Calc. Var. (2010) 37:217–235
DOI 10.1007/s00526-009-0259-9

Calculus of Variations

Optimal boundary control with critical penalization for a PDE model of fluid–solid interactions

Francesca Bucci · Irena Lasiecka

Received: 20 February 2009 / Accepted: 9 June 2009 / Published online: 29 September 2009

Abstract We study the finite-horizon optimal control problem with quadratic functionals for an established fluid-structure interaction model. The coupled PDE system under investigation comprises a parabolic (the fluid) and a hyperbolic (the solid) dynamics; the coupling occurs at the interface between the regions occupied by the fluid and the solid. We establish several trace regularity results for the fluid component of the system, which are then applied to show well-posedness of the Differential Riccati Equations arising in the optimization problem. This yields the feedback synthesis of the unique optimal control, under a very weak constraint on the observation operator; in particular, the present analysis allows general functionals, such as the integral of the *natural energy* of the physical system. Furthermore, this work confirms that the theory developed in Acquistapace et al. (Adv Diff Eq, [2])—crucially utilized here—encompasses widely differing PDE problems, from thermoelastic systems to models of acoustic-structure and, now, fluid-structure interactions.

Mathematics Subject Classification (2000) 35B37 · 49J20 · 74F10 · 49N10 · 35B65 · 35M20 · 93C20

1 Introduction

In this paper we consider the optimal control problem with quadratic functionals for a fluid-structure interaction model. Of major concern is well-posedness of the Riccati equations arising in the minimization problem, along with the feedback synthesis of the

F. Bucci (✉)
Dipartimento di Matematica Applicata, Università degli Studi di Firenze,
Via S. Marta 3, 50139, Firenze, Italy
e-mail: francesca.bucci@unifi.it

I. Lasiecka
Department of Mathematics, University of Virginia, P.O. Box 400137,
Charlottesville, VA 22904-4137, USA
e-mail: il2v@virginia.edu

(unique) optimal control. The fluid-structure interaction is modeled by a system of coupled partial differential equations (PDE) comprising a Stokes system (the fluid) and a three-dimensional system of dynamic elasticity (the solid). The coupling occurs at an interface separating two regions occupied, respectively, by the fluid and the solid. It is assumed that the motion of the solid is driven by infinitesimal displacements with rapid oscillations. Accordingly, the fluid–solid interface is *stationary*; this and other modeling issues are discussed, e.g., in [17]. The mathematical description of the PDE system, that is the boundary control problem (2.1), as well as further literature will be given in the next section. Our main goal is to establish the validity of a Riccati theory that would allow to control the structure, via boundary controls, acting as forces applied to the interface.

It is well known that—even in the case of a single PDE—one of the main difficulties in a rigorous derivation of the feedback synthesis of the optimal control is the presence of *boundary* controls (or, more generally, *unbounded* control actions), combined with the lack of smoothing effects propagated by the dynamics (see, e.g., [10, 23]). In fact, while the linear-quadratic control problem with unbounded control operator has a complete solution in the case of PDE models whose free dynamics is governed by an *analytic* semigroup, this solution may be out of reach in the case of other kind of dynamics. In particular, the case of purely hyperbolic PDE with boundary/point control is peculiarly different[1] from the parabolic case: it would suffice to recall that in finite time horizon problems the Riccati operator (or optimal cost operator) $P(t)$ does *not* satisfy the differential Riccati equations, unless the observation operator possesses a suitable smoothing property.

On the other hand, certain interconnected PDE systems that combine parabolic and hyperbolic effects may give rise to an *abstract* control system $y' = Ay + By$ which yields a *singular estimate* for the operator $e^{At}B$, near $t = 0$. This property—which is an intrinsic feature of control systems ruled by analytic semigroups—has been first identified in the analysis of an acoustic-structure interaction (where the overall semigroup was *not* analytic); see [3]. The essence of such estimates is the following: the parabolic component does induce a singular estimate (as a consequence of analyticity of the corresponding semigroup), while hyperbolicity 'transports' this estimate across the system through the coupling. Thus, if one can show that a singular estimate is valid for the entire system, then the theory in [21,22] ensures a feedback control law with *bounded* (in the state space) gain operator, along with well-posed Riccati equations. This theory has been successfully applied to diverse composite PDE models, including some thermoelastic systems, beside to various acoustic-structure interactions. Several illustrations are contained in [22] and [24]; see also [14,15], and the recent [12].

For the fluid-structure interaction under investigation, which comprises a parabolic and a hyperbolic PDE, it was shown in [26] that a singular estimate (for the corresponding abstract evolution) is satisfied in the *finite energy space*, as long as the penalization in the quadratic functional does not involve the *mechanical energy* at a truly *energy level*. More precisely, the study in [26] established specific singular estimates and hence well-posedness of the Riccati equations in the special case of penalization of the mechanical variables below the energy level (say, *sub-critical* penalization), yet allowing full penalization of the fluid variable.

The situation becomes mathematically much more difficult and more interesting in physical applications when the mechanical variables are penalized at the *critical* level of the energy (see the functional (2.10)). In fact, not only the regularity results of [25] do not apply, but the theory pertaining to control systems which yield singular estimates [24,25] is

[1] In the infinite time horizon case the so called *gain* (or feedback) operator $\mathcal{B}^* P$ is intrinsically unbounded and the analysis of the algebraic Riccati equations is subtle; see [19,33], and the subsequent improvements in [7,32].

no longer valid. (Indeed, if it were so, the gain operator would be bounded on the state space, while we will show that this is not the case; see Remark 2.9.)

The present work addresses the issue of solvability of the optimal control problem with *general* quadratic functionals (i.e. including *critical* penalization) for the PDE model (2.1). As we shall see, we provide (a positive) answer to the question remained open in [26, Remark 6.1]. This will follow in light of the theory introduced in [2], which is shown to cover the present case in view of the set of trace regularity results established and collected in Theorem 2.10. The theory developed in [2] is more effective in capturing the relevant properties of the dynamics, especially the ones which emanate from hyperbolicity. These ultimately allow to define the gain operator as a densely defined—though still *unbounded*—operator acting on a physically relevant (finite energy) state space. Thus, the pathology discovered in [32,33] (where a boundary control problem for a simple hyperbolic equation yields a gain operator which is not densely defined on the state space) does not occur in the present case, and the optimal synthesis resulting from variational principles is consistent with the one generated by the Riccati equation. It is important to emphasize that this is not always the case; see [32] and [33]. (In this respect, the variational aspect of the minimization process along with the theory developed in [2] are critical ingredients in order to justify the arguments leading to well-posedness of Riccati equations and the associated Riccati synthesis.)

Let us recall that the optimal control theory in [2], while relaxing the 'singular estimate requirement', postulates other regularity conditions of global nature. This makes it possible to obtain meaningful solutions to the differential Riccati equations, despite the gain operator is not bounded on the state space. This, however, does not affect the synthesis, as the optimal solution still belongs to the domain of the gain operator. Originally arisen in the study of boundary control problems for an established system of thermoelasticity [1], so far this theory has been shown to apply as well in the case of certain acoustic-structure interaction model including thermal effects [13].

The paper is organized as follows. In Sect. 2 we introduce the boundary control problem under investigation, along with the statements of our main results, namely Theorems 2.6 and 2.10. Moreover, we briefly record some necessary notation and the fundamental well-posedness result pertaining to the uncontrolled counterpart of the PDE system. Section 3 is entirely devoted to the proof of Theorem 2.10, which establishes the novel, distinct boundary regularity properties (of the solutions to the PDE system) which will ultimately result in solvability of the optimization problem, i.e. Theorem 2.6. Section 4 contains the proof of Theorem 2.6, based upon the application of the theory in [2]. Finally, a short Appendix collects the statements of the regularity results pertaining to the elastic component of the system—recently obtained in [8] and [26]—which are crucially utilized in the proof of Theorem 2.10.

2 The PDE model, statement of main results

The PDE model. The PDE model under investigation describes the interaction of a (very slow) viscous, incompressible fluid, with an elastic body in a three dimensional bounded domain. Although the introduction of such models dates back to [27], their PDE analysis has increased significantly only in the past decade. A mathematical description of the composite PDE system is given below. By Ω_f and Ω_s we denote the open smooth domains occupied by the fluid and the solid, respectively. Then $\Omega \subset \mathbb{R}^3$ denotes the entire solid-fluid region, i.e. Ω is the interior of $\overline{\Omega}_f \cup \overline{\Omega}_s$. The boundary of Ω_s is the *interface* between the fluid and the solid, and is denoted by $\Gamma_s = \partial\Omega_s$. We finally denote by Γ_f the outer boundary of

Ω_f, namely $\Gamma_f = \partial\Omega_f \backslash \partial\Omega_s$. It is assumed that the motion of the solid is entirely due to infinitesimal (but 'fast') displacements, and hence that the interface Γ_s is *fixed*.

The velocity field of the fluid is represented by a vector-valued function u, which satisfies a Stokes system in Ω_f; the scalar function p represents, as usual, the pressure. In the solid region Ω_s the displacement w satisfies the equations of linear elasticity. (The density and the kinematic viscosity which usually appear in the Navier–Stokes equation are set equal to one, just to simplify the notation). The coupling takes place on the interface Γ_s. We recall from [17] that the interface condition $u = w_t$ on Γ_s (in place of the usual no-slip boundary condition $u = 0$) accounts for the fact that although the displacement of the elastic body is small, its velocity is not (small, yet rapid oscillations). Thus, the PDE system is given by

$$\begin{cases} u_t - \operatorname{div} \epsilon(u) + \nabla p = 0 & \text{in } Q_f := \Omega_f \times (0, T) \\ \operatorname{div} u = 0 & \text{in } Q_f \\ w_{tt} - \operatorname{div} \sigma(w) = 0 & \text{in } Q_s := \Omega_s \times (0, T) \\ u = 0 & \text{on } \Sigma_f := \Gamma_f \times (0, T) \\ w_t = u & \text{on } \Sigma_s := \Gamma_s \times (0, T) \\ \sigma(w) \cdot \nu = \epsilon(u) \cdot \nu - p\nu - g & \text{on } \Sigma_s \\ u(0, \cdot) = u_0 & \text{in } \Omega_f \\ w(0, \cdot) = w_0, \quad w_t(0, \cdot) = w_1 & \text{in } \Omega_f. \end{cases} \tag{2.1}$$

In the above coupled PDE system, σ and ϵ denote the elastic stress tensor and the strain tensor, respectively, that are

$$\sigma_{ij}(u) = \lambda \sum_{k=1}^{3} \epsilon_{kk}(u)\delta_{ij} + 2\mu\epsilon_{ij}(u), \quad \epsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.2}$$

where λ, μ are the Lamé constants and δ_{ij} is the Kronecker symbol.

Since the present work is focused on the optimization problem, the subtle questions related to the modeling of fluid-structure interaction phenomena, as well as to the analysis of well-posedness of the corresponding coupled PDE systems, will not be discussed here. Yet, well-posedness of the boundary value problem (2.1), with $g \equiv 0$ (that is the uncontrolled system (2.3) below), is a prerequisite for the study of the associated optimal control problems. Thus, although many authors have contributed to the PDE analysis of *nonlinear* fluid-structure interaction models (where the dynamics of the fluid is ruled by a Navier–Stokes equation), existence of finite energy weak solutions—even for the simpler Stokes-Lamé system (2.3)—has been an open question until [8, Theorem 2.2]. The reader is referred to [8,9] for the analysis of well-posedness of the coupled PDE system (2.3); in addition, [9] includes a very clear introduction to the (nonlinear) fluid-structure interaction problem, along with a technical comparison with the previous mathematical literature. In Sect. 2.1 we shall recall, for the reader's convenience, the theory in [8] that is needed for our purposes.

We finally note that while the present study follows the variational approach of [8], exploiting the novel boundary regularity results established therein, semigroup well-posedness and stability properties of the *linear* model have been investigated in [4]; see the survey paper [5] and its references. For the uniform stabilization problem, see [6].

Further references. There is a large literature on coupled fluid-structure evolution problems. Most works address the issue of developing models for specific physical problems and/or their numerical simulation. Two main different scenarios arise from the applications: the case in which the fluid is flowing in a tube with elastic walls, such as the blood through arteries, and the case where one or more elastic bodies are immersed in a fluid flow. The PDE model under investigation pertains to a physical situation falling under the latter category.

A very nice introduction to fluid-structure interaction problems is provided by [17]. Recent treatises with focus on modeling and numerical analysis are [30] and [29]. An in-depth PDE analysis of well-posedness of these nonlinear models has indeed appeared only recently. Relevant contributions to this problem are given (without any claim of completeness) by [18,31], the aforesaid [8,9,11,16,17], and, lastly, [20]. For more information on this subject, see the bibliography therein.

2.1 Variational and semigroup formulation

Before giving the statement of our main results, let us preliminarily recall from [8] some basic notation, and the chief facts which pertain to the uncontrolled problem, that is system (2.1) with $g \equiv 0$. Further technical results obtained in [8] and [26] will be needed in the proof of our main result; these will be recorded in an Appendix for convenience.

The uncontrolled model. Let us introduce the *free* system corresponding to (2.1), namely

$$\begin{cases} u_t - \operatorname{div}\epsilon(u) + \nabla p = 0 & \text{in } Q_f := \Omega_f \times (0,T) \\ \operatorname{div} u = 0 & \text{in } Q_f \\ w_{tt} - \operatorname{div}\sigma(w) = 0 & \text{in } Q_s := \Omega_s \times (0,T) \\ u = 0 & \text{on } \Sigma_f := \Gamma_f \times (0,T) \\ w_t = u & \text{on } \Sigma_s := \Gamma_s \times (0,T) \\ \sigma(w)\cdot\nu = \epsilon(u)\cdot\nu - p\nu & \text{on } \Sigma_s \\ u(0,\cdot) = u_0 & \text{in } \Omega_f \\ w(0,\cdot) = w_0, \quad w_t(0,\cdot) = w_1 & \text{in } \Omega_f. \end{cases} \tag{2.3}$$

The energy space for the PDE problem (2.3) is

$$Y = H \times H^1(\Omega_s) \times L_2(\Omega_s),$$

where H is defined as follows:

$$H := \left\{u \in L_2(\Omega_f) : \ \operatorname{div} u = 0, \ u\cdot\nu|_{\Gamma_f} = 0\right\}.$$

In addition, we denote by V the space defined as follows:

$$V := \left\{v \in H^1(\Omega_f) : \ \operatorname{div} u = 0, \ u|_{\Gamma_f} = 0\right\};$$

we shall use the following distinct notation for the various inner products which will occur throughout the paper:

$$(u,v)_f := \int_{\Omega_f} uv \,\mathrm{d}\Omega_f, \quad (u,v)_s := \int_{\Omega_s} uv\,\mathrm{d}\Omega_s, \quad \langle u,v\rangle := \int_{\Gamma_s} uv\,\mathrm{d}\Gamma_s.$$

The space V is topologized with respect to the inner product given by

$$(u,v)_{1,f} := \int_{\Omega_f} \epsilon(u)\epsilon(v)\mathrm{d}\Omega_f;$$

the corresponding (induced) norm $|\cdot|_{1,f}$ is equivalent to the usual $H^1(\Omega_f)$ norm, in view of Korn inequality and the Poincaré inequality.

Remark 2.1 The norm $\|\cdot\|_{H^r(D)}$ in the Sobolev space $H^r(D)$ will be shortly denoted by $|\cdot|_{r,D}$ throughout the paper. Note that all the Sobolev spaces H^r related to u and w are actually $(H^r)^3$: the exponent is omitted just for the sake of simplicity.

Let us recall from [8] the definition of *weak* solutions to the (uncontrolled) PDE system (2.3).

Definition 2.2 (*Weak solution*) Let $(u_0, w_0, w_1) \in Y$ and $T > 0$. We say that a triple $(u, w, w_t) \in C([0, T], H \times H^1(\Omega_s) \times L_2(\Omega_s))$ is a weak solution to the PDE system (2.1) if

- $(u(\cdot, 0), w(\cdot, 0), w_t(\cdot, 0)) = (u_0, w_0, w_1)$,
- $u \in L_2(0, T; V)$,
- $\sigma(w) \cdot \nu \in L_2(0, T; H^{-1/2}(\Gamma_s))$, $\frac{\mathrm{d}}{\mathrm{d}t} w|_{\Gamma_s} = u|_{\Gamma_s} \in L_2(0, T; H^{1/2}(\Gamma_s))$, and
- the following variational system holds a.e. in $t \in (0, T)$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(u, \phi)_f + (\epsilon(u), \epsilon(\phi))_f - \langle \sigma(w) \cdot \nu + g, \phi \rangle = 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}(w_t, \psi)_s + (\sigma(w), \epsilon(\psi))_s - \langle \sigma(w) \cdot \nu, \psi \rangle = 0, \end{cases} \tag{2.4}$$

for all test functions $\phi \in V$ and $\psi \in H^1(\Omega_s)$.

Remark 2.3 It is important to emphasize that the regularity properties of the normal stresses (see the third item of Definition 2.2) do not follow from the interior regularity of the fluid-structure variables. It is an independent regularity result, showing the exceptional behavior of hyperbolic traces. This regularity property is necessary in order to justify the variational definition of weak solutions (see (2.4)). While there are other definitions of solutions to non-linear PDE models of fluid-structure interactions which do not require additional regularity on the boundary (see, e.g., [4,17,27]), yet these definitions are not adequate to variationally decouple the (finite energy) weak solutions of the two equations. On the other hand, this decoupling is crucially important in the present analysis, aimed at identifying the distinctive regularity properties of the overall dynamics, that play a major role in the study of the associated optimal control problems. Exploiting the distinct features (analyticity and hyperbolicity) of the decoupled dynamics makes it possible to establish the sharpest results for the coupled PDE system. (This fact was recently utilized in [20], as well.) Consequently, the issue of "hidden" regularity of the hyperbolic component is central to the problem studied and its solution.

Existence of weak (global) solutions of a nonlinear generalization of the PDE problem (2.3) has been established in [8].

Theorem 2.4 (Existence of weak solutions, [8]) *Given any initial datum* $(u_0, w_0, w_1) \in Y$ *and any* $T > 0$, *there exists a weak solution* (u, w, w_t) *to the system* (2.3) *such that*

$$\nabla w\big|_{\Gamma_s} \in L_2(0, T; H^{-1/2}(\Gamma_s)), \quad \frac{\mathrm{d}}{\mathrm{d}t} w\Big|_{\Gamma_s} = w_t\big|_{\Gamma_s} \in L_2(0, T; H^{1/2}(\Gamma_s)).$$

The control system, semigroup formulation. Aiming to apply the optimal control theory pertaining to a general class of evolutions—in the present case, the one developed in [2]—it is convenient to recast the boundary value problem (2.1) as an abstract control system in a Hilbert space. Accordingly, let us introduce the fluid dynamic operator $A : V \to V'$, defined by

$$(Au, \phi) = -(\epsilon(u), \epsilon(\phi)) \quad \forall \phi \in V, \tag{2.5}$$

and the (Neumann) map $N : L_2(\Gamma_s) \to H$ defined as follows:

$$Ng = h \iff (\epsilon(h), \epsilon(\phi)) = \langle g, \phi \rangle \quad \forall \phi \in V.$$

We just recall from [26] that the operator A defined by (2.5) may be considered as acting on H with domain $\mathcal{D}(A) := \{ u \in V : |(\epsilon(u), \epsilon(\phi))| \le C|\phi|_H \}$. Thus, $A : \mathcal{D}(A) \subset H \to H$ is a self-adjoint, negative operator and therefore is the infinitesimal generator an *analytic* semigroup e^{At}, $t \ge 0$, on H. Then, the fractional powers of $-A$ are well defined; to simplify the notation, we shall denote them by A^α (rather than by $(-A)^\alpha$) throughout. Other chief properties of the operators A and N will be recalled in the Appendix.

Then, if we set $y = (u, w, w_t)$, the boundary value problem (2.1) reduces to the linear control system

$$\begin{cases} y' = \mathcal{A}y + \mathcal{B}g \quad \text{in} \quad [\mathcal{D}(\mathcal{A}^*)]' \\ y(0) = y_0 \end{cases} \tag{2.6}$$

where the (dynamic) operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset Y \to Y$ is defined by

$$\mathcal{A} = \begin{pmatrix} A & AN\sigma(\cdot)\cdot\nu & 0 \\ 0 & 0 & I \\ 0 & \operatorname{div}\sigma(\cdot) & 0 \end{pmatrix}, \tag{2.7}$$

with domain

$$\mathcal{D}(\mathcal{A}) = \big\{ y = (u, w, z) \in Y : u \in V,\ A(u + N\sigma(w)\cdot\nu) \in H,\ z \in H^1(\Omega_s), \\ \operatorname{div}\sigma(w) \in L_2(\Omega_s),\ z|_{\Gamma_s} = u|_{\Gamma_s} \big\},$$

and the (control) operator $\mathcal{B} : L_2(\Gamma_s) = U \to [\mathcal{D}(\mathcal{A})]'$ reads as

$$\mathcal{B} = \begin{pmatrix} AN \\ 0 \\ 0 \end{pmatrix}. \tag{2.8}$$

With the dynamics (2.6) we associate the following quadratic cost functional over a given time interval $[0, T]$, $0 < T < \infty$:

$$J(g) = \int_0^T \left(|\mathcal{R}y(t)|_Z^2 + |g(t)|_U^2 \right) \mathrm{d}t, \tag{2.9}$$

where Z is another Hilbert (output) space. The optimal control problem (linear-quadratic, or LQ-problem) is to minimize the functional (2.9), over all control functions $g \in L_2(0, T; U)$, with y solution to (2.6) corresponding to g. As pointed out in the Introduction, we aim to include in the present analysis non-smoothing observation operators $\mathcal{R}$, such as the *identity* operator; hence, $\mathcal{R}$ is initially assumed to satisfy just $\mathcal{R} \in \mathcal{L}(Y, Z)$. By doing so we admit natural quadratic functionals such as the following,

$$J(g) = \frac{1}{2} \int_0^T \left\{ |u(t)|_{0,\Omega_f}^2 + (\sigma(w(t)), \epsilon(w(t)))_s + |w_t(t)|_{0,\Omega_s}^2 + |g(t)|_{0,\Gamma_s}^2 \right\} \mathrm{d}t \tag{2.10}$$

which penalizes the full quadratic energy $E(t)$ of the system.

Remark 2.5 We already emphasized that the study performed in [26] did not provide solvability of optimal control problems with general quadratic functionals: in particular, it did not cover the case of natural functionals such as (2.10). On the other hand, the analysis carried out in [26]—despite the final constraint on the observation operator $\mathcal{R}$—included the case of Bolza problems, where the penalization affects also the state at the *final time* $T < \infty$, namely, when the functional to be minimized is given by

$$J(g) = \int_0^T \left(|\mathcal{R}y(t)|_Z^2 + |g(t)|_U^2 \right) \, \mathrm{d}t + (\mathcal{G}y(T), y(T))_W. \tag{2.11}$$

Note that the LQ-problem with Bolza-type quadratical functionals is not discussed here. In fact, the LQ-problem with quadratic functionals of the form (2.11) (with $\mathcal{G} \neq 0$) for the class of control systems (2.6) described by the Assumptions 4.1, has not been investigated yet.

2.2 Statement of the main results

The main result of the present work is the proof of well-posedness of the (differential) Riccati equations corresponding to the optimal control problems associated with the fluid-structure model (2.1), along with all the inherent assertions about solvability of the optimization problem; see Theorem 2.6. This variational result, however, critically relies on the novel trace regularity results established specifically for the (uncontrolled) PDE system (2.3) in Theorem 2.10, which thus constitute the major technical contribution of the present work. As we shall see, the proof of this set of regularity results is based on the interplay between the maximal parabolic regularity of the fluid component with the 'hidden' regularity of the traces of the hyperbolic (solid) component. Indeed, the fact that the coupling is of hyperbolic/parabolic type will be critically utilized.

2.2.1 The solution to the optimization problem

With reference to the PDE model introduced in the previous section, let us consider the optimal control problem (2.6)–(2.9), that is

Minimize the functional $J(g)$ in (2.9), over all $g \in L_2(0, T; L_2(\Gamma_s))$, where $y(\cdot) = y(\cdot; y_0, g)$ solves the control system (2.6).

Then we have the following.

Theorem 2.6 *Consider the optimal control problem* (2.6)–(2.9)*, with $\mathcal{A}$ and $\mathcal{B}$ given by* (2.7) *and* (2.8)*, respectively. Let us assume that the observation operator $\mathcal{R}$ satisfies*

$$\mathcal{R}^*\mathcal{R} \in \mathcal{L}(\mathcal{D}(\mathcal{A}^\epsilon), \mathcal{D}(\mathcal{A}^{*\epsilon})), \tag{2.12}$$

where $\epsilon > 0$ can be taken arbitrarily small. Then the following assertions hold true.

(1) *For any initial state $y_0 \in Y$ there exists a unique optimal control $g^0(\cdot) \in L_2(0, T; L_2(\Gamma_s))$ such that*

$$J(g^0) = \min_{g \in L_2(0,T;L_2(\Gamma_s))} J(g).$$

The optimal pair $(g^0(\cdot), y^0(\cdot))$ has the following additional regularity:

$$\begin{aligned} &y^0(\cdot) = [u^0(\cdot), w^0(\cdot), w_t^0(\cdot)] \in C([0, T]; H \times H^1(\Omega_s) \times L_2(\Omega_s)); \\ &g^0(\cdot) \in \bigcap_{1 \le p < \infty} L_p(0, T; L_2(\Gamma_s)). \end{aligned} \tag{2.13}$$

(2) *There exists a non-negative, selfadjoint operator (the Riccati operator)* $P(t) \in \mathcal{L}(Y)$, $t \in [0, T]$, *defined explicitly in terms of the data, such that*

$$J(g^0) = (P(0)y_0, y_0)_Y;$$

more precisely $P(\cdot) \in \mathcal{L}(Y, C([0, T], Y))$.

(3) *The gain operator* $\mathcal{B}^* P(\cdot)$ *satisfies* $\mathcal{B}^* P(\cdot) \in \mathcal{L}(\mathcal{D}(\mathcal{A}^\epsilon), C([0, T], L_2(\Gamma_s)))$*; moreover, one has (the feedback synthesis of the optimal control) a.e. in* $[0, T]$:

$$g^0(t) = -\mathcal{B}^* P(t) y^0(t), \quad \forall y_0 \in Y. \tag{2.14}$$

(4) *The operator* $P(t)$ *is the unique—within the class of selfadjoint, positive operators such that* $\mathcal{B}^* P(\cdot) \in \mathcal{L}(\mathcal{D}(\mathcal{A}^\epsilon), C([0, T], L_2(\Gamma_s)))$*—solution of the Differential Riccati Equation satisfied for* $0 \le t < T$ *and* $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in D(\mathcal{A})$,

$$\frac{d}{dt}(P(t)x, y)_Y + (\mathcal{A}^* P(t)x, y)_Y + (P(t)\mathcal{A}x, y)_Y + (\mathcal{R}x, \mathcal{R}y)_Y = (\mathcal{B}^* P(t)x, \mathcal{B}^* P(t)y), \tag{2.15}$$

with

$$\lim_{t \to T^-} (P(t)x, x) = 0, \quad \forall x \in Y.$$

Remark 2.7 According to Theorem 2.6, the gain operator $B^* P(t)$ is bounded from $\mathcal{D}(\mathcal{A}^\epsilon)$ to the control space U, hence densely defined on the state space Y. Formula (2.14) shows that the optimal feedback synthesis is consistent with the Riccati feedback synthesis, where the operator $P(t)$ is determined via the Riccati equation (2.15). This is in striking contrast with the counterexample discussed in [32,33], where the feedback representation of the optimal control cannot be obtained by means of the Riccati operator $P(t)$.

Remark 2.8 Since here the dynamics operator $\mathcal{A}$ is the generator of a s.c. contraction semigroup with $\mathcal{A}^{-1} \in \mathcal{L}(Y)$, then the domains of fractional powers $\mathcal{D}(\mathcal{A}^\epsilon)$ in (2.12) may be computed as intermediate spaces between $\mathcal{D}(\mathcal{A})$ and Y. The same holds for $\mathcal{D}(\mathcal{A}^{*\epsilon})$. (For a comprehensive list of cases where the identity $[\mathcal{D}(\mathcal{A}), Y]_{1-\theta} = \mathcal{D}(\mathcal{A}^\theta)$ holds true, see, e.g., [23, § 0.2.1]). Then, it is not difficult to show that in the present case $\mathcal{D}(\mathcal{A}^{*\epsilon}) \equiv \mathcal{D}(\mathcal{A}^\epsilon)$, provided ϵ is sufficiently small. Therefore, assumption (2.12) is satisfied, with a non-smoothing observation operator, such as $\mathcal{R} = I$. This natural situation was indeed left as an open problem in [26].

Remark 2.9 Observe that the optimal pair does not display the typical regularity (in time) exhibited in the case of control systems whose underlying semigroup is analytic (or, more in general, when singular estimates are satisfied). In particular, the optimal control is not continuous. This is not surprising, in view of the influence of both hyperbolic and parabolic effects on the overall behavior of the solutions.

Moreover, the gain operator $\mathcal{B}^* P(t)$ is no longer bounded on the state space Y, but just *densely defined*. However, this does not affect the final result, as the feedback formula holds for any initial state in the finite energy space. Thus, the observation operator $\mathcal{R}$ need not have regularizing effects, and $\mathcal{R}$ can be critical.

The above observations also explain why the previous Riccati theories are intrinsically unapplicable in the critical case, as they lead to *bounded* gain operators, in contrast with the case under examination.

2.2.2 Trace estimates

The fundamental analytic tool which will enable us to show Theorem 2.6 is a complex of boundary regularity results pertaining to the fluid component of the PDE problem (2.3). These traces' regularity estimates of u (and u_t) on the interface Γ_s are the "PDE counterpart" of the abstract regularity properties of the (unbounded) operator $\mathcal{B}^* e^{\mathcal{A}^* t}$ needed to invoke the optimal control theory of [2]. These regularity estimates are, however, also of independent interest.

Theorem 2.10 (Traces' regularity) *Consider the uncontrolled Stokes-Lamé system, namely the PDE system* (2.3). *Let $y(t) = (u(t), w(t), w_t(t))$ be the solution corresponding to initial data $y_0 = (u_0, w_0, w_1)$. Then the fluid component u admits a decomposition $u(t) = u_1(t) + u_2(t)$, and the following statements pertain to the regularity of the traces of u_1, u_2 and u_t on Γ_s, respectively.*

(i) *The component u_1 satisfies a pointwise* (*in time*) *"singular estimate", namely there exists a positive constant C_T such that*

$$|u_1(t)|_{L_2(\Gamma_s)} \le \frac{C_T}{t^{1/4+\delta}} |y_0|_Y \quad \forall y_0 \in Y, \quad \forall t \in (0, T] \tag{2.16}$$

(*with arbitrarily small $\delta > 0$*).

(ii) *The component u_2 satisfies the following regularity*:

(iia) *if $y_0 \in Y$, then $u_2|_{\Gamma_s} \in L_p(0, T; L_2(\Gamma_s))$ for all* (*finite*) $p \ge 1$;

(iib) *Let $y_0 \in \mathcal{D}(\mathcal{A}^\epsilon)$, where $\epsilon > 0$ can be taken arbitrarily small, but positive. Then $u_2|_{\Gamma_s} \in C([0, T], L_2(\Gamma_s))$.*

(iii) *Let now $y_0 \in \mathcal{D}(\mathcal{A}^{1-\theta})$, with $\theta \in (0, \frac{1}{4})$. Then, the fluid component u of corresponding solution satisfies, for some $q \in (1, 2)$,*

$$u_t|_{\Gamma_s} \in L_q(0, T; L_2(\Gamma_s)) \tag{2.17}$$

continuously with respect to y_0, that is there exists a constant C_T such that

$$\|u_t\|_{L_q(0,T;L_2(\Gamma_s))} \le C_T \, \|y_0\|_{\mathcal{D}(\mathcal{A}^{1-\theta})}. \tag{2.18}$$

The exponent q will depend on θ: more precisely, given $\theta \in (0, \frac{1}{4})$, one has

$$1 < q < \frac{4}{3 + 4\theta}. \tag{2.19}$$

The remainder of the paper is devoted to the proof of the two main results stated in Theorem 2.6 and Theorem 2.10. Section 3 deals with the above boundary regularity results, which will be next utilized in Sect. 4 to establish Theorem 2.6.

3 Proof of the trace regularity results

This section is entirely devoted to the proof of our main contribution, that is Theorem 2.10.

Proof of Theorem 2.10 Our starting point is the equation satisfied by $u(\cdot)$, namely $u_t = Au + AN\sigma(w) \cdot \nu$, whose solutions are given by

$$u(t) = e^{At} u_0 + \int_0^t e^{A(t-s)} AN\sigma(w)(s, \cdot)\nu \, \mathrm{d}s; \tag{3.1}$$

the above expression yields the natural splitting $u(t) = u_1(t) + u_2(t)$, with

$$u_1(t) := e^{At}u_0, \quad u_2(t) := \int_0^t e^{A(t-s)} AN\sigma(w)(s,\cdot)\nu \, ds. \tag{3.2}$$

In view of $N^*Au = -u|_{\Gamma_s}$ (see Lemma A.1), the corresponding traces on Γ_s read as

$$\begin{aligned} u_1|_{\Gamma_s} &= -N^*Au_1(t) = -N^*Ae^{At}u_0, \\ u_2|_{\Gamma_s} &= -N^*Au_2(t) = -N^*A\int_0^t e^{A(t-s)}AN\sigma(w)(s,\cdot)\nu \, ds, \end{aligned} \tag{3.3}$$

respectively.

(i) The *singular* estimate in (2.16) follows as an immediate consequence of the well known estimates pertaining to analytic semigroups:

$$\begin{aligned} \left|u_1(t)|_{\Gamma_s}\right|_U &= |N^*Ae^{At}u_0| = |N^*A^{3/4-\delta}A^{1/4+\delta}e^{At}u_0| \\ &\le \|N^*A^{3/4-\delta}\| \, |A^{1/4+\delta}e^{At}u_0|_Y \sim C_T t^{-1/4-\delta}|u_0|. \end{aligned} \tag{3.4}$$

This shows the validity of assertion (i).

(ii) Let initially $y_0 = (u_0, w_0, w_1) \in Y$. In view of (3.3), it is clear that the sharp regularity theory pertaining to the wave component will play a central role in the study of the regularity of the traces of $u_2(t)$ on Γ_s. More precisely, we shall utilize the recent trace results obtained in [8, Theorem 3.3] and refined in [26, Lemma 5.2]; see Lemma A.2 in the Appendix. Accordingly, following the decomposition of $\sigma(w)\cdot\nu$ established in Lemma A.2, it is convenient to introduce a further splitting, namely

$$u_2(t) = \underbrace{\int_0^t e^{A(t-s)}AN\sigma(w_1)(s,\cdot)\nu \, ds}_{u_{21}(t)} + \underbrace{\int_0^t e^{A(t-s)}AN\sigma(w_2)(s,\cdot)\nu \, ds}_{u_{22}(t)}. \tag{3.5}$$

Thus, one has first

$$\begin{aligned} N^*Au_{21}(t) &= N^*A\int_0^t e^{A(t-s)}AN\sigma(w_1)(s)\cdot\nu \, ds \\ &= [N^*A^{3/4-\epsilon}]A^{1/4+\epsilon+1/2}\int_0^t e^{A(t-s)} \underbrace{A^{1/2}N\sigma(w_1)(s)\cdot\nu}_{f(s)} \, ds \end{aligned}$$

where $f \in C([0,T], L_2(\Omega_s))$ in view of Lemma A.2 and Lemma A.1. Consequently,

$$\int_0^t e^{A(t-s)}f(s)\, ds \in C([0,T], D(A^{1-\sigma})), \tag{3.6}$$

with arbitrarily small $\sigma > 0$; see, e.g., [23, Proposition 0.1, p. 4]. Therefore $N^*Au_{21} \in C([0,T], L_2(\Gamma_s))$, and a fortiori we obtain

$$N^*Au_{21} \in L_p(0,T;U) \quad \forall p \ge 1. \tag{3.7}$$

As for the second summand $N^*Au_{22}(t)$, because of the different regularity of $\sigma(w_2)(s)\cdot\nu$ we rewrite in a different fashion:

$$N^*Au_{22}(t) = \left[N^*A^{3/4-\epsilon}\right] A^{1/2+2\epsilon} \int_0^t e^{A(t-s)} \underbrace{[A^{3/4-\epsilon}N\sigma(w_2)(s)\cdot\nu]}_{\varphi(s)}\, \mathrm{d}s.$$

Notice now that the above integral is the convolution

$$\int_0^t K(t-s)\varphi(s)\, \mathrm{d}s,$$

with $\varphi \in L_2(0,T;U)$ and the kernel K such that $||K(s)|| \sim \frac{1}{s^{1/2+2\epsilon}}$, where ϵ can be taken arbitrarily small. Hence $K \in L_{2-\sigma}(0,T;U)$ for arbitrarily small $\sigma > 0$. Thus, the Young inequality yields

$$N^*Au_{22} \in L_p(0,T;U) \quad \forall p \geq 1. \tag{3.8}$$

Thus, (3.8) combined with (3.7) shows the validity of assertion (iia).

Let now $y_0 \in D(\mathcal{A}^\epsilon)$, $\epsilon > 0$. In this case, by Lemma A.3 it follows $u|_{\Gamma_s} \in H^\epsilon(\Sigma_s)$, provided that $\epsilon < \frac{1}{4}$. This enables us to apply the second part of Lemma A.2, which gives

$$\sigma(w_1)\cdot\nu \in C([0,T], H^{-1/2}(\Gamma_s)), \quad \sigma(w_2)\cdot\nu \in H^\epsilon(\Sigma_s). \tag{3.9}$$

Now, the analysis of N^*Au_{21} follows closely the one in item (iia), yielding the conclusion $N^*Au_{21} \in C([0,T], L_2(\Gamma_s))$ (this is justified by the membership (3.6)). Instead, on the basis of the novel regularity of $\sigma(w_2)\cdot\nu$ in (3.9), from parabolic theory it follows that

$$u_{22} \in H^{\epsilon+3/2,\epsilon/2+3/4}(Q_f),$$

so that

$$N^*Au_{22} \in H^{\epsilon+1,\epsilon/2+1/2}(\Sigma_s) \subset H^{1/2+\epsilon/2}(0,T;L_2(\Gamma_s)) \subset C([0,T], L_2(\Gamma_s)).$$

As both $N^*\mathrm{A}u_{21}$ and N^*Au_{22} belong to $C([0,T], L_2(\Gamma_s))$, then $N^*Au_2 \in C([0,T], L_2(\Gamma_s))$ and (iib) is proved.

(iii) In this last step we aim to ascertain the regularity of the boundary traces of the time derivative u_t on Γ_s. We return to the mild solution (3.1) and compute

$$u_t(t) = \underbrace{Ae^{At}u_0}_{v_1(t)} + \underbrace{\overbrace{A\int_0^t e^{A(t-s)}AN\sigma(w)(s,\cdot)\nu\, \mathrm{d}s}^{v_{2a}} + \overbrace{AN\sigma(w)(t,\cdot)\nu}^{v_{2b}}}_{v_2(t)} \tag{3.10}$$

which can also be rewritten as

$$u_t(t) = \underbrace{Ae^{At}u_0}_{v_1(t)} + \underbrace{\overbrace{\int_0^t e^{A(t-s)}AN\sigma(w_s)(s,\cdot)\nu\, \mathrm{d}s}^{v_{21}(t)} + \overbrace{Ae^{At}N\sigma(w)(0,\cdot)\nu}^{v_{22}(t)}}_{v_2(t)}. \tag{3.11}$$

The plan we aim to carry out is to discuss first the regularity of the function $v_2 := \frac{\partial u_2}{\partial t}$ when $y_0 \in D(\mathcal{A})$ by using its expression in (3.11). Next, when $y_0 \in Y$, we would rather utilize (3.10), and then use interpolation arguments to establish the regularity corresponding to initial data in $D(\mathcal{A}^{1-\epsilon})$. Only subsequently we shall derive the trace regularity of v_2 by applying the operator $-N^*A$.

When $y_0 \in D(\mathcal{A})$, by standard semigroup arguments, using the commutativity of the generator with the semigroup, we obtain that $\sigma(w_t) \cdot \nu$ exhibits the same regularity as that of $\sigma(w) \cdot \nu$ when $y_0 \in Y$, i.e. (invoking once again Lemma A.2)

$$\sigma(w_t) \cdot \nu = \sigma_1 + \sigma_2 \in C([0, T], H^{-1/2}(\Gamma_s)) \; + \; L_2(0, T; L_2(\Gamma_s)).$$

To pinpoint the regularity of v_{21}, we now utilize the above splitting and follow the analysis carried out in the proof of (ii). More precisely, combining elliptic regularity (in particular, Lemma A.1), with the analyticity of the semigroup e^{At}, along with the (singular) estimates pertaining to $A^\alpha e^{At}$, we first obtain, for any t and any $\delta < 1/2$,

$$\begin{aligned} \Big|A^{1/2-\delta} \int_0^t e^{A(t-s)} A N \sigma_1(s)\, \mathrm{d}s\Big| &= \Big|A^{1-\delta} \int_0^t e^{A(t-s)} A^{1/2} N \sigma_1(s)\, \mathrm{d}s\Big| \\ &\le C \int_0^t \frac{1}{(t-s)^{1-\delta}} \mathrm{d}s \, \|\sigma_1\|_{C([0,T],H^{-1/2}(\Gamma_s))} \le C \|\sigma_1\|_{C([0,T],H^{-1/2}(\Gamma_s))}. \end{aligned} \tag{3.12}$$

As for the latter term, we apply as well $A^{1/2-\delta}$ and rewrite as follows:

$$A^{1/2-\delta} \int_0^t e^{A(t-s)} A N \sigma_2(s)\, \mathrm{d}s = \int_0^t [A^{3/4-\delta/2} e^{A(t-s)}]\, [A^{3/4-\delta/2} N]\, \sigma_2(s)\, \mathrm{d}s, \tag{3.13}$$

where it is clear now that the integral is the convolution of $L_{4/3}$ and L_2 (in time) functions, respectively. On the strength of the Young's inequality, we get L_4-regularity in time, so that

$$v_{21} \in C([0, T], D(A^{1/2-\delta})) \; + \; L_4(0, T; D(A^{1/2-\delta})), \quad 0 < \delta < \frac{1}{2}.$$

This implies the membership

$$v_{21} \in L_4(0, T; D(A^{1/2-\delta})), \quad 0 < \delta < \frac{1}{2}. \tag{3.14}$$

On the other hand, still with $y_0 \in D(\mathcal{A})$, one has just $\sigma(w) \cdot \nu \in C([0, T], H^{-1/2}(\Gamma_s))$ which suggests us to rewrite v_{22} as follows:

$$v_{22}(t) = A^{1/2} e^{At} \; \left(A^{1/2} N \sigma(w)(0, \cdot) \nu\right);$$

then, again in view of Lemma A.1 and of the usual singular estimates pertaining to analytic semigroups, it follows

$$v_{22} \in L_q(0,T;D(A^{1/2-\delta})), \tag{3.15}$$

provided that $q(1-\delta) < 1$. Therefore, (3.14) combined with (3.15) yields, for any $0 < \delta_1 < \frac{1}{2}$

$$y_0 \in D(\mathcal{A}) \Longrightarrow v_2 \in L_{q_1}(0,T;D(A^{1/2-\delta_1})) \equiv L_{q_1}(0,T;H^{1-2\delta_1}(\Omega_f)), \tag{3.16}$$

where $q_1 \in (1,2)$ depends on δ_1; more precisely,

$$q_1 < \frac{1}{1-\delta_1}. \tag{3.17}$$

Let now $y_0 \in Y$. In this case we use the decomposition (3.10), and begin with the analysis of v_{2a}. Setting $w = w_1 + w_2$ (according with Lemma A.2), one has

$$A\int_0^t e^{A(t-s)} AN\sigma(w_1)(s,\cdot)\nu \,\mathrm{d}s$$

$$= A^{1/2+\epsilon_1} A^{1-\epsilon_1} \int_0^t e^{A(t-s)} A^{1/2} N\sigma(w_1)(s,\cdot)\nu \,\mathrm{d}s \in C([0,T],[D(A^{1/2+\epsilon_1})]'),$$

while

$$A\int_0^t e^{A(t-s)} AN\sigma(w_2)(s,\cdot)\nu \,\mathrm{d}s$$

$$= A^{1/4+\epsilon_2} A \int_0^t e^{A(t-s)} A^{3/4-\epsilon_2} N\sigma(w_2)(s,\cdot)\nu \,\mathrm{d}s \in L_2(0,T;[D(A^{1/4+\epsilon_2})]')$$

where both ϵ_1 and ϵ_2 can be taken arbitrarily small. As a result,

$$v_{2a} \in L_2(0,T;[D(A^{1/2+\epsilon})]'), \quad 0 < \epsilon < \frac{1}{2}. \tag{3.18}$$

As for the term v_{2b}, readily

$$AN\sigma(w_1)(t,\cdot)\nu = A^{1/2}A^{1/2}N\sigma(w_1)(t,\cdot)\nu \in C([0,T],[D(A^{1/2})]')$$

while

$$AN\sigma(w_2)(t,\cdot)\nu = A^{1/4+\epsilon}[A^{3/4-\epsilon}N]\sigma(w_2)(t,\cdot)\nu \in L_2(0,T;[D(A^{1/4+\epsilon})]'),$$

and since ϵ can be taken arbitrarily small, we deduce as well

$$v_{2b} = AN\sigma(w)(t,\cdot)\nu \in L_2(0,T;[D(A^{1/2})]'). \tag{3.19}$$

On the basis of (3.18) and (3.19), we obtain

$$y_0 \in Y \Longrightarrow v_2 = v_{2a} + v_{2b} \in L_2(0,T;[D(A^{1/2+\delta_2})]') \equiv L_2(0,T;[H^{1+2\delta_2}(\Omega_f)]'), \tag{3.20}$$

if $0 < \delta_2 < \frac{1}{4}$.

Thus, (3.20), combined with (3.16), gives by interpolation

$$y_0 \in D(\mathcal{A}^{1-\theta}) \Longrightarrow v_2 \in L_{q_1}(0, T; W), \tag{3.21}$$

where q_1 is as in (3.17) and W is the interpolation space

$$W = (H^{1-2\delta_1}(\Omega_f), [H^{1+2\delta_2}(\Omega_f)]')_\theta \equiv H^s(\Omega_f),$$

if

$$s = (1-2\delta_1)(1-\theta) - \theta(1+2\delta_2) = 1 - 2\delta_1 - 2\theta(1+\delta_2-\delta_1) \geq 0;$$

see [28, Theorem 12.5]. Notice that by taking, for instance, $\delta_1 = \delta_2 =: \delta$, one has $s \geq 1/2$ provided that

$$\theta + \delta \leq \frac{1}{4}. \tag{3.22}$$

In this case $v_2 \in H^s(\Omega_f)$ with $s \geq 1/2$ and hence its trace on Γ_s is well defined. Notice that, in view of the constraint (3.22), we need to require $0 < \theta < \frac{1}{4}$. Consequently, given any θ such that $0 < \theta < \frac{1}{4}$, choosing, e.g., $\delta = 1/4 - \theta$ in view of (3.22), from (3.21) it follows

$$y_0 \in D(\mathcal{A}^{1-\theta})) \Longrightarrow N^* A v_2 \in L_{q_1}(0, T; L_2(\Gamma_s)) \quad \forall q_1 < \frac{4}{3+4\theta}. \tag{3.23}$$

It remains to establish the regularity of the first summand $N^* A v_1(t) = N^* A e^{At} A u_0$ when $y_0 \in D(\mathcal{A}^{1-\theta})$. In this case $u_0 \in (H^1(\Omega_f), L_2(\Omega_f))_\theta = H^{1-\theta}(\Omega_f)$, and from

$$N^* A v_1(t) := N^* A e^{At} A u_0 = [N^* A^{3/4-\epsilon}] A^{1/4+\epsilon+1/2+\theta/2} e^{At} A^{(1-\theta)/2} u_0,$$

it immediately follows

$$y_0 \in D(\mathcal{A}^{1-\theta}) \Longrightarrow N^* A v_1 \in L_{q_2}(0, T; L_2(\Gamma_s)) \quad \forall q_2 < \frac{4}{3+2\theta+4\epsilon}. \tag{3.24}$$

Notice that in the above membership the Sobolev exponent q_2 belongs to $(1, 2)$, as well. In conclusion, since ϵ in (3.24) can be taken arbitrarily small, the regularity $L_{q_1}(0, T; L_2(\Gamma_s))$ combined with $L_{q_2}(0, T; L_2(\Gamma_s))$ (in (3.23) and (3.24), respectively) imply the membership

$$y_0 \in D(\mathcal{A}^{1-\theta}) \Longrightarrow u_t|_{\Gamma_s} =: v|_{\Gamma_s} \in L_q(0, T; L_2(\Gamma_s)) \quad \forall q < \frac{4}{3+4\theta}, \tag{3.25}$$

which concludes the proof. □

4 Proof of Theorem 2.6

The conclusions stated in Theorem 2.6 will follow from [1, Theorem 2.3], once we verify the standing assumptions, which are recorded below for the reader's convenience.

Assumption 4.1 For each $t \in [0, T]$, the operator $B^* e^{A^* t}$ can be represented as

$$\mathcal{B}^* e^{\mathcal{A}^* t} y_0 = F(t) y_0 + G(t) y_0, \quad t \geq 0, \quad y_0 \in \mathcal{D}(\mathcal{A}^*), \tag{4.1}$$

where $F(t) : Y \to U$ and $G(t) : \mathcal{D}(\mathcal{A}^*) \to U$, $t > 0$, are bounded linear operators satisfying the following assumptions:

1. there is $\gamma \in (\frac{1}{2}, 1)$ such that $\|F(t)\|_{\mathcal{L}(Y,U)} \leq C\, t^{-\gamma}$ for all $t \in (0, T]$;

2. the operator $G(\cdot)$ belongs to $\mathcal{L}(Y, L^p(0, T; U))$ for all $p \in [1, \infty)$, with

$$\|G(\cdot)\|_{\mathcal{L}(Y, L^p(0,T;U))} \le c_p < \infty \quad \forall p \in [1, \infty); \tag{4.2}$$

3. there is $\epsilon > 0$ such that:
   (a) the operator $G(\cdot)\mathcal{A}^{*-\epsilon}$ belongs to $\mathcal{L}(Y, C([0, T], U))$, with

$$\|\mathcal{A}^{-\epsilon} G(t)^*\|_{\mathcal{L}(U,Y)} \le c < \infty \quad \forall t \in [0, T]; \tag{4.3}$$

   (b) the operator $\mathcal{R}^*\mathcal{R}$ belongs to $\mathcal{L}(\mathcal{D}(\mathcal{A}^\epsilon), \mathcal{D}(\mathcal{A}^{*\epsilon}))$;
   (c) there is $q = q(\epsilon) \in (1, 2)$ such that the operator $\mathcal{B}^* e^{\mathcal{A}^* \cdot} \mathcal{R}^*\mathcal{R}\mathcal{A}^\epsilon$ has an extension belonging to $\mathcal{L}(Y, L^q(0, T; U))$.

Remark 4.2 Aiming to apply Theorem 2.6, the main task is to find a value of ϵ which conforms to the potentially conflicting requirements in $3(a)$ and $3(c)$. In the present case, we shall show that while condition $3(a)$ is satisfied for arbitrarily small (positive) ϵ, when $\epsilon \to 0$ condition $3(c)$ yields $q \to 3/4 \in (1, 2)$.

Remark 4.3 Note that the set of requirements in Assumptions 4.1 involves the regularity in time of the operator $\mathcal{B}^* e^{\mathcal{A}^* t}$, both locally at the origin (with singularity controlled by γ), and globally (in L_p sense). It is this latter regularity that results from the hyperbolic propagation of the analytic singular estimate.

We now prove that the regularity results established in Theorem 2.10 imply all the requirements in Assumptions 4.1, with suitable values of γ, ϵ and q.

Verification of Assumptions 4.1. Following [26, Proof of Theorem 5.1], it is not difficult to verify that given any initial state $y_0 = (u_0, w_0, w_1) \in \mathcal{D}(\mathcal{A}^*)$, one has $\mathcal{B}^* e^{\mathcal{A}^* t} y_0 = N^* A\hat{u}(t) = -\hat{u}(t)|_{\Gamma_s}$, where $\hat{u}(t)$ is the first component of the solution $\hat{y} := (\hat{u}, \hat{w}, \hat{w}_t)$ to the (homogeneous) *adjoint* system

$$\begin{cases} \hat{y}'(t) = \mathcal{A}^* \hat{y}(t) \\ \hat{y}(0) = y_0. \end{cases}$$

The abstract adjoint system yields a system of coupled PDE which is essentially the same as the original boundary value problem (2.3), except for few changes of sign in the equations. As a consequence, the regularity results established by Lemma A.2 and Lemma A.3 readily produce analogues, by replacing $\mathcal{D}(\mathcal{A})$ and $y = (u, w, w_t)$ by $\mathcal{D}(\mathcal{A}^*)$ and $\hat{y} = (\hat{u}, \hat{w}, \hat{w}_t)$, respectively. Similarly, the fluid component $\hat{u}$ of the solution $\hat{y}$ to the dual PDE system satisfies—*mutatis mutandis*—the regularity properties in Theorem 2.10.

1. In light of the decomposition of $\hat{u}$ found in Theorem 2.10, let us set

$$F(t)y_0 := \hat{u}_1(t)\big|_{\Gamma_s}, \quad G(t)y_0 := \hat{u}_2(t)\big|_{\Gamma_s}.$$

   Then, the first statement in Theorem 2.10, along with the estimate (2.16), provides us with the sought-after singular estimate, with (optimal) exponent $\gamma = 1/4 + \delta$, and the first of Assumptions 4.1 is satisfied.
2. Assertion (iia) in Theorem 2.10 is nothing but the regularity condition 2. of the Assumptions 4.1, valid for all $p \in [1, \infty)$.
3. Condition (iib) of Theorem 2.10 translates into $G(t)\mathcal{A}^{*-\epsilon} y_0 \in C([0, T], L_2(\Gamma_s))$, which in turn gives the assertion (3a) of the Assumption 4.1, with no constraints on ϵ. It remains to verify the tricky assertion (3c) of Assumption 4.1. This will be implied by condition (iii) in Theorem 2.10. We first claim that the estimate (2.18) in (iii) of Theorem 2.10 yields, for any $\theta \in (0, 1/4)$, the following one:

$$\|\mathcal{B}^* e^{\mathcal{A}^* t} \mathcal{A}^{*\theta} y_0\|_{L^q(0,T;L_2(\Gamma_s))} \le C|y_0|_Y, \quad y_0 \in \mathcal{D}(A^{*\theta}), \quad 1 < q < \frac{4}{3+4\theta}. \tag{4.4}$$

This is easily seen if one just observes that if $y_0 \in \mathcal{D}(\mathcal{A}^{*\theta})$ one has

$$\mathcal{B}^* e^{\mathcal{A}^* t} \mathcal{A}^{*\theta} y_0 = \mathcal{B}^* e^{\mathcal{A}^* t} \mathcal{A}^* \, (\mathcal{A}^{*\theta-1} y_0) = \mathcal{B}^* \frac{\mathrm{d}}{\mathrm{d}t} e^{\mathcal{A}^* t} z_0 = \hat{u}_t|_{\Gamma_s},$$

where now $z_0 := \mathcal{A}^{*\theta-1} y_0 \in \mathcal{D}(\mathcal{A}^{*1-\theta})$. Then, in view of the assumption (3b) on the observation operator $\mathcal{R}$, one concludes

$$\begin{aligned} \|\mathcal{B}^* e^{\mathcal{A}^* t} \mathcal{R}\mathcal{R}^* \mathcal{A}^{\theta} y_0\|_{L^q(0,T;L_2(\Gamma_s))} &= \|\mathcal{B}^* e^{\mathcal{A}^* t} \mathcal{A}^{*\theta} \mathcal{A}^{*-\theta} \mathcal{R}\mathcal{R}^* \mathcal{A}^{\theta} y_0\|_{L^q(0,T;L_2(\Gamma_s))} \\ &\le C|y_0|_Y, \quad y_0 \in \mathcal{D}(\mathcal{A}^{\theta}), \end{aligned}$$

i.e. condition (3c) is satisfied with $\epsilon = \theta$, $0 < \theta < 1/4$, for any q such that $1 < q < 4/(3+4\theta)$. This completes the proof of Theorem 2.6. □

Acknowledgements This research was started while the second author was visiting the *Centro di Ricerca Matematica Ennio De Giorgi* of the Scuola Normale Superiore in Pisa, whose support is acknowledged. The first author acknowledges partial support of the Università degli Studi di Firenze, within the 2008 Project "Calcolo delle variazioni e teoria del controllo", as well as partial support of the Italian MIUR, within the PRIN Project 2007WECYEA. The second author acknowledges support of the National Science Foundation under Grant DMS-060666882.

Appendix

For completeness' sake and for the reader's convenience we record the statements of some results which are used frequently or critically in the proof of our main result. Lemma A.1 is Proposition 4.3 in [26]. Lemma A.2 and Lemma A.3 are specialized versions of Lemma 5.2 and Lemma 5.3 in [26].

Lemma A.1 ([26]) *The Green (Neumann) map $N : L_2(\Gamma_s) \to H$ satisfies the following regularity results.*

(i) *One has $N^* A u = -u|_{\Gamma_s}$, $u \in V$, where the adjoint is computed with respect to the L_2-topology.*
(ii) $N \in \mathcal{L}(L_2(\Gamma_s), \mathcal{D}(A^{3/4-\delta})) \cap \mathcal{L}(H^{-1/2}(\Gamma_s), \mathcal{D}(A^{1/2}))$ *for any δ, $0 < \delta < \frac{3}{4}$.*

Lemma A.2 *Let $y_0 = (u_0, w_0, w_1) \in \mathcal{D}(\mathcal{A}^{\alpha})$ with $0 \le \alpha \le \frac{1}{4}$, and let $(u(t), w(t), w_t(t))$ be the corresponding strong solution with $f \equiv u|\Gamma_s \in L_2(0, T; H^{1/2}(\Gamma_s))$. Then, the solution of the initial/boundary value problem*

$$\begin{cases} w_{tt} - \operatorname{div} \sigma(w) = 0 & \text{in } Q_s \\ \dfrac{d}{dt} w|_{\Gamma_s} = f & \text{on } \Sigma_s \\ w(0,\cdot) = w_0, \;\; w_t(0,\cdot) = w_1 & \text{in } \Omega_s \end{cases} \tag{A.1}$$

can be decomposed as $w = w_1 + w_2$, where $\sigma(w_1) \cdot \nu \in C([0,T], H^{-1/2}(\Gamma_s))$, while $\sigma(w_2) \cdot \nu \in L_2(0,T; L_2(\Gamma_s))$. If, in addition, $f \in H^{\alpha}(\Sigma_s)$, then $\sigma(w_2) \cdot \nu \in H^{\alpha}(\Sigma_s)$. Moreover, the following estimates hold true.

$$\begin{aligned} \|\sigma(w_1) \cdot \nu\|^2_{C([0,T], H^{-1/2}(\Gamma_s))} &\le C_1 \left(|w_0|^2_{1,\Omega_s} + |w_1|^2_{0,\Omega_s} + |f|_{L_2(0,T;H^{1/2}(\Gamma_s))} \right) \\ \|\sigma(w_2) \cdot \nu\|^2_{H^{\alpha}(\Sigma_s)} &\le C_2 \left(|y_0|_{\mathcal{D}(\mathcal{A}^{\alpha})} + |f|_{H^{\alpha}(\Sigma_s)} \right) \end{aligned}$$

Lemma A.3 ([26]) *Consider the uncontrolled counterpart of the PDE problem* (2.1)*, that is* (2.1) *with* $g \equiv 0$. *Let initial data satisfy* $y_0 = (u_0, w_0, w_1) \in \mathcal{D}(\mathcal{A}^\alpha)$, $0 \le \alpha \le \frac{1}{4}$. *Then, for any* $T < \infty$ *we have* $u|_{\Gamma_s} \in H^\alpha(\Sigma_s)$ *and the following estimate holds true:*

$$|u|_{H^\alpha(\Sigma_s)} \le C|y_0|_{\mathcal{D}(\mathcal{A}^\alpha)}. \tag{A.2}$$

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