_n)) \to 0$$

strongly in $L^1(\mathbb{R}^N)$ as $n \to \infty$. Consequently, by the periodicity of g_0,

$$\begin{aligned} &\liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{1}{2}g(x, f(v_n))f'(v_n)v_n - G(x, f(v_n))\right] \\ &= \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{1}{2}g_0(x, f(u_n))f'(u_n)u_n - G_0(x, f(u_n))\right]. \end{aligned} \tag{4.22}$$

Now, from (g_3), (g_4) and the Lemma 1-(6), we have, for $s \geq 0$,

$$\begin{aligned} &\frac{1}{2}g_0(x, f(s))f'(s)s - G_0(x, f(s)) \\ &\geq \frac{1}{4}[g_0(x, f(s)) - g(x, f(s))]f(s) + \frac{1}{4}g(x, f(s))f(s) - G(x, f(s)) \\ &\quad + [G(x, f(s)) - G_0(x, f(s))] \\ &\geq -\frac{1}{4}h_3(x)|f(s)|^{q_3} - h_1(x) - h_2(x)|f(s)|^{q_2}. \end{aligned}$$

Moreover, from Lemma 10, it follows that

$$\int_{\mathbb{R}^N} h_i(x)|f(u_n)|^{q_i} \to \int_{\mathbb{R}^N} h_i(x)|f(u)|^{q_i}, \quad \text{as} \quad n \to \infty, \ i = 2, 3. \tag{4.23}$$

Thus, since $g_0(x, s) = 0$ if $s < 0$, by Fatou's Lemma, it follows

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{1}{2} g_0(x, f(u_n)) f'(u_n) u_n - G_0(x, f(u_n)) + h_2 |f(u_n)|^{q_2} + \frac{1}{4} h_3 |f(u_n)|^{q_3} + h_1\right]$$
$$\geq \int_{\mathbb{R}^N} \left[\frac{1}{2} g_0(x, f(u)) f'(u) u - G_0(x, f(u)) + h_2 |f(u)|^{q_2} + \frac{1}{4} h_3 |f(u)|^{q_3} + h_1\right]. \quad (4.24)$$

From (4.22)–(4.24) we obtain (4.20). The proof of Theorem 1 is complete. ⊓⊔

4.2 Proof of Theorem 2

We argue as in the initial steps of the proof of Theorem 1. Since g_0 satisfies (g_1) and (g_2), applying Corollary 2, we find a sequence $(v_n) \subset H^1(\mathbb{R}^N)$ such that

$$I_0(v_n) \to c \geq \alpha > 0 \quad \text{and} \quad \|I_0'(v_n)\|(1 + \|v_n\|) \to 0, \quad \text{as } n \to \infty, \quad (4.25)$$

where c is given by Theorem 3. By Lemma 4, we may suppose, without loss of generality, that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$. From this and (2.8), we have that v is a critical point of I_0, that is, $I_0'(v) = 0$. Hence, in order to prove Theorem 2, it suffices to assume that $v = 0$.

In view of Proposition 2, it follows that $0 < \alpha \leq c < \frac{1}{2N} \|K\|_\infty^{(2-N)/2} S^{N/2}$. Furthermore, by Proposition 1, there exist a sequence $(y_n) \subset \mathbb{R}^N$ and $r, \eta > 0$ such that $|y_n| \to \infty$, as $n \to \infty$, and

$$\limsup_{n\to\infty} \int_{B_r(y_n)} |v_n|^2 \geq \eta > 0, \quad \text{for all } n \in \mathbb{N}. \quad (4.26)$$

As in the proof of Theorem 1, we may assume that $(y_n) \subset \mathbb{Z}^N$. Then, defining $u_n(x) = v_n(x + y_n)$, $n \in \mathbb{N}$, we have that $\|u_n\|_0 = \|v_n\|_0$ for all $n \in \mathbb{N}$. Consequently, taking a subsequence if necessary, there exists $u \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $u_n \to u$ strongly in $L^2_{loc}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ almost everywhere in $\mathbb{R}^N$. We claim that u is a critical point of I_0. Indeed, given $\varphi \in H^1(\mathbb{R}^N)$, by (V), (K), (g_1) and (g_2), we get

$$\langle I_0'(u_n), \varphi\rangle \to \langle I_0'(u), \varphi\rangle, \quad \text{as } n \to \infty. \quad (4.27)$$

On the other hand, considering $\varphi_n(x) = \varphi(x - y_n)$ for all $n \in \mathbb{N}$, in view of the periodicities of V_0, K_0 and g_0, we get

$$\langle I_0'(u_n), \varphi\rangle = \langle I_0'(v_n), \varphi_n\rangle, \quad \text{for all } n \in \mathbb{N}.$$

Consequently, from (4.25) and the fact that $\|\varphi_n\|_0 = \|\varphi\|_0$ for all $n \in \mathbb{N}$, we conclude that

$$\langle I_0'(u_n), \varphi\rangle \to 0, \quad \text{as } n \to \infty.$$

This limit together with (4.27) shows that u is a critical point of I_0. The claim is proved. Furthermore, (4.26) implies that $u \neq 0$, and as in Theorem 1, $u > 0$. The Theorem 2 is proved.

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