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Calculus of Variations

Differentiability and higher integrability results for local minimizers of splitting-type variational integrals in 2D with applications to nonlinear Hencky-materials

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Abstract We prove higher integrability and differentiability results for local minimizers $u\colon \mathbb{R}^2 \supset \Omega \to \mathbb{R}^M$, $M \geq 1$, of the splitting-type energy $\int_\Omega [h_1(|\partial_1 u|) + h_2(|\partial_2 u|)]\,\mathrm{d}x$. Here h_1, h_2 are rather general N-functions and no relation between h_1 and h_2 is required. The methods also apply to local minimizers $u\colon \mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$ of the functional $\int_\Omega [h_1(|\mathrm{div}\, u|) + h_2(|\varepsilon^D(u)|)]\,\mathrm{d}x$ so that we can include some variants of so-called nonlinear Hencky-materials. Further extensions concern non-autonomous problems.

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1 Introduction

Over the last two decades increasing attention has been paid to the question of interior regularity (i.e. higher integrability of the gradient or even continuity of the first weak derivatives) of local minimizers $u\colon \mathbb{R}^n \supset \Omega \to \mathbb{R}^M$ of variational integrals

$$I[u,\Omega] = \int_\Omega H(\nabla u)\,\mathrm{d}x \tag{1.1}$$

with anisotropic energy density $H\colon \mathbb{R}^{nM} \to [0,\infty)$. Here, roughly speaking, H is called an anisotropic integrand if we have

$$\lambda(|Z|)|Y|^2 \leq D^2H(Z)(Y,Y) \leq \Lambda(|Z|)|Y|^2 \tag{1.2}$$

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with functions λ, Λ: $[0, \infty) \to [0, \infty)$, which do **not** satisfy an estimate of the form

$$c_1 \leq \frac{\Lambda}{\lambda} \leq c_2 \tag{1.3}$$

with positive constants c_1, c_2. Much of the literature is devoted to the investigation of the scalar case (i.e. $M = 1$) and to the closely related situation that $M > 1$ together with the requirement that H depends on the modulus of the gradient. We refer to the papers of Choe [17], of Fusco and Sbordone [24], of Marcellini [28,29], of Marcellini and Papi [30], of Mingione and Siepe [31] as well as to the work [3] and the references quoted therein, where the interested reader will find interior regularity theorems for a variety of anisotropic energies. If $n \geq 3$ together with $M \geq 2$, then mainly the partial regularity of local minimizers is discussed as done for example by Acerbi and Fusco [1], Cupini et al. [19], Esposito et al. [20], Passarelli Di Napoli and Siepe [32] and the authors [5,10]. In these papers so-called anisotropic (p, q)-growth is considered, which means that

$$\lambda(|Z|) \approx |Z|^{p-2}, \quad \Lambda(|Z|) \approx |Z|^{q-2} \tag{1.4}$$

holds for exponents $1 < p \leq q < \infty$, and almost everywhere regularity follows if in addition to (1.4) we have an estimate of the form

$$q < c(n)p, \tag{1.5}$$

where $c(n)$ is large for low dimensions n, but $c(n) \to 1$ as $n \to \infty$. Let us note that under some extra assumptions on the structure of H (1.5) can be replaced by weaker restrictions at least if the case of locally bounded local minimizers is considered. For an overview on the history as well as for a collection of recent contributions mainly concerning anisotropic (p, q)-growth we refer to [4].

A very natural class of anisotropic problems arises if we consider integrands $H(\nabla u)$ which split into a sum of strictly convex functions, each of them depending on different partial derivatives, for example

$$H(\nabla u) = H_1(\tilde{\nabla} u) + H_2(\partial_n u), \quad \tilde{\nabla} u := (\partial_1 u, \ldots, \partial_{n-1} u), \tag{1.6}$$

where H_1 and H_2 might be of power growth with different growth rates $\bar{p}$ and $\bar{q}$ in the sense that

$$D^2 H_1(\tilde{\xi}) \approx |\tilde{\xi}|^{\bar{p}-2}, \quad D^2 H_2(\xi_n) \approx |\xi_n|^{\bar{q}-2}, \quad \xi = (\tilde{\xi}, \xi_n) \in \mathbb{R}^{nM}. \tag{1.7}$$

Let $2 < \bar{p} < \bar{q}$. Then from (1.6) and (1.7) we deduce the validity of (1.2) and (1.4) with $p := 2$ and $q := \bar{q}$, and (1.5) reads as $\bar{q} < 2c(n)$, which means that we cannot benefit in any way from the value of $\bar{p}$ if we reduce the setting described above through (1.6) and (1.7) to the unstructured requirement (1.2) together with (1.4) and (1.5). In the papers [9–11,13] and [14] we showed how to get much better results by working with techniques based on the splitting structure of the integrand, for example in the scalar case and under the natural hypothesis that the local minimizer is locally bounded we could show interior $C^{1,\alpha}$-regularity for local minimizers of the energy with density $H(\nabla u) = \sum_{i=1}^{n}(1+|\partial_i u|^2)^{p_i/2}$ independent of the choices of $p_i > 1$.

In the present paper we now concentrate on splitting integrals (1.6) in two dimensions including the vectorial situation (i.e. $M > 1$) and working with the following hypotheses: let for $Z \in \mathbb{R}^{2M}$

$$H(Z) = h_1(|Z_1|) + h_2(|Z_2|) \tag{1.8}$$

with functions h_1, h_2: $[0, \infty) \to [0, \infty)$ of class C^2 s.t. for $h = h_1$ and $h = h_2$ it holds

$$h \text{ is strictly increasing and convex together with} \qquad \text{(A1)}$$
$$h''(0) > 0 \quad \text{and} \quad \lim_{t \to 0} \frac{h(t)}{t} = 0\,;$$

$$\text{there is a constant } \bar{k} > 0 \text{ s.t. } h(2t) \leq \bar{k} h(t) \quad \text{for all } t \geq 0; \qquad \text{(A2)}$$

$$\text{for an exponent } \omega \geq 0 \text{ and a constant } a \geq 0 \text{ it holds} \qquad \text{(A3)}$$
$$\frac{h'(t)}{t} \leq h''(t) \leq a(1+t^2)^{\frac{\omega}{2}} \frac{h'(t)}{t} \quad \text{for all } t \geq 0.$$

Let us draw some conclusions from (A1)–(A3):

(i) (A1) implies that $h(0) = 0 = h'(0)$ and $h'(t) > 0$ for $t > 0$. From (A3) it follows that $t \mapsto h'(t)/t$ is increasing, moreover we get $h(t) \geq h''(0)t^2/2$. In particular h is a N-function (see [2]) of at least quadratic growth.

(ii) The ($\Delta 2$)-property stated in (A2) implies

$$h(t) \leq c(t^m + 1)$$

for some exponent $m \geq 2$, hence by the convexity of h

$$h'(t) \leq c(t^{m-1} + 1),$$

where here and in the following "c" denotes a constant whose value may vary from line to line.

(iii) Combining (A2) with the convexity of h we see that

$$\frac{1}{\bar{k}} h'(t)t \leq h(t) \leq t h'(t), \quad t \geq 0. \qquad (1.9)$$

(iv) For $Y = (Y_1, Y_2)$, $Z = (Z_1, Z_2) \in \mathbb{R}^{2M}$ we have

$$\sum_{i=1}^{2} \min\left[h_i''(|Z_i|), \frac{h_i'(|Z_i|)}{|Z_i|}\right] |Y_i|^2 \leq D^2 H(Z)(Y, Y)$$
$$\leq \sum_{i=1}^{2} \max\left[h_i''(|Z_i|), \frac{h_i'(|Z_i|)}{|Z_i|}\right] |Y_i|^2,$$

so that by (A3)

$$\sum_{i=1}^{2} \frac{h_i'(|Z_i|)}{|Z_i|} |Y_i|^2 \leq D^2 H(Z)(Y, Y) \leq \sum_{i=1}^{2} a(1 + |Z_i|^2)^{\frac{\omega}{2}} \frac{h_i'(|Z_i|)}{|Z_i|} |Y_i|^2, \qquad (1.10)$$

and for a suitable exponent $\bar{q} > 2$ it follows

$$c|Y|^2 \leq D^2 H(Z)(Y, Y) \leq C(1 + |Z|^2)^{\frac{\bar{q}-2}{2}} |Y|^2, \qquad (1.11)$$

the first inequality being a consequence of (i).

Definition 1.1 Let $\Omega \subset \mathbb{R}^2$ and let H from (1.8) satisfy (A1)–(A3). Then a function $u \in W^1_{1,loc}(\Omega; \mathbb{R}^M)$ is called a local minimizer of the functional I from (1.1) iff $I[u, \Omega'] < \infty$ and $I[u, \Omega'] \leq I[v, \Omega']$ for all $v \in W^1_{1,loc}(\Omega; \mathbb{R}^M)$ such that $\mathrm{spt}(u - v) \subset \Omega'$, where Ω' is any subdomain of Ω with compact closure in Ω.

For a definition of the Sobolev classes $W^k_{p,loc}(\Omega; \mathbb{R}^M)$ and related spaces we refer the reader to [2]. Our first result is the following

Theorem 1.1 *Let $n = 2$ and let H satisfy (1.8) together with (A1)–(A3). Suppose further that $u \in W^1_{1,loc}(\Omega; \mathbb{R}^M)$ locally minimizes the functional I from (1.1). Then we have:*

(i) *∇u belongs to $L^t_{loc}(\Omega; \mathbb{R}^{2M})$ for any finite t.*
(ii) *If (A3) holds with $\omega < 2$, then $u \in C^{1,\alpha}(\Omega; \mathbb{R}^M)$ for any $0 < \alpha < 1$.*

Remark 1.1 We emphasize that in (i) no restriction on the value of ω is required.

Remark 1.2 From the proof it will become clear that the results of Theorem 1.1 are also true for local minimizers of $\int_\Omega [h_1(|\nabla u|) + h_2(|\partial_2 u|)\,\mathrm{d}x$ or of $\int_\Omega [h_1(|\partial_1 u|) + h_2(|\nabla u|)]\,\mathrm{d}x$ provided (A1)–(A3) hold for h_1 and h_2.

Remark 1.3 Let us compare Theorem 1.1 to our previous works on splitting functionals on plane domains:

(i) In [8] we discussed the case of densities $H_1(\partial_1 u) + H_2(\partial_2 u)$ with functions H_i: $\mathbb{R}^M \to [0, \infty)$ s.t. for $i = 1, 2$ and $Y, Z \in \mathbb{R}^M$

$$\lambda(1 + |Z|^2)^{\frac{p_i - 2}{2}} |Y|^2 \leq D^2 H_i(Z)(Y, Y) \leq \Lambda(1 + |Z|^2)^{\frac{p_i - 2}{2}} |Y|^2$$

for exponents $2 \leq p_1 \leq p_2 < \infty$ and proved part (ii) of Theorem 1.1 under the assumption $p_2 < 2p_1$.

(ii) This result was improved in [9], Theorem 1, (c) and Remark 4, by showing that the hypothesis $p_2 < 2p_1$ can be dropped in case that $2 < p_1 \leq p_2 < \infty$.

(iii) In [11], Theorem 2.2, we considered the density $H(\nabla u) = h_1(|\partial_1 u|) + h_2(|\partial_2 u|)$, where h_1, h_2 satisfy (A1)–(A3) with $\omega = 0$ and where $h_1(t) \leq h_2(t)$ for large values of t is required. Then we obtained the result of Theorem 1.1(i). Now, in the present setting, we impose no ordering relation like $h_1 \leq h_2$ on h_1 and h_2, moreover—at least for part (i) of the theorem—there is also no limitation on the value of ω.

Next we pass to non-autonomous densities of the form $H(x, Z) = h_1(x, |Z_1|) + h_2(x, |Z_2|)$, $x \in \overline{\Omega}$, $Z = (Z_1, Z_2) \in \mathbb{R}^{2M}$, with functions $h_i(x, t)$ satisfying (A1)–(A3) uniformly in $x \in \overline{\Omega}$ (replacing h'_i by $\frac{\partial}{\partial t} h_i$, etc.) and for which ($\alpha, i = 1, 2$)

$$\left| \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial t} h_i(x, t) \right| \leq c \frac{\partial}{\partial t} h_i(x, t)\,, \quad x \in \overline{\Omega}, \quad t \geq 0 \tag{A4}$$

holds. Then we have

Theorem 1.2 *Let $H(x, Z)$ satisfy the modified set of assumptions (1.8), (A1)–(A3) and let (A4) hold. Then, if $u \in W^1_{1,loc}(\Omega; \mathbb{R}^M)$ locally minimizes*

$$\int_\Omega H(x, \nabla u)\,\mathrm{d}x = \int_\Omega h_1(x, |\partial_1 u|)\,\mathrm{d}x + \int_\Omega h_2(x, |\partial_2 u|)\,\mathrm{d}x,$$

the statements of Theorem 1.1 continue to hold.

Remark 1.4 An example, to which Theorem 1.2 applies, is the energy $\int_\Omega [h_1(x, |\partial_1 u|) + h_2(x, |\partial_2 u|)]\,dx$ with densities $h_i(x, t)$ of the form

$$h_i(x,t) = (a_i(x) + t^2)^{\frac{p_i}{2}} - a_i(x)^{\frac{p_i}{2}} \quad \text{or}$$
$$h_i(x,t) = (1 + a_i(x)t^2)^{\frac{p_i}{2}} - 1.$$

Here p_i are exponents in the interval $[2, \infty)$ and $a_i : \overline{\Omega} \to (0, \infty)$ denote sufficiently regular functions. In the more interesting situation that $h_i(x, t) = (1 + t^2)^{p_i(x)/2} - 1$ with variable exponents $p_i : \overline{\Omega} \to [2, \infty)$ condition (A4) is violated, and we can not deduce regularity from Theorem 1.2. However it should be noted that for the isotropic variant, i.e. for energies like $\int_\Omega |\nabla u|^{p(x)}\,dx$, smoothness results are due to Coscia/Mingione [18].

Remark 1.5 Since we deal with local minimizers and discuss interior regularity, it is sufficient to know that in the non-autonomous case the bounds (A1)–(A4) are uniform in $x \in \Omega'$ for subdomains $\Omega' \Subset \Omega$.

As an application of the arguments used for the proof of Theorem 1.1 we also obtain regularity results for a certain class of nonlinear elastic materials in 2D. Let $n = M = 2$. Then, according to [33], the energy functional of a nonlinear Hencky material is given by

$$E[u, \Omega] := \int_\Omega \left[\frac{\lambda}{2} (\operatorname{div} u)^2 + \varphi(|\varepsilon^D(u)|) \right] dx,$$

where λ denotes a positive constant and where $\varepsilon(u)$ is the symmetric part of the gradient of the deformation $u : \Omega \to \mathbb{R}^2$. $\varepsilon^D(u) := \varepsilon(u) - \frac{1}{2} \operatorname{div} u \mathbf{1}$ is the deviatoric part of $\varepsilon(u)$, and since the above model is used as an approximation for plasticity, the density φ usually is of nearly linear growth which means $\varphi(t) = t \ln(1 + t)$ or $\varphi(t) = (1 + t^2)^{s/2} - 1$ for some $s > 1$ close to 1. From the work of Frehse and Seregin [22] the interior $C^{1,\alpha}$-regularity of local minimizers of the functional E follows for the logarithmic case as well as for the power growth case with $s \le 2$. In [7] we gave a slight extension up to $s < 4$ and for any s under the additional hypothesis that (for some reason) we have the information $\operatorname{div} u \in L^s_{loc}(\Omega)$. Now we can remove these restrictions, which enables us to discuss energies having rather general growth w.r.t. $\operatorname{div} u$ and $\varepsilon^D(u)$, precisely:

Theorem 1.3 *Let $n = M = 2$, let (A1)–(A3) hold for the functions h_1, h_2, and consider a local minimizer u of the energy*

$$\int_\Omega \left[h_1(|\operatorname{div} u|) + h_2(|\varepsilon^D(u)|) \right] dx.$$

Then ∇u is in the space $L^t_{loc}(\Omega; \mathbb{R}^{2\times 2})$ for any finite exponent t. If $\omega < 2$ holds in (A3), then this can be improved to $u \in C^{1,\alpha}(\Omega; \mathbb{R}^2)$ for any $\alpha < 1$. In particular we have interior differentiability for the choices $h_1(t) = t^2$, $h_2(t) = (1 + t^2)^{s/2} - 1$ with $s \ge 2$.

Remark 1.6 Of course a "non-autonomous" variant of Theorem 1.3 can be obtained in the spirit of Theorem 1.2.

Our paper is organized as follows: in Sect. 2, we give the proof of Theorem 1.1, the necessary adjustments concerning the non-autonomous case are presented in Sect. 3. In Sect. 4, we briefly sketch the situation for functionals related to the energy modeling nonlinear Hencky-materials. A class of energies satisfying our hypotheses is shortly discussed in the Appendix.

2 Proof of Theorem 1.1

Let (A1)–(A3) hold and consider a local minimizer u of the functional I from (1.1). As outlined for example in [8] the following calculations can be justified by working with a local regularization with exponent $\bar{q}$ introduced in (1.11) having a sufficient degree of regularity, which follows from the results of [15] or [26].

Let $\eta \in C_0^\infty(\Omega)$. Then we have (from now on summation w.r.t. indices repeated twice and this convention is used both for Greek and for Latin indices)

$$0 = \int_\Omega \partial_\alpha [DH(\nabla u)] : \nabla(\eta^2 \partial_\alpha u)\, \mathrm{d}x,$$

hence an integration by parts yields

$$\begin{aligned}\int_\Omega D^2 H(\nabla u)(\partial_\alpha \nabla u, \partial_\alpha \nabla u)\eta^2 \,\mathrm{d}x &= -\int_\Omega \partial_\alpha [DH(\nabla u)] : (\nabla \eta^2 \otimes \partial_\alpha u)\,\mathrm{d}x \\ &= \int_\Omega DH(\nabla u) : \partial_\alpha [\nabla \eta^2 \otimes \partial_\alpha u]\,\mathrm{d}x. \qquad (2.1)\end{aligned}$$

Here ":" is the scalar product of matrices and "⊗" denotes the tensor product of vectors. From the first inequality in (1.11) we deduce

$$\text{l.h.s. of (2.1)} \geq c \int_\Omega |\nabla^2 u|^2 \eta^2 \,\mathrm{d}x. \qquad (2.2)$$

For the r.h.s. of (2.1) we observe (w.l.o.g. $0 \leq \eta \leq 1$)

$$\begin{aligned}&\left| \int_\Omega DH(\nabla u) : \partial_\alpha [\nabla \eta^2 \otimes \partial_\alpha u]\,\mathrm{d}x \right| \\ &\leq c \left[\int_\Omega h_i'(|\partial_i u|)|\nabla^2 u|\eta|\nabla \eta|\,\mathrm{d}x + \int_\Omega h_i'(|\partial_i u|)|\nabla u||\nabla^2 \eta^2|\,\mathrm{d}x \right] \\ &\leq \varepsilon \int_\Omega \eta^2 |\nabla^2 u|^2\,\mathrm{d}x + c(\varepsilon)\int_\Omega |\nabla \eta|^2 \left(h_1'(|\partial_1 u|)^2 + h_2'(|\partial_2 u|)^2 \right)\,\mathrm{d}x \\ &\quad + c\int_\Omega \left[h_1'(|\partial_1 u|)^2 + h_2'(|\partial_2 u|)^2 + |\nabla u|^2 \right] |\nabla^2 \eta^2|\,\mathrm{d}x,\end{aligned}$$

where $\varepsilon > 0$ is arbitrary and where we have used Young's inequality several times. If ε is small enough and if we use (2.2), the ε-term can be absorbed in the l.h.s. of (2.1). Recalling the lower bound $h_i(t) \geq ct^2$, we arrive at

$$\begin{aligned}&\int_\Omega \eta^2 D^2 H(\nabla u)(\partial_\alpha \nabla u, \partial_\alpha \nabla u)\,\mathrm{d}x \\ &\leq c\int_\Omega (|\nabla \eta|^2 + |\nabla^2 \eta|)\left(h_1'(|\partial_1 u|)^2 + h_2'(|\partial_2 u|)^2 + H(\nabla u) \right)\,\mathrm{d}x. \qquad (2.3)\end{aligned}$$

The r.h.s. of (2.3) is handled using ideas of [23]: let us fix a subdomain $\Omega^* \Subset \Omega$ and consider discs $B_r(z) \subset B_R(z) \subset \Omega^*$. Let further $\eta \equiv 1$ on $B_r(z)$, $\operatorname{spt}\eta \subset B_R(z)$ and $|\nabla^l \eta| \le c(R-r)^{-l}$, $l = 1, 2$. Denoting by $c(\Omega^*)$ constants depending on the (finite) energy of u over Ω^*, we get from (2.3)

$$\int\limits_{B_r(z)} D^2 H(\nabla u)(\partial_\alpha \nabla u, \partial_\alpha \nabla u)\,\mathrm{d}x \le (R-r)^{-2}\left[c(\Omega^*) + c\int\limits_{B_R(z)} \left(h_1'(|\partial_1 u|)^2 + h_2'(|\partial_2 u|)^2\right)\mathrm{d}x\right]. \tag{2.4}$$

For any $L > 0$ we have using (1.9)

$$\begin{aligned}\int\limits_{B_R(z)} h_1'(|\partial_1 u|)^2\,\mathrm{d}x &= \int\limits_{B_R(z)\cap[|\partial_1 u|\le L]} h_1'(|\partial_1 u|)^2\,\mathrm{d}x + \int\limits_{B_R(z)\cap[|\partial_1 u|> L]} h_1'(|\partial_1 u|)^2\,\mathrm{d}x \\ &\le h_1'(L)^2 \pi R^2 + cL^{-2}\int\limits_{B_R(z)\cap[|\partial_1 u|> L]} h_1(|\partial_1 u|)^2\,\mathrm{d}x \\ &\le \pi R^2 h_1'(L)^2 + cL^{-2}\int\limits_{B_R(z)} h_1(|\partial_1 u|)^2\,\mathrm{d}x,\end{aligned}$$

and the same estimate is true for h_2. Let $L := \frac{1}{\lambda}\frac{1}{R-r}$ for some $0 < \lambda \le 1$ and with $R \le 1$. Recalling that in this case

$$h_i'(L)^2 \le cL^{2m-2},$$

we deduce from (2.4) and the above inequalities for a suitable positive exponent β

$$\int\limits_{B_r(z)} D^2 H(\nabla u)(\partial_\alpha \nabla u, \partial_\alpha \nabla u)\,\mathrm{d}x \le c(\Omega^*, \lambda)(R-r)^{-\beta} + c\lambda^2 \int\limits_{B_R(z)} \left(h_1(|\partial_1 u|)^2 + h_2(|\partial_2 u|)^2\right)\mathrm{d}x, \tag{2.5}$$

and (2.5) is valid for all $0 < \lambda \le 1$ and all discs $B_R(z) \subset \Omega^*$, $R \le 1$.

Let $\rho \in (0, R)$ and define $r = (\rho + R)/2$. With $\eta \in C_0^\infty(B_r(z))$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_\rho(z)$ and $|\nabla \eta| \le c/(r-\rho)$ $(= 2c/(R-\rho))$ we find with Sobolev's inequality

$$\begin{aligned}&\int\limits_{B_\rho(z)} \left(h_1(|\partial_1 u|)^2 + h_2(|\partial_2 u|)^2\right)\mathrm{d}x \\ &\le \int\limits_{B_r(z)} (\eta h_1(|\partial_1 u|))^2\,\mathrm{d}x + \int\limits_{B_r(z)} (\eta h_2(|\partial_2 u|))^2\,\mathrm{d}x \\ &\le c\left[\int\limits_{B_r(z)} |\nabla \eta| h_i(|\partial_i u|)\,\mathrm{d}x + \int\limits_{B_r(z)} h_1'(|\partial_1 u|)|\nabla \partial_1 u|\,\mathrm{d}x + \int\limits_{B_r(z)} h_2'(|\partial_2 u|)|\nabla \partial_2 u|\,\mathrm{d}x\right]^2\end{aligned}$$

$$\le c(R-\rho)^{-2}\left[\int_{B_R(z)} H(\nabla u)\,\mathrm{d}x\right]^2$$
$$+c\left[\int_{B_r(z)} h_1'(|\partial_1 u|)|\nabla\partial_1 u|\,\mathrm{d}x+\int_{B_r(z)} h_2'(|\partial_2 u|)|\nabla\partial_2 u|\,\mathrm{d}x\right]^2$$
$$\le c(\Omega^*)(R-\rho)^{-2}+c[\dots]^2.$$

In $[\dots]^2$ we can apply Hölder's inequality to get (again using (1.9))

$$[\dots]^2\le\int_{B_r(z)}\frac{h_1'(|\partial_1 u|)}{|\partial_1 u|}|\nabla\partial_1 u|^2\,\mathrm{d}x\int_{B_r(z)}|\partial_1 u|h_1'(|\partial_1 u|)\,\mathrm{d}x$$
$$+\int_{B_r(z)}\frac{h_2'(|\partial_2 u|)}{|\partial_2 u|}|\nabla\partial_2 u|^2\,\mathrm{d}x\int_{B_r(z)}|\partial_2 u|h_2'(|\partial_2 u|)\,\mathrm{d}x$$
$$\le c\int_{B_r(z)}H(\nabla u)\,\mathrm{d}x\left\{\int_{B_r(z)}\frac{h_1'(|\partial_1 u|)}{|\partial_1 u|}|\nabla\partial_1 u|^2\,\mathrm{d}x+\int_{B_r(z)}\frac{h_2'(|\partial_2 u|)}{|\partial_2 u|}|\nabla\partial_2 u|^2\,\mathrm{d}x\right\}$$
$$\le c(\Omega^*)\{\dots\}.$$

If we use the first inequality in (1.10) with the choices $Z=\nabla u$ and $Y=\partial_1\nabla u,\partial_2\nabla u$, then

$$\{\dots\}\le\int_{B_r(z)}D^2H(\nabla u)(\partial_\alpha\nabla u,\partial_\alpha\nabla u)\,\mathrm{d}x,$$

and from (2.5) we finally deduce

$$\int_{B_\rho(z)}\left(h_1(|\partial_1 u|)^2+h_2(|\partial_2 u|)^2\right)\,\mathrm{d}x$$
$$\le c(\Omega^*)(R-\rho)^{-2}+c(\Omega^*,\lambda)(R-r)^{-\beta}+c(\Omega^*)\lambda^2\int_{B_R(z)}\left(h_1(|\partial_1 u|)^2+h_2(|\partial_2 u|)^2\right)\,\mathrm{d}x.$$

Since $R-r=\frac{1}{2}(R-\rho)$ and since we may assume that $\beta\ge 2$, the above inequality implies after appropriate choice of λ

$$\int_{B_\rho(z)}\left(h_1(|\partial_1 u|)^2+h_2(|\partial_2 u|)^2\right)\,\mathrm{d}x\le c(\Omega^*)(R-\rho)^{-\beta}+\frac{1}{2}\int_{B_R(z)}\left(h_1(|\partial_1 u|)^2+h_2(|\partial_2 u|)^2\right)\mathrm{d}x,$$

which means (see [25, Lemma 3.1, p. 161]) that $h_1(|\partial_1 u|)^2+h_2(|\partial_2 u|)^2$ is in the space $L^1_{loc}(\Omega)$ (uniformly w.r.t. the "hidden" approximation). But then (2.5) shows the same for $D^2H(\nabla u)(\partial_\alpha\nabla u,\partial_\alpha\nabla u)$ and as remarked before (2.2) this yields $u\in W^2_{2,loc}(\Omega;\mathbb{R}^M)$ (again uniform w.r.t. the approximation). Sobolev's theorem finally implies part (i) of Theorem 1.1.

For proving (ii) we proceed similar to Theorem 1, (c) in [9] by reducing the situation to a "lemma on higher integrability" established in [12]. With $\eta\in C_0^\infty(\Omega)$ and $P\in\mathbb{R}^{2M}$ we have

$$0 = \int_{\Omega} \partial_\alpha [DH(\nabla u)] : \nabla(\eta^2 \partial_\alpha [u - P(x)])\, dx$$

and from this equation we obtain

$$\int_{\Omega} \eta^2 \Phi^2\, dx = -2 \int_{\Omega} \eta D^2 H(\nabla u)(\partial_\alpha \nabla u, \partial_\alpha [u - P(x)] \otimes \nabla \eta)\, dx, \tag{2.6}$$

where we have abbreviated

$$\Phi := D^2 H(\nabla u)(\partial_\alpha \nabla u, \partial_\alpha \nabla u)^{\frac{1}{2}}.$$

Note that by the foregoing calculations Φ is in the space $L^2_{loc}(\Omega)$. On the r.h.s. of (2.6) we apply the Cauchy-Schwarz inequality to the bilinear form $D^2 H(\nabla u)$ and get from (2.6) after choosing η s.t. $\eta \equiv 1$ on $B_r(z_0)$, $0 \le \eta \le 1$, $\operatorname{spt} \eta \subset B_{2r}(z_0)$, $|\nabla \eta| \le c/r$ for a disc $B_{2r}(z_0) \subset \Omega^* \Subset \Omega$

$$\int_{B_r(z_0)} \Phi^2\, dx \le \frac{c}{r} \int_{B_{2r}(z_0)} \Phi |D^2 H(\nabla u)|^{\frac{1}{2}} |\nabla u - P|\, dx. \tag{2.7}$$

The second inequality in (1.10) shows

$$\begin{aligned} |D^2 H(\nabla u)|^{\frac{1}{2}} &\le c \left[(1 + |\partial_1 u|^2)^{\frac{\omega}{4}} \sqrt{\frac{h_1'(|\partial_1 u|)}{|\partial_1 u|}} + (1 + |\partial_2 u|^2)^{\frac{\omega}{4}} \sqrt{\frac{h_2'(|\partial_2 u|)}{|\partial_2 u|}} \right] \\ &=: c \left[\tilde{\psi}_1 + \tilde{\psi}_2 \right], \end{aligned}$$

and if we let $\tilde{\psi} := (\tilde{\psi}_1^2 + \tilde{\psi}_2^2)^{1/2}$, then exactly the same arguments leading to (30) in [9] enable us to derive from (2.7) the inequality

$$\left[\fint_{B_r(z_0)} \Phi^2\, dx \right]^{\frac{1}{2}} \le c \left[\fint_{B_{2r}(x_0)} (\tilde{\psi} \Phi)^{\frac{4}{3}}\, dx \right]^{\frac{3}{4}}. \tag{2.8}$$

Note that during the proof of (2.8) one needs the information that $|\nabla^2 u| \le c\Phi \le c\Phi\tilde{\psi}$ which follows from our assumptions concerning h_1, h_2. In order to proceed as in [9] we have to check that

$$\exp(\beta \tilde{\psi}^2) \in L^1_{loc}(\Omega^*) \tag{2.9}$$

is true for any $\beta > 0$. Let us define

$$\psi_1 := \int_0^{|\partial_1 u|} \sqrt{\frac{h_1'(t)}{t}}\, dt, \quad \psi_2 := \int_0^{|\partial_2 u|} \sqrt{\frac{h_2'(t)}{t}}\, dt.$$

The first inequality in (1.10) shows

$$|\nabla \psi_1|^2 + |\nabla \psi_2|^2 \le c\Phi^2,$$

so that ψ_1 and ψ_2 belong to $W^1_{2,loc}(\Omega)$ and therefore $\psi := (\psi_1^2 + \psi_2^2)^{1/2}$ is in the same space. By Trudinger's inequality (see Theorem 7.15 of [27]) we find $\beta_0 > 0$ s.t. for discs $B_\rho \subset \Omega^*$ we have

$$\int_{B_\rho} \exp(\beta_0 \psi^2)\,\mathrm{d}x \le c(\rho). \tag{2.10}$$

We have a.e. on $[|\partial_1 u| \ge 1]$ (recalling (1.9))

$$\begin{aligned}\tilde{\psi}_1 &\le c|\partial_1 u|^{\frac{\omega}{2}} \sqrt{\frac{h_1'(|\partial_1 u|)}{|\partial_1 u|}} = c|\partial_1 u|^{\frac{\omega}{2}-1} \left(|\partial_1 u| h_1'(|\partial_1 u|)\right)^{\frac{1}{2}}\\ &\le c|\partial_1 u|^{\frac{\omega}{2}-1} h_1(|\partial_1 u|)^{\frac{1}{2}},\end{aligned}$$

whereas

$$\psi_1 \ge \int_{|\partial_1 u|/2}^{|\partial_1 u|} \sqrt{\frac{h_1'(t)}{t}}\,\mathrm{d}t \ge c h_1(|\partial_1 u|)^{\frac{1}{2}}$$

[see (A2), (1.9)], hence $\tilde{\psi}_1 \le c|\partial_1 u|^{\frac{\omega}{2}-1}\psi_1$ on $[|\partial_1 u| \ge 1]$. At the same time it holds

$$\psi_1 \le c h_1(|\partial_1 u|)^{\frac{1}{2}} \le c|\partial_1 u|^{\frac{m}{2}},$$

and for small δ we obtain

$$\tilde{\psi}_1 \le c\psi_1^{1-\delta} |\partial_1 u|^{\frac{\omega}{2}-1+\delta\frac{m}{2}} \tag{2.11}$$

on $[|\partial_1 u| \ge 1]$. Since we assume $\omega < 2$ in part (ii) of Theorem 1.1, we can fix δ s.t. we have $\frac{\omega}{2} - 1 + \delta\frac{m}{2} < 0$. Young's inequality applied on the r.h.s. of (2.11) then gives for any $\mu > 0$

$$\tilde{\psi}_1^2 \le \mu\psi_1^2 + c(\mu) \quad \text{on} \quad [|\partial_1 u| \ge 1]. \tag{2.12}$$

On $[|\partial_1 u| \le 1]$ we just observe

$$\tilde{\psi}_1^2 \le c \le c + \psi_1 \le \mu\psi_1^2 + c(\mu),$$

hence the inequality (2.12) holds on Ω, and obviously the same arguments apply to ψ_2, $\tilde{\psi}_2$. This shows

$$\tilde{\psi}^2 \le \mu\psi^2 + c(\mu) \quad \text{a.e. on } \Omega \tag{2.13}$$

for any $\mu > 0$. Let us fix $\beta > 0$. Then (by (2.13))

$$\int_{B_\rho} \exp(\beta\tilde{\psi}^2)\,\mathrm{d}x \le c(\mu,\beta) \int_{B_\rho} \exp(\beta\mu\psi^2),$$

and if we choose $\mu = \beta_0/\beta$, then the desired claim (2.9) follows from (2.10). Now we can complete the proof of Theorem 1.1, (ii) as done in [9]. □

3 Proof of Theorem 1.2

Let us first assume that u is sufficiently regular so that we do not have to argue with solutions of regularized problems or with difference quotients.

Then, with the notation $H = H(x, P)$, the counterpart of (2.1) reads as

$$\int_\Omega D_p^2 H(\cdot, \nabla u)(\partial_\alpha \nabla u, \partial_\alpha \nabla u)\eta^2 \, dx + \int_\Omega \left[\frac{\partial}{\partial x_\alpha} D_p H\right](\cdot, \nabla u) : \eta^2 \partial_\alpha \nabla u \, dx$$
$$= \int_\Omega D_p H(\cdot, \nabla u) : \partial_\alpha(\nabla \eta^2 \otimes \partial_\alpha u) \, dx,$$

where the second term on the l.h.s. is the new one. However, due to assumption (A4), the behavior of this term is of the same quality as of the r.h.s. and we therefore have (2.5). The next step in Sect. 2 is to make use of Sobolev's inequality, which in the non-autonomous case just gives uncritical new terms and as before we arrive at $|\nabla u| \in L^t_{loc}(\Omega)$ for all $t < \infty$.

Now, following the arguments of Sect. 2 leading to part (ii) of Theorem 1.1, we again obtain some extra terms in the non-autonomous case under consideration.

But in Section 4 of [12] and Section 6 of [8] it is described in detail how these extra terms can be handled leading to a generalized version of (2.8) to which Lemma 1.2 of [12] still is applicable. Thus, as sketched in Sect. 2, the proof of Theorem 1.2 would be complete if our "smoothness assumption" can be guaranteed.

As outlined in [21] the usual local regularization procedure cannot be applied, which means that if we fix a disk B compactly contained in Ω and consider the mollification $(u)_\varepsilon$ of our local minimizer, then the convergence

$$\int_B H(x, \nabla(u)_\varepsilon) \, dx \to \int_B H(x, \nabla u) \, dx \quad \text{as} \quad \varepsilon \to 0 \tag{3.1}$$

may fail to hold due to the possibility of the occurrence of Lavrentiev's phenomenon. In the autonomous case (3.1) easily follows from Jensen's inequality and enables us to study the regularized problems (as done in Sect. 2)

$$\int_B H_\delta(\nabla w) \, dx \to \min \quad \text{in} \ (u)_\varepsilon + \overset{\circ}{W}{}^1_{\bar{q}}(B; \mathbb{R}^M),$$

where $H_\delta = \delta(1 + |\cdot|^2)^{\bar{q}/2} + H$ with $\bar{q}$ from (1.11) and $\delta = \delta(\varepsilon)$ being defined in a suitable way. In fact, (3.1) is the key ingredient for proving that the (regular) solutions u_δ of the auxiliary problems converge towards our local minimizer u on the disk B so that all uniform estimates obtained for the sequence $\{u_\delta\}$ finally continue to hold for u.

In the non-autonomous case we now follow ideas of Marcellini [29] and of Cupini et al. [19] by introducing a "regularization from below", which means the integrand $H(x, P)$ is replaced by an appropriate sequence $H_\Theta(x, P)$ of integrands having quadratic growth w.r.t. $P \in \mathbb{R}^{2M}$ and s.t. $H_\Theta(x, P) \uparrow H(x, P)$ as $\Theta \to \infty$. We note that a related type of approximations also occurs in Section 3 of [6] but we cannot refer to this since now the setting is different.

Let us pass to the details by recalling that $H(x, P) = h_1(x, |P_1|) + h_2(x, |P_2|)$, where $h(x, t) := h_i(x, t)$, $i = 1, 2$, satisfies (A1)–(A4). Since the approximation procedure is done w.r.t. the second variable t we just write $h(t)$. Then we define

$$g(t) := \frac{h'(t)}{t}, \quad \text{i.e.} \quad h(t) = \int_0^t s g(s)\, \mathrm{d}s,$$

and by (A1) and (A3) g is increasing and satisfies $g(0) = h''(0) > 0$. Now we fix $\Theta > 0$ and consider $\eta = \eta_\Theta \in C^1([0,\infty))$ s.t. $0 \le \eta \le 1$, $\eta' \le 0$, $|\eta'| \le c/\Theta$, $\eta \equiv 1$ on $[0, 3\Theta/2]$ and $\eta \equiv 0$ on $[2\Theta, \infty)$. Moreover, for all $t \ge 0$ we let

$$g_\Theta(t) := g(0) + \int_0^t \eta(s) g'(s)\, \mathrm{d}s \le g(t),$$

$$h_\Theta(t) := \int_0^t s g_\Theta(s)\, \mathrm{d}s \le h(t).$$

We claim the validity of (A1)–(A4) for the functions h_Θ with constants being independent of Θ.

The properties $h_\Theta \in C^2([0,\infty))$, $h_\Theta(t) = h(t)$ for all $t \le 3\Theta/2$ and $\lim_{\Theta\to\infty} h_\Theta(t) = h(t)$ for any fixed $t \in [0,\infty)$ are easily verified, and it is immediate that (A1) holds for h_Θ.

Ad (A3). We have for all $t \ge 0$

$$h''_\Theta(t) = g_\Theta(t) + t g'_\Theta(t) \ge g_\Theta(t) = \frac{h'_\Theta(t)}{t},$$

hence the first inequality of (A3) is true. For proving the second one we observe that $g_\Theta(t) = h'_\Theta(t)/t$ gives

$$\begin{aligned} h''_\Theta(t) &= \frac{h'_\Theta(t)}{t} + t g'_\Theta(t) = \frac{h'_\Theta(t)}{t} + t\eta(t) g'(t) = \frac{h'_\Theta(t)}{t} + t\eta(t)\frac{t h''(t) - h'(t)}{t^2} \\ &= \frac{h'_\Theta(t)}{t} + \eta(t)\left[h''(t) - \frac{h'(t)}{t}\right] \le \frac{h'_\Theta(t)}{t} + \eta(t) a (1+t^2)^{\frac{\omega}{2}} \frac{h'(t)}{t}, \end{aligned}$$

where the r.h.s. of (A3) for h and the non-negativity of $\eta(t)$, $h'(t)/t$ are used for the last estimate. The r.h.s. of (A3) for h_Θ then follows (with constant $1 + a$ and unchanged exponent) from

$$\eta(t)\frac{h'(t)}{t} = g(0) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(\eta(s) g(s))\, \mathrm{d}s = g(0) + \int_0^t \eta(s) g'(s)\, \mathrm{d}s + \int_0^t \eta'(s) g(s)\, \mathrm{d}s$$

together with the observation that the second integral on the r.h.s. is non-positive by the sign of η', i.e. we have

$$\eta(t)\frac{h'(t)}{t} \le g_\Theta(t) = \frac{h'_\Theta(t)}{t}.$$

Ad (A2). Here it is to show that h_Θ satisfies the (Δ_2)-property with a constant not depending on Θ. We first write

$$h_\Theta(2t) = \int_0^{2t} s g_\Theta(s)\, \mathrm{d}s = 4 \int_0^t s g_\Theta(2s)\, \mathrm{d}s \tag{3.2}$$

and

$$g_\Theta(s) = g(0) + \int_0^s \frac{\mathrm{d}}{\mathrm{d}u}(\eta g)\,\mathrm{d}u - \int_0^s \eta'(u)g(u)\,\mathrm{d}u$$
$$= \eta(s)g(s) - \int_0^s \eta'(u)g(u)\,\mathrm{d}u,$$

and from the monotonicity of η we deduce

$$g_\Theta(2s) = \eta(2s)g(2s) - \int_0^{2s} \eta'(u)g(u)\,\mathrm{d}u$$
$$\leq \eta(s)g(2s) + \int_0^s (-\eta'(u))g(u)\,\mathrm{d}u + \int_s^{2s} (-\eta'(u))g(u)\,\mathrm{d}u, \tag{3.3}$$

where both integrals on the r.h.s. have a positive sign. Now observe recalling (1.9) and (A2) (valid for h)

$$g(2s) = \frac{h'(2s)}{2s} = \frac{h'(2s)2s}{(2s)^2} \leq c\frac{h(2s)}{(2s)^2} \leq c\frac{h(s)}{s^2} \leq c\frac{h'(s)s}{s^2} = cg(s),$$

which gives using (3.3)

$$g_\Theta(2s) \leq c\left[\eta(s)g(s) + \int_0^s (-\eta'(u))g(u)\,\mathrm{d}u\right] + \int_s^{2s} (-\eta'(u))g(u)\,\mathrm{d}u$$
$$= cg_\Theta(s) + \int_s^{2s} (-\eta'(u))g(u)\,\mathrm{d}u.$$

Returning to (3.2) it is finally shown that

$$h_\Theta(2t) \leq c\underbrace{\int_0^t sg_\Theta(s)\,\mathrm{d}s}_{=h_\Theta(t)} + 4\underbrace{\int_0^t s\int_s^{2s} (-\eta'(u))g(u)\,\mathrm{d}u\,\mathrm{d}s}_{=:\xi}, \tag{3.4}$$

with ξ satisfying

$$\xi \leq c\int_0^t s\int_s^{2s} \frac{1}{\Theta}\chi_{[3\Theta/2,2\Theta]}(u)g(u)\,\mathrm{d}u\,\mathrm{d}s$$
$$\leq c\int_0^t s\frac{1}{\Theta}g(2s)\,|[3\Theta/2, 2\Theta] \cap [s, 2s]|\,\mathrm{d}s =: \xi^*.$$

In case that $t \leq 3\Theta/4$ we have $2s \leq \frac{3\Theta}{2}$ for all $s \in [0, t]$, hence ξ^* vanishes and we have $h_\Theta(2t) \leq ch_\Theta(t)$ on account of (3.4).

If $t \in [3\Theta/4, 2\Theta]$, then (1.9) and (A2) give

$$\xi^* \le c \int_0^{2\Theta} s \frac{1}{\Theta} g(2s)\Theta \, \mathrm{d}s \le c\Theta^2 g(4\Theta) \le ch'(4\Theta)4\Theta \le ch(4\Theta) \le ch(\Theta/2)$$
$$= ch_\Theta(\Theta/2) \le ch_\Theta(t)$$

and again we are done.

Finally for $t > 2\Theta$ we note

$$\begin{aligned}
\xi^* &= c \int_{3\Theta/4}^{t} s \frac{1}{\Theta} g(2s) \, |[3\Theta/2, 2\Theta] \cap [s, 2s]| \, \mathrm{d}s \\
&= c \int_{3\Theta/4}^{2\Theta} s \frac{1}{\Theta} g(2s) \, |[3\Theta/2, 2\Theta] \cap [s, 2s]| \, \mathrm{d}s + \underbrace{c \int_{2\Theta}^{t} s \frac{1}{\Theta} g(2s) \, |[3\Theta/2, 2\Theta] \cap [s, 2s]| \, \mathrm{d}s}_{=0} \\
&\le c\Theta^2 g(4\Theta),
\end{aligned}$$

and as in the second case we have $\Theta^2 g(4\Theta) \le ch(\Theta) = ch_\Theta(\Theta) \le ch_\Theta(t)$. This finally proves (A2) for h_Θ with a uniform constant.

Ad (A4). Returning to the full notation and recalling the definition of $h_\Theta = h_\Theta(x, t)$ we have

$$\left| \nabla_x \frac{\partial}{\partial t} h_\Theta(x, t) \right| = \left| t \nabla_x \left[g(x, 0) + \int_0^t \eta(s) \frac{\partial}{\partial s} g(x, s) \, \mathrm{d}s \right] \right|$$

and

$$\begin{aligned}
\nabla_x \left[g(x, 0) + \int_0^t \eta(s) \frac{\partial}{\partial s} g(x, s) \, \mathrm{d}s \right] &= \nabla_x \left[\eta(t) g(x, t) - \int_0^t \eta'(s) g(x, s) \, \mathrm{d}s \right] \\
&= \eta(t) \nabla_x g(x, t) - \int_0^t \eta'(s) \nabla_x g(x, s) \, \mathrm{d}s \\
&= \eta(t) \frac{1}{t} \nabla_x \frac{\partial}{\partial t} h(x, t) - \int_0^t \eta'(s) \frac{1}{s} \nabla_x \frac{\partial}{\partial s} h(x, s) \, \mathrm{d}s.
\end{aligned}$$

The sign of η' implies together with (A4)

$$\begin{aligned}
\left| \nabla_x \frac{\partial}{\partial t} h_\Theta(x, t) \right| &\le \eta(t) \left| \nabla_x \frac{\partial}{\partial t} h(x, t) \right| - t \int_0^t \eta'(s) \frac{1}{s} \left| \nabla_x \frac{\partial}{\partial s} h(x, s) \right| \mathrm{d}s \\
&\le c \left[\eta(t) \frac{\partial}{\partial t} h(x, t) - t \int_0^t \eta'(s) \frac{1}{s} \frac{\partial}{\partial s} h(x, s) \, \mathrm{d}s \right]
\end{aligned}$$

$$= c\left[g(x,0) + \int_0^t \eta(s)\frac{\partial}{\partial s} g(x,s)\,\mathrm{d}s \right] t$$
$$= c\frac{\partial}{\partial t} h_\Theta(x,t),$$

which is (A4) uniformly for $h_\Theta(x,t)$. □

We now let (with an obvious meaning of $h_{i,\Theta}$, $i = 1, 2$)

$$H_\Theta(x,P) := h_{1,\Theta}(x,|P_1|) + h_{2,\Theta}(x,|P_2|), \quad x \in \overline{\Omega}, \quad P = (P_1, P_2) \in \mathbb{R}^{2M},$$

and observe $H_\Theta \le H$ as well as

$$\lim_{\Theta\to\infty} H_\Theta(x,P) = H(x,P).$$

Moreover, we have the ellipticity estimate (in the sense of bilinear forms)

$$c \le D^2 H_\Theta(x,P) \le \Lambda(\Theta), \quad c := \inf_{x\in\overline{\Omega}} \min_{i=1,2} \frac{\partial^2 h_i}{\partial t^2}(x,0) > 0,$$

which follows from (A3) and the definition of H_Θ. Therefore H_Θ is of quadratic growth, and since u belongs to the class $W^1_{2,loc}(\Omega;\mathbb{R}^M)$, the problem

$$\int_B H_\Theta(x,\nabla w)\,\mathrm{d}x \to \min \quad \text{in } u + \overset{\circ}{W}{}^1_2(B;\mathbb{R}^M)$$

admits a unique solution u_Θ on each fixed disk $B \Subset \Omega$, whose interior differentiability can be deduced from Campanato's work [16], comments after Theorem 3, which clearly extends to the non-autonomous case. Alternatively, the smoothness of u_Θ follows from the results in Section 6 of [8]. Thus we can carry out the calculations described at the beginning of this section for the functions u_Θ with the results ($B' \Subset B$)

$$\|\nabla u_\Theta\|_{L^t(B')} \le c\left(t, B', \int_B H(x,\nabla u)\,\mathrm{d}x\right) < \infty \tag{3.5}$$

for any finite t, and – assuming $\omega < 2$ –

$$\|u_\Theta\|_{C^{1,\alpha}(B')} \le c\left(\alpha, B', \int_B H(x,\nabla u)\,\mathrm{d}x\right) < \infty \tag{3.6}$$

for all $\alpha \in (0,1)$, where of course the constant c on the r.h.s. of (3.5) and (3.6) also depends on the uniform constants occurring in (A1)–(A4). From the construction it is immediate that

$$\sup_\Theta \|u_\Theta\|_{W^1_2(B)} < \infty,$$

hence $u_\Theta \rightharpoonup \bar{u}$ in $W^1_2(B;\mathbb{R}^M)$ for some function $\bar{u} \in u + \overset{\circ}{W}{}^1_2(B;\mathbb{R}^M)$. We claim that $\bar{u} = u$. Let $\Theta \ge 2k$. Then $\eta_\Theta \ge \eta_k$, hence $h_{i,\Theta} \ge h_{i,k}$ and in conclusion $H_\Theta \ge H_k$. For k fixed the lower semicontinuity of H_k implies

$$\int_B H_k(\cdot,\nabla\bar{u})\,\mathrm{d}x \le \liminf_{\Theta\to\infty} \int_B H_k(\cdot,\nabla u_\Theta)\,\mathrm{d}x,$$

and at the same time

$$\liminf_{\Theta\to\infty} \int_B H_k(\cdot, \nabla u_\Theta)\,\mathrm{d}x \le \liminf_{\Theta\to\infty} \int_B H_\Theta(\cdot, \nabla u_\Theta)\,\mathrm{d}x.$$

The minimality of u_Θ shows

$$\int_B H_\Theta(\cdot, \nabla u_\Theta)\,\mathrm{d}x \le \int_B H_\Theta(\cdot, \nabla u)\,\mathrm{d}x \le \int_B H(\cdot, \nabla u)\,\mathrm{d}x,$$

hence

$$\int_B H_k(\cdot, \nabla \bar{u})\,\mathrm{d}x \le \int_B H(\cdot, \nabla u)\,\mathrm{d}x,$$

and Fatou's lemma implies

$$\int_B H(\cdot, \nabla \bar{u})\,\mathrm{d}x \le \int_B H(\cdot, \nabla u)\,\mathrm{d}x,$$

from which our claim follows. Since (3.5) and (3.6) obviously extend to the weak limit $\bar{u}$, the proof of Theorem (1.2) is complete. □

4 Proof of Theorem 1.3

For symmetric (2×2)-matrices ε we write

$$H(\varepsilon) = h_1(|\mathrm{tr}\,\varepsilon|) + h_2(|\varepsilon^D|)$$

and obtain for any $\psi \in C_0^\infty(\Omega; \mathbb{R}^2)$

$$0 = \int_\Omega \partial_\alpha(DH(\varepsilon(u))) : \varepsilon(\psi)\,\mathrm{d}x,$$

in particular by letting $\psi = \eta^2 \partial_\alpha u$, $\eta \in C_0^\infty(\Omega)$,

$$0 = \int_\Omega \partial_\alpha(DH(\varepsilon(u))) : \varepsilon(\eta^2 \partial_\alpha u)\,\mathrm{d}x, \tag{4.1}$$

where again the sum is taken w.r.t. indices repeated twice. Here we remark that (4.1) can be justified along the same lines as inequality (10) in [7]. Alternatively we may use a regularization from below as done in the previous section. (4.1) yields

$$\int_\Omega \eta^2 D^2 H(\varepsilon(u))(\partial_\alpha \varepsilon(u), \partial_\alpha \varepsilon(u))\,\mathrm{d}x = \int_\Omega DH(\varepsilon(u)) : \partial_\alpha[\nabla\eta^2 \odot \partial_\alpha u]\,\mathrm{d}x, \tag{4.2}$$

"⊙" being the symmetric product of vectors. We remark that by (1.10)

$$\frac{h_1'(|\mathrm{div}\, u|)}{|\mathrm{div}\, u|}|\nabla \mathrm{div}\, u|^2 + \frac{h_2'(|\varepsilon^D(u)|)}{|\varepsilon^D(u)|}|\nabla \varepsilon^D(u)|^2 \le D^2 H(\varepsilon(u))(\partial_\alpha \varepsilon(u), \partial_\alpha \varepsilon(u)), \tag{4.3}$$

and using the inequality

$$|\nabla^2 u| \le c|\nabla \varepsilon(u)| \le c\left[|\nabla \mathrm{div}\, u| + |\nabla \varepsilon^D(u)|\right]$$

as well as the properties of h_i, we see that $|\nabla^2 u|^2$ is bounded by the l.h.s. of (4.3), hence

$$|\nabla^2 u|^2 \leq c D^2 H(\varepsilon(u))(\partial_\alpha \varepsilon(u), \partial_\alpha \varepsilon(u)). \tag{4.4}$$

As in Sect. 2 we estimate

$$\begin{aligned} |\text{r.h.s. of } (4.2)| &\leq c \left[\int_\Omega \eta |\nabla \eta| |\nabla^2 u| \left(h_1'(|\operatorname{div} u|) + h_2'(|\varepsilon^D(u)|) \right) \mathrm{d}x \right. \\ &\qquad \left. + \int_\Omega |\nabla^2 \eta^2| |\nabla u| \left(h_1'(|\operatorname{div} u|) + h_2'(|\varepsilon^D(u)|) \right) \mathrm{d}x \right] \\ &\leq \tau \int_\Omega \eta^2 |\nabla^2 u|^2 \,\mathrm{d}x + \int_\Omega \left(c(\tau)|\nabla \eta|^2 + c|\nabla^2 \eta^2| \right) \\ &\qquad \times \left(h_1'(|\operatorname{div} u|)^2 + h_2'(|\varepsilon^D(u)|)^2 \right) \mathrm{d}x + c \int_\Omega |\nabla^2 \eta^2| |\nabla u|^2 \,\mathrm{d}x, \end{aligned}$$

where we have used Young's inequality and where $\tau > 0$ is arbitrary. We note that due to the growth of h_i we have $H(\varepsilon(u)) \geq c|\varepsilon(u)|^2$, and therefore Korn's inequality shows that $|\nabla u| \in L^2_{loc}(\Omega)$. Choosing τ sufficiently small and quoting (4.4), we deduce from (4.2), (4.4) and the above estimates the following variant of (2.4)

$$\begin{aligned} &\int_{B_r(z)} D^2 H(\varepsilon(u))(\partial_\alpha \varepsilon(u), \partial_\alpha \varepsilon(u)) \,\mathrm{d}x \\ &\quad \leq (R - r)^{-2} \left[c(\Omega^*) + c \int_{B_R(z)} \left(h_1'(|\operatorname{div} u|)^2 + h_2'(|\varepsilon^D(u)|)^2 \right) \mathrm{d}x \right], \end{aligned}$$

and exactly the same arguments as applied after (2.4) turn this inequality into the appropriate version of (2.5). Using the same notation as after (2.5) we obtain

$$\begin{aligned} &\int_{B_\rho(z)} \left(h_1(|\operatorname{div} u|)^2 + h_2(|\varepsilon^D(u)|)^2 \right) \mathrm{d}x \\ &\quad \leq c(\Omega^*)(R - \rho)^{-2} + c \left[\int_{B_r(z)} h_1'(|\operatorname{div} u|) |\nabla \operatorname{div} u| \,\mathrm{d}x + \int_{B_r(z)} h_2'(|\varepsilon^D(u)|) |\nabla \varepsilon^D(u)| \,\mathrm{d}x \right]^2 \end{aligned}$$

and with Hölder's inequality it follows

$$\begin{aligned} [\ldots]^2 &\leq c(\Omega^*) \int_{B_r(z)} \left(\frac{h_1'(|\operatorname{div} u|)}{|\operatorname{div} u|} |\nabla \operatorname{div} u|^2 + \frac{h_2'(|\varepsilon^D(u)|)}{|\varepsilon^D(u)|} |\nabla \varepsilon^D(u)|^2 \right) \mathrm{d}x \\ &\leq c(\Omega^*) \int_{B_r(z)} D^2 H(\varepsilon(u))(\partial_\alpha \varepsilon(u), \partial_\alpha \varepsilon(u)) \,\mathrm{d}x, \end{aligned}$$

where (4.3) is used to derive the latter estimate. As in Sect. 2 this gives

$$h_1(|\operatorname{div} u|), \quad h_2(|\varepsilon^D(u)|) \in L^2_{loc}(\Omega),$$

hence $|\nabla\varepsilon(u)| \in L^2_{loc}(\Omega)$ and Korn's inequality shows $u \in W^2_{2,loc}(\Omega;\mathbb{R}^2)$. This proves that $|\nabla u| \in L^t_{loc}(\Omega)$ for any finite t. Let us consider the case $\omega < 2$. (2.7) reads as

$$\int_{B_r(z_0)} \Phi^2 \,\mathrm{d}x \le \frac{c}{r} \int_{B_{2r}(z_0)} \Phi |DH(\varepsilon(u))||\nabla u - P| \,\mathrm{d}x,$$

$$\Phi := D^2H(\varepsilon(u))(\partial_\alpha\varepsilon(u), \partial_\alpha\varepsilon(u))^{\frac{1}{2}}.$$

The auxiliary functions $\psi_1, \dots$ have to be modified in an obvious way, and during the calculations leading now to (2.8) we again benefit from the fact that $|\nabla^2 u| \le c|\nabla\varepsilon(u)|$. (2.9) then follows without further changes, and the proof of $C^{1,\alpha}$-regularity can be completed by repeating the arguments after (15) in [7]. □

Appendix A: An example of a function h with (A1)–(A3)

Here we are going to construct an example of a function h with (A1)–(A3) based on ideas already used in Sect. 3: let g be a function $[0,\infty) \to [0,\infty)$ of class C^1 satisfying $g(0) > 0$ and $g'(t) \ge 0$ for all $t \ge 0$. Then we immediately have (A1) for the function

$$h(t) := \int_0^t s g(s) \,\mathrm{d}s.$$

For the first inequality in (A3) we just observe $h''(t) \ge g(t) = h'(t)/t$. Given $\omega \ge 0$, the second inequality of (A3) is satisfied if and only if

$$t g'(t) \le c t^\omega g(t) \quad \text{for all} \quad t \gg 1. \tag{A.1}$$

Now suppose that we have sequences of positive numbers $\{a_i\}$, $\{\varepsilon_i\}$ s.t. $g' = 0$ outside the union of the intervals $I_i = (a_i - \varepsilon_i, a_i + \varepsilon_i)$, $g'(a_i) = a_i^{\omega-1}$ and g' is linear on $(a_i - \varepsilon_i, a_i)$ as well as on $(a_i, a_i + \varepsilon_i)$. Moreover, it is supposed that $a_i \to \infty$ and that the intervals I_i are disjoint. Then we have

$$\begin{aligned} g(t) &= g(0) + \int_0^t g'(s) \,\mathrm{d}s \le g(0) + \int_0^\infty g'(s)\,\mathrm{d}s \\ &\approx g(0) + \sum_{i=1}^\infty \varepsilon_i a_i^{\omega-1} \end{aligned} \tag{A.2}$$

and with an appropriate choice of ε_i the r.h.s. of (A.2) is bounded. This clearly implies the validity of (A.1), and it is not possible to replace ω by an exponent $\tilde{\omega} < \omega$ in (A.1). If the r.h.s. of (A.2) is bounded, i.e.

$$0 < g(0) \le g(t) \le c,$$

then we have

$$g(2t) \le \frac{c}{g(0)} g(t)$$

and by the definition of h

$$h(2t) = \int_0^{2t} sg(s)\,\mathrm{d}s = 4\int_0^{t} sg(2s)\,\mathrm{d}s \le c\int_0^{t} sg(s)\,\mathrm{d}s = ch(t),$$

i.e. the function h has the (Δ_2) property.

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