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Calculus of Variations

Radial and non radial solutions for Hardy–Hénon type elliptic systems

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Abstract We discuss existence and non-existence of positive solutions for the following system of Hardy and Hénon type:

$$\begin{cases} -\Delta v = |x|^{\alpha} u^{p}, \ -\Delta u = |x|^{\beta} v^{q} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \ni 0$ is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, $p, q > 1$, and $\alpha, \beta > -N$. We also study symmetry breaking for ground states when Ω is the unit ball in $\mathbb{R}^N$.

Mathematics Subject Classification (2000) 35J47 · 35J60 · 35J20

1 Introduction

In this paper we consider the following system of superlinear elliptic equations of Hardy and Hénon type:

$$\begin{cases} -\Delta v = |x|^{\alpha} u^{p} & \text{in } \Omega, \\ -\Delta u = |x|^{\beta} v^{q} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where Ω is a bounded domain in $\mathbb{R}^N$ with $0 \in \Omega$, $N \geq 3$, $p, q > 1$, and $\alpha, \beta > -N$. In particular, we will investigate existence, multiplicity and qualitative properties (such as radial symmetry in the case Ω a ball) of solutions.

The case of a single equation (especially the case $\alpha > 0$) has been widely studied (see for instance [2–4,7,24,26], and the references therein). Recall that the equation

$$\begin{cases} -\Delta u = |x|^\alpha u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

is called of Hardy type if $\alpha < 0$ (because of its relation to the Hardy-Sobolev inequality) and it is of Hénon type if $\alpha > 0$ (this equation was introduced by Hénon in 1973 [15] for the study of stellar systems). One has the following results:
Hardy type: By variational methods one obtains the existence of a nontrivial solution of (2) in $H_0^1(\Omega)$ ($\Omega \subset R^N$ an arbitrary bounded domain) provided that $2 < p+1 < \frac{2(N-|\alpha|)}{N-2}$, by an application of the celebrated Caffarelli-Kohn-Nirenberg estimates (CKN, [6]), and due to a generalized Pohozaev type identity one proves non-existence of nontrivial solutions in starshaped domains if $0 \geq \alpha > -N$ and $p+1 = \frac{2(N-|\alpha|)}{N-2}$.
Hénon type: For $\alpha > 0$ one obtains for Ω arbitrary (bounded) the existence of a solution for $2 < p+1 < \frac{2N}{N-2}$. On the other hand, if Ω is a ball, then one has the existence of a radial solution in a larger range, namely for $2 < p+1 < \frac{2(N+\alpha)}{N-2}$ (see [23]), and the non-existence of nontrivial solutions in the range: $p+1 \geq \frac{2(N+\alpha)}{N-2}$ by a Pohozaev-type identity for radial functions.

Concerning the symmetry of solutions one has the following result: if Ω is a ball, one proves by moving plane techniques [12] that the *minimal energy* solution is positive and *radially symmetric* if $\alpha \leq 0$ (Hardy case). One can pose the question if this symmetry continues to be present also for positive α. In an interesting paper Smets et al. [26] showed that this is not the case: they proved that for $\alpha > 0$ and sufficiently large a *symmetry-breaking* occurs, that is, to the minimal energy level (which is attained for $p+1 < 2^* = \frac{2N}{N-2}$) corresponds a solution which is *not radially symmetric*. In a related result, Cao and Peng proved in [5] that for $\alpha > 0$ and $p+1$ sufficiently close to 2^* the ground-state solutions of (2) are not radial since they possess a unique maximum point which tends to $\partial\Omega$ as $p+1 \to 2^*$.

Turning to the system (1), we first recall the case $\alpha = \beta = 0$ which has been studied by many authors. Here the natural restriction on the exponents p and q for existence/non-existence of solutions is given by the *critical hyperbola*, that is

$$\frac{N}{p+1} + \frac{N}{q+1} = N-2; \tag{3}$$

this hyperbola was first introduced by Mitidieri [21] who proved non-existence of solutions for (p, q) lying on or above the hyperbola, using a Pohozaev-type identity. Existence of solutions for $(p+1, q+1)$ below the critical hyperbola was proved by de Figueiredo and Felmer (see [9]) and by Hulshoff and van der Vorst (see [16,17]) by using a variational set-up with fractional Sobolev spaces. A different approach, working with Sobolev-Orlicz spaces (which allows a generalization to non-polynomial nonlinearities), can be found in [8,10].

Recently, the general case $\alpha \neq 0$, and/or $\beta \neq 0$ has been investigated independently by de Figueiredo et al. [11] and Liu and Yang [20]; in both papers an approach via fractional Sobolev spaces is used. As in the scalar case, the presence of the weight functions $|x|^\alpha$ and $|x|^\beta$ affects the range of p and q for which the problem may have solutions. Indeed, in [11] and [20] it is shown that the dividing line between existence and non-existence is given by the following "weighted" critical hyperbola

$$\frac{N+\alpha}{p+1}+\frac{N+\beta}{q+1}=N-2. \tag{4}$$

For future reference, we call the hyperbola (3) the M-hyperbola (for *Mitidieri hyperbola*), and the hyperbola (4) the $\alpha\beta$-hyperbola).

We remark that systems of type (1) are closely related to the double weighted Hardy–Littlewood–Sobolev inequality (see e.g. Stein and Weiss [27] and Lieb [18]). This becomes clear by the approach we use for system (1), which is different from the one proposed in the previously cited papers. Following for instance Wang [28] we (formally) deduce from the second equation in (1)

$$v=(-\Delta u)^{\frac{1}{q}}|x|^{-\frac{\beta}{q}},$$

and inserting this into the first equation we obtain the following scalar equation for the u-component

$$-\Delta\left((-\Delta u)^{1/q}|x|^{-\beta/q}\right)=|x|^{\alpha}u^{p} \tag{5}$$

We intend to investigate the rôle played by the weights α and β when dealing with the existence and symmetry of *ground state* (or *minimal energy*) solutions of Eq. 5, that is, minimizers u of the corresponding Raleigh quotient

$$R(u)=\frac{\int_{\Omega}|x|^{-\beta(r-1)}|\Delta u|^{r}}{\left(\int_{\Omega}|x|^{\alpha}|u|^{p+1}\,dx\right)^{\frac{r}{p+1}}},\quad r:=\frac{q+1}{q}, \tag{6}$$

on the weighted Sobolev space

$$W^{2,r}(\Omega,|x|^{-\beta(r-1)}dx)\cap W_0^{1,r}(\Omega)$$

Here we denote with $W^{2,r}(\Omega,|x|^{-\beta(r-1)}dx)$ the set of functions $u\in W^{2,1}_{\text{loc}}(\Omega)$ such that

$$\int_{\Omega}(|u|^{r}+|\nabla u|^{r}+\sum_{|\xi|=2}|D^{\xi}u|^{r}|x|^{-\beta(r-1)})\,dx<+\infty,$$

endowed with the norm

$$\|u\|_{W^{2,r}(\Omega,|x|^{-\beta/q}dx)}:=\left(\int_{\Omega}(|u|^{r}+|\nabla u|^{r}+\sum_{|\xi|=2}|D^{\xi}u|^{r}|x|^{-\beta(r-1)}dx)\,dx\right)^{1/r};$$

also, we denote with

$$W^{2,r}_{rad}(\Omega,|x|^{-\beta(r-1)}dx)$$

the subspace of $W^{2,r}(\Omega,|x|^{-\beta(r-1)}dx)$ of radial functions.

Furthermore, let

$$W^{2,r}_{D}(\Omega,|x|^{-\beta(r-1)}dx)$$

denote the closure of $\{\phi\in C^{\infty}(\Omega):\ \phi=0 \text{ on } \partial\Omega\}$ in $W^{2,r}(\Omega,|x|^{-\beta(r-1)}dx)$, i.e. the closure of the smooth functions in Ω with Dirichlet boundary conditions, and with

$$W^{2,r}_{D,rad}(\Omega,|x|^{-\beta(r-1)}dx)$$

the corresponding subspace of radial functions (for Ω a ball).

For the definition of this space and related properties we refer the reader to the Appendix. In particular, we prove there the following generalization of the Meyers-Serrin denseness result:

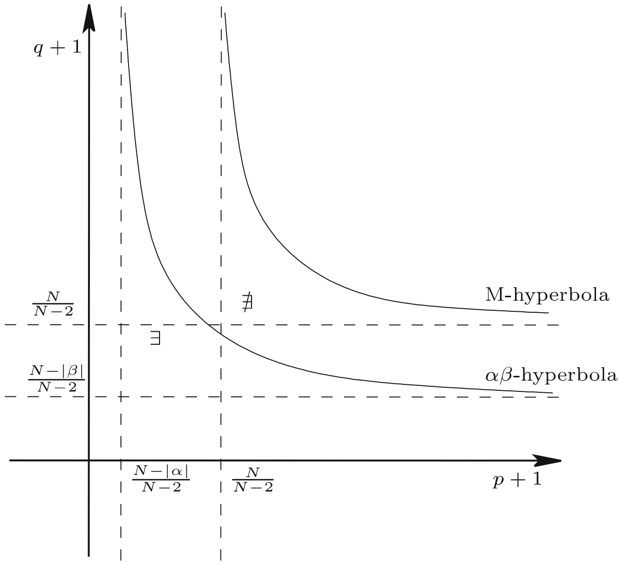

Fig. 1 Hardy type system

Proposition *Let Ω a domain with smooth boundary. Suppose that $\alpha, \beta > -N$ and $p, q > 1$ with $q > \frac{\beta}{N}$. Then*

$$W_D^{2,r}(\Omega, |x|^{-\beta/q} dx) = W^{2,r}(\Omega, |x|^{-\beta/q} dx) \cap W_0^{1,r}(\Omega).$$

It is not difficult to prove that critical points of $R(u)$ on $W_D^{2,r}(\Omega, |x|^{-\beta/q} dx)$ are (up to rescaling) weak solutions of (5), i.e. verifying

$$\begin{cases} \int\limits_{\Omega} (-\Delta u)^{1/q} |x|^{-\beta/q} (-\Delta\varphi) dx = \int\limits_{\Omega} |x|^{\alpha} u^p \varphi \, dx, \\ \text{for all } \varphi \in W_D^{2,r}(\Omega, |x|^{-\beta/q} dx) \end{cases}$$

and, moreover, if $v = (-\Delta u)^{\frac{1}{q}} |x|^{-\frac{\beta}{q}}$, then $v \in W^{2,\frac{p+1}{p}}(\Omega, |x|^{-\frac{\alpha}{p}} dx) \cap W_0^{1,\frac{p+1}{p}}(\Omega)$. In accordance, by a *strong solution* of the system we mean a couple (u, v) of weak-solutions such that

$$(u, v) \in W^{2,r}(\Omega, |x|^{-\frac{\beta}{q}} dx) \cap W_0^{1,r}(\Omega) \times W^{2,\frac{p+1}{p}}(\Omega, |x|^{-\frac{\alpha}{p}} dx) \cap W_0^{1,\frac{p+1}{p}}(\Omega).$$

In what follows, we will denote by E the space $E = W_D^{2,r}(\Omega, |x|^{-\beta(r-1)} dx) = W^{2,r}(\Omega, |x|^{-\beta/q} dx) \cap W_0^{1,r}(\Omega)$ (if the values β and r are clear from the context), and by E_{rad} the radial component of E.

In this paper we investigate solvability and symmetry properties of the solutions for general exponents α and β. We will see that the solvability and the qualitative properties of the solutions depend on the location of the exponents p, q with respect to the M-hyperbola and the $\alpha\beta$-hyperbola.

Note that if $\alpha, \beta < 0$, then the $\alpha\beta$-hyperbola lies below the M-hyperbola, see Fig. 1. In this case, we have the following result:

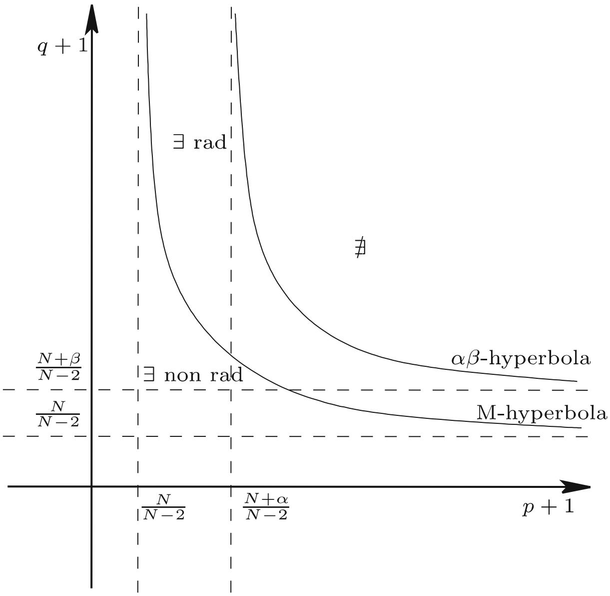

Fig. 2 Hénon type system

Theorem 1 (Hardy-type system) *Let* $0 \geq \alpha, \beta > -N$, *and* $p, q > 1$.

(a) *If* $\frac{N-|\alpha|}{p+1} + \frac{N-|\beta|}{q+1} = N-2$ *(i.e. on the* $\alpha\beta$*-hyperbola), and if* Ω *is starshaped, then system* (1) *has no non-trivial solution, and hence* $\inf_E R(u)$ *is not attained.*

(b) *If* $\frac{N-|\alpha|}{p+1} + \frac{N-|\beta|}{q+1} > N-2$ *(i.e. below the* $\alpha\beta$*-hyperbola), then*

 (b_1) $\inf_E R(u)$ *is attained, and therefore system* (1) *has a nontrivial solution* $\bar{u}$ *(the minimal energy solution);*

 (b_2) *if* Ω *is a ball, then* $\bar{u}$ *is radially symmetric*

Next, we consider the case $\alpha, \beta > 0$, i.e. the Hénon-type system. Note that then the $\alpha\beta$-hyperbola lies above the M-hyperbola, and there are three regions which characterize the behavior of the system, see Fig. 2.

Theorem 2 (Hénon type system) *Let* $\alpha, \beta > 0$, *and suppose that* $p, q > 1$.

(a) *If* $\frac{N+\alpha}{p+1} + \frac{N+\beta}{q+1} \leq N-2$ *(i.e., on or above the* $\alpha\beta$*-hyperbola) and* Ω *is starshaped, then system* (1) *has no non-trivial solution;*

(b) *Suppose that* $\Omega = B_1(0)$. *If* $q > \max\{1, \frac{\beta}{N}\}$ *and* $\frac{N+\alpha}{p+1} + \frac{N+\beta}{q+1} > N-2$ *(i.e. below the* $\alpha\beta$*-hyperbola), then system* (1) *has a radial solution (not necessarily of minimal energy);*

(c) *If* $\frac{N}{p+1} + \frac{N}{q+1} > N-2$ *(i.e. below the M-hyperbola), then* $\inf_E R(u)$ *is attained and hence system* (1) *has a ground state solution; furthermore, if* $\alpha > 0$ *is sufficiently large, then the ground state solution is* not radially symmetric.

Remarks (1) Note that (c) of the previous theorem can be interpreted as a symmetry breaking: for $\alpha, \beta \leq 0$ we have by Theorem 1 that the ground state solution in a ball is radial, while (c) says that for $\alpha, \beta > 0$ and α large the ground state solution is non radial.

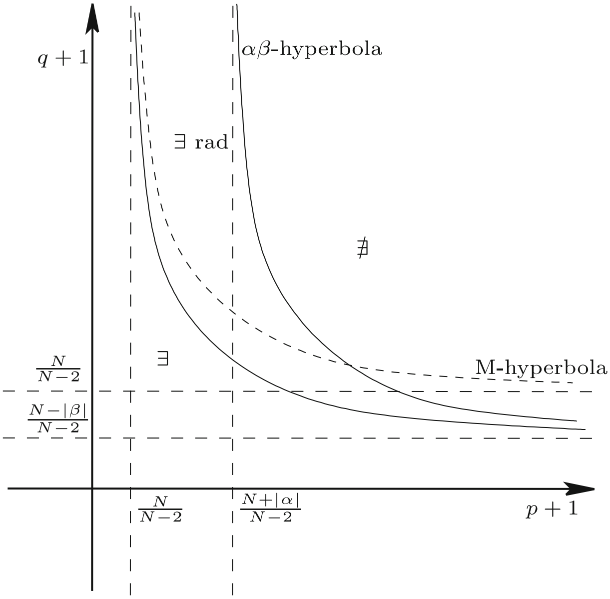

Fig. 3 Hardy-Hénon type system

(2) By (b) and (c) of Theorem 2 we get the existence of at least two solutions for $(p+1, q+1)$ below the M-hyperbola and α large: one radial solution (obtained as minimum of $R(u)$ on E_{rad}), and the minimal energy solution which is non-radial.

In the Hénon case there is a very recent symmetry breaking result, due to He and Yang ([14]): they prove that if (for q fixed) p goes towards the M-hyperbola, then the ground state solution is non radial; this result extends to systems the corresponding result for the equation by Cao and Peng [5]).

Finally, we consider the case of a mixed *Hénon-Hardy type system*, i.e. one exponent (say α) is positive and the other exponent (i.e. β) is negative. In this case the M-hyperbola and the $\alpha\beta$-hyperbola intersect, see Fig. 3. We show that in this case a third hyperbola comes into play.

Theorem 3 (Hénon-Hardy type system) *Let $\alpha > 0$, $0 > \beta > -N$, and suppose that $p, q > 1$.*

(a) *If $\frac{N+\alpha}{p+1} + \frac{N-|\beta|}{q+1} = N-2$ (i.e. on the $\alpha\beta$-hyperbola) and Ω is starshaped, then system* (1) *has no solution;*
(b) *Assume $\Omega = B_1(0)$. If $\frac{N+\alpha}{p+1} + \frac{N-|\beta|}{q+1} > N-2$ (i.e. below the $\alpha\beta$-hyperbola), then system* (1) *has a radial solution (not necessarily of minimal energy);*
(c) *If (p, q) satisfies $\frac{N}{p+1} + \frac{N-|\beta|}{q+1} > N-2$, then $\inf_E R(u)$ is attained and hence system* (1) *has a ground state solution; furthermore, if $\alpha > 0$ is sufficiently large, then the ground state solution is* not radially symmetric.

Remark Suppose that Ω is an arbitrary bounded domain: if $(p+1, q+1)$ lies below the $\alpha\beta$-hyperbola and above the M-hyperbola, then it is *not known* whether $\inf_E R(u)$ is attained.

2 The Hardy case: proof of Theorem 1

Proof of Theorem 1 (a): this is obtained via a generalized identity of Pohozaev-type, see Proposition 7 Sect. 6 below. Note that this case is somewhat delicate due to the singular weights.

Proof of Theorem 1 (b_1): To prove the existence of a positive solution we minimize the Rayleigh quotient $R(u)$ given in (6). By the compactness of the embedding in Lemma 4 (see Sect. 5) the infimum is attained by a positive function, which is a (strong) solution of problem (10).

Proof of Theorem 1 (b_2): for $\alpha = \beta = 0$ and if Ω is a ball, it was proved by X.J. Wang [28] that the ground state of (6) is a radial and radially decreasing positive function. By adapting his argument (moving planes technique and maximum principle) one can extend this result also to the values $\alpha, \beta < 0$ (noting that the weights do not interfere with the moving planes technique). □

3 The Hénon case: proof of Theorem 2

Proof of Theorem 2 (a): this follows again by a Pohozaev-type identity proved in Proposition 6 in Sect. 6 below.

Proof of Theorem 2 (b): The existence of radial solutions under the hypotheses of Theorem 2(b) follows from the embedding result for radial functions in Lemma 9 in Sect. 7 below, by considering again the Rayleigh quotient $R(u)$ on the weighted space $W^{2,r}_{rad}(\Omega, |x|^{-\beta/q} dx)$.

Proof of Theorem 2 (c): We have to show that $m := \inf_E R(u)$ is attained. Let $\{u_n\} \subset E$ be a minimizing sequence. We may assume that

$$\int_\Omega |x|^\alpha |u_n|^{p+1} = 1, \quad \text{and} \quad \int_\Omega |x|^{-\beta/N} |\Delta u_n|^r dx \to m > 0.$$

Then clearly $\int_\Omega |\Delta u_n|^r dx \le c$, and by the assumption and the Rellich-Kondrachov compactness theorem it follows that $\{u_n\}$ has a convergent subsequence in $L^{p+1}(\Omega)$, and hence also in $L^{p+1}(\Omega, |x|^\alpha dx)$. This is sufficient to conclude that $\inf_E R(u)$ is attained. □

Finally, we show that if $\alpha > 0$ is sufficiently large, then the radial ground state level lies above the ground state level: indeed, by Proposition 10 below we have the following lower estimate for the radial ground state level:

$$S^{rad}_{\alpha,\beta} \ge C\, \alpha^{2r + \frac{r}{p+1} - 1}, \quad \text{for } \alpha \ge \alpha_0$$

On the other hand, for the ground state level the following upper estimate holds (see Proposition 11 below): there exist $C > 0$ and α_0 such that for $\alpha \ge \alpha_0$

$$S_{\alpha,\beta} \le C\, \alpha^{2r - N + N\frac{r}{p+1}}$$

From these two inequalities it follows that the ground state is non radial for α sufficiently large, since

$$\frac{r}{p+1} - 1 > -N + N\frac{r}{p+1} \Longleftrightarrow \frac{r}{p+1} < 1\,,$$

which is clearly the case.

4 The mixed case: proof of Theorem 3

Proof of Theorem 3 (a) and (b): as in Theorem 2

Proof of Theorem 3 (c): Let $\{u_n\} \subset E$ be a minimizing sequence for $m = \inf R(u)$. We may assume that

$$\int_\Omega |x|^\alpha |u_n|^{p+1} = 1, \quad \text{and} \quad \int_\Omega |x|^{|\beta|/N} |\Delta u_n|^r dx \to m.$$

We apply the embedding result in Lemma 4 (see Sect. 5 below) for $\alpha = 0$, i.e. under the hypotheses of Theorem 3 c). By the compactness of the embedding we have that for a subsequence $u_n \to u$ in $L^{p+1}(\Omega)$, and since $\alpha > 0$ clearly also in $L^{p+1}(\Omega, |x|^\alpha dx)$. Thus is sufficient to conclude that $\inf_E R(u)$ is attained.

Proof of Theorem 3 (c): one proves as in Theorem 2 c) that if $\alpha > 0$ is sufficiently large, then the ground state level is non radial.

□

5 An embedding result of Caffarelli-Kohn-Nirenberg type

We first prove a preliminary embedding result:

Lemma 4 *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain with $0 \in \Omega$. For given α, β, q let $p^\star$ such that*

$$\frac{N - |\alpha|}{p^\star + 1} + \frac{N - |\beta|}{q + 1} = N - 2. \tag{7}$$

Then we have the following continuous embedding

$$W^{2,r}(\Omega, |x|^{-\beta(r-1)} dx) \hookrightarrow L^{p+1}(\Omega, |x|^\alpha), \quad \textit{for} \quad 0 \le p \le p^\star;$$

furthermore, if

$$\frac{N - |\alpha|}{p + 1} + \frac{N - |\beta|}{q + 1} > N - 2, \quad \textit{i.e.} \;\; p < p^\star, \tag{8}$$

then the embedding is compact.

Proof This follows from the generalization due to Lin (see [19]) of the Caffarelli-Kohn-Nirenberg inequality [6]: if (7) holds, then there exists a constant C such that

$$\left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{(p^\star+1)} \right)^{\frac{1}{(p^\star+1)}} \le C \left(\int_{\mathbb{R}^N} |x|^{-\beta(r-1)} |D^2 u|^r \right)$$

for all $u \in C_0^\infty(\mathbb{R}^N)$.

Via extension theorems (see for instance [1]) we have

$$W^{2,r}(\Omega, |x|^{-\beta(r-1)} dx) \hookrightarrow L^{p^\star+1}(\Omega, |x|^\alpha),$$

and for any $0 \le p < p^\star$ the embedding is compact. □

Another consequence of the CKN inequalities is the following

Lemma 5 *If* $(u, v) \in W_D^{2,\frac{q+1}{q}}(\Omega, |x|^{|\beta|/q}dx) \times W_D^{2,\frac{p+1}{p}}(\Omega, |x|^{|\alpha|/p}dx)$*, with p and q as in* (7)*, then* $\nabla u \nabla v \in L^1(\Omega)$.

Proof If $u \in W_D^{2,\frac{q+1}{q}}(\Omega, |x|^{|\beta|/q}dx)$ and $v \in W_D^{2,\frac{p+1}{p}}(\Omega, |x|^{|\alpha|/p}dx)$ then from the CKN inequality applied to ∇u and ∇v, one has $\nabla u \in L^s(\Omega)$ with $s = \frac{N(q+1)}{Nq+|\beta|-(q+1)}$, and $\nabla v \in L^t(\Omega)$ with $t = \frac{N(p+1)}{Np+|\alpha|-(p+1)}$. It is not difficult to prove that $\frac{1}{s} + \frac{1}{t} = 1$. So the assertion follows by the Hölder inequality. □

6 A generalized identity of Pohozaev-type

In this section we prove, via a generalized identity of Pohozaev-type, the non-existence of "strong" solutions on or above the critical $\alpha\beta$-hyperbola.

We consider first the Hénon case $\alpha \geq 0$, $\beta \geq 0$. In this case we can suppose that the solutions (u, v) are of class $C^2(\Omega) \cap C^1(\overline{\Omega})$.

Proposition 6 *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded, smooth, starshaped domain with respect to* $0 \in \mathbb{R}^N$. *If* $\alpha, \beta \geq 0$ *and*

$$\frac{N+\alpha}{p+1} + \frac{N+\beta}{q+1} \leq N-2 \tag{9}$$

then the problem

$$\begin{cases} -\Delta v = |x|^\alpha |u|^{p-1}u & \text{in } \Omega, \\ -\Delta u = |x|^\beta |v|^{q-1}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

has no nontrivial strong positive solutions.

Proof This follows by adapting the argument of Mitidieri in [21]. Let

$$G(x, u, v) = \frac{1}{p+1}|x|^\alpha |u|^{p+1} + \frac{1}{q+1}|x|^\beta |v|^{q+1} \tag{11}$$

so (10) becomes

$$\begin{cases} -\Delta v = \dfrac{\partial G}{\partial u}(x, u, v) & \text{in } \Omega \\ -\Delta u = \dfrac{\partial G}{\partial v}(x, u, v) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \tag{12}$$

We multiply the equations respectively by $(x \cdot \nabla u)$ and by $(x \cdot \nabla v)$, add the two equations and integrate. By an application of the divergence theorem (see Proposition 2.1 and Corollary 2.1 in [21]) we have for the left sides

$$\int_\Omega \{\Delta u\,(x \cdot \nabla v) + \Delta v\,(x \cdot \nabla u)\}dx$$
$$= \int_{\partial\Omega} \left\{\frac{\partial u}{\partial n}\frac{\partial v}{\partial n}(x \cdot n)\right\} ds + (N-2)\int_\Omega (\nabla u \nabla v)dx \tag{13}$$

(since $u = v = 0$ on $\partial\Omega$, we have $(x \cdot \nabla u) = x \cdot n\frac{\partial u}{\partial n}$ and $(x \cdot \nabla v) = x \cdot n\frac{\partial v}{\partial n}$ on $\partial\Omega$).

For the right sides we have, since

$$\frac{\partial G}{\partial u}(x\cdot\nabla u)+\frac{\partial G}{\partial v}(x\cdot\nabla v)=\operatorname{div}\{xG(x,u,v)\}$$
$$-NG(x,u,v)-\alpha\frac{|x|^{\alpha}}{p+1}|u|^{p+1}-\beta\frac{|x|^{\beta}}{q+1}|v|^{q+1} \tag{14}$$

and taking into account that $G(x,u,v)=0$ on $\partial\Omega$:

$$\int_{\Omega}\left\{\frac{\partial G}{\partial u}(x\cdot\nabla u)+\frac{\partial G}{\partial v}(x\cdot\nabla v)\right\}dx$$
$$=-\frac{N+\alpha}{p+1}\int_{\Omega}|x|^{\alpha}|u|^{p+1}dx-\frac{N+\beta}{q+1}\int_{\Omega}|x|^{\beta}|v|^{q+1}dx. \tag{15}$$

Therefore

$$\int_{\partial\Omega}\left\{\frac{\partial u}{\partial n}\frac{\partial v}{\partial n}(x\cdot n)\right\}ds+(N-2)\int_{\Omega}(\nabla u\nabla v)dx$$
$$=\frac{N+\alpha}{p+1}\int_{\Omega}|x|^{\alpha}|u|^{p+1}dx+\frac{N+\beta}{q+1}\int_{\Omega}|x|^{\beta}|v|^{q+1}dx. \tag{16}$$

Now, multiplying the first equation by u, the second by v and integrating, one obtains

$$\int_{\Omega}(\nabla u\nabla v)dx=\int_{\Omega}|x|^{\alpha}|u|^{p+1}dx=\int_{\Omega}|x|^{\beta}|v|^{q+1}dx$$

So (16) becomes *(Pohozaev identity)*

$$\int_{\partial\Omega}\left\{\frac{\partial u}{\partial n}\frac{\partial v}{\partial n}(x\cdot n)\right\}ds$$
$$=\left\{-(N-2)+\frac{N+\alpha}{p+1}+\frac{N+\beta}{q+1}\right\}\int_{\Omega}|x|^{\alpha}|u|^{p+1}dx \tag{17}$$

Since Ω is starshaped and u, v are positive, we have $\int_{\partial\Omega}\left\{\frac{\partial u}{\partial n}\frac{\partial v}{\partial n}(x\cdot n)\right\}ds>0$, and hence by (16)

$$0<\int_{\partial\Omega}\left\{\frac{\partial u}{\partial n}\frac{\partial v}{\partial n}(x\cdot n)\right\}ds$$
$$=\left\{-(N-2)+\frac{N+\alpha}{p+1}+\frac{N+\beta}{q+1}\right\}\int_{\Omega}|x|^{\alpha}|u|^{p+1}dx \tag{18}$$

which gives a contradiction for the choice of p and q. □

Next, we consider the Hardy case.

Proposition 7 *Let Ω as in Proposition 6 and assume that $0\geq\alpha,\ \beta>-N$ and*

$$\frac{N-|\alpha|}{p+1}+\frac{N-|\beta|}{q+1}=N-2; \tag{19}$$

then there exists no positive strong solution (u,v) of (12).

Proof For this case we follow the idea developed by B. Xuan (see Appendix in [29]).

Let (u, v) a positive solution of system (12). Due to the Hardy weights this solution may be singular in the origin, but standard regularity results imply that for every δ small, u and v belong to $C^2(\Omega \setminus B_\delta(0)) \cap C^0(\overline{\Omega \setminus B_\delta(0)})$. We multiply the equations respectively by $(x \cdot \nabla u)$ and by $(x \cdot \nabla v)$, add the two equations and integrate over $\Omega_\delta = \Omega \backslash B_\delta(0)$

$$
\begin{aligned}
&-\int_{\Omega_\delta} \{\Delta u\,(x \cdot \nabla v) + \Delta v\,(x \cdot \nabla u)\} dx \\
&\quad = \int_{\Omega_\delta} \left\{ \frac{\partial G}{\partial u}(x \cdot \nabla u) + \frac{\partial G}{\partial v}(x \cdot \nabla v) \right\} dx,
\end{aligned} \tag{20}
$$

where $G = G(x, u, v)$ is as in (11).

We apply the Divergence Theorem to xG, so that one has for the right side of (20)

$$
\begin{aligned}
&\int_{\Omega_\delta} \left\{ \frac{\partial G}{\partial u}(x \cdot \nabla u) + \frac{\partial G}{\partial v}(x \cdot \nabla v) \right\} dx \\
&\quad = -\frac{N - |\alpha|}{p+1} \int_{\Omega_\delta} \frac{u^{p+1}}{|x|^\alpha} dx - \frac{N - |\beta|}{q+1} \int_{\Omega_\delta} \frac{v^{q+1}}{|x|^\beta} dx + \int_{|x|=\delta} G(u, v, x)(x \cdot n) ds,
\end{aligned} \tag{21}
$$

while for the left side of (20) (see [21], Corollary 2.1) one has

$$
\begin{aligned}
&\int_{\Omega_\delta} \{\Delta u\,(x \cdot \nabla v) + \Delta v\,(x \cdot \nabla u)\}\, dx \\
&\quad = \int_{\partial \Omega_\delta} \left\{ \frac{\partial u}{\partial n}(x \cdot \nabla v) + \frac{\partial v}{\partial n}(x \cdot \nabla u) - (\nabla u \nabla v)(x \cdot n) \right\} ds + (N-2) \int_{\Omega_\delta} (\nabla u \nabla v) dx \\
&\quad = \int_{|x|=\delta} \left\{ \frac{\partial u}{\partial n}(x \cdot \nabla v) + \frac{\partial v}{\partial n}(x \cdot \nabla u) - (\nabla u \nabla v)(x \cdot n) \right\} ds \\
&\qquad + \int_{\partial \Omega} \left\{ \frac{\partial u}{\partial n} \frac{\partial v}{\partial n}(x \cdot n) \right\} ds + (N-2) \int_{\Omega_\delta} (\nabla u \nabla v) dx.
\end{aligned} \tag{22}
$$

Now, multiplying the first equation by u, the second by v and integrating, one obtains

$$
\int_{\Omega_\delta} (\nabla u \nabla v) dx - \int_{|x|=\delta} v \nabla u \cdot x = \int_{\Omega_\delta} |x|^\beta |v|^{q+1} dx
$$

and

$$
\int_{\Omega_\delta} (\nabla u \nabla v) dx - \int_{|x|=\delta} u \nabla v \cdot x = \int_{\Omega_\delta} |x|^\alpha u^{p+1} dx
$$

The Pohozaev identity (17) follows if we prove that all the integrals along $\{|x| = \delta\}$ go to zero, at least for a subsequence $\delta_k \to 0$. But this follows by the mean value theorem, since by the Lemmas in Sect. 5 the integrals

$$\int_\Omega G(u,v,x)dx,\ \int_\Omega |u\nabla v|dx\,,\ \int_\Omega |v\nabla u|dx\,,\ \int_\Omega |\nabla u \nabla v|dx$$

are finite. Indeed, if ψ is a positive function in $L^1(\Omega)$, then $\epsilon_k = \int_{B_{1/k}(0)} \psi(x)dx \to 0$ as $k \to +\infty$. Moreover, if $\psi \in C(\Omega\backslash\{0\})$ then

$$\epsilon_k = \int_{B_{1/k}(0)} \psi(x)dx = \int_0^{1/k} \int_{|x|=\delta} \psi(x)ds\, d\delta$$

By the Mean Value Theorem there exists $\delta_k \in (0, 1/k)$ such that

$$\epsilon_k = \frac{1}{k} \int_{|x|=\delta_k} \psi(x)ds$$

Therefore

$$\int_{|x|=\delta_k} \psi(x)(x\cdot n)ds \;=\; \int_{|x|=\delta_k} \psi(x)\delta_k ds = k\epsilon_k\delta_k \le \epsilon_k \to 0.$$

□

Proposition 8 *Let Ω as in Proposition 6, and assume that $\alpha \ge 0\,, 0 \ge \beta > -N$ and*

$$\frac{N+\alpha}{p+1} + \frac{N-|\beta|}{q+1} = N-2 \tag{23}$$

then there exists no positive strong solution (u, v) of (12).

Proof By combining the previous methods one obtains the result. □

7 An embedding theorem for radial functions

Proposition 9 *Let $\Omega \subset \mathbb{R}^N$ be the ball $\Omega = B_1(0)$. Let $\alpha,\ \beta > -N$ and let p and q such that $q > \frac{\beta}{N}$ and*

$$\frac{\alpha+N}{p+1} + \frac{N+\beta}{q+1} > N-2. \tag{24}$$

Then the embedding

$$E_{rad} = W^{2,r}_{D,rad}(\Omega, |x|^{-\beta/q}dx) \hookrightarrow L^{p+1}(\Omega, |x|^\alpha dx), \qquad r = \frac{q+1}{q}$$

is continuous and compact.

Proof By the density result (Theorem 16 in the Appendix) it is sufficient to prove the assertion for radial $u \in C^\infty(\Omega) \cap E_{rad}$ with $u = 0$ on $\partial\Omega$. For such u we have

$$\Delta u = t^{1-N}(u'(t)t^{N-1})'$$

It is sufficient to prove that there exists a constant C such that

$$\left(\int_0^1 t^\alpha |u(t)|^{p+1} t^{N-1} dt\right)^{\frac{1}{p+1}} \le C\left(\int_0^1 |(u'(s)s^{N-1})'|^r s^{r-rN+N-1-\beta/q} ds\right)^{1/r} =: C\|u\|_*$$

Set $w(t) = u'(t)t^{N-1}$. Then, since $w(0) = 0$

$$|u(t)| = \left|\int_1^t u'(s)ds\right| = \left|\int_1^t w(s)s^{1-N}ds\right| = \left|\int_t^1 \left[\int_0^s w'(\xi)d\xi\right] s^{1-N}ds\right|$$

$$= \left|\int_1^t \left[\int_0^s w'(\xi)\, \xi^{\frac{r-rN+N-1-\beta/q}{r}}\, \xi^{-\frac{r-rN+N-1-\beta/q}{r}}\, d\xi\right] s^{1-N}ds\right|$$

(by Hölder inequality with exponents $r = 1 + \frac{1}{q}$ and $r' = q + 1$)

$$\leq \int_t^1 \left[\int_0^s |w'(\xi)|^r\, \xi^{r-rN+N-1-\beta/q}\right]^{1/r} \left[\int_0^s \xi^{-\frac{r-rN+N-1-\beta/q}{r}(q+1)}\, d\xi\right]^{\frac{1}{q+1}} s^{1-N}ds$$

$$\leq \|u\|_* \int_t^1 \left[\int_0^s \xi^{N+\beta-1}\, d\xi\right]^{\frac{1}{q+1}} s^{1-N}ds$$

$$\leq \|u\|_* \int_t^1 s^{\frac{N+\beta}{q+1}+1-N}ds$$

Now three cases may occur:

Case 1 $\frac{N+\beta}{q+1} > N-2$ that is $q+1 < \frac{N+\beta}{N-2}$. In this case we have

$$\int_\Omega |x|^\alpha |u(x)|^{p+1}dx \leq C\|u\|_*^{p+1};$$

Case 2 For $q + 1 = \frac{N+\beta}{N-2}$

$$\int_\Omega |x|^\alpha |u(x)|^{p+1}dx \leq C\|u\|_* \int_\Omega |x|^\alpha |\log(|x|)|^{p+1}dx \leq C\|u\|_*^{p+1},$$

since, for $\alpha > -N$, $|x|^\alpha |\log(|x|)|^{p+1}$ is integrable.

Case 3 Finally for $q + 1 > \frac{N+\beta}{N-2}$

$$\int_\Omega |x|^\alpha |u(x)|^{p+1}dx = \omega_{N-1}\int_0^1 t^{\alpha+N-1}|u(t)|^{p+1}dt$$

$$\leq C\|u\|_*^{p+1}\int_0^1 t^{\alpha+N-1}t^{(p+1)\left(\frac{N+\beta}{q+1}-N+2\right)}dt \leq C\|u\|_*^{p+1}$$

for α such that

$$\alpha + N + (p+1)\left(\frac{N+\beta}{q+1} - N + 2\right) > 0\,,$$

that is

$$\frac{\alpha+N}{p+1}+\frac{N+\beta}{q+1}>N-2.$$

Finally, the proof of the compactness is standard. □

8 Estimates for ground states

8.1 The radial ground state level (β fixed, $\alpha \to +\infty$)

We give now an estimate from below for the radial level

$$S^{rad}_{\alpha,\beta}=\inf_{u\in E_{rad}\setminus\{0\}}\frac{\int_\Omega |x|^{-\beta/q}|\Delta u|^r\,dx}{\left(\int_\Omega |x|^\alpha |u|^{p+1}\,dx\right)^{\frac{r}{p+1}}}$$

Proposition 10 *There exist $C>0$ and α_0 such that*

$$S^{rad}_{\alpha,\beta}\ge C\alpha^{2r+\frac{r}{p+1}-1},\qquad \alpha\ge\alpha_0$$

Proof Let $\varepsilon=\frac{N}{N+\alpha}$ and $u(x)=u(|x|)$ a smooth radial function such that $u=0$ on $\partial\Omega$. Let $v(\rho)=u(\rho^\varepsilon)$. We have

$$v'(\rho)=\varepsilon u'(\rho^\varepsilon)\rho^{\varepsilon-1}\quad\text{and}\quad v''(\rho)=\varepsilon^2u''(\rho^\varepsilon)\rho^{2\varepsilon-2}+\varepsilon(\varepsilon-1)u'(\rho^\varepsilon)\rho^{\varepsilon-2}$$

so that

$$u'(\rho^\varepsilon)=\rho^{1-\varepsilon}\varepsilon^{-1}v'(\rho)\quad\text{and}\quad u''(\rho^\varepsilon)=\varepsilon^{-2}\rho^{2-2\varepsilon}[v''(\rho)-(\varepsilon-1)\rho^{-1}v'(\rho)]$$

Therefore, by the change of variable $t=\rho^\varepsilon$,

$$\begin{aligned}
\int_\Omega |x|^{-\beta/q}|\Delta u|^r dx &= \omega_{N-1}\int_0^1\left|u''(t)+\frac{N-1}{t}u'(t)\right|^r t^{N-1-\beta/q}dt\\
&=\omega_{N-1}\int_0^1 \varepsilon\rho^{\varepsilon N-\frac{\varepsilon\beta}{q}-1}\left|u''(\rho^\varepsilon)+\frac{N-1}{\rho^\varepsilon}u'(\rho^\varepsilon)\right|^r d\rho\\
&=\omega_{N-1}\int_0^1 \varepsilon\rho^{\varepsilon N-\frac{\varepsilon\beta}{q}-1}\Big|\varepsilon^{-2}\rho^{2-2\varepsilon}\left[v''(\rho)-(\varepsilon-1)\rho^{-1}v'(\rho)\right]\\
&\quad+\frac{N-1}{\rho^\varepsilon}\varepsilon^{-1}\rho^{1-\varepsilon}v'(\rho)\Big|^r d\rho\\
&=\omega_{N-1}\int_0^1 \varepsilon^{1-2r}\rho^{\varepsilon N-\frac{\varepsilon\beta}{q}-1}\Big|\rho^{2-2\varepsilon}\left[v''(\rho)-(\varepsilon-1)\rho^{-1}v'(\rho)\right]\\
&\quad+(N-1)\varepsilon\rho^{1-2\varepsilon}v'(\rho)\Big|^r d\rho\\
&=\omega_{N-1}\int_0^1 \varepsilon^{1-2r}\rho^{\varepsilon N-\frac{\varepsilon\beta}{q}-1+2r-2r\varepsilon}\left|v''(\rho)+\frac{N-2\varepsilon+1}{\rho}v'(\rho)\right|^r d\rho
\end{aligned}$$

$$= \omega_{N-1}\varepsilon^{1-2r}\int_0^1 \rho^{\varepsilon N-\frac{\varepsilon\beta}{q}-1+2r-2r\varepsilon}\rho^{-(N-2\varepsilon+1)r}\left|\left(\rho^{N-2\varepsilon+1}v'(\rho)\right)'\right|^r d\rho$$

$$= \omega_{N-1}\varepsilon^{1-2r}\int_0^1 \rho^{\varepsilon N-\frac{\varepsilon\beta}{q}+r-Nr-1}\left|\left(\rho^{N-2\varepsilon+1}v'(\rho)\right)'\right|^r d\rho$$

$$= \omega_{N-1}\varepsilon^{1-2r}\int_0^1 \rho^{\varepsilon N-\frac{\varepsilon\beta}{q}+r-Nr-1}\left|\left(\rho^{\gamma}v'(\rho)\right)'\right|^r d\rho.$$

where $\gamma = N - 2\varepsilon + 1$. Moreover, by the choice of ε,

$$\int_\Omega |x|^\alpha |u(x)|^{p+1}\,dx = \omega_{N-1}\,\varepsilon\int_0^1 |v(\rho)|^{p+1}\rho^{N-1}d\rho.$$

Thus, we get the following estimate for the radial level:

$$S_{\alpha,\beta}^{rad} = \varepsilon^{-2r-\frac{r}{p+1}+1}\inf_{v\in E_{rad}\setminus\{0\}}\frac{\int_0^1 \rho^{\varepsilon N-\frac{\varepsilon\beta}{q}+r-Nr-1}\left|\left(\rho^{\gamma}v'(\rho)\right)'\right|^r d\rho}{\left(\int_0^1 |v(\rho)|^{p+1}\rho^{N-1}\right)^{\frac{r}{p+1}}} \tag{25}$$

It is now sufficient to show that there exists $\eta > 0$ such that

$$\inf_{v\in E_{rad}\setminus\{0\}}\frac{\int_0^1 \rho^{\varepsilon N-\frac{\varepsilon\beta}{q}+r-Nr-1}\left|\left(\rho^{\gamma}v'(\rho)\right)'\right|^r d\rho}{\left(\int_0^1 |v(\rho)|^{p+1}\rho^{N-1}\right)^{\frac{r}{p+1}}} \geq \eta > 0 \qquad \text{uniformly as } \varepsilon \to 0$$

We proceed as in the embedding result setting $w(\rho) = v'(\rho)\rho^\gamma$. Then

$$|v(t)| = \left|\int_1^t v'(\rho)d\rho\right| = \left|\int_1^t w(\rho)\rho^{-\gamma}d\rho\right| = \left|\int_1^t \rho^{-\gamma}\left(\int_0^\rho w'(s)ds\right)d\rho\right|$$

$$= \left|\int_1^t \rho^{-\gamma}\left[\int_0^\rho w'(s)s^{\frac{\varepsilon N-\frac{\varepsilon\beta}{q}+r-Nr-1}{r}}s^{-\frac{\varepsilon N-\frac{\varepsilon\beta}{q}+r-Nr-1}{r}}ds\right]d\rho\right|$$

(Hölder inequality)

$$
\begin{aligned}
&\leq \left| \int_1^t \rho^{-\gamma} \left(\int_0^1 |w'(s)|^r s^{\varepsilon N - \frac{\varepsilon\beta}{q} + r - Nr - 1} ds \right)^{1/r} \left(\int_0^\rho s^{(-\varepsilon N + \frac{\varepsilon\beta}{q} - r + Nr + 1)q} ds \right)^{\frac{1}{q+1}} d\rho \right| \\
&\leq \left| \int_1^t \rho^{-\gamma} \left(\int_0^1 |w'(s)|^r s^{\varepsilon N - \frac{\varepsilon\beta}{q} + r - Nr - 1} ds \right)^{1/r} \left(\int_0^\rho s^{-\varepsilon Nq + \varepsilon\beta - 1 + N(q+1)} ds \right)^{\frac{1}{q+1}} d\rho \right| \\
&\leq \left| \int_1^t \rho^{-\gamma} \left(\int_0^1 |w'(s)|^r s^{\varepsilon N - \frac{\varepsilon\beta}{q} + r - Nr - 1} ds \right)^{1/r} \rho^{-\frac{\varepsilon Nq}{q+1} + \frac{\varepsilon\beta}{q+1} + N} d\rho \right| \\
&= \left(\int_0^1 |w'(s)|^r s^{\varepsilon N - \frac{\varepsilon\beta}{q} + r - Nr - 1} ds \right)^{1/r} \left| \int_1^t \rho^{2\varepsilon - \frac{Nq\varepsilon}{q+1} + \frac{\varepsilon\beta}{q+1} - 1} d\rho \right| \\
&= \left(\int_0^1 |w'(s)|^r s^{\varepsilon N - \frac{\varepsilon\beta}{q} + r - Nr - 1} ds \right)^{1/r} \left| \int_1^t \rho^{\varepsilon\left(2 - N + \frac{N+\beta}{q+1}\right) - 1} d\rho \right| =: \Im
\end{aligned}
$$

For $q + 1 \neq \frac{N+\beta}{N-2}$ one has

$$
\Im = \left(\int_0^1 |w'(s)|^r s^{\varepsilon N - \frac{\varepsilon\beta}{q} + r - Nr - 1} ds \right)^{1/r} \left| \frac{t^{\varepsilon(2 - N + \frac{N+\beta}{q+1})} - 1}{\varepsilon\left(N - 2 - \frac{N+\beta}{q+1}\right)} \right|
$$

Therefore

$$
\begin{aligned}
&\left(\int_0^1 |v(t)|^{p+1} t^{N-1} dt \right)^{\frac{r}{p+1}} \\
&\quad \leq \left(\int_0^1 |w'(s)|^r s^{\varepsilon N - \frac{\varepsilon\beta}{q} + r - Nr - 1} ds \right) \left(\int_0^1 \left| \frac{t^{\varepsilon(2 - N + \frac{N+\beta}{q+1})} - 1}{\varepsilon\left(N - 2 - \frac{N+\beta}{q+1}\right)} \right|^{p+1} t^{N-1} dt \right)^{\frac{r}{p+1}}
\end{aligned}
$$

Now we prove that the last term is uniformly bounded as $\varepsilon \to 0$. Let

$$
g_\varepsilon(t) = \left| \frac{t^{\varepsilon(2 - N + \frac{N+\beta}{q+1})} - 1}{\varepsilon\left(N - 2 - \frac{N+\beta}{q+1}\right)} \right|^{p+1} t^{N-1}
$$

We have that

$$
g_\varepsilon(t) \to (-\log t)^{p+1} t^{N-1} \text{ on } (0, 1), \quad \text{as } \varepsilon \to 0
$$

and

$$
g_\varepsilon(t) \leq (-\log t)^{p+1} t^{N-1} \quad \text{on } (0, 1).
$$

Then, by the Dominated Convergence Theorem

$$\int_0^1 g_\varepsilon(t)dt \to \int_0^1 (-\log t)^{p+1} t^{N-1} dt$$

which is finite.

The case $q+1 = \frac{N+\beta}{N-2}$ is easier and left to the reader. This ends the proof. □

8.2 The ground state level

Following the ideas of Smetz, Su and Willem in [26], we give an upper bound for the level

$$S_{\alpha,\beta} = \inf_{W^r(\Omega)\setminus\{0\}} R(u) \tag{26}$$

Proposition 11 *Let p, q as in (4), such that*

$$\frac{N}{p+1} + \frac{N}{q+1} > N - 2. \tag{27}$$

Then there exist $C > 0$ and α_0 such that for $\alpha \geq \alpha_0$

$$S_{\alpha,\beta} \leq C\, \alpha^{2r-N+N\frac{r}{p+1}} \tag{28}$$

Proof Let ψ a positive smooth function with support in Ω. Let us consider the rescaled function $\psi_\alpha(x) = \psi(\alpha(x - x_\alpha))$, where $x_\alpha = (1 - \frac{1}{\alpha}, 0, \ldots, 0)$. Since ψ_α has support in the ball $B(x_\alpha, \frac{1}{\alpha})$, by the change of variable $y = \alpha(x - x_\alpha)$ we obtain for $\beta > 0$

$$\begin{aligned}\int_\Omega |x|^{-\beta/q} |\Delta\psi_\alpha|^r dx &= \int_{B(x_\alpha, \frac{1}{\alpha})} |x|^{-\beta/q} |\Delta\psi_\alpha|^r dx \\ &\leq \alpha^{2r-N} \int_\Omega \left(1 - \frac{2}{\alpha}\right)^{-\beta/q} |\Delta\psi|^r dy \ ,\end{aligned}$$

while for $\beta \leq 0$

$$\begin{aligned}\int_\Omega |x|^{-\beta/q} |\Delta\psi_\alpha|^r dx &= \int_{B(x_\alpha, \frac{1}{\alpha})} |x|^{-\beta/q} |\Delta\psi_\alpha|^r dx \\ &\leq \int_{B(x_\alpha, \frac{1}{\alpha})} |\Delta\psi_\alpha|^r dx = \alpha^{2r-N} \int_\Omega |\Delta\psi|^r dy.\end{aligned}$$

Furthermore,

$$\int_\Omega |x|^\alpha \psi_\alpha^{p+1}(x)dx = \int_{B(x_\alpha, \frac{1}{\alpha})} |x|^\alpha \psi_\alpha^{p+1}(x)dx \geq \left(1 - \frac{2}{\alpha}\right)^\alpha \int_\Omega \alpha^{-N} \psi^{p+1}(y)dy.$$

This implies

$$S_\alpha \leq C\alpha^{2r-N+N\frac{r}{p+1}} \frac{\int_\Omega |\Delta\psi|^r dx}{(\int_\Omega \psi^{p+1}(x)dx)^{\frac{r}{p+1}}}$$

We remark that $2r - N + N\frac{r}{p+1} > 0$ by (27). □

9 Appendix: Sobolev spaces with A_r weights

9.1 Some definitions

Let $r > 1$, $\lambda > 0$ be a r-weight, i.e. a function on $\mathbb{R}^N$ such that

$$\lambda > 0 \text{ a.e. on } \mathbb{R}^N, \quad \lambda \quad \text{and} \quad \lambda^{-1/(r-1)} \in L^1_{\text{loc}}(\mathbb{R}^N)$$

Let $\Omega \in \mathbb{R}^N$ a bounded smooth domain. We denote with $L^r(\Omega, \lambda)$ the set of functions $u \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega |u|^r \lambda \, dx < +\infty$$

and with $W^{2,r}(\Omega, \lambda)$ the set of functions $u \in W^{2,1}_{\text{loc}}(\Omega)$ such that

$$\int_\Omega (|u|^r + |\nabla u|^r + \sum_{|\xi|=2} |D^\xi u|^r \lambda) \, dx < +\infty$$

One can easily prove that endowed with the norm

$$\|u\|_{W^{2,r}(\Omega,\lambda)} := \left(\int_\Omega (|u|^r + |\nabla u|^r + \sum_{|\xi|=2} |D^\xi u|^r \lambda) \, dx \right)^{1/r}$$

$W^{2,r}(\Omega, \lambda)$ is a Banach Space. We also denote with $\tilde{W}^{2,r}_0(\Omega, \lambda)$ the closure of $\{\phi \in C^\infty(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$ in $W^{2,r}(\Omega, \lambda)$. We are interested to study some "density" property of these Sobolev spaces with weights. To this aim we introduce the following

Definition 12 *(Muckenhoupt Class A_r)* We say that a r-weight λ on $\mathbb{R}^N$ is in the Muckenhoupt class A_r if

$$\left(\frac{1}{|B|} \int_B \lambda dx \right) \left(\frac{1}{|B|} \int_B \lambda^{-1/(r-1)} dx \right)^{r-1} \leq C \tag{29}$$

for every ball B contained in $\mathbb{R}^N$ (here $|B|$ denotes the Lebesgue measure of the ball B).

Example 13 $\lambda(x) = |x|^\gamma \in A_r$ iff $-N < \gamma < N(r-1)$

This class of weights is strictly related to the Hardy-Littlewood maximal function

Definition 14 Let $f \in L^1_{\text{loc}}(\mathbb{R}^N)$. The maximal function of f is defined by

$$(Mf)(x) = \sup_{R>0} \frac{1}{|B_R(x)|} \int_{B_R(x)} |f(y)| dy. \tag{30}$$

In fact the "condition" A_r was introduced by B. Muckenhoupt in the following Theorem

Theorem 15 (Muckenhoupt [22]) *Let λ be a r-weight. The following conditions are equivalent:*

(i) *there is a constant C such that*

$$\int_{\mathbb{R}^N} [(Mf)]^r \lambda \, dx \;\leq C \int_{\mathbb{R}^N} |f|^r \lambda dx \qquad \forall f \in L^r(\mathbb{R}^N) \tag{31}$$

(ii) $\lambda \in A_r$

9.2 Approximation by smooth functions on Ω

The central part of this section is to prove the following extension of the celebrated Meyer-Serrin result

Theorem 16

$$\tilde{W}_0^{2,r}(\Omega, \lambda) = W^{2,r}(\Omega, \lambda) \cap W_0^{1,r}(\Omega)$$

In order to prove Theorem 16 we need some preliminary results

Theorem 17 *Let $\lambda \in L^1(\Omega)$ a positive measure on Ω. Then $C_0(\Omega)$ is dense in $L^r(\Omega, \lambda)$ $(1 \leq r < +\infty)$.*

Proof (Theorem 2.19 in [1]) It is sufficient to prove that for every $\varepsilon > 0$ and a nonnegative function u there exists $\varphi \in C_0(\Omega)$ such that $\|u - \varphi\|_{L^r(\Omega,\lambda)} < \varepsilon$.

For u measurable and nonnegative there exists a monotonically increasing sequence $\{s_n\}$ of nonnegative simple functions converging point-wise to u on Ω and strongly in $L^r(\Omega, \lambda)$ (since $0 \leq s_n(x) \leq u(x)$, we have $s_n \in L^r(\Omega, \lambda)$ and $(u(x) - s_n(x))^r \lambda(x) \leq u(x)^r \lambda(x)$, so that by the Dominated Convergence theorem $s_n \to u$ in $L^r(\Omega, \lambda)$). Thus there exists $s \in \{s_n\}$ such that $\|u - s\|_{L^r(\Omega,\lambda)} < \varepsilon/2$. By Lusin's theorem there exists for all $\delta > 0$ a $\varphi \in C_0(\mathbb{R}^N)$ such that

$$|\varphi(x)| \leq \|s\|_\infty$$

and

$$\text{Vol} E < \delta, \quad E = \{x \in \mathbb{R}^N : \quad \varphi(x) \neq s(x)\}.$$

Therefore, by the absolute continuity of the integral, we can choose $\delta = \delta(\varepsilon)$ such that

$$\|s - \varphi\|_{L^r(\Omega,\lambda)} \leq \|s - \varphi\|_\infty (\int_E \lambda(x) \, dx)^{1/r} < \varepsilon/2$$

□

Lemma 18 [25] *Let J be a nonnegative, real-valued function in $C_0^\infty(\mathbb{R}^N)$ with the following properties*

$$J(x) = 0 \quad if \quad |x| \geq 1, \quad and \quad \int_{\mathbb{R}^N} J(x) = 1.$$

We consider the sequence of "mollifiers" $J_\epsilon(x) = \epsilon^{-N} J(x/\epsilon)$. Then

(i) $J_\epsilon(x) = 0$ *if* $|x| \geq 1$

(ii) *There exists a positive constant* $C = C(N, \sup J)$ *such that, if*

$$J_\epsilon * u(x) = \int_{\mathbb{R}^N} J_\epsilon(x-y)u(y)dy \ ,$$

then

$$|J_\epsilon * u(x)| \leq CM(u)(x), \qquad \forall u \in L^1_{\text{loc}}(\mathbb{R}^N).$$

Proof From (i) and the definition of maximal function one has

$$|J_\epsilon * u(x)| \leq \frac{\sup J}{\epsilon^N} \int_{B_\epsilon(x)} |u(y)|dy \leq C \frac{1}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} |u(y)|dy \leq CM(u)(x)$$

□

Theorem 19 ([1]) *Let u be a function which is defined on $\mathbb{R}^N$ and vanishes identically outside Ω. Let λ a r-weight ($1 \leq r < +\infty$) belonging to the Muckenhoupt class A_r.*

(a) *If $u \in L^1_{\text{loc}}(\mathbb{R}^N)$, then $J_\epsilon * u(x) \in C^\infty(\mathbb{R}^N)$*
(b) *If $u \in L^1_{\text{loc}}(\Omega)$ and supp$(u) \subset\subset \Omega$, then, for $\epsilon <$ dist (supp$(u), \partial\Omega$), $J_\epsilon * u(x) \in C_0^\infty(\Omega)$*
(c) *If $u \in L^r(\Omega, \lambda)$, then $J_\epsilon * u(x) \in L^r(\Omega, \lambda)$. Moreover there exists a positive constant $C = C(N, \sup J)$ such that*

$$\|J_\epsilon * u\|_{L^r(\Omega,\lambda)} \leq C\|u\|_{L^r(\Omega,\lambda)}$$

(d) *If $u \in L^r(\Omega, \lambda)$, then*

$$\|J_\epsilon * u - u\|_{L^r(\Omega,\lambda)} \to 0, \quad \epsilon \to 0^+$$

Proof (For (a) and (b) see [1] Theorem 2.29).
If $u \in L^r(\Omega, \lambda)$ then ($u \in L^1_{\text{loc}}(\mathbb{R}^N)$) from Lemma 18 we have

$$|J_\epsilon * u(x)| \leq CM(u)(x).$$

Hence since λ is in the Muckenhoupt class, by (31) (Theorem 15)

$$\int_\Omega |J_\epsilon * u(x)|^r \lambda(x)dx \leq C \int_\Omega |M(u)|^r(x)\lambda(x)\, dx \leq C_1 \int_\Omega |u(x)|^r \lambda(x)\, dx.$$

In particular $\|J_\epsilon * u\|_{L^r(\Omega,\lambda)} \leq C\|u\|_{L^r(\Omega,\lambda)}$ (here $C = C(N, \sup J)$). Now, let $\eta > 0$ be given. By Theorem 17 there exists $\varphi \in C^0(\Omega)$ such that $\|u - \varphi\|_{L^r(\Omega,\lambda)} < \frac{\eta}{2(C+1)}$.

Now, since $\int_{\mathbb{R}^N} J_\epsilon(y)dy = 1$, by the uniform continuity of φ there exists ϵ_0 such that for all $0 < \epsilon < \epsilon_0$

$$|J_\epsilon * \varphi(x) - \varphi(x)| = \left| \int_{\mathbb{R}^N} J_\epsilon(x-y)(\varphi(y) - \varphi(x))dy \right|$$
$$\leq \sup_{|y-x|<\epsilon} |\varphi(y) - \varphi(x)| < \frac{\eta}{2(\int_\Omega \lambda(x)dx)^{1/r}}$$

This is sufficient to obtain

$$\|J_\epsilon * \varphi - \varphi\|_{L^r(\Omega,\lambda)} < \eta/2$$

Finally, from (c) one has

$$\begin{aligned}\|J_\epsilon * u - u\|_{L^r(\Omega,\lambda)} &\le \|J_\epsilon * u - J_\epsilon * \varphi\|_{L^r(\Omega,\lambda)} + \|J_\epsilon * \varphi - \varphi\|_{L^r(\Omega,\lambda)} + \|u-\varphi\|_{L^r(\Omega,\lambda)}\\ &\le (1+C)\|u-\varphi\|_{L^r(\Omega,\lambda)} + \|J_\epsilon * \varphi - \varphi\|_{L^r(\Omega,\lambda)} < \eta\end{aligned}$$

□

Lemma 20 *Let $u \in W^{2,r}(\Omega,\lambda)$. If $\Omega' \subset\subset \Omega$, then $J_\epsilon * u \to u$ in $W^{2,r}(\Omega',\lambda)$*

Proof Let $\epsilon < \mathrm{dist}(\Omega', \partial\Omega)$, then $D^\alpha J_\epsilon * u = J_\epsilon * D^\alpha u$ in the distributional sense in Ω' (see [1], lemma 3.16). Since $Du \in L^r(\Omega)$, and $D^\alpha u \in L^r(\Omega,\lambda)$ for $|\alpha| = 2$ we have, by Theorem 19(c)

$$\|D^\alpha J_\epsilon * u - D^\alpha u\|_{L^r(\Omega',\lambda)} = \|J_\epsilon * D^\alpha u - D^\alpha u\|_{L^r(\Omega',\lambda)} \to 0 \quad \text{as } \epsilon \to 0^+$$

and

$$\|DJ_\epsilon * u - Du\|_{L^r(\Omega')} = \|J_\epsilon * Du - u\|_{L^r(\Omega')} \to 0 \quad \text{as } \epsilon \to 0^+$$

so that

$$\|J_\epsilon * u - u\|_{W^{2,r}(\Omega',\lambda)} \to 0 \quad \text{as } \epsilon \to 0^+ \tag{32}$$

□

Now we can prove Theorem 16

Proof (we simplify the proof considering the case $\Omega = B_1(0)$) (see also [1] and [13]).

If $u \in W^{2,r}(\Omega,\lambda)$ and $\varepsilon > 0$ we prove that there exists $\varphi \in C^\infty(\Omega)$ such that $\|\varphi - u\|_{W_0^{2,r}(\Omega',\lambda)} < \varepsilon$. For $k = 1, 2, \ldots$ let

$$\Omega_k = \left\{x \in \Omega : \ |x| < 1 - \frac{1}{k}\right\}, \quad \Omega_0 = \Omega_{-1} = \emptyset$$

and

$$U_1 = \Omega_2, \ U_k = \left\{x : \ \frac{k-2}{k-1} < |x| < \frac{k}{k+1}\right\}, \quad k = 2, \ldots$$

Then

$$\mathcal{O} = \{U_k : \ k = 1, 2, \ldots\}$$

is a collection of open subsets of Ω that covers Ω. Let Ψ be a C^∞-partition of unity for Ω subordinate to $\mathcal{O}$. Let ψ_k denote the sum of the finitely many functions $\psi \in \Psi$ whose support are contained in U_k. Then $\psi_k \in C_0^\infty(U_k)$ and $\sum_{k=1}^\infty \psi_k(x) = 1, \ \forall x \in \Omega$.

Let $0 < \epsilon < \frac{1}{(k+1)(k+2)}$ and $V_k = \{x : \ \frac{k-3}{k-2} \le |x| < \frac{k+1}{k+2}, \ k = 3, \ldots\}$, $V_1 = \Omega_3$, $V_2 = \Omega_4$. Then $\mathrm{supp}(J_\epsilon * (\psi_k u)) \subset V_k \subset\subset \Omega$. Since $\psi_k u \in W^{2,r}(\Omega,\lambda)$, by Lemma 20 we may choose $0 < \epsilon_k < \frac{1}{(k+1)(k+2)}$ such that

$$\|J_\epsilon * (\psi_k u) - (\psi_k u)\|_{W^{2,r}(\Omega,\lambda)} = \|J_\epsilon * (\psi_k u) - (\psi_k u)\|_{W^{2,r}(V_k,\lambda)} < \frac{\epsilon}{2^k}.$$

Let $\varphi = \sum_{j=1}^{+\infty} J_{\epsilon_k} * (\psi_k u)$. Since on any $\Omega' \subset\subset \Omega$ only finitely many terms in the sum can be nonzero, one has $\varphi \in C^\infty(\Omega)$ and $\varphi = 0$ on $\partial\Omega$. For $x \in \Omega_k$, we have

$$u(x) = \sum_{j=1}^{k+2} \psi_j(x) u(x) \quad \text{and} \quad \varphi(x) = \sum_{j=1}^{k+2} J_{\epsilon_k} * (\psi_j u)(x)$$

Thus

$$\|u - \varphi\|_{W^{2,r}(\Omega_k,\lambda)} \leq \sum_{j=1}^{k+2} \|J_{\epsilon_k} * (\psi_j u) - \psi_j u\|_{W_0^{2,r}(\Omega,\lambda)} < \varepsilon$$

By the monotone convergence theorem $\|u - \varphi\|_{W_0^{2,r}(\Omega,\lambda)} < \varepsilon$. □

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