Calculus of Variations

The harmonic mean curvature flow of nonconvex surfaces in $\mathbb{R}^3$

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Abstract We consider a compact star-shaped mean convex hypersurface $\Sigma^2 \subset \mathbb{R}^3$. We prove that in some cases the flow exists until it shrinks to a point. We also prove that in the case of a surface of revolution which is star-shaped and mean convex, a smooth solution always exists up to some finite time $T < \infty$ at which the flow shrinks to a point asymptotically spherically.

Mathematics Subject Classification (2000) 53C44

1 Introduction

We will consider in this work the deformation of a compact hyper-surface Σ_t in $\mathbb{R}^3$ with no boundary under the harmonic mean curvature flow (HMCF) namely the flow

$$\frac{\partial P}{\partial t} = -\frac{G}{H}\nu \tag{1.1}$$

which evolves each point P of the surface in the direction of its normal unit vector with speed equal to the *harmonic mean curvature* of the surface G/H, with G denoting the Gaussian curvature of Σ_t and H its mean curvature. Here ν denotes the outer unit normal to the surface

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at P. This flow remains weakly parabolic without the condition that Σ_t is strictly convex. However, it becomes degenerate at points where the Gaussian curvature G vanishes.

The existence of solutions to the HMCF with strictly convex smooth initial data was first shown by Andrews [3] who also showed that under the HMCF strictly convex smooth surfaces converge to round points in finite time. In [8], Diëter established the short time existence of solutions to the HMCF with weakly convex smooth initial data. More precisely, Diëter showed that if at time $t = 0$ the surface Σ_0 satisfies $G \geq 0$ and $H > 0$, then there exists a unique strictly convex smooth solution Σ_t of the HMCF defined on $0 < t < \tau$, for some $\tau > 0$. By the results of Andrews, the solution will exist up to the time where its enclosed volume becomes zero.

Caputo and the first author [5] considered the highly degenerate case where the initial surface is weakly convex with flat sides, where the parabolic equation describing the motion of the surface becomes highly degenerate at points where both curvatures G and H become zero. The solvability and optimal regularity of the surface Σ_t, for $t > 0$, was addressed and studied by viewing the flow as a *free-boundary* problem. It was shown that a surface Σ_0 of class $C^{k,\gamma}$ with $k \geq 1$ and $0 < \gamma \leq 1$ at $t = 0$, will remain in the same class for $t > 0$. In addition, the strictly convex parts of the surface become instantly C^∞ smooth up to the flat sides on $t > 0$, and the boundaries of the flat sides evolve by the curve shortening flow.

The case $G < 0$ was recently studied by the first author and Hamilton [7], under the assumption that the initial surface is a surface or revolution with boundary, and has $G < 0$ and $H < 0$ everywhere. It was shown in [7] that under certain boundary conditions, there exists a time $T_0 > 0$ for which the HMCF admits a unique solution Σ_t up to T_0, such that $H < 0$ for all $t < T_0$ and $H(\cdot, T_0) \equiv 0$ on some set of sufficiently large measure. In addition, the boundary of the surface evolves by the curve shortening flow.

In this work we address the questions of short time and long time existence and regularity of the HMCF under the assumption that Σ_0 is star-shaped with $H > 0$ but with G changing sign.

Let M^2 be a smooth, compact surface without boundary and $F_0 : M^2 \to \mathbb{R}^3$ be a smooth immersion of M^2. Let us consider a smooth family of immersions $F(\cdot, t) : M^2 \to \mathbb{R}^3$ satisfying

$$\frac{\partial F(p, t)}{\partial t} = -\kappa(p, t) \cdot \nu(p, t) \tag{HMCF}$$

where $\kappa = G/H$ denotes the harmonic mean curvature of $\Sigma_t := F(M^2, t)$ and ν its outer unit normal at every point. This is an equivalent formulation of the HMCF.

For any compact two-dimensional surface M^2 which is smoothly embedded in $\mathbb{R}^3$ by $F : M^2 \to \mathbb{R}^3$, let us denote by $g = (g_{ij})$ the induced metric, and by ∇ the induced Levi–Civita connection. The second fundamental form $A = \{h_{ij}\}$ is a symmetric bilinear form $A(p) : T_p\Sigma \times T_pM \to \mathbb{R}$, defined by $A(u, v) = \langle \nabla_u \nu, v \rangle$. The Weingarten map $W(p) : T_p\Sigma \to T_p\Sigma$ of T_pM given by the immersion F with respect to the normal ν, can be computed as $h^i_j = g^{ik}h_{kj}$. The eigenvalues of $W(p)$ are called the principal curvatures of F at p and are denoted by $\lambda_1 = \lambda_1(p)$ and $\lambda_2 = \lambda_2(p)$. The mean curvature $H := \text{trace}(W) = \lambda_1 + \lambda_2$, the total curvature $|A|^2 := \text{trace}(W^t W) = \lambda_1^2 + \lambda_2^2$ and the Gauss curvature $G = \det W = \lambda_1 \lambda_2$.

Remark 1.1 We will recall some standard facts about homogeneous of degree one functions of matrices that can be found in [1]. The speed κ of the interface evolving by the HMCF can be viewed as a function of the Weingarten map W and therefore, more generally, as a function $\kappa : S \to \mathbb{R}$, where S denotes the set of all symmetric, positive transformations of

$T\Sigma^2$ with strictly positive trace. Let λ_1, λ_2 be the eigenvalues of $A \in S$. We can then define the symmetric function $f(\lambda_1, \lambda_2) := \kappa(A)$. We have:

- If f is concave (convex) and $\lambda_i > \lambda_j$, then $\frac{\partial f}{\partial \lambda_i} - \frac{\partial f}{\partial \lambda_j}$ is negative (positive).
- Let $\ddot{\kappa} \in T\Sigma \otimes T^*\Sigma \otimes T\Sigma \otimes T^*\Sigma$ denote the second derivative of κ at the point $A \in S$. If A is diagonal, then

$$
\ddot{\kappa}(\xi, \eta) = \sum_{p,q} \frac{\partial^2 \kappa}{\partial \lambda_p \lambda_q} \xi_p^p \eta_q^q + \sum_{p \neq q} \frac{\frac{\partial \kappa}{\partial \lambda_p} - \frac{\partial \kappa}{\partial \lambda_q}}{\lambda_p - \lambda_q} \xi_p^q \eta_p^q. \tag{1.2}
$$

The outline of the paper is as follows:

i.   In Sect. 2, we will establish the short time existence of (HMCF), under the assumption that the initial surface Σ_0 is compact of class $C^{2,1}$ and has $H > 0$. To do so we will have to bound H from below away from zero. This does not follow naturally from the evolution of H. To obtain such a bound we need to combine the evolution of H with the evolution of the gradient of the second fundamental form. This explains our assumption that $\Sigma \in C^{2,1}$.

ii. In Sect. 3, we will study the long time existence of the regularized flow (HMCF$_\epsilon$) (defined in the next section). We will show that there exists a maximal time of existence T_ϵ of a smooth solution Σ_t^ϵ of (HMCF$_\epsilon$) such that either $H(P_t, t) \to 0$, as $t \to T_\epsilon$ at some points $P_t \in \Sigma_t^\epsilon$, or Σ_t^ϵ shrinks to a point as $t \to T_\epsilon$. In addition, we will establish uniform in ϵ curvature bounds and curvature pinching estimates. In the special case where the initial data is a surface of revolution, we will show that the flow always exists up to the time when the surface shrinks to a point.

iii. In Sect. 4, we will pass to the limit, $\epsilon \to 0$, to establish the long time existence of (HMCF).

2 Short time existence

Our goal in this section is to show the following short time existence result for the HMCF.

Theorem 2.1 *Let Σ_0 be a compact hyper-surface in $\mathbb{R}^3$ which is of class $C^{2,1}$ and has strictly positive mean curvature $H > 0$. Then, there exists $T > 0$ for which the harmonic mean curvature flow (HMCF) admits a unique $C^{2,1}$ solution Σ_t, such that $H > 0$ on $t \in [0, T]$.*

Because the HMCF becomes degenerate when the Gauss curvature of the surface Σ_t changes sign, we will show the short time existence for (HMCF) by considering its ϵ-regularization of the flow defined by

$$
\frac{\partial F_\epsilon}{\partial t} = -\left(\frac{G}{H} + \epsilon H \right) \cdot \nu \tag{HMCF$_\epsilon$}
$$

and starting at Σ_0. We will denote by Σ_t^ϵ the surfaces obtained by evolving the initial surface Σ_0 along the flow (HMCF$_\epsilon$).

Since the right-hand side of (HMCF$_\epsilon$) can be viewed as a function of the second fundamental form matrix A, a direct computation shows that its linearization is given by

$$
\mathcal{L}_\epsilon(u) = \frac{\partial}{\partial h_k^i} \left(\frac{G}{H} + \epsilon H \right) \nabla_i \nabla^k u = a_\epsilon^{ik} \nabla_i \nabla^k u
$$

with

$$a_\epsilon^{ik} = \frac{\partial}{\partial h_k^i} \left(\frac{G}{H} + \epsilon H \right). \tag{2.1}$$

Notice that if we compute a_ϵ^{ik} in geodesic coordinates around the point (at which the matrix A is diagonal) we get

$$a_\epsilon^{ik} = \begin{pmatrix} \frac{\lambda_2^2}{(\lambda_1+\lambda_2)^2} + \epsilon & 0 \\ 0 & \frac{\lambda_1^2}{(\lambda_1+\lambda_2)^2} + \epsilon \end{pmatrix} \tag{2.2}$$

which is strictly positive definite, no matter what the principal curvatures are.

The following short time existence for the regularized flow (HMCF$_\epsilon$) follows from the standard theory on the existence of solutions to strictly parabolic equations.

Proposition 2.2 *Let Σ_0 be a compact hyper-surface in $\mathbb{R}^3$ which is of class $C^{1,1}$ and has strictly positive mean curvature $H > 0$. Then, there exists $T_\epsilon > 0$, for which the harmonic mean curvature flow (HMCF$_\epsilon$) admits a smooth solution Σ_t^ϵ, such that $H > 0$ on $t \in [0, T_\epsilon)$.*

Our goal is to show that if the initial surface Σ_0 is of class $C^{2,1}$, then there is a $T_0 > 0$, so that $T_\epsilon \geq T_0$ for every ϵ and that we have uniform estimates on F_ϵ, independent of ϵ, so that we can take a limit of F_ϵ as $\epsilon \to 0$ and obtain a solution of (HMCF) that is of class $C^{2,1}$. The main obstacle here is to exclude that $H_\epsilon(P_\epsilon, t_\epsilon) \to 0$, as $\epsilon \to 0$, for some points $P_\epsilon \in \Sigma_{t_\epsilon}^\epsilon$ and times $t_\epsilon \to 0$. Notice that our flow cannot be defined at points where $H = 0$.

Notation

- When there is no possibility of confusion, we will use the letters c, C and T_0 for various constants which are independent of ϵ but change from line to line.
- Throughout this section we will denote by λ_1, λ_2 the two principal curvatures of the surface Σ_t^ϵ at a point P and will assume that $\lambda_1 \geq \lambda_2$.
- When there is no possibility of confusion we will drop the index ϵ from H, G, A, g_{ij}, h_{ij} etc.

The next lemma follows directly from the computations of B. Andrews in [1] (Chapter 3).

Lemma 2.3 *If Σ_t^ϵ moves by (HMCF$_\epsilon$), with speed $\kappa_\epsilon := \frac{G}{H} + \epsilon H$, the computation in [1] gives us the evolution equations*

i. $\frac{\partial}{\partial t} H = \mathcal{L}_\epsilon H + \frac{\partial^2 \kappa_\epsilon}{\partial h_q^p \partial h_m^l} \nabla^i h_q^p \nabla_i h_m^l + \frac{\partial \kappa_\epsilon}{\partial h_m^l} h_p^l h_m^p H$

ii. $\frac{\partial}{\partial t} \kappa_\epsilon = \mathcal{L}_\epsilon \kappa_\epsilon + \frac{\partial \kappa_\epsilon}{\partial h_j^i} h_{il} h_{lj} \kappa_\epsilon.$

Note that if $\kappa := \frac{G}{H}$, we have

$$\frac{\partial \kappa}{\partial h_p^q} h_m^q h_p^m = \sum_{i=1}^2 \frac{\partial \kappa}{\partial \lambda_i} \lambda_i^2 = 2\kappa^2 \tag{2.3}$$

hence

$$\frac{\partial \kappa_\epsilon}{\partial h_p^q} h_m^q h_p^m = \sum_{i=1}^2 \frac{\partial \kappa_\epsilon}{\partial \lambda_i} \lambda_i^2 = 2\kappa^2 + \epsilon |A|^2 \tag{2.4}$$

with $|A|^2 = \lambda_1^2 + \lambda_2^2$. We then conclude from the above lemma that H and κ_ϵ satisfy the evolution equations

$$\frac{\partial}{\partial t} H = \mathcal{L}_\epsilon H + \frac{\partial^2 \kappa_\epsilon}{\partial h_q^p \partial h_m^l} \nabla^i h_q^p \nabla_i h_m^l + (2\kappa^2 + \epsilon |A|^2) H \tag{2.5}$$

and

$$\frac{\partial \kappa_\epsilon}{\partial t} = \mathcal{L}_\epsilon \kappa_\epsilon + (2\kappa^2 + \epsilon |A|^2) \kappa_\epsilon. \tag{2.6}$$

We will now combine the above evolution equations to establish the following uniform bound on the second fundamental form.

Proposition 2.4 *There exist uniform constants C and T_0 so that*

$$\max_{\Sigma_t^\epsilon} |A| \leq C, \qquad \forall t \in [0, \min(T_\epsilon, T_0)). \tag{2.7}$$

Proof Recall that H satisfies (2.5). If we multiply this equation by H, we get

$$\frac{\partial H^2}{\partial t} = \mathcal{L}_\epsilon(H^2) - 2a_\epsilon^{ik} \nabla_i H \nabla_k H + \frac{\partial^2 \kappa_\epsilon}{\partial h_q^p \partial h_m^l} \nabla^i h_q^p \nabla_i h_m^l \cdot H + 2(2\kappa^2 + \epsilon |A|^2) H^2$$

with $\kappa = G/H$. Notice that the definiteness of the matrix $D^2\kappa_\epsilon = \left[\frac{\partial^2 \kappa_\epsilon}{\partial h_q^p \partial h_m^l}\right]$ depends on the sign of H. It is easier to check this in geodesic coordinates around a point at which the Weingarten map is diagonalized. In those coordinates, by (1.2), we have

$$\sum_{p,q,m,l} \frac{\partial^2 \kappa_\epsilon}{\partial h_q^p \partial h_m^l} \nabla^i h_q^p \nabla_i h_m^l = \sum_{p,q} \frac{\partial^2 \kappa_\epsilon}{\partial \lambda_p \lambda_q} \nabla^i h_p^p \nabla_i h_q^q + \sum_{p \neq q} \frac{\frac{\partial \kappa_\epsilon}{\partial \lambda_p} - \frac{\partial \kappa_\epsilon}{\partial \lambda_q}}{\lambda_p - \lambda_q} (\nabla_i h_p^q)^2$$

where the matrix $D^2\kappa_\epsilon := \left[\frac{\partial^2 \kappa_\epsilon}{\partial \lambda_p \lambda_q}\right]$ is given by

$$D^2\kappa_\epsilon = \begin{pmatrix} -\frac{2\lambda_2^2}{(\lambda_1+\lambda_2)^3} & \frac{2\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^3} \\ \frac{2\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^3} & -\frac{2\lambda_1^2}{(\lambda_1+\lambda_2)^3} \end{pmatrix} = -\frac{2}{H^3} \begin{pmatrix} \lambda_2^2 & -\lambda_1\lambda_2 \\ -\lambda_1\lambda_2 & \lambda_1^2 \end{pmatrix} \tag{2.8}$$

and for $p \neq q$,

$$\frac{\frac{\partial \kappa_\epsilon}{\partial \lambda_p} - \frac{\partial \kappa_\epsilon}{\partial \lambda_q}}{\lambda_p - \lambda_q} = \frac{\lambda_q^2 - \lambda_p^2}{\lambda_p - \lambda_q} = -\frac{1}{H}.$$

It is now easy to see that

$$\frac{\partial^2 \kappa_\epsilon}{\partial h_q^p \partial h_m^l} \nabla^i h_q^p \nabla_i h_m^l \cdot H \leq 0 \tag{2.9}$$

hence

$$\frac{\partial H^2}{\partial t} \leq \mathcal{L}_\epsilon(H^2) + 2(2\kappa^2 + \epsilon |A|^2) H^2. \tag{2.10}$$

Similarly, from the evolution of κ_ϵ, namely (2.6), we obtain

$$\frac{\partial \kappa_\epsilon^2}{\partial t} \leq \mathcal{L}_\epsilon(\kappa_\epsilon^2) + 2(2\kappa^2 + \epsilon |A|^2) \kappa_\epsilon^2. \tag{2.11}$$

We observe that because of the appearance of the second fundamental form $|A|^2$ in the zero-order term of (2.10) and (2.11), we cannot estimate the maximum of H^2 and κ_ϵ^2 directly from each equation using the maximum principle. This is because the surface is not convex. However, it is possible to estimate the maximum of $H^2 + \kappa_\epsilon^2$ by combining the two evolution equations. To this end, we set $M = H^2 + \kappa_\epsilon^2$ and compute, by adding the last two equations, that

$$\frac{\partial M}{\partial t} \leq \mathcal{L}_\epsilon M + 2 \left(2\kappa^2 + \epsilon |A|^2\right) M. \tag{2.12}$$

It is easy to verify that

$$\kappa_\epsilon^2 + H^2 \geq (1 - \epsilon)(\kappa^2 + H^2),$$

which immediately implies

$$2\kappa^2 + \epsilon |A|^2 = 2\kappa^2 + \epsilon(H^2 - 2\kappa H) \leq (2 + \epsilon)\left(\kappa^2 + H^2\right)$$
$$\leq \frac{2 + \epsilon}{1 - \epsilon}\left(\kappa_\epsilon^2 + H^2\right). \tag{2.13}$$

Combining (2.13) and (2.12) we conclude

$$\frac{\partial M}{\partial t} \leq \mathcal{L}_\epsilon M + \theta\, M^2$$

for some uniform constant θ. The maximum principle then implies the differential inequality

$$\frac{d M_{\max}}{dt} \leq \theta\, M_{\max}^2$$

which readily implies that

$$\max_{\Sigma_t^\epsilon} M \leq C, \qquad \forall t \in [0, \min(T_\epsilon, T_0))$$

for some uniform in ϵ constants C and T_0. This bound combined with (2.13) implies bounds on κ^2 and H^2 from which (2.7) easily follows.   □

To establish the short existence of the flow (HMCF) on $(0, T_0)$ for some $T_0 > 0$, we still need to bound H from below away from zero independently of ϵ. This does not follow naturally from the evolution of H, because the equation (2.5) carries a quadratic negative term which depends on the derivatives of the second fundamental form. Hence to establish the lower bound on H we need to combine the evolution of H with the evolution of the gradient of the second fundamental form. This is shown in the next proposition.

Proposition 2.5 *There exist uniform in ϵ positive constants T_0, C and δ, so that*

$$|\nabla A| \leq C \quad \text{and} \quad H \geq \delta, \quad \text{on } \Sigma_t^\epsilon$$

for $t \in [0, \min(T_\epsilon, T_0))$.

Proof We will first compute the evolution equation for $\sum_{i,j} |\nabla h_i^j|^2$. Lets first see how h_i^j evolves. We have

$$\frac{\partial}{\partial t} h_i^j = \mathcal{L}_\epsilon(h_i^j) + \frac{\partial^2 \kappa_\epsilon}{\partial h_q^p \partial h_m^l} \nabla^i h_q^p \nabla^j h_m^l + \frac{\partial \kappa_\epsilon}{\partial h_m^l} h_p^l h_m^p h_j^i.$$

From the previous equation, commuting derivatives we get

$$
\frac{\partial}{\partial t} \nabla_r h_i^j = \mathcal{L}_\epsilon(\nabla_r h_i^j) + \frac{\partial^2 \kappa_\epsilon}{\partial h_p^q \partial h_n^s} \nabla_r h_n^s \nabla_p \nabla_q h_i^j + \frac{\partial \kappa_\epsilon}{\partial h_p^q} R_{rpqm} \nabla_m h_i^j
$$

$$
+ \frac{\partial^3 \kappa_\epsilon}{\partial h_q^p \partial h_m^l \partial h_n^s} \nabla_r h_n^s \nabla^i h_q^p \nabla_j h_m^l + \frac{\partial^2 \kappa_\epsilon}{\partial h_q^p \partial h_m^l} \nabla_r \nabla^i h_p^q \nabla_j h_m^l
$$

$$
+ \frac{\partial^2 \kappa_\epsilon}{\partial h_q^p \partial h_m^l} \nabla^i h_q^p \nabla_r \nabla_j h_m^l + \frac{\partial^2 \kappa_\epsilon}{\partial h_m^l \partial h_n^s} \nabla_r h_n^s h_p^l h_m^p h_j^i
$$

$$
+ \frac{\partial \kappa_\epsilon}{\partial h_m^l} \nabla_r h_p^l h_m^p h_i^j + \frac{\partial \kappa_\epsilon}{\partial h_m^l} h_p^l \nabla_r h_m^p h_i^j + \frac{\partial \kappa_\epsilon}{\partial h_m^l} h_p^l h_m^p \nabla_r h_i^j. \qquad (2.14)
$$

Let $w = \sum_{i,j} |\nabla h_i^j|^2$. Since $|\nabla h_i^j|^2 = g^{pq} \nabla_p h_i^j \nabla_q h_i^j$ and $\frac{\partial g_{ij}}{\partial t} = 2\kappa_\epsilon h_{ij}$, we get

$$
\frac{\partial w}{\partial t} = -4 g^{pa} g^{qb} \kappa_\epsilon h_{ab} \nabla_p h_i^j \nabla_q h_i^j + \mathcal{L}_\epsilon(w) - 2\dot{\kappa}_\epsilon(\nabla^2 h_i^j, \nabla^2 h_i^j)
$$

$$
+ g^{pq} \frac{\partial \kappa_\epsilon}{\partial h_a^b} R_{pabs} \nabla_s h_i^j \nabla_q h_i^j + \frac{\partial^2 \kappa_\epsilon}{\partial h_p^q \partial h_m^l} g^{ra} \nabla_r h_m^l \nabla_p \nabla_q h_i^j \nabla_a h_i^j
$$

$$
+ \frac{\partial^3}{\partial h_p^q \partial h_m^l \partial h_n^s} g^{ra} \nabla_r h_n^s \nabla^i h_p^q \nabla_j h_m^l \nabla_a h_i^j + \frac{\partial^2 \kappa_\epsilon}{\partial h_p^q \partial h_m^l} g^{ra} \nabla_r \nabla^i h_p^q \nabla_j h_m^l \nabla_a h_i^j
$$

$$
+ \frac{\partial^2 \kappa_\epsilon}{\partial h_p^q \partial h_m^l} g^{ra} \nabla^i h_p^q \nabla_r \nabla_j h_m^l \nabla_a h_i^j + \frac{\partial^2 \kappa_\epsilon}{\partial h_m^l \partial h_n^s} g^{ra} \nabla_r h_n^s h_p^l h_m^p h_i^j
$$

$$
+ \frac{\partial \kappa_\epsilon}{\partial h_m^l} g^{ra} \nabla_r h_p^l h_m^p h_i^j \nabla_a h_i^j + \frac{\partial \kappa_\epsilon}{\partial h_m^l} g^{ra} h_p^l \nabla_r h_m^p h_i^j \nabla_a h_i^j
$$

$$
+ \frac{\partial \kappa_\epsilon}{\partial h_m^l} g^{ra} h_p^l h_m^p \nabla_r h_i^j \nabla_a h_i^j.
$$

Whenever we see i and j in the previous equation we assume that we are summing over all indices i and j. Also,

$$
\dot{\kappa}_\epsilon(\nabla^2 h_i^j, \nabla^2 h_i^j) = \frac{\partial \kappa_\epsilon}{\partial h_p^q} g^{pq} g^{cd} \nabla_q \nabla_c h_i^j \nabla_p \nabla_d h_i^j.
$$

Notice that since $|A| \leq C$ for all $t \in [0, \min(T_\epsilon, T_0))$, we have

$$
\left| \frac{\partial^2 \kappa_\epsilon}{\partial h_p^q \partial h_m^l} \right| \leq \frac{C_1}{H^3} \quad \text{and} \quad \left| \frac{\partial^3 \kappa_\epsilon}{\partial h_p^q \partial h_m^l \partial h_n^s} \right| \leq \frac{C_1}{H^4}
$$

for a uniform constant C.

We next compute the evolution equation for $1/H$ from the evolution of H, namely (2.5). By direct computation we get that

$$
\frac{\partial}{\partial t}\left(\frac{1}{H}\right) = \mathcal{L}_\epsilon\left(\frac{1}{H}\right) - \frac{2}{H^3} \frac{\partial \kappa_\epsilon}{\partial h_p^q} \nabla_p H \nabla_q H - \frac{1}{H^2} \frac{\partial^2 \kappa_\epsilon}{\partial h_p^q \partial h_m^l} \nabla^i h_p^q \nabla_i h_m^l - \frac{1}{H} \frac{\partial \kappa_\epsilon}{\partial h_m^l} h_p^l h_m^p.
$$

Taking away the second negative term on the right-hand side we easily conclude the differential inequality

$$
\frac{\partial}{\partial t}\left(\frac{1}{H}\right) \leq \mathcal{L}_\epsilon\left(\frac{1}{H}\right) + \frac{C\,w}{H^5} + \frac{C}{H}.
$$

Combining the evolution equations of w and $1/H$ we will now compute the evolution equation for

$$\mathcal{V} := w + \frac{1}{H}.$$

We look at the point (P, t) at which $\mathcal{V}$ achieves its maximum at time t and choose coordinates around P so that both, the second fundamental form and the metric matrix are diagonal at P. Using the exact form of coefficients $a_\epsilon^{ik} = \frac{\partial \kappa_\epsilon}{\partial h_p^q}$ computed in (2.1) we get

$$-2 \sum_{i,j} \dot{\kappa}_\epsilon (\nabla^2 h_i^j, \nabla^2 h_i^j) = -2 \sum_{i,j} \frac{\partial \kappa_\epsilon}{\partial h_p^q} g^{cd} \nabla_q \nabla_c h_i^j \nabla_p \nabla_d h_i^j$$

$$= -\frac{2}{H^2} \sum_{i,j} [\lambda_2^2 (\nabla_1 \nabla_1 h_i^j)^2 + \lambda_1^2 (\nabla_2 \nabla_2 h_i^j)^2$$

$$+ (\lambda_1^2 + \lambda_2^2) (\nabla_1 \nabla_2 h_i^j)^2]. \tag{2.15}$$

Our goal is to absorb all the remaining terms that contain the second order derivatives, appearing in the evolution equation for $\mathcal{V}$, in the good term (2.15). By looking at the evolution equation of w, we see that those second order terms are

$$\mathcal{O} = \frac{\partial^2 \kappa_\epsilon}{\partial h_p^q \partial h_m^l} g^{ra} \nabla_r h_m^l \nabla_p \nabla_q h_i^j \nabla_a h_i^j,$$

$$\mathcal{P} = \frac{\partial^2 \kappa_\epsilon}{\partial h_p^q \partial h_m^l} g^{ra} \nabla_r \nabla^i h_p^q \nabla_j h_m^l \nabla_a h_i^j$$

and

$$\mathcal{R} = \frac{\partial^2 \kappa_\epsilon}{\partial h_p^q \partial h_m^l} g^{ra} \nabla^i h_p^q \nabla_r \nabla_j h_m^l \nabla_a h_i^j$$

where we understand summing over all indices. Denote by $\xi_m^l := \nabla_r h_m^l$ and by $\eta_p^q := \nabla_p \nabla_q h_i^j$. If we specify the coordinates around the maximum point P in which W and g are diagonal, by (1.2)

$$\mathcal{O} = g^{ra} \nabla_r h_i^j \left(\sum_{p,q} \frac{\partial^2 \kappa}{\partial \lambda_p \lambda_q} \xi_p^p \eta_q^q + \sum_{p \neq q} \frac{\frac{\partial \kappa}{\partial \lambda_p} - \frac{\partial \kappa}{\partial \lambda_q}}{\lambda_p - \lambda_q} \xi_p^q \eta_p^q \right)$$

$$= \nabla_r h_i^j \left(-\frac{2\lambda_2^2}{H^3} \nabla_r h_1^1 \nabla_1 \nabla_1 h_i^j - \frac{2\lambda_1^2}{H^3} \nabla_r h_2^2 \nabla_2 \nabla_2 h_i^j + \frac{2\lambda_1 \lambda_2}{H^3} \nabla_r h_1^1 \nabla_2 \nabla_2 h_i^j \right.$$

$$\left. + \frac{2\lambda_1 \lambda_2}{H^3} \nabla_r h_2^2 \nabla_1 \nabla_1 h_i^j - \frac{1}{H} (\nabla_1 \nabla_2 h_i^j \nabla_r h_1^2 + \nabla_2 \nabla_1 h_i^j \nabla_r h_2^1) \right). \tag{2.16}$$

Since $|A| \leq C$ and

$$\frac{1}{H} = \frac{\lambda_1 + \lambda_2}{H^2} \leq \frac{|\lambda_1| + |\lambda_2|}{H^2}$$

by the Cauchy–Schwartz inequality, we can estimate $\mathcal{O}$ term by term, namely

$$\left|2\nabla_r h_i^j \frac{\lambda_2^2}{H^3}\nabla_r h_1^1 \nabla_1 \nabla_1 h_i^j\right| = \left|2\nabla_r h_i^j \frac{\lambda_2}{H^2}\nabla_r h_1^1\right|\left|\frac{\lambda_2}{H}\nabla_1\nabla_1 h_i^j\right|$$

$$\leq C\frac{w^2}{H^4} + \beta_1 \frac{\lambda_2^2}{H^2}|\nabla_1\nabla_1 h_i^j|^2$$

$$\leq C\frac{w^2}{H^4} + \beta_1 \sum_{ij}\dot{\kappa}_\epsilon(\nabla^2 h_i^j \nabla^2 h_i^j)$$

and

$$\left|2\frac{\lambda_1\lambda_2}{H^3}\nabla_r h_i^j \nabla_r h_1^1 \nabla_1 \nabla_1 h_i^j\right| = \left|2\frac{\lambda_1}{H^2}\nabla_r h_1^1 \nabla_r h_i^j\right|\left|\frac{\lambda_2}{H}\nabla_1\nabla_i h_i^j\right|$$

$$\leq C\frac{w^2}{H^4} + \beta_1 \frac{\lambda_2^2}{H^2}|\nabla_1\nabla_1 h_i^j|$$

$$\leq C\frac{w^2}{H^4} + \beta_1 \sum_{ij}\dot{\kappa}_\epsilon(\nabla^2 h_i^j \nabla^2 h_i^j)$$

and

$$\left|\frac{1}{H}\nabla_1\nabla_2 h_i^j \nabla_r h_1^2 \nabla_r h_i^j\right| \leq \left|\frac{1}{H^2}\nabla_r h_1^2 \nabla_r h_i^j\right|\left|\frac{\lambda_1+\lambda_2}{H}\nabla_1\nabla_2 h_i^j\right|$$

$$\leq C\frac{w^2}{H^4} + \beta_1(\lambda_1^2+\lambda_2^2)|\nabla_1\nabla_2 h_i^j|^2$$

$$\leq C\frac{w^2}{H^4} + \beta_1 \sum_{ij}\dot{\kappa}_\epsilon(\nabla^2 h_i^j, \nabla^2 h_i^j)$$

where $\beta_1 > 0$ is a uniform small number. We can estimate other terms in $\mathcal{O}$ the same way and combining all those estimates yield

$$|\mathcal{O}| \leq C\frac{w^2}{H^4} + \beta \sum_{i,j}\dot{\kappa}_\epsilon(\nabla^2 h_i^j, \nabla^2 h_i^j)$$

where $\beta > 0$ is a small fixed number.

In order to estimate $\mathcal{P}$, we would like to be able somehow to switch the pair of indices $\{i, r\}$ with $\{p, q\}$ so that we reduce estimating $\mathcal{P}$ to the previous case of $\mathcal{O}$. We will use Gauss–Codazzi equations in the form

$$\nabla_l h_{ij} = \nabla_i h_{lj}.$$

In our special coordinates at the point we have

$$\nabla_r\nabla^i h_p^q = \nabla_r\nabla^i(h_{ps}g^{qs}) = \nabla_r(g^{ij}\nabla_j(h_{ps}g^{qs}))$$
$$= \nabla_r(g^{ij}g^{qs})\cdot\nabla_j h_{ps} + g^{ij}g^{qs}\nabla_r\nabla_j h_{ps} + h_{ps}\nabla_r(g^{ij}\nabla_j g^{qs})$$
$$= \nabla_p\nabla_q h_{rj} + \nabla_r(g^{ij}g^{qs})\cdot\nabla_j h_{ps} + h_{pp}\nabla_r(g^{ij}\nabla_j g^{pq}). \qquad (2.17)$$

We have the following:

Claim There is a uniform constant $\tilde{C}$ depending on C, so that

$$|g(\cdot, t)|_{C^2} \leq \tilde{C} \tag{2.18}$$

as long as $|A| \leq C$.

To prove (2.18) we observe that in geodesic coordinates $\{x_i\}$ around a point p, which corresponds to the origin in geodesic coordinates, we have

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3} R_{ipqj} x^p x^q + O(|x|^3) \tag{2.19}$$

and that an easy computation shows that

$$\nabla_p \nabla_q g_{ij}(0) = -\frac{1}{3} R_{ipqj}.$$

By the Gauss equations, we have $R_{ipqj} = h_{iq} h_{pj} - h_{ij} h_{pq}$, which yields to $|\nabla_p \nabla_q g_{ij}| \leq \tilde{C}$ as long as $|A| \leq C$. This together with (2.19) proves the Claim.

Combining (2.17)–(2.18), we obtain as in the estimate of $\mathcal{O}$, the bound

$$|\mathcal{P}| \leq \frac{Cw^2}{H^4} + \beta \sum_{i,j} \dot{\kappa}_\epsilon (\nabla^2 h_i^j, \nabla^2 h_i^j).$$

Similarly, we get the estimate for $\mathcal{R}$. We conclude that

$$|\mathcal{O}| + |\mathcal{P}| + |\mathcal{R}| \leq \frac{Cw^2}{H^4} + 3\beta \sum_{i,j} \dot{\kappa}_\epsilon (\nabla^2 h_i^j, \nabla^2 h_i^j). \tag{2.20}$$

Choosing $\beta > 0$ so that $3\beta < 2$ in (2.20) and analyzing the right-hand side of the evolution of w term by term, we obtain the following estimate at the maximum point P of $\mathcal{V}$ at time t

$$\frac{d\mathcal{V}_{\max}}{dt} \leq Cw + \frac{Cw^2}{H^4} + \frac{C\sqrt{w}}{H^3} + \frac{Cw}{H^5} + \frac{C}{H}.$$

Young's inequality, implies the estimates

$$\frac{w^2}{H^4} \leq w^6 + \frac{1}{H^6} \leq \mathcal{V}^6$$

and

$$\frac{w}{H^5} \leq w^6 + \frac{1}{H^6} \leq \mathcal{V}^6$$

and

$$\frac{\sqrt{w}}{H^3} \leq w + \frac{1}{H^6} \leq \mathcal{V} + \mathcal{V}^6.$$

Hence, denoting by $f(t) = \mathcal{V}_{\max}(t)$ we obtain

$$\frac{df}{dt} \leq C(f + f^6) \tag{2.21}$$

which implies the existence of uniform constants $\bar{C}$ and T_0, depending only on C and $f(0)$, so that

$$\sup_{\Sigma_t^\epsilon} \left(\frac{1}{H} + \sum_{i,j} |\nabla h_i^j|^2 \right) \leq \bar{C}, \qquad \text{for all } t \in [0, \min(T_\epsilon, T_0)).$$

This finishes the proof of the lemma.   □

Having all the curvature estimates (that are proved above), we can justify the short time existence of the $C^{2,1}$-solution to the (HMCF).

Proof of Theorem 2.1 For every $\epsilon > 0$, let T_ϵ be the maximal time so that

$$|A|_{C^1(\Sigma_t^\epsilon)} \le C, \quad \text{and} \quad H \ge \delta > 0$$

where C, δ are constants taken from Proposition 2.5. Take now $\epsilon_i \to 0$. We have that $|A|_{C^1(\Sigma_t^{\epsilon_i})} \le C$, which implies $|F_{\epsilon_i}|_{C^{2,1}} \le C$, for all $t \in [0, T_0]$. By the Arzela-Ascoli theorem there is a subsequence so that $F_{\epsilon_i}(\cdot, t) \overset{C^{2,1}}{\to} F(\cdot, t)$, where $F(\cdot, t)$ is a $C^{2,1}$ solution to (HMCF). Since we have a comparison principle for $C^{2,1}$ solutions to (1.1) as discussed above, the uniqueness of a $C^{2,1}$ solution immediately follows. □

3 Long time existence for the ϵ-flow

In this section we will study the long time existence for the ϵ-regularized flow (HMCF$_\epsilon$) assuming that Σ_0 is an arbitrary smooth surface with mean curvature $H > 0$, Euler characteristic $\chi(\Sigma_0) > 0$ and it is star-shaped with respect to the origin. Throughout the section we fix $\epsilon > 0$ sufficiently small, we denote by Σ_t^ϵ the surface evolving by (HMCF$_\epsilon$) and, to simplify the notation, we drop the index ϵ from $F, \nu, H, G, \kappa, A, g_{ij}, h_{ij}$ etc. The ϵ-flow has one obvious advantage over (1.1), it is not degenerate and therefore it has smoothing properties. Indeed, it follows from the Krylov and Schauder estimates that a $C^{1,1}$ solution of (HMCF$_\epsilon$) is C^∞ smooth.

Assume that Σ_t^ϵ is a solution of (HMCF$_\epsilon$) on $[0, T_\epsilon)$ and let us consider the evolution equation for the area form $d\mu_t$, namely

$$\frac{\partial}{\partial t} d\mu_t = -2 \left(\frac{G}{H} + \epsilon H \right) \frac{H}{2} d\mu_t = -(G + \epsilon H^2) d\mu_t.$$

Integrating it over the surface Σ_t^ϵ we obtain the following ODE for the total area $\mu_t(\Sigma_t^\epsilon)$ of the surface Σ_t^ϵ

$$\frac{d}{dt} \mu_t(\Sigma_t^\epsilon) = -\int_{\Sigma_t^\epsilon} (G + \epsilon H^2) \, d\mu_t.$$

By the Gauss-Bonnet formula we have

$$\int_{\Sigma_t^\epsilon} G \, d\mu_t = 2\pi \, \chi(\Sigma_t).$$

Since Σ_0 is a surface with positive Euler characteristic, then by the uniformization theorem $\chi(\Sigma_t) = 2$ and therefore we conclude the equation

$$\frac{d}{dt} \mu_t(\Sigma_t^\epsilon) = -4\pi - \epsilon \int_{\Sigma_t^\epsilon} H^2 \, d\mu_t. \tag{3.1}$$

Denote by T_ϵ the maximum time of existence of (HMCF$_\epsilon$). Integrating (3.1) in time from 0 to T_ϵ, solving with respect of T_ϵ and using that $\mu_t(\Sigma_t^\epsilon) \geq 0$, gives

$$T_\epsilon \leq \frac{1}{4\pi}\mu_0(\Sigma_0) - \frac{\epsilon}{4\pi}\int\limits_0^{T_\epsilon}\int\limits_{\Sigma_t^\epsilon} H^2 \, d\mu_t.$$

This, in particular shows that

$$T_\epsilon \leq \frac{1}{4\pi}\mu_0(\Sigma_0) \tag{3.2}$$

where $\mu_0(\Sigma_0)$ is the area of the initial surface Σ_0.

Our goal is to prove the following result, concerning the long time existence of the flow (HMCF$_\epsilon$). We will also establish curvature bounds and curvature pinching estimates which are independent of ϵ.

Theorem 3.1 *Let Σ_0 be a compact star-shaped hyper-surface in $\mathbb{R}^3$ which is of class $C^{1,1}$ and has strictly positive mean curvature $H > 0$. Then, there exists a maximal time of existence T_ϵ of a smooth (HMCF$_\epsilon$) flow Σ_ϵ^t such that either:*

(i) *$H(P_t, t) \to 0$, as $t \to T_\epsilon$ at some points $P_t \in \Sigma_t^\epsilon$, or*
(ii) *Σ_ϵ^t shrinks to a point as $t \to T_\epsilon$ and T_ϵ is given explicitly by*

$$T_\epsilon = \frac{1}{4\pi}\mu_0(\Sigma_0) - \frac{\epsilon}{4\pi}\int\limits_0^{T_\epsilon}\int\limits_{\Sigma_t^\epsilon} H^2 \, \mu_t \tag{3.3}$$

where $\mu_0(\Sigma_0)$ is the total area of Σ_0. Moreover, $\int_{\Sigma_t^\epsilon} H^2 \, d\mu_t$ is uniformly bounded for all $t \in [0, T_\epsilon)$, independently of ϵ.

Assume that (i) does not happen in Theorem 3.1. Then, we have

$$\min_{\Sigma_\epsilon^t} H(\cdot, t) \geq \delta > 0, \quad \text{for all } t \in [0, T_\epsilon) \tag{3.4}$$

where T_ϵ is the maximal existence time of a smooth flow Σ_ϵ^t.

Proposition 3.2 *Assuming that (i) does not happen in Theorem 3.1, then the maximal time of existence T of the flow (HMCF$_\epsilon$) satisfies $T \leq \mu_0(\Sigma_0)/4\pi$ and*

$$\limsup_{t \to T} |A| = \infty.$$

Proof The bound $T \leq M_0/4\pi$ is proven above. Assume that $\max_{\Sigma_\epsilon^t} |A| \leq C$ for all $t \in [0, T)$. Then we want to show that the surfaces Σ_t^ϵ converge, as $t \to T$, to a smooth limiting surface Σ_T^ϵ. Similarly as in [8], using the curvature bounds we have, for all $0 < t_t < t_2 < T$, the bounds

$$|F(p, t_1) - F(p, t_2)| \leq C |t_2 - t_1| \quad \text{and} \quad \left|\frac{\partial}{\partial t} g_{ij}\right|^2 \leq C$$

for a uniform in t constant C, which imply that $F(\cdot, t)$ converges, as $t \to T$ to some continuous surface $\tilde{\Sigma}_\epsilon^T$. We get uniform C^2-bounds on F out of the bound on $|A|$. Since our equation is uniformly parabolic and the operator κ is concave, by Krylov and Schauder estimates we obtain all higher derivative bounds.

We have just shown that the surface Σ_T^ϵ is C^∞ smooth. Also from our assumption

$$H(\cdot, T) \geq \delta > 0, \quad \text{on } \Sigma_T^\epsilon.$$

By Proposition 2.2 there exists $\tau_\epsilon > 0$ for which a smooth flow can be continued on $[T, T + \tau_\epsilon)$, which contradicts our assumption that T is maximal. Hence, $\limsup_{t \to T_\epsilon} |A| = \infty$ and the result follows. □

3.1 Monotonicity formula

We will now show the monotonicity property of the quantity

$$Q_\epsilon = \langle F_\epsilon, \nu \rangle + 2t \, \kappa_\epsilon$$

along the flow (HMCF_ϵ). This will play an essential role in establishing the long time existence. Similar quantity was considered by Smoczyk [16].

Lemma 3.3 *Assuming that $q_\epsilon(0) := \min_{\Sigma_0} \langle F_\epsilon, \nu \rangle \geq 0$, the quantity*

$$q_\epsilon(t) := \min_{\Sigma_t^\epsilon}(\langle F_\epsilon, \nu \rangle + 2t \, \kappa_\epsilon)$$

is increasing in time for as long as the solution Σ_t^ϵ exists. Hence,

$$q_\epsilon(t) := \min_{\Sigma_t^\epsilon}(\langle F_\epsilon, \nu \rangle + 2t \, \kappa_\epsilon) \geq q_\epsilon(0) \geq 0.$$

Proof We will compute the evolution of Q_ϵ and apply the maximum principle. By the computation in [17] we have

$$\mathcal{L}_\epsilon \nu + a_\epsilon^{ik} h_{ij} h_{jk} \nu = \nabla \kappa_\epsilon.$$

Since $\mathcal{L}_\epsilon F_\epsilon = -\kappa_\epsilon \nu$, an easy computation yields

$$\mathcal{L}_\epsilon(\langle F_\epsilon, \nu \rangle) = \kappa_\epsilon - a^{ik} h_{ij} h_{jk} \langle F_\epsilon, \nu \rangle + \langle F_\epsilon, \nabla \kappa_\epsilon \rangle.$$

On the other hand, since $\frac{\partial \nu}{\partial t} = \nabla \kappa_\epsilon$, it follows

$$\frac{\partial}{\partial t} \langle F_\epsilon, \nu \rangle = -\kappa_\epsilon + \langle F_\epsilon, \nabla \kappa_\epsilon \rangle$$

which yields

$$\frac{\partial}{\partial t} \langle F_\epsilon, \nu \rangle - \mathcal{L}_\epsilon(\langle F_\epsilon, \nu \rangle) = -2\kappa_\epsilon + a_\epsilon^{ik} h_{ij} h_{jk} \langle F_\epsilon, \nu \rangle.$$

We also have

$$\frac{\partial \kappa_\epsilon}{\partial t} = \mathcal{L}_\epsilon(\kappa_\epsilon) + a_\epsilon^{ik} h_{ij} h_{jk} \kappa_\epsilon.$$

Hence $Q_\epsilon = \langle F_\epsilon, \nu \rangle + 2t \, \kappa_\epsilon$ satisfies

$$\frac{\partial Q_\epsilon}{\partial t} = \mathcal{L}_\epsilon Q_\epsilon + a_\epsilon^{ik} h_{ij} h_{jk} Q_\epsilon. \tag{3.5}$$

Notice that the right-hand side of (3.5) is a strictly elliptic operator and

$$a_\epsilon^{ik} h_{ij} h_{jk} \geq 0.$$

We conclude by the maximum principle that

$$q'_\epsilon(t) = \frac{d}{dt}(\langle F_\epsilon, \nu\rangle + 2t\,\kappa_\epsilon\,)_{\min} \geq 0$$

assuming that $q_\epsilon(0) \geq 0$. This implies that $q_\epsilon(t) \geq q_\epsilon(0)$ finishing the proof of the lemma. $\square$

Notice that if instead of Q_ϵ we take the quantity

$$Q_{\eta,\epsilon} = \langle F_\epsilon, \nu\rangle + 2(t+\eta)\,\kappa_\epsilon$$

for any constant $\eta \in \mathbb{R}$, the same computation as above yields to that $Q_{\eta,\epsilon}$ satisfies

$$\frac{\partial Q_{\eta,\epsilon}}{\partial t} = \mathcal{L}_\epsilon Q_{\eta,\epsilon} + a_\epsilon^{ik} h_{ij} h_{jk} Q_{\eta,\epsilon}.$$

Assume that at time $t = 0$, we have

$$q_{\eta,\epsilon}(0) = \min_{\Sigma_0}(\langle F, \nu\rangle + 2\,\eta\,\kappa_\epsilon) \geq 0$$

for some $\eta \in \mathbb{R}$ (notice that $F_\epsilon = F$ at $t = 0$). Then, the maximum principle to the above equation, gives:

Proposition 3.4 *For any $\eta \in \mathbb{R}$, such that $q_{\eta,\epsilon}(0) := \min_{\Sigma_0}(\langle F, \nu\rangle + 2\,\eta\,\kappa_\epsilon) \geq 0$ the quantity*

$$q_{\eta,\epsilon}(t) := \min_{\Sigma_t^\epsilon}(\langle F_\epsilon, \nu\rangle + 2(t+\eta)\,\kappa_\epsilon\,)$$

is increasing in time. Hence

$$q_{\epsilon,\eta}(t) := \min_{\Sigma_t^\epsilon}(\langle F_\epsilon, \nu\rangle + 2(t+\eta)\,\kappa_\epsilon\,) \geq q_{\epsilon,\eta}(0) \geq 0. \tag{3.6}$$

Since the initial surface Σ_0 is star-shaped, we may choose $\eta > 0$ so that we have $q_{\eta,\epsilon}(0) > 0$. This is possible by continuity, since $\langle F, \nu\rangle > 0$. By Proposition 3.4 we have

$$Q_{\eta,\epsilon}(\cdot, t) \geq q_{\eta,\epsilon}(t) > 0$$

which implies the lower bound

$$\kappa_\epsilon = \frac{G}{H} + \epsilon H \geq -\frac{\langle F_\epsilon, \nu\rangle}{2(t+\eta)}. \tag{3.7}$$

We will show next that (3.7) implies a uniform lower bound on κ_ϵ, independently of ϵ. To this end, we need to bound $\langle F_\epsilon, \nu\rangle$, independently of ϵ. This bound follows from the comparison principle for curvature flows with the property that the speed is an increasing function of the principal curvatures. More precisely, we have the following lemma.

Lemma 3.5 *There exists a constant C, independent of ϵ, so that*

$$\kappa_\epsilon := \frac{G}{H} + \epsilon H \geq -C, \quad \forall t \in [0, T_\epsilon). \tag{3.8}$$

Proof We claim that there is a uniform constant C so that $|F_\epsilon| \leq C$, for all $t \in [0, T_\epsilon)$. To see that, let $\psi_0 : S^2 \to \mathbb{R}^3$ denote the parametrization of a sphere that encloses the initial hypersurface Σ_0. By the result of Andrews [1], the solution $\psi_\epsilon(\cdot, t)$ of (HMCF$_\epsilon$) with initial condition ψ_0 shrinks to a point in some finite time $\tilde{T}_\epsilon$. Moreover,

$$\tilde{T}_\epsilon \leq \tilde{T} < \infty \tag{3.9}$$

for a uniform constant $\tilde{T}$.

The standard comparison principle shows that the images of F_ϵ and ψ_ϵ stay disjoint for all the time of their existence. To see this, we consider the evolution of

$$d(p, q, t) := |F_\epsilon(p, t) - \psi_\epsilon(q, t)|, \quad (p, q) \in \Sigma_t^\epsilon \times S^2.$$

Assume that the minimum of d at time t occurs at (p_0, q_0). If W denotes the Weingarten map, at that minimum point $W(p_0) \geq W(q_0)$, so by the monotonicity of our speed κ_ϵ, $\kappa_\epsilon(W(p_0)) \geq \kappa_\epsilon(W(q_0))$. The maximum principle tells us that $d_{min}(t)$ is non-decreasing and therefore the distance between the images of F_ϵ and ψ_ϵ is non-decreasing. Hence, they stay disjoint in time. As a consequence of that, our hypersurfaces $F_\epsilon(\cdot, t)$ stay enclosed by the sphere ψ_0 for all times of their existence (since $\psi_\epsilon(\cdot, t)$ are enclosed by ψ_0) and therefore $|F_\epsilon|(\cdot, t) \leq C$ for a uniform constant C.

The above bound implies that $\langle F_\epsilon, \nu \rangle \leq C$, for a uniform in ϵ constant C and all $t \in [0, T_\epsilon)$. This together with (3.7) yield to (3.8). □

3.2 Curvature pinching estimates for the ϵ-flow

Define

$$\mathcal{F}_\epsilon := \langle F_\epsilon, \nu \rangle + 2t\, \kappa_\epsilon.$$

Notice that division by $\mathcal{F}_\epsilon$ makes sense since by Lemma 3.3, $(\mathcal{F}_\epsilon)_{\min}$ is increasing in time and $(\mathcal{F}_\epsilon)_{\min}(0) \geq \delta > 0$ due to star-shapedness. As we showed above, $\sup_{\Sigma_t} |F_\epsilon| \leq C$ for a uniform constant C. Rewrite the evolution equation for H from Lemma 2.3 in the form

$$\frac{\partial}{\partial t} H = \mathcal{L}_\epsilon(H) + \ddot{\kappa}_\epsilon(\nabla W, \nabla W) + \dot{\kappa}_\epsilon(W^2)H$$

where W is the Weingarten map and

$$\ddot{\kappa}_\epsilon(\nabla W, \nabla W) = \frac{\partial^2 \kappa_\epsilon}{\partial h_q^p \partial h_m^l} \nabla^i h_q^p \nabla_j h_m^l \quad \text{and} \quad \dot{\kappa}_\epsilon(W^2) = \frac{\partial \kappa_\epsilon}{\partial h_m^l} h_p^l h_m^p.$$

Then, by direct computation we have

$$\frac{\partial}{\partial t}\left(\frac{H}{\mathcal{F}_\epsilon}\right) = \mathcal{L}_\epsilon\left(\frac{H}{\mathcal{F}_\epsilon}\right) + \frac{2}{\mathcal{F}_\epsilon} \dot{\kappa}_\epsilon\left(\nabla \mathcal{F}, \nabla \frac{H}{\mathcal{F}_\epsilon}\right) + \frac{1}{2\mathcal{F}_\epsilon} \text{trace}_g \ddot{\kappa}_\epsilon(\nabla W, \nabla W). \quad (3.10)$$

By (2.9), the last term in this equation is negative. Hence, by the maximum principle, the supremum of $H/\mathcal{F}_\epsilon$ is decreasing. In particular, we have:

Lemma 3.6 *Assume that Σ_t^ϵ is a solution of* HMCF$_\epsilon$ *on $[0, T_\epsilon)$ with Σ_0 as in Theorem 2.1. Then,*

$$\sup_{\Sigma_t^\epsilon \times [0, \tau)} \frac{H}{\langle F_\epsilon, \nu \rangle + 2t\, \kappa_\epsilon} \leq C, \quad \text{on } [0, T_\epsilon) \quad (3.11)$$

for a uniform constant C that depends only on Σ_0.

Denote by λ_1, λ_2 the two principal curvatures of the surface Σ_t^ϵ at some time t and point P.

Lemma 3.7 *If there is some time t_0 so that $\liminf_{t \to t_0} H(\cdot, t) = 0$, then*

$$\liminf_{t \to t_0} (\lambda_1^2 + \lambda_2^2) = 0. \quad (3.12)$$

Proof Assume $\liminf_{t \to t_0} H(\cdot, t) = 0$. We distinguish the following two cases:

  (i) $\lambda_1 > 0$ and $\lambda_2 \geq 0$. In this case (3.12) immediately follows.
  (ii) $\lambda_1 > 0$ and $\lambda_2 < 0$. By Lemma 3.3,

$$\kappa_\epsilon := \frac{G}{H} + \epsilon H \geq -C$$

uniformly in time, which implies

$$\lambda_1 |\lambda_2| \leq C H + \epsilon H^2.$$

Since $\liminf_{t \to t_0} H = 0$, at least for one of the two principal curvatures must tend to zero, i.e.

$$\liminf_{t \to t_0} |\lambda_i| = 0. \tag{3.13}$$

Since $\liminf_{t \to t_0} H = 0$, (3.12) readily follows. $\square$

Lemma 3.8 *There exist uniform (in time t and ϵ) constants $C > 0$ and $\epsilon_0 > 0$, such that for every $0 < \epsilon \leq \epsilon_0$, if $\lambda_2 \leq 0$ at P, then*

$$\lambda_1 \leq C.$$

Proof Since $\lambda_2 \leq 0$, we have $G/H \leq 0$. Hence, from (3.11) and the bound $|\langle F_\epsilon, \nu \rangle| \leq C_0$, for a uniform in time constant C_0, we conclude that

$$H \leq C + C \epsilon H$$

for a constant C that depends only on the initial data. We conclude that for $\epsilon \leq \epsilon_0$, with ϵ_0 sufficiently small depending only on the initial data Σ_0, we have

$$H := \lambda_1 + \lambda_2 \leq C$$

from which the desired bound on λ_1 follows with the aid of the previous lemma. $\square$

Lemma 3.9 *There exist uniform (in t and ϵ) constants $C > 0$ and $\epsilon_0 > 0$, such that for every $0 < \epsilon \leq \epsilon_0$ we have*

$$\lambda_2 \geq -C.$$

Proof Assume that $\lambda_2 < 0$ (otherwise the bound is obvious). Then, $\lambda_1 > 0$ (since $H = \lambda_1 + \lambda_2 > 0$) and by Lemma 3.5, we have

$$\kappa_\epsilon := \frac{G}{H} + \epsilon H \geq -C$$

for a uniform in time constant C. Also, by the previous lemma $H \leq \lambda_1 \leq C$. Hence,

$$|\lambda_2| \leq C (1 + \epsilon) \frac{\lambda_1 + \lambda_2}{\lambda_1} \leq \tilde{C}.$$

                                                                                              $\square$

Remark 3.10 Lemma 3.8 implies that if the flow terminates because of the blowing up of the second fundamental form, that could only happen in the convex region of Σ_t^ϵ where $\lambda_1 \geq 0$, $\lambda_2 \geq 0$.

Lemma 3.11 *There exist uniform (in time t and ϵ) constants $C > 0$, $C_0 > 0$ and ϵ_0, such that for every $\epsilon \geq \epsilon_0$ if $\lambda_1 \geq C_0$ at P, then $\lambda_2 > 0$ at P and*

$$1 \leq \frac{\lambda_1}{\lambda_2} \leq C.$$

Proof From the previous lemma, $\lambda_2 > 0$ if C_0 is chosen sufficiently large. Hence, from the bound (3.11) we conclude

$$\begin{aligned}(\lambda_1 + \lambda_2)^2 &\leq C_1 (\lambda_1 + \lambda_2) + 2T_\epsilon [\lambda_1 \lambda_2 + \epsilon (\lambda_1 + \lambda_2)] \\ &\leq \tilde{C}_1 (\lambda_1 + \lambda_2) + \tilde{C}_2 \lambda_1 \lambda_2 \end{aligned} \tag{3.14}$$

for some uniform in ϵ and t constants $\tilde{C}_1$ and $\tilde{C}_2$. By taking C_0 sufficiently large, we can make

$$(\lambda_1 + \lambda_2)^2 - C_1 (\lambda_1 + \lambda_2) \geq \frac{1}{2} (\lambda_1 + \lambda_2)^2.$$

Hence, (3.14) implies the bound

$$\lambda_1^2 + \lambda_2^2 \leq 2 \tilde{C}_2 \lambda_1 \lambda_2$$

from which the desired estimate readily follows. □

To facilitate future references we combine the previous three lemmas in the following proposition:

Proposition 3.12 *There exist $\epsilon_0 > 0$ and positive constants C_1, C_2, uniform in $0 < \epsilon < \epsilon_0$ and t, so that for every $0 < \epsilon < \epsilon_0$, we have*

  i. $\lambda_2 \geq -C_1$, *and*
  ii. $\lambda_1 \leq C_1 \lambda_2 + C_2$.

In the proof of Theorem 3.1 we will also need the following bound.

Lemma 3.13 *There is a uniform constant C, independent of ϵ and t so that*

$$\int_{\Sigma_t^\epsilon} H^2 \, d\mu_t \leq C.$$

Proof We begin by noticing that that $\int_{\Sigma_t^\epsilon} G \, d\mu_t$ is a topological invariant, equal to $2\pi \chi$, where χ is the Euler charactersistic of Σ_0. Since $\chi = 2$ we then have

$$\int_{\Sigma_t^\epsilon} G \, d\mu_t = 4\pi. \tag{3.15}$$

At any point we can choose the coordinates in which the second fundamental form is diagonal, with eigenvalues λ_1 and λ_2 as before and $\lambda_1 \geq \lambda_2$. By Lemma 3.12 we have $\lambda_1 \leq C_1 \lambda_2 + C_2$ which gives the inequality

$$G := \lambda_1 \lambda_2 \geq \frac{1}{C_1} \lambda_1^2 - \frac{C_2}{C_1} \lambda_1.$$

Using Cauchy–Scwartz we conclude the bound

$$\lambda_1 \lambda_2 \geq \tilde{C}_1 \lambda_1^2 - \tilde{C}_2$$

where $\tilde{C}_1$, $\tilde{C}_2$ are some uniform constants independent of ϵ and time. This yields to the estimate

$$|A|^2 = \lambda_1^2 + \lambda_2^2 \leq C_1\, G + C_2$$

which after integrated over Σ_t^ϵ implies the bound

$$\int\limits_{\Sigma_t^\epsilon} |A|^2\, d\mu_t \leq C_1 \int\limits_{\Sigma_t^\epsilon} G\, d\mu_t + C_2\, \mu_t(\Sigma_t^\epsilon)$$

with $\mu_t(\Sigma_t^\epsilon)$ denoting, as above, the surface area of Σ_t^ϵ. By (3.1), $\mu_t(\Sigma_t^\epsilon) \leq \mu_0(\Sigma_0)$, where $\mu_0(\Sigma_0)$ denotes the surface area of Σ_0. Hence, the lemma readily follows from (3.15). □

3.3 The proof of Theorem 3.1

Having all the ingredients from the previous sections we will finish the proof of Theorem 3.1.

Proof of Theorem 3.1 Fix an ϵ and let $T = T_\epsilon$ be a maximal time up to which the flow exists. To simplify the notation we will omit the ϵ-scripts from everything, including T_ϵ and the surface Σ_t^ϵ, denoting them by T and Σ_t respectively. Because of Proposition 3.2, the second fundamental form blows up at time T. Hence, there are sequences $t_i \to T$ and $p_i \in \Sigma_{t_i}$ so that

$$Q_i := |A|(p_i, t_i) = \max_{t \in [0, t_i]} \max_{\Sigma_{t_i}} |A|(\cdot, t_i) \to \infty, \qquad \text{as } i \to \infty.$$

Consider the sequence $\tilde{\Sigma}_t^i$ of rescaled solutions defined by

$$\tilde{F}_i(\cdot, t) := Q_i \left(F\left(\cdot, t_i + \frac{t}{Q_i^2}\right) - p_i \right). \tag{3.16}$$

Notice that under the above rescaling all points p_i are shifted to the origin. If g, H and $A := \{h_{jk}\}$ are the induced metric, the mean curvature and the second fundamental form of Σ_t, respectively, then the corresponding rescaled quantities are given by

$$\tilde{g}_i = Q_i^2 g, \qquad \tilde{H}_i = \frac{H}{Q_i}, \qquad |\tilde{A}_i|^2 = \frac{|A|^2}{Q_i^2}.$$

Consider a sequence of rescaled solutions $\tilde{\Sigma}_t^i$. They have a property that

$$\max_{\tilde{\Sigma}_t^i} |\tilde{A}_i| \leq 1, \quad \text{for } t \in [-1, 0] \text{ and } |\tilde{A}_i|(0, 0) = 1.$$

The above uniform estimates on the second fundamental form yield uniform higher order estimates on $\tilde{F}_i(\cdot, t)$ and the Theorem of Arzela-Ascoli gives us a uniformly convergent subsequence $\tilde{F}_{i_k}(\cdot, t)$ on compact subsets, converging to a smooth $\tilde{F}(\cdot, t)$ for every $t \in [0, 1]$. Notice that

$$\tilde{\kappa}_i = \frac{\tilde{G}_i}{\tilde{H}_i} + \epsilon\, \tilde{H}_i = \frac{\lambda_1 \lambda_2}{Q_i(\lambda_1 + \lambda_2)} + \epsilon\, \frac{\lambda_1 + \lambda_2}{Q_i}$$

and therefore by Proposition 3.12,

$$|\tilde{\kappa}_i| \leq \begin{cases} \frac{C}{Q_i} & \text{if } \lambda_1, \lambda_2 << Q_i \\ C, & \text{if } \lambda_1, \lambda_2 \sim Q_i \end{cases} \tag{3.17}$$

since $\lambda_2 \geq -C$ and λ_1 is big, comparable to the rescaling constant Q_i, if and only if λ_2 is big and comparable to Q_i (both λ_1 and λ_2 are computed at time $t_i + t/Q_i^2$). This implies that $\tilde{F}(\cdot, t)$ solves $\frac{\partial}{\partial t}\tilde{F}(\cdot, t) = -\tilde{\kappa}_\epsilon \nu$, where

$$\tilde{\kappa}_\epsilon = \begin{cases} 0, & \text{if } \tilde{\lambda}_1 = 0, \ \tilde{\lambda}_2 = 0 \\ \frac{\tilde{\lambda}_1 \tilde{\lambda}_2}{\tilde{\lambda}_1 + \tilde{\lambda}_2} + \epsilon\,(\tilde{\lambda}_1 + \tilde{\lambda}_2), & \text{if } \tilde{\lambda}_1 > 0, \ \tilde{\lambda}_2 > 0. \end{cases} \tag{3.18}$$

By Proposition 3.12 there are uniform constants C_1, C_2 so that

$$\lambda_1 \leq C_1 \lambda_2 + C_2$$

which holds uniformly on Σ_t, for all $t \geq 0$ for which the flow exists, which after rescaling yields

$$\tilde{\lambda}_1^i \leq C_1 \tilde{\lambda}_2^i + \frac{C_2}{Q_i}. \tag{3.19}$$

The previous estimate implies that the limiting surface (which we denote by $\tilde{\Sigma}_0$) is convex (possibly not strictly convex). There are two possibilities for $\tilde{\Sigma}_0$: either it is a flat plane or it is a non-flat complete weakly convex smooth hypersurface in $\mathbb{R}^3$. Let $\tilde{F}_0$ be a smooth embedding of $\tilde{\Sigma}_0$ into $\mathbb{R}^3$. Due to our rescaling, the norm of the second fundamental form of rescaled surfaces is 1 at the origin and therefore $\tilde{\Sigma}_0$ is not a plane, but is strictly convex at least somewhere. It has the property that

$$\sup_{\tilde{\Sigma}_0} |\tilde{A}| \leq C.$$

By the results in [9] there is a smooth complete solution $\bar{\Sigma}_t$ to the mean curvature flow

$$\begin{cases} \frac{\partial}{\partial t}\bar{F}(p, t) = -\bar{H}\nu(p, t), & p \in \bar{\Sigma}_t, \ t > 0 \\ \bar{F}(p, 0) = \tilde{F}_0. \end{cases} \tag{3.20}$$

The results in [9] (see Theorems 2.1, 2.3, 3.1 and 3.4, which provide with curvature estimates and are of local nature) imply that the curvature of $\bar{\Sigma}_t$ stays uniformly bounded for some short time $t \in [0, T_0)$. The evolution for $\bar{H}$ along the mean curvature flow is given by

$$\frac{\partial}{\partial t}\bar{H} = \Delta\bar{H} + \bar{H}|\bar{A}|^2.$$

As in [9], due to the curvature bounds, the mean curvature $\bar{H}$ satisfies the conditions of Theorem 4.3 in [9] (the maximum principle for parabolic equations on complete hypersurfaces) and therefore nonnegative mean curvature is preserved along the flow. This together with the strong maximum principle implies that if $\bar{H}$ is not identically zero at $t = 0$, then it becomes strictly positive at $t > 0$. We also know that $\bar{\Sigma}_0$ satisfies $\bar{\lambda}_1 \leq C\bar{\lambda}_2$ for a uniform constant C, which follows from (3.19) after taking the limit as $i \to \infty$. Since we are assuming $\bar{\lambda}_1 \geq \bar{\lambda}_2$ this can be written as

$$\bar{h}_{ij} \geq \eta\,\bar{H}\,\bar{g}_{ij}, \tag{3.21}$$

for some uniform constant $\eta > 0$ and we will say the second fundamental form of $\bar{\Sigma}$ is η-pinched.

By the curvature bounds, the maximum principle for complete hypersurfaces and the evolution for $\bar{h}_{ij} - \eta\bar{H}\bar{g}_{ij}$ it follows that the pinching estimate (3.21) is preserved by the mean curvature flow (as in [14]). In particular, this implies that $\bar{h}_{ij}$ is strictly positive definite, which

means $\bar{\Sigma}_t$ is strictly convex for $t > 0$. The result of R. Hamilton in [13] states that a smooth strictly convex and complete hypersurface with its second fundamental form η-pinched must be compact. Hence, it follows that $\bar{\Sigma}_t$ has to be compact for $t > 0$. In this case, the initial data $\tilde{\Sigma}_0$ has to be compact as well.

We recall that $\tilde{\Sigma}_0$ is the limit of the hyper-surfaces $\tilde{\Sigma}_0^i$ which are obtained via re-scaling from the surfaces Σ_{t_i}. Hence, since $\tilde{\Sigma}_0$ is compact, there are constants i_0, C so that for $i \geq i_0$, we have

$$\text{diam}(\Sigma_{t_i}) < \frac{C}{Q_i} \to 0 \quad \text{as } i \to \infty, \tag{3.22}$$

and therefore $\Sigma_{t_i} \to \{\bar{p}\}$.

Claim 3.14 For any point $q \in \mathbb{R}^3$, we have

$$\frac{\partial}{\partial t} |F - q|^2 = \mathcal{L}_\epsilon(|F - q|^2) - 2\frac{|A|^2}{H^2}.$$

Proof Follows by a simple computation. □

By Claim 3.14, $|F - \tilde{p}|_{\max}(t)$ is decreasing along (HMCF$_\epsilon$) and therefore

$$\Sigma_t \to \{\tilde{p}\}, \quad \text{as } t \to T$$

which implies that the surface Σ_t shrinks to a point as as $t \to T$. Hence, $\mu_t(\Sigma_t) \to 0$ as $t \to T$. It follows by (3.1) that T must be given by (3.3). □

4 Passing to the limit $\epsilon \to 0$

We will assume in this section that Σ_t^ϵ are solutions of the flow (HMCF$_\epsilon$) which satisfy the condition (3.4) uniformly in ϵ, with T_ϵ given by (3.3). We shall show that we can pass to the limit $\epsilon \to 0$ to obtain a solution of the (HMCF) which is defined up to time

$$T := \lim_{\epsilon \to 0} T_\epsilon = \frac{\mu_0(\Sigma_0)}{4\pi}.$$

The key result is the following uniform bound on the second fundamental form A of Σ_ϵ.

Proposition 4.1 *Under assumption (3.4), for any $\tau < T$, there is a uniform constant $C = C(\tau)$ so that*

$$\max_{\Sigma_t^\epsilon} |A|(\cdot, t) \leq C, \quad \forall \epsilon > 0 \quad \text{and} \quad \forall t \in [0, \tau]. \tag{4.1}$$

where A denotes the second fundamental form of the surface Σ_t^ϵ.

Proof Assume there is $\tau < T$ for which (4.1) does not hold. Then, there exist sequences $t_i \to \tau$, $\epsilon_i \to 0$ and $p_i \in \Sigma_{t_i}^{\epsilon_i}$ so that

$$Q_i := |A|(p_i, t_i) = \max_{\Sigma_t^{\epsilon_i} \times [0, t_i]} |A| \to \infty \quad \text{as } j \to \infty.$$

Consider, as before, the rescaled sequence of solutions $\tilde{\Sigma}_t^i$ defined by the immersions $\tilde{F}_i(\cdot, t) : M^2 \to \mathbb{R}^3$,

$$\tilde{F}_i(\cdot, t) := Q_i \left(F_{\epsilon_i} \left(\cdot, t_i + \frac{t}{Q_i^2} \right) - p_i \right).$$

Due to our rescaling, the second fundamental form of rescaled surfaces is uniformly bounded in i. This uniform estimates on the second fundamental form yield uniform C^2-bounds on $\tilde{F}_i(\cdot, 0)$ and the Theorem of Arzela–Ascoli gives us a uniformly convergent subsequence on compact subsets, converging in the in $C^{1,1}$-topology to a $C^{1,1}$ surface $\tilde{\Sigma}$ defined by the immersion $\tilde{F}$.

By Lemma 3.12, there are uniform constants C_1, C_2 so that the estimate

$$\lambda_1 \leq C_1\lambda_2 + C_2$$

holds uniformly on Σ_t^ϵ, for all $t \geq 0$ for which the flow exists, and all ϵ, which after rescaling yields to the estimate

$$\tilde{\lambda}_1^i \leq C_1\tilde{\lambda}_2^i + \frac{C_2}{Q_i}.$$

Hence, the limiting surface $\tilde{\Sigma}$ is convex. There are two possibilities for $\tilde{\Sigma}$, either it is a flat plane, or it is a complete convex $C^{1,1}$-hypersurface.

Due to our rescaling, the curvatures of the rescaled surfaces $\tilde{\Sigma}_t^i$ are uniformly bounded in i. This in particular implies a uniform local Lipschitz condition on $\tilde{F}_i(M^2, 0)$. This means that there are fixed numbers r_0 and C_0 so that for every $q \in \tilde{F}_i(M^2)$, $\tilde{F}_i(U_{r_0,q})$ (where $U_{r_0,q}$ is a component of $\tilde{F}_i^{-1}(B_{r_0}(\tilde{F}_i(q)))$ containing q, and B_{r_0} is a ball of radius r_0 in $\mathbb{R}^3$) can be written as the graph of a Lipschitz function over a hyperplane in $\mathbb{R}^3$ through $\tilde{F}_i(q)$ with Lipschitz constant less than C_0. Notice that both C_0 and r_0 are independent of i, they both depend on a uniform upper bound on the second fundamental form. This means the limiting surface $\tilde{\Sigma}$ will satisfy a uniform local Lipschitz condition.

Lemma 4.2 *The limiting hypersurface $\tilde{\Sigma}$ is not a plane.*

Proof Assume that the limiting hypersurface $\tilde{\Sigma}$ is a plane. Then, for each i we can write $\tilde{\Sigma}_i$ in a neighbourhood which is a ball $B(0, 1)$ of radius 1 around the origin as a graph of a C^2-function $\tilde{u}_i$, over some hyperplane $\mathcal{H}_i$. In particular, we can choose one that is tangent to $\tilde{\Sigma}_i$ at the origin. Then

$$\tilde{h}_{jk}^i = \frac{D_{jk}\tilde{u}_i}{(1 + |D\tilde{u}_i|^2)^{\frac{1}{2}}}. \tag{4.2}$$

We can choose a coordinate system in each hyperplane so that the second fundamental form and also $D^2 u_i$ are diagonal at the origin. The function u_i is a height function that measures the distance of our surface from the hyperplane $\mathcal{H}_i$. We also have that $u_i \overset{C^{1,1}}{\to} \tilde{u}$ as $i \to \infty$ and $u_i(0) = 0$ for all i. If $\tilde{\Sigma}$ were a plane then $\tilde{u} \equiv 0$ and $|D\tilde{u}| \equiv 0$ which would imply $|\tilde{u}|_{C^{1,1}} \equiv 0$. Take $\epsilon > 0$ very small. Then there would exist i_0 so that for $i \geq i_0$, $|u_i|_{C^{1,1}} < \epsilon$ on $B(0, 1) \subset \tilde{\Sigma}_i$. Since we have (4.2), the last estimate would contradict the fact $|\tilde{A}_i|(0, 0) = 1$, that is valid by the way we rescaled our solution. □

It follows from the previous lemma and the discussion above that $\tilde{\Sigma}$ is a complete convex, non-flat $C^{1,1}$-surface that satisfies $\tilde{\lambda}_1 \leq C\tilde{\lambda}_2$, whenever those quantities are defined (since a surface is $C^{1,1}$, the principal curvatures are defined almost everywhere). Because of our uniform curvature estimates of the rescaled sequence, $\tilde{\Sigma}$ is a uniformly locally Lipschitz surface. By the results in [9] there is a solution $\bar{F}_t$ of the Mean Curvature flow (3.20) with initial data $\tilde{\Sigma}$ on some time interval $[0, T_1)$ and $\bar{F}_t$ is smooth for $t > 0$. We can now carry out

the same argument as in the proof of Theorem 3.1 to show that $\tilde{\Sigma}$ has to be compact. That would mean that for $j >> 1$,

$$\text{diam}\,(\Sigma_{t_j}^{\epsilon_j}) \leq \frac{C}{Q_j} \quad \text{and} \quad \text{area}\,(\Sigma_{t_j}^{\epsilon_j}) \leq \frac{C}{Q_j} \tag{4.3}$$

for a uniform constant C. Since $T_j \to \tau < T$, (4.3) and Lemma 3.13 contradict (3.1). This shows that (4.1) holds true, therefore finishing our proof. $\qquad\square$

We will now show that because of (4.1) we can pass along subsequences $\epsilon_i \to 0$ and show that the solutions $\Sigma_t^{\epsilon_i}$ converge to a solution Σ_t of (HMCF).

Observe first that since $\partial F_\epsilon / \partial t = -(\kappa + \epsilon H)\nu$, by Proposition 4.1 we have that $|\partial F_\epsilon / \partial t| \leq C$, uniformly in ϵ. Hence, F_ϵ is uniformly Lipshitz in t. Combining this with Proposition 4.1 and the assumption (3.4), we conclude that for every $\tau < T$ there is a subsequence $\epsilon_i \to 0$ and a 1-parameter family of $C^{1,1}$ surfaces $F(\cdot, t)$, so that $F_{\epsilon_i} \to F$ in the $C^{1,1}$ norm, $\partial F_{\epsilon_i}/\partial t \to \partial F/\partial t$ in the weak sense and F satisfies

$$\frac{\partial F}{\partial t} = -\kappa\,\nu. \tag{4.4}$$

Due to (3.4) our solution has the property that

$$\text{ess inf}_{\Sigma_t \times [0,T)}\,H \geq \delta. \tag{4.5}$$

Claim 4.3 The limiting solution of (4.4) does not depend on the sequence $\epsilon_i \to 0$.

Proof Consider the evolution of a surface Σ_t by a fully-nonlinear equation of the form

$$\frac{\partial F}{\partial t} = -\mathcal{F}(h_{ij})\,\nu \tag{4.6}$$

where h_{ij} is the second fundamental form and $\mathcal{F}$ is a function of the eigenvalues of $\{h_{ij}\}$, which we denote by λ_1, λ_2 and assume that $\lambda_1 \geq \lambda_2$. Let $\mu = \lambda_2/\lambda_1$ and take

$$\mathcal{F}(\lambda_2, \mu) = \begin{cases} \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} = \frac{\lambda_2}{1+\mu}, & \text{for } \mu \geq -\delta_1 \\ \frac{\lambda_2}{1-\delta_1}, & \text{otherwise} \end{cases} \tag{4.7}$$

which we can be written as

$$\mathcal{F}(h_{ij}) = \begin{cases} \kappa, & \text{for } H g_{ij} \geq (1-\delta_1)h_{ij} \\ \frac{H + \sqrt{H^2 - 4G}}{1-\delta_1}, & \text{otherwise.} \end{cases} \tag{4.8}$$

We can also consider solutions of (1.1) in the viscosity sense (defined in [6] and [10]). In that case (4.6) can be written in the form

$$u_t = \begin{cases} \dfrac{\det\left(D_i\left(\frac{D_j u}{|Du|}\right)\right)}{\text{div}\left(\frac{Du}{|Du|}\right)}, & \text{when div}\left(\frac{Du}{|Du|}\right)\delta_{ij} \geq (1-\delta_1)D_i\left(\frac{D_j u}{|Du|}\right) \\[4mm] \dfrac{\text{div}\left(\frac{Du}{|Du|}\right) + \sqrt{\text{div}\left(\frac{Du}{|Du|}\right) - 4\det\left(D_i\left(\frac{D_j u}{|Du|}\right)\right)}}{1-\delta_1} & \text{otherwise.} \end{cases} \tag{4.9}$$

Equation (4.9) can be expressed as

$$u_t + \mathcal{F}_1(t, \nabla u, \nabla^2 u) = 0, \tag{4.10}$$

with

$$\mathcal{F}_1(t, p, X) = -\frac{|p| \det\left(X - \frac{p}{|p|} \otimes \left(X \cdot \frac{p}{|p|}\right)\right)}{\text{trace}\left(\left(I - \frac{p}{|p|} \otimes \frac{p}{|p|}\right) \cdot X\right)},$$

if

$$I \cdot \text{trace}\left(\left(I - \frac{p}{|p|} \otimes \frac{p}{|p|}\right) \cdot X\right) \geq (1 - \delta_1)\left(X - X \cdot \frac{p}{|p|} \otimes \frac{p}{|p|}\right)$$

and

$$\mathcal{F}_1(t, p, X) = \frac{1}{1 - \delta_1}\left(\frac{1}{|p|}\text{trace}\left(\left(\delta_{ij} - \frac{p}{|p|} \otimes \frac{p}{|p|}\right)X\right)\right)$$

$$+ \sqrt{\frac{1}{|p|^2}\left[\left(\text{trace}\left(\left(\delta_{ij} - \frac{p}{|p|} \otimes \frac{p}{|p|}\right)X\right)\right)^2 - \frac{4}{|p|}\det\left(X - \frac{(X \cdot p)}{|p|} \otimes \frac{p}{|p|}\right)\right]}\right),$$

otherwise.

Notice that the lower bound (4.5) together with our curvature pinching estimates (that follow from the Proposition 3.12) imply that

$$H\, g_{ij} \geq (1 - \delta_1)h_{ij}$$

for some $1 > \delta_1 > 0$. This implies that we can view a solution to (4.4) as a solution to (4.6) with F as in (4.8). The function $\mathcal{F}_1(t, p, X)$ is continuous on $(0, T) \times \mathbb{R}^2\setminus\{0\} \times S^{2\times2}$, it satisfies the conditions of Theorem 7.1 in [6] and (4.10) is a degenerate parabolic geometric equation in the sense of Definition 5.1 in [6]. Theorem 7.1 in [6] shows the uniqueness of viscosity solutions to (4.10). The $C^{1,1}$ solution on $[0, T)$ constructed above is a viscosity solution to (4.10) and by the uniqueness result it is the unique $C^{1,1}$ solution to (4.4). This means that the limiting solution of (4.4) does not depend on the sequence $\epsilon_i \to 0$. □

5 Radial case

In this section we will employ the results from the previous section to completely describe the long time behaviour of (1.1) in the case of surfaces of revolution, $r = f(x, t)$ around the x-axis. For such a surface of revolution the two principal curvatures are given by

$$\lambda_1 = \frac{1}{f(1 + f_x^2)^{\frac{1}{2}}} \quad \text{and} \quad \lambda_2 = -\frac{f_{xx}}{(1 + f_x^2)^{\frac{3}{2}}}. \tag{5.1}$$

Therefore,

$$H = \lambda_1 + \lambda_2 = \frac{-ff_{xx} + f_x^2 + 1}{f(1 + f_x^2)^{\frac{3}{2}}} > 0$$

and

$$G = \lambda_1\, \lambda_2 = \frac{-f_{xx}}{f(1 + f_x^2)^2}.$$

When the surface evolves by (1.1), $f(x, t)$ evolves by

$$f_t = \frac{f_{xx}}{-ff_{xx} + f_x^2 + 1}. \tag{5.2}$$

We will consider solutions $f(\cdot, t)$ on an interval $I_t = [a_t, b_t] \subset [0, 1]$ such that $f(a_t, t) = f(b_t, t) = 0$, $f > 0$ and $\tilde{H} = -ff_{xx} + f_x^2 + 1 > 0$. From (5.1) we see that $\lambda_1 > 0$ and λ_2 changes its sign, depending on the convexity of f. The linearization of (5.2) around a point f is

$$\tilde{f}_t = \frac{1 + f_x^2}{\tilde{H}^2} \, \tilde{f}_{xx} - \frac{2 f_x f_{xx}}{\tilde{H}^2} \, \tilde{f}_x + \frac{f_{xx}^2}{\tilde{H}^2} \tilde{f} \tag{5.3}$$

which is uniformly parabolic when $\tilde{H}$ is away from zero, no matter what is the sign of the smaller eigenvalue λ_2.

Theorem 5.1 *Assume that at time $t = 0$, Σ_0 is a $C^{1,1}$ star-shaped surface of revolution $r = f(x, 0)$, for $x \in [0, 1]$, $f(0, 0) = f(1, 0) = 0$, $f(\cdot, 0) > 0$ and $H > 0$. Then, the flow exists up to the maximal time*

$$T = \frac{\mu_0(\Sigma_0)}{4\pi}$$

when the surface Σ_t contracts to a point. Moreover, the surface becomes strictly convex at time $t_1 < T$ and asymptotically spherical at its extinction time T.

Since the equation is strictly parabolic when $\tilde{H} > 0$, the short time existence of a smooth solution on some time interval $[0, \tau]$, follows by classical results. Having a smooth solution to (1.1) on $[0, \tau]$ implies that we have a smooth solution $f(\cdot, t)$ to (5.2). By the comparison principle, $f(x, t)$ is defined on $I_t = [a_t, b_t] \subset [0, 1]$ and $f(a_t, t) = f(b_t, t) = 0$. Since the surface is smooth and $H > 0$ on $[0, \tau]$, the expressions for λ_1 and λ_2 in (5.1) yield to the bounds

$$\limsup_{x \to a_t} f \, |f_x| \le C_1(t) \quad \text{and} \quad \limsup_{x \to b_t} f \, |f_x| \le C_2(t), \quad \text{for } 0 \le t \le \tau.$$

In the next lemma we will show that the above bounds do not depend on the lower bound on H, but only on the initial data.

Lemma 5.2 *Assume that the solution f is smooth on $[0, t_0)$, for some $t_0 \le T$ and $H > 0$ on $[0, t_0)$. Then, there exists a uniform constant C, depending only on initial data, so that*

$$f^2 f_x^2 \le C, \quad \text{for all } t \in [0, t_0). \tag{5.4}$$

Proof We will bound $f^2 f_x^2$ from above by the maximum principle. Let us compute its evolution equation. We first compute the evolution of f_x by differentiating (5.2) in x. We get

$$(f_x)_t = \frac{f_{xxx} (1 + f_x^2) - f_x f_{xx}^2}{\tilde{H}^2}, \tag{5.5}$$

which yields the following equation for f_x^2:

$$(f_x^2)_t = \frac{2 f_{xxx} f_x (1 + f_x^2) - 2 f_x^2 f_{xx}^2}{\tilde{H}^2} = \frac{((f_x^2)_{xx} - 2 f_{xx}^2)(1 + f_x^2) - 2 f_x^2 f_{xx}^2}{\tilde{H}^2}$$

$$= \frac{(f_x^2)_{xx} (1 + f_x^2) - 4 f_x^2 f_{xx}^2 - 2 f_{xx}^2}{\tilde{H}^2}.$$

The function f^2 satisfies the equation

$$(f^2)_t = \frac{(f^2)_{xx} - 2f_x^2}{\tilde{H}^2}.$$

Combining the last two equations we obtain

$$
\begin{aligned}
(f^2 f_x^2)_t &= \frac{(f_x^2)_{xx}(1 + f_x^2) - 4f_x^2 f_{xx}^2 - 2f_{xx}^2}{\tilde{H}^2} f^2 + 2\frac{f_{xx}f f_x^2}{\tilde{H}^2} \\
&= \frac{(f^2 f_x^2)_{xx}(1 + f_x^2)}{\tilde{H}^2} - \frac{(1 + f_x^2)((f^2)_{xx} f_x^2 - 2(f_x^2)_x (f^2)_x)}{\tilde{H}^2} + 2\frac{f_{xx}f f_x^2}{\tilde{H}^2}.
\end{aligned}
$$

Let $t < t_0$. We distinguish the following two cases:

Case 1 The $(f^2 f_x^2)_{\max}(t)$ is attained in the interior of (a_t, b_t). Then, at that point $(f^2 f_x^2)_x = 0$, which implies (since $f(\cdot, t) > 0$ in the interior) that

$$f_x^3 = -ff_x f_{xx}. \tag{5.6}$$

Hence, the maximum principle implies the differential inequality

$$
\begin{aligned}
\frac{d}{dt}(f^2 f_x^2)_{\max}(t) &\le -\frac{1 + f_x^2}{\tilde{H}^2}\left((f^2)_{xx} f_x^2 - 2(f_x^2)_x (f^2)_x\right) + 2\frac{f_{xx}f f_x^2}{\tilde{H}^2} \\
&= -8\frac{(1 + f_x^2)f_x^4}{\tilde{H}^4} - 2\frac{f_x^4}{\tilde{H}^2} \le 0. \tag{5.7}
\end{aligned}
$$

Case 2 The $(f^2 f_x^2)_{\max}(t)$ is attained at one of the tips $\{a_t, b_t\}$. Assume it is attained at a_t. The point of the surface Σ_t that arises from $x = a_t$ can be viewed as the interior point of Σ_t around which our surface is convex. We can solve locally, around the point $x = a_t$ (say for $x \in [a_t, x_t]$) the equation $y = f(x, t)$ with respect to x, yielding to the map $x = g(y, t)$. Notice that $ff_x = y/g_y$ and that $x = a_t$ corresponds to $y = 0$. Since $\{f(x, t) | x \in [a_t, x_t]\} \cup \{-f(x, t) | x \in [a_t, x_t]\}$ is a smooth curve, we have that $x = g(y, t)$ is a smooth graph for $y \in [-f(x_t, t), f(x_t, t)]$. If $f^2 f_x^2(\cdot, t)$ attains its maximum somewhere in $[a_t, x_t)$, then y^2/g_y^2 attains its maximum in the interior of $(-f(x_t, t), f(x_t, t))$.

We will now compute the evolution of y^2/g_y^2 from the evolution of $f^2 f_x^2$. Since

$$f_x(x, t) = \frac{1}{g_y(y, t)}$$

from the evolution of $f^2 f_x^2$ we get

$$\left(\frac{y^2}{g_y^2}\right)_t = \frac{(1 + g_y^2)}{g_y^2 \tilde{H}^2}\left(\frac{y^2}{g_y^2}\right)_{xx} - \frac{(1 + g_y^2)}{g_y^2 \tilde{H}^2}\left((y^2)_{xx}\frac{1}{g_y^2} - 2\left(\frac{1}{g_y^2}\right)_x (y^2)_x\right) + \frac{2\,y_{xx}y}{g_y^2 \tilde{H}^2}.$$

By direct computation we have

$$\left(\frac{1}{g_y^2}\right)_x = -\frac{2g_{yy}}{g_y^3} \quad \text{and} \quad (y^2)_x = \frac{2y}{g_y} \quad \text{and} \quad y_{xx} = -\frac{g_{yy}}{g_y^3}$$

and

$$\left(\frac{y^2}{g_y^2}\right)_{xx} = \left(\frac{y^2}{g_y^2}\right)_{yy} - \left(\frac{y^2}{g_y^2}\right)_y \frac{g_{yy}}{g_y^3} \quad \text{and} \quad (y^2)_{xx} = \frac{2}{g_y^2} - \frac{2yg_{yy}}{g_y^3}.$$

Combining the above yields to

$$\left(\frac{y^2}{g_y^2}\right)_t = \frac{(g_y^2+1)}{g_y^2 \tilde{H}^2}\left(\frac{y^2}{g_y^2}\right)_{yy} - \left(\frac{y^2}{g_y^2}\right)_y \frac{g_{yy}\,(1+g_y^2)}{g_y^5 \tilde{H}^2}$$
$$- \frac{(1+g_y^2)}{g_y^2 \tilde{H}^2}\left(\frac{2}{g_y^4}+2\frac{y g_{yy}}{g_y^5}\right) - \frac{2\,y g_{yy}}{g_y^5 \tilde{H}^2}$$

which can be re-written it as

$$\left(\frac{y^2}{g_y^2}\right)_t = \frac{(g_y^2+1)}{g_y^2 \tilde{H}^2}\left(\frac{y^2}{g_y^2}\right)_{yy} - \left(\frac{y^2}{g_y^2}\right)_y \frac{g_{yy}(1+g_y^2)}{g_y^2 \tilde{H}^5}$$
$$- \frac{(1+g_y^2)}{g_y^2 \tilde{H}^2}\left(\frac{2}{g_y^4}+\frac{2\,y^2 g_y g_{yy}}{y g_y^6}\right) - \frac{2\,y^2 g_y g_{yy}}{y g_y^6 \tilde{H}^2}. \tag{5.8}$$

At the maximum point of y^2/g_y^2 we have

$$y^2 g_y g_{yy} = y g_y^2.$$

This together with the maximum principle applied to (5.8) yield to the differential inequality

$$\frac{d}{dt}\left(\frac{y^2}{g_y^2}\right)_{\max}(t) \leq -\frac{4\,(1+g_y^2)}{g_y^6 \tilde{H}^2} - \frac{2}{\tilde{H}^2 g_y^4} \leq 0. \tag{5.9}$$

Estimates (5.7) and (5.9) imply that $(f^2 f_x^2)(x,t) \leq C$, for all $x \in [a_t, b_t]$ and all $t \leq t_0$, where C is a uniform constant independent of time. This finishes the proof of the lemma. □

Corollary 5.3 *Let $T = \mu_0(\Sigma_0)/4\pi$ be as in Theorem 5.1. Then, there exists a uniform constant δ, depending only on the initial data, so that $H \geq \delta > 0$, for all $t \in [0, T)$.*

Proof It is enough to show that if $H > 0$ on $[0, t_0)$, then $H \geq \delta > 0$ there. We recall that $\lambda_1 = 1/f(1 + f_x^2)^{1/2}$. Hence, the estimate (5.4) yields to the bound

$$\lambda_1 \geq c > 0 \qquad \text{on } \Sigma_t, \quad \text{for } t \in [0, t_0). \tag{5.10}$$

Since $H = \lambda_1 + \lambda_2$, if $\lambda_2 \geq 0$, then $H \geq \lambda_1 \geq c$. If $\lambda_2 < 0$ and $H \leq c/2$ (otherwise we are done) by (5.10) we have

$$\lambda_1 - |\lambda_2| \leq \frac{c}{2} \Rightarrow |\lambda_2| \geq \frac{c}{2}.$$

Observe next that Lemma 3.5 implies the bound

$$\frac{\lambda_1 |\lambda_2|}{H} \leq C, \quad \text{for a uniform constant } C.$$

Hence

$$H \geq \frac{\lambda_1 |\lambda_2|}{C} \geq \frac{c^2}{2C}.$$

In any case, we have

$$H \geq \min\left\{\frac{c}{2}, \frac{c^2}{2C}\right\}$$

which shows our lemma with $\delta := \min\{\frac{c}{2}, \frac{c^2}{2C}\}$.   □

Lemma 5.4 *Let $[0, T)$ be the maximal interval of existence of a solution to (1.1). Then, $\max_{\Sigma_t} |A|$ becomes unbounded as $t \to T$.*

Proof Assume that $\sup_{\Sigma_t} |A| \leq C$, for all $t \in [0, T)$ and write

$$H = \frac{\tilde{H}}{f(1 + f_x^2)^{3/2}}$$

with $\tilde{H} = -ff_{xx} + f_x^2 + 1$. Then $H \leq C$ (since $|A|$ is bounded) and $H \geq \delta > 0$ (by the previous result). Hence,

$$c_1 \leq \frac{f(f_x^2 + 1)^{3/2}}{\tilde{H}} \leq c_2$$

which implies

$$\frac{c_1}{f(1 + f_x^2)^{1/2}} \leq \frac{1 + f_x^2}{\tilde{H}} \leq \frac{c_2}{f(1 + f_x^2)^{1/2}}.$$

We can rewrite it as

$$c_1 \lambda_1 \leq \frac{1 + f_x^2}{\tilde{H}} \leq c_2 \lambda_1 \tag{5.11}$$

which together with (5.10) and $|A| \leq C$ imply the bounds

$$C_1 \leq \frac{1 + f_x^2}{\tilde{H}} \leq C_2 \tag{5.12}$$

for uniform constants C_1, C_2, for all $t \in [0, T)$. This means the linearization (5.3) of (5.2) is uniformly elliptic on time interval $[0, T)$. If our surface of revolution at time t is given by an embedding $F(\Sigma, t)$, which is a solution to (1.1), $|A| \leq C$ implies $|F|_{C^2} \leq C$ on the time interval $[0, T)$ and the speed $|\kappa| \leq C$ (we will use the same symbol C to denote different uniform constants). It is easy to see that $F(\cdot, t)$ converges to a continuous limit $F(\cdot, T)$ as $t \to T$, since

$$|F(x, t_1) - F(x, t_2)| \leq \int_{t_1}^{t_2} |\kappa| \, dt \leq C|t_1 - t_2|.$$

Due to

$$\left| \frac{\partial}{\partial t} g_{ij} \right|^2 = \left| 2h_{ij}\kappa \right|^2 \leq 4 \left| A \right|^2 \kappa^2 \leq C$$

and [12] we have that $F(\cdot, T)$ represents a surface. It is a $C^{1,1}$ surface of revolution $r = f(x, T)$ around the x-axis that comes as a limit as $t \to T$ of surfaces of revolution $r = f(x, t)$. Take $0 < \epsilon << b_T - a_T$ arbitrarily small. Consider $f(r, t)$ on $x \in [a_t + \epsilon, b_t - \epsilon]$, that is, away from the tips $x = a_t$ and $x = b_t$ where $f = 0$ and f_x becomes unbounded. Since our solution is $C^{1,1}$, $c_1 \leq f(r, T) \leq c_2$ and $|f_x| \leq c_3$, at time $t = T$ and for $x \in [a_T + \epsilon, b_T - \epsilon]$, where c_1, c_2, c_3 all depend on ϵ. Due to (5.12), equation (5.2) is uniformly parabolic and standard parabolic estimates yield

$$|f(\cdot, T)|_{C^k} \leq C(\epsilon, k), \qquad \text{for every } k > 0 \text{ and } x \in [a_T + \epsilon, b_T - \epsilon]. \tag{5.13}$$

We can repeat the previous discussion to every $\epsilon > 0$ to conclude that our surface Σ_T is smooth for $x \in (a_T, b_T)$. By writing our surface locally as a graph $x = g(y, t)$ around the

tips (at which our surface is strictly convex), we can show that our surface is smooth at the tips as well (similar methods to those discussed above apply in this case). □

The same proof as the one for the flow (HMCF$_\epsilon$) which was presented in the previous section, shows that our radial surface Σ_t shrinks to a point at $T = \frac{\mu_0(\Sigma_0)}{4\pi}$, where $\mu_0(\Sigma_0)$ is the total area of Σ_0. In particular, this means $f(x, t) \to 0$ as $t \to T$.

We will show next that at some time $t_1 < T$ the surface Σ_{t_1} becomes strictly convex. This will follow from the next lemma.

Lemma 5.5 *Assume that f is a solution of the HMCF on $[0, T)$. Then, there exists a constant $c > 0$, independent of t, such that $f(x, t) \geq c$, at all points (x, t), with $0 \leq t < T$ and $f_{xx}(x, t) \geq 0$.*

Proof Fix $t < T$. Since our surface Σ_t is convex around the tip $x = a_t$ we have $f_{xx} \leq 0$ there. Let c_t be the largest number in $[a_t, b_t]$ so that Σ_t is strictly convex for $x \in [a_t, c_t]$. If $c_t = b_t$, then Σ_t is convex and we have nothing to show. Otherwise, $f_{xx}(x, t) \leq 0$ for $a_t \leq x \leq c_t$ and $f_{xx}(x, t) > 0$ in $(c_t, c_t + \epsilon_t)$ for some $\epsilon_t > 0$. Hence, $f_x(\cdot, t)$ is increasing in x, for $x \in (c_t, c_t + \epsilon_t)$.

Consider the function $f_x(\cdot, t)$ on the interval $x \in [c_t, b_t)$. From the above discussion and the fact that $\lim_{x \to b_t} f_x(x, t) = -\infty$, we conclude that the maximum

$$M(t) := \max \{ f_x(x, t), \ x \in [c_t, b_t] \}$$

is attained in the interior of $[c_t, b_t]$. Recall the evolution equation for f_x to be

$$(f_x)_t = \frac{f_{xxx}(1 + f_x^2)}{\tilde{H}^2} - \frac{f_x f_{xx}^2}{\tilde{H}^2}.$$

Hence, assuming that $M(t) \geq 0$, the maximum principle implies that $M'(t) \leq 0$. This shows that f_x is uniformly bounded from above on $[c_t, b_t]$. Since a similar argument can be applied near the other tip b_t, we finally conclude that $|f_x|$ is uniformly bounded in the non-convex part (if it exists) away from the tips.

We will now conclude the proof of the lemma. Assume that $f_{xx}(x, t) \geq 0$, which holds in a non-convex part of our evolving surface. At that point, we have

$$\lambda_2 := -\frac{f_{xx}}{(1 + f_x^2)^{3/2}} \leq 0.$$

Since $\lambda_2 \leq 0$, Lemma 3.8 implies the bound

$$\lambda_1 := \frac{1}{f(1 + f_x^2)^{1/2}} \leq C$$

which reduces to the the bound

$$f \geq \frac{1}{C(1 + f_x^2)^{1/2}} \geq \frac{1}{\tilde{C}} =: c$$

in the non-convex part where $f_x^2 \leq C$, uniformly in t. This finishes the proof of the lemma. □

We will now conclude the proof of Theorem 5.1: Since $f(x, t) \to 0$ as $t \to T$, with $T = \frac{\mu_0(\Sigma_0)}{4\pi}$, there is some time $t_1 < T$ so that

$$f(x, t) < \frac{c}{2}, \qquad \text{for all } x \in [a_t, b_t]$$

where $c > 0$ is the constant taken from Lemma 5.5. Hence, by Lemma 5.5 the surface Σ_t is convex for $t \geq t_1$. Since $H \geq \delta > 0$ for all $t < T$, the surface Σ_{t_1} is strictly convex. The result of Andrews [1], implies that Σ_t shrinks asymptotically spherically to a point as $t \to T$.

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