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Calculus of Variations

Ricci flow of negatively curved incomplete surfaces

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Abstract We show uniqueness of Ricci flows starting at a surface of uniformly negative curvature, with the assumption that the flows become complete instantaneously. Together with the more general existence result proved in [10], this settles the issue of well-posedness in this class.

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1 Introduction

In 1982, Hamilton [5] introduced the study of Ricci flow, which evolves a Riemannian metric g on a manifold $\mathcal{M}$ under the nonlinear evolution equation

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}[g(t)]. \tag{1.1}$$

Hamilton proved that if $\mathcal{M}$ is closed (i.e. compact and without boundary) then for any initial metric g_0, there exist $T > 0$ and a smooth Ricci flow $g(t)$ for $t \in [0, T]$, with $g(0) = g_0$. We also have uniqueness: even if T is reduced, there can be no other such flow. (See also [4].) Shi [7] and Chen-Zhu [2] generalised this to the case of noncompact $\mathcal{M}$ in the case that the initial metric and all flows are assumed to be complete and with bounded curvature.

This theory left open the problem of starting a Ricci flow in the more general situation that the initial metric is incomplete. This possibility springs out when one contemplates, for example, restarting a Ricci flow after a finite-time singularity has occurred in the case that $\mathcal{M}$ has dimension at least 3.

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In [10], the second author developed a very general existence theorem for Ricci flows which produces (as a special case) a Ricci flow starting at any initial Riemannian surface of Gauss curvature bounded above—whether complete or not—which distinguishes itself by being complete at any strictly positive time. Evidence was given in [10] to support the idea that this *instantaneous completeness* should be the right condition to guarantee uniqueness also.

In this paper, we show that this is the case under the additional assumption that the upper bound for the Gauss curvature is negative. We also demonstrate how the existence issue is simpler in this case.

Theorem 1.1 *Suppose $\mathcal{M}$ is any surface (i.e. a 2-dimensional manifold without boundary) equipped with a smooth Riemannian metric g_0 whose Gauss curvature satisfies $K[g_0] \leq -\eta < 0$, but which need not be complete. Then there exists a* **unique** *smooth Ricci flow $g(t)$ for $t \in [0, \infty)$ with the following properties:*

(i) $g(0) = g_0$;
(ii) $g(t)$ *is complete for all* $t > 0$;
(iii) *the curvature of $g(t)$ is bounded above for any compact time interval within* $[0, \infty)$;
(iv) *the curvature of $g(t)$ is bounded below for any compact time interval within* $(0, \infty)$.

Moreover, this solution satisfies $K[g(t)] \leq -\frac{\eta}{1+2\eta t}$ for $t \geq 0$ and $-\frac{1}{2t} \leq K[g(t)]$ for $t > 0$.

Some discussion of what such flows look like can be found in [10]. Generally, as $t \downarrow 0$, they blow up in a manner reminiscent of reverse bubbling in the harmonic map heat flow (see [8] and [1]).

The main difficulty in proving the new uniqueness part of this result is that we do not assume directly any control on the behaviour of any competing Ricci flow near spatial infinity. This control needs to be built up by appealing to geometric results which can exploit our completeness assumption, and combining the results with a direct analysis of the conformal factor of the flows more in the spirit of the literature on the logarithmic fast-diffusion equation. The main input from the previous literature comes from [10] and Yau's version of the Schwarz Lemma, Theorem 2.3 (see [11]). We will also have to juggle two subtly different comparison principles: We prove a 'geometric comparison principle,' Theorem 4.2 which compares two Ricci flows under the hypotheses that one of them is complete, and certain curvature bounds are satisfied, and will also repeatedly appeal to a standard 'direct comparison principle,' Theorem A.1 which compares certain Ricci flows without looking beyond the equation satisfied by their conformal factors, and in particular without noticing their geometry.

2 A priori estimates on solutions

On a surface, the Ricci curvature of a metric g takes the simple form $\mathrm{Ric}[g] = K[g]g$, where $K[g]$ represents the Gauss curvature. Therefore the Ricci flow is the conformally invariant flow $\frac{\partial g}{\partial t} = -2K[g]g$ (which coincides with the Yamabe flow in this dimension).

If we choose a local complex coordinate $z = x + iy$ and write the metric locally as $g = \mathrm{e}^{2u}\,|\mathrm{d}z|^2$ (where $|\mathrm{d}z|^2 = \mathrm{d}x^2 + \mathrm{d}y^2$) then $K[g] = -\,\mathrm{e}^{-2u}\,\Delta u$ (where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is defined in terms of the local coordinates) and we can write the Ricci flow as

$$\frac{\partial u}{\partial t} = \mathrm{e}^{-2u}\,\Delta u = -K[u], \tag{2.1}$$

where we abuse notation here and in the sequel by abbreviating $K[\mathrm{e}^{2u}\,|\mathrm{d}z|^2]$ by $K[u]$.

The first observation to make about Theorem 1.1 is that without loss of generality, we may assume that g_0 is a conformal metric $\mathrm{e}^{2u_0}\,|\mathrm{d}z|^2$ on the unit disc $\mathcal{D} \subset \mathbb{C}$. Indeed, we can lift g_0 to the universal cover of $\mathcal{M}$, and since g_0 has uniformly negative Gauss curvature, the conformal type of this cover must be $\mathcal{D}$ (rather than S^2 or $\mathbb{C}$ which could be ruled out using the Gauss-Bonnet Theorem or by applying Corollary 2.5 below to large discs within $\mathbb{C}$, respectively). If we can establish both existence *and* uniqueness on the disc, then we can be sure to be able to quotient the solution to give ultimately a unique solution on the original surface.

Next we observe that by dilating g_0, and parabolically rescaling $g(t)$ (see [9, §1.2.3] for a discussion of parabolic rescaling) we may assume that $\eta = 1$ in Theorem 1.1. These considerations motivate the following:

Definition 2.1 A smooth Ricci flow $g(t)$ on $\mathcal{D}$ for $t \in [0, T]$ is called **admissible**, provided

(i) $g(t)$ is complete for $t > 0$;
(ii) $K[g] \leq C$ on $[0, T] \times \mathcal{D}$;
(iii) $K[g] \geq -C_\varepsilon$ on $[\varepsilon, T] \times \mathcal{D}$ for all $\varepsilon \in (0, T]$.

Lemma 2.2 *Suppose* $\mathrm{e}^{2u_0}\,|\mathrm{d}z|^2$ *is a smooth metric on the disc* $\mathcal{D}$ *with* $K[u_0] \leq -1$ *and* $\mathrm{e}^{2w(t)}\,|\mathrm{d}z|^2$ *is an admissible Ricci flow on* $[0, T] \times \mathcal{D}$ *with initial condition* $w(0) = u_0$. *Then* w *satisfies*

(A) $K[w] \geq -\frac{1}{2t}$,
(B) $w(t, x) \geq \ln \frac{2}{1-|x|^2} + \frac{1}{2}\ln(2t)$,

on $(0, T] \times \mathcal{D}$, *while on* $[0, T] \times \mathcal{D}$ *we have*

(C) $w(t, x) \leq \ln \frac{2}{1-|x|^2} + \frac{1}{2}\ln(2t + 1)$,
(D) $w(t, x) \geq u_0(x) - Ct$.

The proof relies on the following special case of the Schwarz lemma of S.-T. Yau. For convenience we give a proof in Appendix B.

Theorem 2.3 (Schwarz-Pick-Ahlfors-Yau [11]) *Let* $(\mathcal{M}_1, g_1)$ *and* $(\mathcal{M}_2, g_2)$ *be two Riemannian surfaces without boundary. If*

(a) $(\mathcal{M}_1, g_1)$ *is complete,*
(b) $K[g_1] \geq -a_1$ *for some number* $a_1 \geq 0$, *and*
(c) $K[g_2] \leq -a_2 < 0$,

then any conformal map $f : \mathcal{M}_1 \to \mathcal{M}_2$ *satisfies*

$$f^*(g_2) \leq \frac{a_1}{a_2} g_1.$$

Setting $\mathcal{M}_1 = \mathcal{M}_2$ to be a disc, $a_1 = a_2 = C > 0$, $f = \mathrm{id}$ and either g_1 or g_2 to be H, the complete metric of constant curvature $-C$, one obtains barriers for metrics of uniformly negative curvature. The most significant consequence is the following lower bound.

Corollary 2.4 *Let* g *be a complete conformal Riemannian metric on a disc, whose Gauss curvature is bounded from below by a constant* $-C < 0$. *If* H *is the complete conformal metric of constant curvature* $-C$ *on that disc, then*

$$H \leq g.$$

A further consequence (which also follows by a more elementary comparison argument) is an upper bound for (possibly incomplete) negatively curved surfaces:

Corollary 2.5 *Let g be a conformal Riemannian metric on a disc, whose curvature is bounded from above by a constant $-C < 0$. If H is the complete conformal metric of constant curvature $-C$ on that disc, then*

$$g \leq H.$$

Proof of Lemma 2.2 The Gauss curvature obeys the equation

$$\frac{\partial K}{\partial t} = \Delta K + 2K^2,$$

under Ricci flow (see for example [9, Proposition 2.5.4]) and one may apply the comparison principle (for example [3, Theorem 12.14]) if the flow is complete and its curvature is bounded. For $\varepsilon > 0$ the conditions (ii) and (iii) of Definition 2.1 give such a uniform bound on $K[w(t)]$ restricted to the time interval $t \in [\varepsilon, T]$. Comparing to the solution of the ODE $\frac{\partial}{\partial t} k = 2k^2$ with the lower curvature bound $-C_\varepsilon$ as initial condition at time $t = \varepsilon$, yields

$$K[w|_{[\varepsilon,T]}] \geq -\frac{1}{2(t-\varepsilon) + C_\varepsilon^{-1}} \geq -\frac{1}{2(t-\varepsilon)}.$$

Letting $\varepsilon \to 0$, one obtains (A).

For any time $t > 0$, the metric $\mathrm{e}^{2w(t)}\,|\mathrm{d}z|^2$ is complete (i) and has curvature bounded from below (A). Using Corollary 2.4 one obtains directly (B), since the conformal metric of constant curvature $-\frac{1}{2t}$ on the disc is

$$2t\left(\frac{2}{1-|x|^2}\right)^2 |\mathrm{d}z|^2.$$

For small $\delta > 0$, consider $w|_{\overline{\mathcal{D}_{1-\delta}}}$ and write the conformal factor of the Ricci flow on the disc of radius $1-\delta$ with Gauss curvature initially -1 as

$$h_\delta(t,x) := \ln\frac{2(1-\delta)}{(1-\delta)^2 - |x|^2} + \frac{1}{2}\ln(2t+1).$$

By Corollary 2.5 we have $w|_{\mathcal{D}_{1-\delta}}(0,\cdot) \leq h_\delta(0,\cdot)$. Furthermore $w|_{\overline{\mathcal{D}_{1-\delta}}}$ and h_δ fulfil the requirements for the direct comparison principle (Theorem A.1), thus $w|_{\mathcal{D}_{1-\delta}} \leq h_\delta$ holds throughout $[0,T] \times \mathcal{D}_{1-\delta}$. Since h_δ is continuous in δ, letting $\delta \to 0$ yields (C).

Finally, the upper curvature bound (ii) and the evolution equation of w gives

$$\frac{\partial}{\partial t} w(t,x) = -K[w(t,x)] \geq -C,$$

which integrates to give (D). □

3 Existence

The existence of the solution given by Theorem 1.1 is a special case of the more general existence theory developed in [10]. However, the proof can be streamlined in the case that $(\mathcal{M}, g_0)$ is conformally hyperbolic (by which we mean that it can be made hyperbolic by a conformal change of metric) and in this section we sketch this simplified proof in the particular case that $(\mathcal{M}, g_0)$ has $K[g_0] \leq -1$ and is conformally the disc $\mathcal{D}$. (In this paper we can reduce to this case by virtue of our uniqueness result as described in Sect. 2.)

Theorem 3.1 (Existence, special case of [10, Theorem 1.1]) *Let g_0 be a smooth conformal metric on $\mathcal{D}$ (possibly incomplete) with $K[g_0] \leq -1$. Then there exists a smooth Ricci flow $G(t)$ on $\mathcal{D}$, for $t \in [0, \infty)$ with $G(0) = g_0$, such that $G(t)$ is complete for every $t > 0$. The Gauss curvature of this instantaneously complete solution satisfies*

$$-\frac{1}{2t} < K[G(t)] \leq -\frac{1}{2t+1} \qquad \text{for } t > 0.$$

Moreover, $G(t)$ is maximal in the sense that if $g(t)$ for $t \in [0, \varepsilon] \subset [0, \infty)$ is another Ricci flow with $g(0) = g_0$, then

$$g(t) \leq G(t)$$

for all $t \in [0, \varepsilon]$.

The properties described in Lemma 2.2 also apply to these solutions.

Proof We follow the basic strategy of [10], constructing $G(t)$ as a limit of approximating Ricci flows on smaller base manifolds. We make some simplification of the convergence, and exploit what we know about the conformal type to simplify the proof of instantaneous completeness in this special case.

Let $u_0 : \mathcal{D} \to \mathbb{R}$ be the conformal factor of g_0, that is, $g_0 = \mathrm{e}^{2u_0} \, |\mathrm{d}z|^2$.

For each $k \in \mathbb{N}$, define $D_k := \mathcal{D}_{1-\frac{1}{k+1}}$ to be the disc of radius $1-\frac{1}{k+1}$, and let $h_k : D_k \to \mathbb{R}$ defined by

$$h_k(x) := \ln \frac{\frac{2}{k+1}}{\left(1 - \frac{1}{k+1}\right)^2 - |x|^2}$$

be the conformal factor of the complete conformal metric of curvature $-k^2$ on D_k. Note that h_k is pointwise (weakly) decreasing in the sense that for all $x \in \mathcal{D}$, and k sufficiently large so that $x \in D_k$, the sequence $h_k(x)$ is weakly decreasing.

Loosely following [10], given $\eta > 0$ we choose a smooth cut-off function $\Psi : \mathbb{R} \to \mathbb{R}$ with the properties that $\Psi(s) = 0$ for $s \leq -\eta$, $\Psi(s) = s$ for $s \geq \eta$, and $\Psi''(s) \geq 0$ for all s. Then $0 \leq \Psi' \leq 1$ and $\Psi(s) \geq s$ for all s. We use Ψ to define the metric

$$\bar{g}_k = \mathrm{e}^{2\Psi(h_k - u_0)} \, g_0$$

on D_k, which can be viewed as a smoothed-out 'pointwise maximum' of the metrics represented by u_0 and h_k. Writing $\bar{u}_k : D_k \to \mathbb{R}$ for the conformal factor of $\bar{g}_k$, that is, $\bar{g}_k = \mathrm{e}^{2\bar{u}_k} \, |\mathrm{d}z|^2$, we have

$$\bar{u}_k \geq u_0|_{D_k} \quad \text{and } \bar{u}_k \geq h_k.$$

Just as in [10, §4], we see that $K[\bar{g}_k]$ is bounded below (with lower bound dependent on k) and abbreviating $w_k := h_k - u_0|_{D_k}$, we compute the uniform upper curvature bound

$$\begin{aligned}
K[\bar{g}_k] &= -\,\mathrm{e}^{-2(\Psi(w_k)+u_0)}\,\Delta(\Psi(w_k) + u_0) \\
&= -\,\mathrm{e}^{-2(\Psi(w_k)+u_0)}(\Psi''(w_k)|\nabla w_k|^2 + \Psi'(w_k)\Delta(h_k - u_0) + \Delta u_0) \\
&\leq -\,\mathrm{e}^{-2(\Psi(w_k)+u_0)}(\Psi'(w_k)\Delta h_k + (1 - \Psi'(w_k))\Delta u_0) \\
&= \mathrm{e}^{-2(\Psi(w_k)+u_0)}(\Psi'(w_k)(\mathrm{e}^{2h_k}\,K[h_k]) + (1 - \Psi'(w_k))(\mathrm{e}^{2u_0}\,K[u_0])) \\
&= \Psi'(w_k)\,\mathrm{e}^{-2(\Psi(w_k)-w_k)}\,K[h_k] + (1 - \Psi'(w_k))\,\mathrm{e}^{-2\Psi(w_k)}\,K[u_0] \\
&\leq \mathrm{e}^{-2\eta}(\Psi'(w_k)K[h_k] + (1 - \Psi'(w_k))K[u_0]) \\
&\leq -\,\mathrm{e}^{-2\eta}\,.
\end{aligned}$$

In the last-but-one line we used the facts that both $K[h_k]$ and $K[u_0]$ are negative, and also that $\Psi(s) - s \leq \eta$ where $\Psi'(s) \neq 0$ (i.e. for $s \geq -\eta$) and $\Psi(s) \leq \eta$ where $\Psi'(s) \neq 1$ (i.e. for $s \leq \eta$). The last line follows from the fact that both $K[h_k] \leq -1$ and $K[u_0] \leq -1$.

The conformal factors $\bar{u}_k$ are (weakly) decreasing (as the h_k are decreasing) and

$$\lim_{k\to\infty} \bar{u}_k(x) = u_0(x).$$

Let $g_k(t)$ be the Ricci flow as given by Shi [7] with $g_k(0) = \bar{g}_k$, on D_k over a maximal time interval $[0, T)$. Since these Ricci flows are each complete with bounded curvature, the maximum principle (as in Lemma 2.2) tells us that $K[g_k(t)] \geq -\frac{1}{2t}$ for $t > 0$ but also that $K[g_k(t)] \leq -\frac{1}{2t+e^{2\eta}}$ for $t \geq 0$. In particular, we must have $T = \infty$ – i.e. long-time existence for each $g_k(t)$ – since the curvature is known to blow up at a singularity of a Ricci flow.

Let $u_k : [0, \infty) \times D_k \to \mathbb{R}$ be the conformal factor of g_k, that is, $g_k = e^{2u_k} |dz|^2$. Since the conformal factors $u_k(0) = \bar{u}_k$ are decreasing in k, we can compare $u_k|_{\overline{D_{k-1}}}(t)$ and $u_{k-1}(t)$ using Theorem A.1 to find that the sequence $u_k(t)$ is (weakly) decreasing in the sense that at each point $x \in \mathcal{D}$ and $t \geq 0$, for sufficiently large k so that $x \in D_k$ we have $u_k(t, x)$ (weakly) decreasing.

By virtue of (2.1) we have

$$\frac{\partial u_k(t)}{\partial t} = -K[u_k(t)] \geq 0,$$

and hence $u_0(x) \leq \bar{u}_k(x) \leq u_k(t, x)$ for all $x \in D_k$ and $t > 0$, so it makes sense to define $u : [0, \infty) \times \mathcal{D} \to \mathbb{R}$ by

$$u(t, x) = \lim_{k\to\infty} u_k(t, x),$$

and consider the corresponding metric flow $G(t) := e^{2u(t)} |dz|^2$.

By parabolic regularity theory we can see that $G(t)$ will be a smooth Ricci flow inheriting the curvature estimates of $g_k(t)$ and satisfying $G(0) = g_0$.

To see the instantaneous completeness of $G(t)$, we compare u with the conformal factor $h(t, x) := \ln \frac{2}{1-|x|^2} + \frac{1}{2}\ln(2t)$ of the 'big-bang' Ricci flow $e^{2h(t)} |dz|^2$ which is the complete metric on $\mathcal{D}$ of constant curvature $-\frac{1}{2t}$ at time $t > 0$. Indeed, the comparison principle of Theorem A.1 tells us that $h(t, x) \leq u_k(t, x)$ for all $x \in D_k$ and $t > 0$, and therefore, by taking the limit $k \to \infty$, $h(t, x) \leq u(t, x)$ for all $x \in \mathcal{D}$ and $t > 0$. Consequently, for all $t > 0$, we have $G(t) \geq e^{2h(t)} |dz|^2$. The completeness of $G(t)$ for $t > 0$ then follows from the completeness of $e^{2h(t)} |dz|^2$.

The maximality of $G(t)$ follows from a similar comparison argument. If $\tilde{u} : [0, \varepsilon]\times\mathcal{D} \to \mathbb{R}$ is the conformal factor of any other Ricci flow with $\tilde{u}(0, \cdot) = u_0$, then the comparison principle of Theorem A.1 tells us that $\tilde{u}|_{D_k}(t, x) \leq u_k(t, x)$ for all $x \in D_k$ and $t \in [0, \varepsilon]$, and therefore (taking $k \to \infty$) $\tilde{u}(t, x) \leq u(t, x)$ for all $x \in \mathcal{D}$ and $t \in [0, \varepsilon]$.

We have almost finished, except that we appear to have constructed a flow $G(t)$ for each $\eta > 0$, and each of these is guaranteed only to have its Gauss curvature bounded above by $-\frac{1}{2t+e^{2\eta}}$. However, it is not hard to see that there can exist only one *maximal* solution, and so all of the flows $G(t)$ must be identical. At this point we may take the limit $\eta \downarrow 0$ and deduce that $K[G(t)] \leq -\frac{1}{2t+1}$ for $t \geq 0$, which completes the proof. □

4 Uniqueness

The following theorem states the uniqueness part of the main Theorem 1.1 and concludes its proof.

Theorem 4.1 *Let* $e^{2u_0}|dz|^2$ *be a smooth metric on the unit disk* $\mathcal{D}$ *satisfying the upper curvature bound* $K[u_0] \leq -1$. *For some* $T > 0$ *let* $e^{2v(t)}|dz|^2$ *be an admissible Ricci flow (Definition 2.1) with* $v(0) = u_0$. *Then* $e^{2v(t)}|dz|^2$ *is unique among such instantaneously complete solutions.*

The proof relies on the following *geometric* comparison result.

Theorem 4.2 (Geometric comparison principle) *Suppose* $(\mathcal{M}, g_1(t))$ *and* $(\mathcal{M}, g_2(t))$ *are two conformally equivalent Ricci flows on some time interval* $[0, T]$, *and define* $Q : [0, T] \times \mathcal{M} \to \mathbb{R}$ *to be the function for which* $g_1(t) = e^{2Q(t)} g_2(t)$. *Suppose further that* $g_2(t)$ *is complete for each* $t \in [0, T]$ *and that for some constant* $C \geq 0$ *we have*

$$\text{(i)}|K[g_2]| \leq C, \qquad \text{(ii)}K[g_1] \leq C, \qquad \text{(iii)}Q \leq C$$

on $[0, T] \times \mathcal{D}$. *If* $g_1(0) \leq g_2(0)$, *then* $g_1(t) \leq g_2(t)$ *for all* $t \in [0, T]$.

Proof With respect to a local complex coordinate z, let us write $g_1(t) = e^{2u(t)}|dz|^2$ and $g_2(t) = e^{2\tilde{v}(t)}|dz|^2$ for some locally defined functions $u(t)$ and $\tilde{v}(t)$, and note then that $Q = u - \tilde{v}$. Since $g_1(t)$ and $g_2(t)$ are Ricci flows, we get

$$\frac{\partial(u-\tilde{v})}{\partial t} = e^{-2u}\Delta u - e^{-2\tilde{v}}\Delta\tilde{v} = (e^{-2u} - e^{-2\tilde{v}})\Delta u + e^{-2\tilde{v}}\Delta(u-\tilde{v}).$$

Writing $\Delta_{g_2(t)}$ for the Laplace-Beltrami operator with respect to the metric $g_2(t)$, we obtain, where $Q > 0$ (i.e. where $u > \tilde{v}$)

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_{g_2(t)}\right) Q &= \left(\frac{\partial}{\partial t} - e^{-2\tilde{v}}\Delta\right)(u-\tilde{v}) \\ &= (e^{-2u} - e^{-2\tilde{v}})\Delta u = (u-\tilde{v})\frac{e^{-2u} - e^{-2\tilde{v}}}{u-\tilde{v}}\Delta u \\ &= (u-\tilde{v})(-2)e^{-2\xi}\Delta u = 2(u-\tilde{v})e^{2(u-\xi)}(-e^{-2u}\Delta u) \\ &= 2e^{2(u-\xi)}K[g_1](u-\tilde{v}) \\ &\leq 2e^{2C}C(u-\tilde{v}) = (2Ce^{2C})Q,\end{aligned}$$

where at each point, ξ was chosen between u and $\tilde{v}$ according to the mean value theorem. Applying the weak maximum principle to Q (see for example [3, Theorem 12.10] with $g_2(0)$ as complete background metric with bounded curvature and $g_2(t)$ as one-parameter family of complete metrics) keeping in mind that $Q(0,\cdot) \leq 0$, we conclude that $Q \leq 0$ throughout $[0, T] \times \mathcal{M}$ as desired. □

Proof of Theorem 4.1 Theorem 3.1 provides the existence of such an admissible solution $e^{2u(t)}|dz|^2$ with $u(0) = u_0$. Let $e^{2v(t)}|dz|^2$ be any another admissible solution with the same initial condition $v(0) = u_0$. From Theorem 3.1 we know that $u(t)$ is maximal among such instantaneously complete solutions, and in particular, $v(t) \leq u(t)$ for all $t \in [0, T]$. Hence it remains to show the converse inequality $u(t) \leq v(t)$.

Let $C > 0$ be the uniform upper bound of the curvature of v: $K[v(t)] \leq C$ for all $t \in [0, T]$, which exists since v is admissible. For small $\delta \in (0, T)$ define

$$\tilde{v}(t, x) := v(e^{-2C\delta}(t+\delta), x) + C\delta \qquad \text{for } (t, x) \in [0, T-\delta] \times \mathcal{D},$$

which is a slight adjustment of v, again a solution to the Ricci flow:

$$\left(\frac{\partial}{\partial t}\tilde{v} - \mathrm{e}^{-2\tilde{v}}\,\Delta\tilde{v}\right)(t,x) = \mathrm{e}^{-2C\delta}\left(\frac{\partial}{\partial t}v - \mathrm{e}^{-2v}\,\Delta v\right)(\mathrm{e}^{-2C\delta}(t+\delta),x) = 0.$$

Our aim is to show that u is a lower bound for $\tilde{v}$, and hence (by taking $\delta \downarrow 0$) also a lower bound for v as desired. To do this, we wish to apply Theorem 4.2 to the Ricci flows $g_1(t)$ and $g_2(t)$ generated by the conformal factors u and $\tilde{v}$ respectively.

First, note that $g_2(t)$ is complete for all $t \in [0, T-\delta]$ since $\mathrm{e}^{2v}\,|\mathrm{d}z|^2$ is an admissible Ricci flow and is therefore complete for all $t \in (0, T]$. Furthermore, $g_2(t)$ has upper and lower curvature bounds:

$$|K[\tilde{v}]| \leq \sup_{[\mathrm{e}^{-2C\delta}\,\delta, T]\times\mathcal{D}} \mathrm{e}^{-2C\delta}\,|K[v]| < \infty, \tag{4.1}$$

so hypothesis (i) of Theorem 4.2 is satisfied. The upper bound for the curvature of $g_1(t) = \mathrm{e}^{2u}\,|\mathrm{d}z|^2$ required by hypothesis (ii) of Theorem 4.2 follows since $g_1(t)$ was constructed to be admissible.

Next we verify hypothesis (iii) of Theorem 4.2, namely that $u - \tilde{v}$ is bounded from above. Applying (C) of Lemma 2.2 to $\mathrm{e}^{2u}\,|\mathrm{d}z|^2$, we find that

$$u(t,x) \leq \ln\frac{2}{1-|x|^2} + \frac{1}{2}\ln(2t+1), \tag{4.2}$$

for $t \in [0, T]$, while (B) of Lemma 2.2 applied to $\mathrm{e}^{2v}\,|\mathrm{d}z|^2$ gives

$$v(t,x) \geq \ln\frac{2}{1-|x|^2} + \frac{1}{2}\ln(2t),$$

for $t \in (0, T]$ and hence that

$$\begin{aligned}\tilde{v}(t,x) &\geq \ln\frac{2}{1-|x|^2} + \frac{1}{2}\ln(2\,\mathrm{e}^{-2C\delta}(t+\delta)) + C\delta \\ &= \ln\frac{2}{1-|x|^2} + \frac{1}{2}\ln(2(t+\delta))\end{aligned} \tag{4.3}$$

for $t \in [0, T-\delta]$. Subtracting (4.3) from (4.2), we find that

$$u - \tilde{v} \leq \frac{1}{2}\ln(2T+1) - \frac{1}{2}\ln(2\delta)$$

as desired.

The final hypothesis of Theorem 4.2 to verify is that $g_1(0) \leq g_2(0)$, i.e. that $u(0,\cdot) \leq \tilde{v}(0,\cdot)$. But

$$\tilde{v}(0,\cdot) = v(\mathrm{e}^{-2C\delta}\,\delta,\cdot) + C\delta \geq u_0 - C\,\mathrm{e}^{-2C\delta}\,\delta + C\delta \geq u_0 = u(0,\cdot)$$

by part (D) of Lemma 2.2 as desired.

We may therefore apply Theorem 4.2 over the time interval $[0, T-\delta]$ to deduce that $u(t) \leq \tilde{v}(t)$ for all $t \in [0, T-\delta]$. Hence, given any $(t,x) \in [0,T) \times \mathcal{D}$, we conclude

$$u(t,x) \leq \lim_{\delta\downarrow 0}\tilde{v}(t,x) = v(t,x).$$

□

Appendix A: Comparison principle

In this appendix we clarify the statement and proof of one of the many variants of the standard weak maximum principle.

Theorem A.1 (Direct comparison principle) *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain and for some $T > 0$ let $u \in C^{1,2}((0,T)\times\Omega) \cap C([0,T]\times\bar{\Omega})$ and $v \in C^{1,2}((0,T)\times\Omega) \cap C([0,T]\times\Omega)$ both be solutions of the Ricci flow equation (2.1) for the conformal factor of the metric. Furthermore, suppose that for each $t \in [0,T]$ we have $v(t,x) \to \infty$ as $x \to \partial\Omega$. If $v(0,x) \geq u(0,x)$ for all $x \in \Omega$, then $v \geq u$ on $[0,T]\times\Omega$.*

Proof For every $\varepsilon > 0$ consider

$$v_\varepsilon(t,x) := v\left(\frac{1}{\varepsilon}\ln(\varepsilon t+1), x\right) + \frac{1}{2}\ln(\varepsilon t+1) \qquad \text{for all } (t,x) \in [0,T]\times\Omega,$$

which is well-defined since $\frac{1}{\varepsilon}\ln(\varepsilon t+1) \leq t$ for all $t \geq 0$. Observe that v_ε is a slight modification of v, with $v_\varepsilon(0,\cdot) = v(0,\cdot)$, and v_ε converges pointwise to v as $\varepsilon \to 0$, but in contrast to v it is a strict supersolution of the Ricci flow (2.1):

$$\begin{aligned}\left(\frac{\partial}{\partial t}v_\varepsilon - \mathrm{e}^{-2v_\varepsilon}\,\Delta v_\varepsilon\right)(t,x) &= \frac{1}{\varepsilon t+1}\left(\frac{\partial}{\partial t}v - \mathrm{e}^{-2v}\,\Delta v\right)\left(\frac{1}{\varepsilon}\ln(\varepsilon t+1), x\right) + \frac{\varepsilon}{2(\varepsilon t+1)} \\ &= \frac{\varepsilon}{2(\varepsilon t+1)} > 0 \quad \text{for } (t,x) \in [0,T]\times\Omega. \end{aligned} \tag{A.1}$$

We are going to prove $(v_\varepsilon - u) \geq 0$ on $[0,T]\times\Omega$ and conclude the theorem's statement by letting $\varepsilon \to 0$. Since by hypothesis u is continuous on $[0,T]\times\bar{\Omega}$ and $v_\varepsilon(t,\cdot)$ blows up near the boundary $\partial\Omega$ for each $t \in [0,T]$, we have

$$(v_\varepsilon - u)(t,x) \to \infty \quad \text{as} \quad x \to \partial\Omega,$$

for every time $t \in [0,T]$ and hence $(v_\varepsilon - u)(t,\cdot)$ attains its infimum in Ω. Now assume that $(v_\varepsilon - u)$ becomes negative in $[0,T]\times\Omega$, and define the time t_0 at which $(v_\varepsilon - u)$ first becomes negative by

$$t_0 := \inf\{t \in [0,T] : \min_{x\in\Omega}(v_\varepsilon - u)(t,x) < 0\} \in [0,T).$$

Picking any minimum $x_0 \in \Omega$ of $(v_\varepsilon - u)(t_0,\cdot)$, we have

$$(v_\varepsilon - u)(t_0,x_0) = 0, \quad \Delta(v_\varepsilon - u)(t_0,x_0) \geq 0 \quad \text{and} \quad \frac{\partial}{\partial t}(v_\varepsilon - u)(t_0,x_0) \leq 0.$$

Subtracting the Ricci flow equation (2.1) from (A.1) at this point (t_0,x_0), we find

$$\begin{aligned} 0 &< \left(\frac{\partial}{\partial t}v_\varepsilon - \mathrm{e}^{-2v_\varepsilon}\,\Delta v_\varepsilon\right)(t_0,x_0) - \left(\frac{\partial}{\partial t}u - \mathrm{e}^{-2u}\,\Delta u\right)(t_0,x_0) \\ &= \frac{\partial}{\partial t}(v_\varepsilon - u)(t_0,x_0) - \mathrm{e}^{-2u(t_0,x_0)}\,\Delta(v_\varepsilon - u)(t_0,x_0) \leq 0, \end{aligned}$$

which is a contradiction. Therefore $v_\varepsilon \geq u$ on $[0,T]\times\Omega$, and the corresponding statement for v follows by letting $\varepsilon \to 0$. □

Appendix B: Yau's Schwarz Lemma

For convenience, we prove now the Schwarz lemma of Yau (Theorem 2.3). The proof uses the following generalised maximum principle by Omori, whose proof was simplified by Yau in [12, Theorem 1, p. 206]:

Theorem B.1 *[6, Theorem A′, p. 211] On a complete Riemannian surface $(\mathcal{M}, g)$ with Gaussian curvature bounded from below, let f be a C^2-function which is bounded above. Then, for an arbitrarily point $p \in \mathcal{M}$ and for any $\varepsilon > 0$, there exists a point q depending on p such that*

$$\text{(i) } \Delta_g f(q) < \varepsilon; \qquad \text{(ii) } |\nabla f(q)|_g < \varepsilon; \qquad \text{(iii) } f(q) \geq f(p).$$

The essential idea [12] to find q is to imagine the point $(q, f(q))$ on the graph of f in $\mathcal{M} \times \mathbb{R}$ which is closest to the point (p, k) for some enormous $k \gg 1$.

Proof of Theorem 2.3 By dilating g_1 and g_2, we may assume $a_1 = 1 = a_2$, that is $K[g_1] \geq -1 \geq K[g_2]$. Since we only need the theorem in the case that f is strictly conformal, we will assume this in the proof and leave the minor adjustments required for the full theorem to the reader.[1] Define $w \in C^\infty(\mathcal{M}_1)$ by

$$f^*(g_2) = \mathrm{e}^{2w}\, g_1. \tag{B.1}$$

It remains to show that $w \leq 0$. Assume instead that there exists $p \in \mathcal{M}_1$ with $w(p) > 0$. Then we can choose an $\varepsilon \in (0, 1)$ such that

$$\varepsilon < \frac{\mathrm{e}^{w(p)} - \mathrm{e}^{-w(p)}}{1 + \mathrm{e}^{w(p)}}. \tag{B.2}$$

Now define $\tilde{w}(x) := -\,\mathrm{e}^{-w(x)}$ for all $x \in \mathcal{M}_1$. Since $(\mathcal{M}_1, g_1)$ is complete with curvature bounded from below and $\tilde{w}$ is bounded above, we may apply Theorem B.1 to find a point $q \in \mathcal{M}_1$ with

$$\Delta_{g_1} \tilde{w}(q) < \varepsilon, \qquad |\nabla \tilde{w}|^2_{g_1}(q) < \varepsilon \quad \text{and} \quad \tilde{w}(q) \geq \tilde{w}(p). \tag{B.3}$$

Since $x \mapsto -\,\mathrm{e}^{-x}$ is strictly increasing, we also have $w(q) \geq w(p) > 0$. Now compute

$$\begin{aligned} \Delta_{g_1} \tilde{w} &= \mathrm{e}^{-w}\, \Delta_{g_1} w - \mathrm{e}^{-w}\, |\nabla w|^2_{g_1} \\ &= \mathrm{e}^{-w}\, \Delta_{g_1} w - \mathrm{e}^{w}\, |\nabla \tilde{w}|^2_{g_1}. \end{aligned} \tag{B.4}$$

By computing with respect to a local complex coordinate, we find that

$$\Delta_{g_1} w = -\,\mathrm{e}^{2w}\, K[g_2] \circ f + K[g_1],$$

which together with the curvature estimates $-K[g_2] \geq 1$ and $K[g_1] \geq -1$ gives $\mathrm{e}^{-w}\, \Delta_{g_1} w \geq \mathrm{e}^{w} - \mathrm{e}^{-w}$, and so (B.4) improves to

$$\Delta_{g_1} \tilde{w} \geq \mathrm{e}^{w} (1 - |\nabla \tilde{w}|^2_{g_1}) - \mathrm{e}^{-w}.$$

Evaluating at q, using (B.3) and the fact that $w(q) \geq w(p)$, we obtain

$$\varepsilon > \Delta_{g_1} \tilde{w}(q) \geq \mathrm{e}^{w(q)}(1 - \varepsilon) - \mathrm{e}^{-w(q)} \geq \mathrm{e}^{w(p)}(1 - \varepsilon) - \mathrm{e}^{-w(p)}$$

[1] In the weakly conformal case f might either be constant (nothing to prove) or have isolated singular points $P := \{p_1, p_2, \ldots\}$. The function w we define in (B.1) will then have logarithmic singularities on P, but will be strictly negative close to such singularities and the $\tilde{w}$ of the proof could be adjusted to a smooth function on the whole of $\mathcal{M}_1$ (including P) without altering anything where w is positive.

and hence

$$\varepsilon > \frac{e^{w(p)} - e^{-w(p)}}{1 + e^{w(p)}}$$

which contradicts (B.2). □

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References

1. Bertsch, M., Dal Passo, R., van der Hout, R.: Nonuniqueness for the heat flow of harmonic maps on the disk. Arch. Ration. Mech. Anal. **161**, 93–112 (2002)
2. Chen, B.-L., Zhu, X.-P.: Uniqueness of the Ricci flow on complete noncompact manifolds. J. Differ. Geom. **74**, 119–154 (2006)
3. Chow, B., Chu, S.-C., Glickenstein, D., Guenther, C., Isenberg, J., Ivey, T., Knopf, D., Lu, P., Luo, F., Ni, L.: The Ricci Flow: Techniques and Applications: Part II: Analytic Aspects. Volume **144** of Mathematical Surveys and Monographs. American Mathematical Society, 2008
4. DeTurck, D.: Deforming metrics in the direction of their Ricci tensors. In: Collected papers on Ricci flow. Cao, H.D., Chow, B., Chu, S.C., Yau, S.T. (eds.) Series in Geometry and Topology, **37**. International Press, 2003
5. Hamilton, R.S.: Three-manifolds with positive Ricci curvature. J. Differ. Geom. **17**, 255–306 (1982)
6. Omori, H.: Isometric immersions of Riemannian manifolds. J. Math. Soc. Jpn. **19**(2), 205–214 (1967)
7. Shi, W.-X.: Deforming the metric on complete Riemannian manifolds. J. Differ. Geom. **30**, 223–301 (1989)
8. Topping, P.M.: Reverse bubbling and nonuniqueness in the harmonic map flow. Int. Math. Res. Notes **10**, 505–520 (2002)
9. Topping, P.M.: Lectures on the Ricci flow. L.M.S. Lecture notes series **325** C.U.P. (2006) http://www.warwick.ac.uk~maseq/RFnotes.html
10. Topping, P.M.: Ricci flow compactness via pseudolocality, and flows with incomplete initial metrics. J. Eur. Math. Soc. (JEMS) (to appear)
11. Yau, S.-T.: Remarks on conformal transformations. J. Differ. Geom. **8**, 369–381 (1973)
12. Yau, S.-T.: Harmonic functions on complete Riemannian manifolds. Comm. Pure Appl. Math. **28**, 201–228 (1975)