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Calculus of Variations

Affine Moser–Trudinger and Morrey–Sobolev inequalities

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Abstract An affine Moser–Trudinger inequality, which is stronger than the Euclidean Moser–Trudinger inequality, is established. In this new affine analytic inequality an affine energy of the gradient replaces the standard L^n energy of gradient. The geometric inequality at the core of the affine Moser–Trudinger inequality is a recently established affine isoperimetric inequality for convex bodies. Critical use is made of the solution to a normalized version of the L^n Minkowski Problem. An affine Morrey–Sobolev inequality is also established, where the standard L^p energy, with $p > n$, is replaced by the affine energy.

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1 Introduction and main results

The standard Sobolev inequality in $\mathbb{R}^n$, $n \geq 2$, for $p \in [1, n)$, provides an upper bound for the $L^{\frac{np}{n-p}}(\mathbb{R}^n)$ norm of functions f from the Sobolev space $W^{1,p}(\mathbb{R}^n)$ in terms of the $L^p(\mathbb{R}^n)$

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norm of the Euclidean length of their gradient,

$$\|\nabla f\|_p = \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p dx \right)^{1/p}. \tag{1.1}$$

An optimal form of the Sobolev inequality, with sharp constant, reads

$$a_{p,n}\|f\|_{\frac{np}{n-p}} \leq \|\nabla f\|_p, \quad \text{for } f \in W^{1,p}(\mathbb{R}^n), \tag{1.2}$$

and goes back to Federer and Fleming [21] and Maz'ya [42] for $p = 1$, and to Aubin [5] and Talenti [52] for $1 < p < n$. Here, $a_{p,n} = n^{\frac{1}{p}}(\frac{n-p}{p-1})^{1-\frac{1}{p}}[\omega_n \Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})/\Gamma(n)]^{\frac{1}{n}}$, where Γ is the gamma function, and $\omega_n = \pi^{\frac{n}{2}}/\Gamma\left(1+\frac{n}{2}\right)$ is the n-dimensional volume enclosed by the unit sphere $\mathbf{S}^{n-1}$.

In [38,56], a strengthened version of the sharp Sobolev inequality was established (see [41] for further extensions), where the customary norm $\|\nabla f\|_p$ is replaced by a new invariant (of functions) defined by

$$\mathcal{E}_p(f) = c_{n,p} \left(\int_{\mathbf{S}^{n-1}} \|D_v f\|_p^{-n} dv \right)^{-1/n}. \tag{1.3}$$

Here, $c_{n,p} = (\frac{n\omega_n\omega_{p-1}}{2\omega_{n+p-2}})^{1/p}(n\omega_n)^{1/n}$, and for each vector $v \in \mathrm{S}^{n-1}$ the expression $\|D_v f\|_p$ stands for the $L^p(\mathbb{R}^n)$ norm of the directional derivative $D_v f$ of f along v, namely

$$\|D_v f\|_p = \left(\int_{\mathbb{R}^n} |v \cdot \nabla f(x)|^p dx \right)^{1/p}, \tag{1.4}$$

where "$\cdot$" denotes the usual inner product in $\mathbb{R}^n$. An important fact is that $\mathcal{E}_p(f)$ is invariant under affine transformations of $\mathbb{R}^n$ of the form $x \mapsto Ax + x_0$, with $x_0 \in \mathbb{R}^n$ and $A \in \mathrm{SL}(n)$. Note that by contrast $\|\nabla f\|_p$ is invariant only for $A \in \mathrm{SO}(n)$, rather than $A \in \mathrm{SL}(n)$. We call the invariant $\mathcal{E}_p(f)$ the L^p *affine energy of* f.

The affine Sobolev inequality, established in [56] for $p = 1$ and in [38] for $1 < p < n$, states that

$$a_{p,n}\|f\|_{\frac{np}{n-p}} \leq \mathcal{E}_p(f), \quad \text{for } f \in W^{1,p}(\mathbb{R}^n). \tag{1.5}$$

Observe that (1.5) is an affine inequality whereas the Sobolev inequality (1.2) is only Euclidean. Inequality (1.5) improves the Sobolev inequality (1.2) because, by the Hölder inequality and Fubini's theorem, one easily sees that

$$\mathcal{E}_p(f) \leq \|\nabla f\|_p \tag{1.6}$$

for every $f \in W^{1,p}(\mathbb{R}^n)$ and $p \geq 1$ [38, Inequality (7.1)]. In fact, the affine Sobolev inequality (1.5) is essentially stronger than the Euclidean Sobolev inequality, because the ratio $\mathcal{E}_p(f)/\|\nabla f\|_p$ is not uniformly bounded from below by any positive constant, as f ranges in $W^{1,p}(\mathbb{R}^n)$. This is demonstrated, for instance, by considering functions f having the form $f(x) = \varphi(Ax)$ for some fixed function φ and letting A vary in $\mathrm{SL}(n)$.

It is the aim of this note to complete the picture of affine Sobolev inequalities given in [38,56], and deal with both the limiting case $p = n$ and super-limiting case $p > n$.

Functions $f \in W^{1,n}(\mathbb{R}^n)$, whose support sprt f has Lebesgue measure $|\text{sprt } f|$ that is finite, are known to be not merely in $L^q(\mathbb{R}^n)$ for every $q < \infty$, but to be even exponentially summable [46,54,55]. The Moser–Trudinger inequality is the statement that there exists $m_n > 0$ such that

$$\frac{1}{|\text{sprt } f|} \int_{\mathbb{R}^n} e^{(n\omega_n^{1/n}|f(x)|/\|\nabla f\|_n)^{n'}} dx \le m_n \tag{1.7}$$

for every $f \in W^{1,n}(\mathbb{R}^n)$ with $0 < |\text{sprt } f| < \infty$, where $n' = n/(n-1)$ is the Hölder conjugate of n. The constant $n\omega_n^{1/n}$ is best possible, in that inequality (1.7) would fail for any real number m_n if $n\omega_n^{1/n}$ were to be replaced by a larger number. Although not explicitly known, the best constant m_n on the right-hand side of (1.7) can be characterized as

$$m_n = \sup_{\phi} \int_0^{\infty} e^{\phi(t)^{n'} - t} dt, \tag{1.8}$$

where ϕ ranges among all non-decreasing locally absolutely continuous functions in $[0, \infty)$ such that $\phi(0) = 0$ and $\int_0^{\infty} \phi'(t)^n dt \le 1$.

The Moser–Trudinger inequality and close variants have attracted the attention of specialists in both the theory of function spaces and in partial differential equations; see e.g., [3–7,10,11,15,16,19,20,23,24,26,30,47].

Our first result deals with a stronger, in light of (1.6), affine version of (1.7):

Theorem 1.1 *Suppose $n > 1$. Then for every $f \in W^{1,n}(\mathbb{R}^n)$ with $0 < |\text{sprt } f| < \infty$,*

$$\frac{1}{|\text{sprt } f|} \int_{\mathbb{R}^n} e^{(n\omega_n^{1/n}|f(x)|/\mathcal{E}_n(f))^{n'}} dx \le m_n. \tag{1.9}$$

The constant $n\omega_n^{1/n}$ is best possible in that (1.9) would fail for any real number m_n if $n\omega_n^{1/n}$ were to be replaced by a larger number.

Carleson and Chang [10] proved that (spherically symmetric) extremals do exist for the Moser–Trudinger inequality (1.7). As a consequence, since, by (1.6),

$$\frac{1}{|\text{sprt } f|} \int_{\mathbb{R}^n} e^{(n\omega_n^{1/n}|f(x)|/\|\nabla f\|_n)^{n'}} dx \le \frac{1}{|\text{sprt } f|} \int_{\mathbb{R}^n} e^{(n\omega_n^{1/n}|f(x)|/\mathcal{E}_n(f))^{n'}} dx \le m_n \tag{1.10}$$

for each $f \in W^{1,n}(\mathbb{R}^n)$ with $0 < |\text{sprt } f| < \infty$, extremals for the affine Moser–Trudinger inequality (1.9) exist as well. Moreover, if f is an extremal for the Moser–Trudinger inequality (1.7), then not only is f also an extremal for the affine Moser–Trudinger inequality, but since we are dealing with an affine inequality, composing f with any element of $\mathrm{GL}(n)$ will also yield an extremal for the affine Moser–Trudinger inequality.

Let us now turn to the case when $p > n$. For these values of p, the Morrey–Sobolev embedding theorem tells us that any function from $W^{1,p}(\mathbb{R}^n)$ is essentially bounded. An optimal bound for $\|f\|_\infty$ in terms of $\|\nabla f\|_p$ is available, and states that

$$\|f\|_\infty \le b_{n,p} |\text{sprt } f|^{\frac{1}{n} - \frac{1}{p}} \|\nabla f\|_p \tag{1.11}$$

for every $f \in W^{1,p}(\mathbb{R}^n)$ such that $|\mathrm{sprt}\, f| < \infty$. Here, $b_{n,p} = n^{-\frac{1}{p}} \omega_n^{-\frac{1}{n}} (\frac{p-1}{p-n})^{1/p'}$, where p' is the Hölder conjugate of p. See Talenti [53]. The affine counterpart of (1.11) is contained in the following result.

Theorem 1.2 *If $p > n$, then for every $f \in W^{1,p}(\mathbb{R}^n)$ such that $|\mathrm{sprt}\, f| < \infty$,*

$$\|f\|_\infty \le b_{n,p} |\mathrm{sprt}\, f|^{\frac{1}{n}-\frac{1}{p}} \mathcal{E}_p(f). \tag{1.12}$$

Equality holds in (1.12) whenever

$$f(x) = a\left(1 - |A(x - x_0)|^{\frac{p-n}{p-1}}\right)_+ \tag{1.13}$$

for some $a \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, and $A \in \mathrm{GL}(n)$.

Here, the subscript "+" stands for the "positive part".

2 A symmetrization inequality for the affine energy

A key tool in our approach to Theorems 1.1 and 1.2 is an affine version of the Pólya–Szegö principle regarding the decrease of gradient norms under symmetrization. Recall that, given any measurable function $f : \mathbb{R}^n \to \mathbb{R}$ such that $|\{x \in \mathbb{R}^n : |f(x)| > t\}| < \infty$ for every $t > 0$, its distribution function $\mu_f : (0, \infty) \to [0, \infty)$ is defined by

$$\mu_f(t) = |\{|f| > t\}| \quad \text{for } t > 0, \tag{2.1}$$

its decreasing rearrangement $f^* : [0, \infty) \to [0, \infty]$ is defined by

$$f^*(s) = \sup\{t > 0 : \mu_f(t) > s\} \quad \text{for } s \ge 0, \tag{2.2}$$

and its spherically symmetric rearrangement $f^\star : \mathbb{R}^n \to [0, \infty]$ is defined by

$$f^\star(x) = f^*(\omega_n |x|^n) \quad \text{for } x \in \mathbb{R}^n. \tag{2.3}$$

Note that

$$\mu_f = \mu_{f^*} = \mu_{f^\star}, \tag{2.4}$$

and hence

$$|\mathrm{sprt}\, f| = |\mathrm{sprt}\, f^*| = |\mathrm{sprt}\, f^\star|, \tag{2.5}$$

$$\|f\|_\infty = f^*(0) = \|f^\star\|_\infty, \tag{2.6}$$

and

$$\int_{\mathbb{R}^n} \Phi(|f(x)|)dx = \int_0^\infty \Phi(f^*(s))ds = \int_{\mathbb{R}^n} \Phi(f^\star(x))dx \tag{2.7}$$

for every continuous increasing function $\Phi : [0, \infty) \to [0, \infty)$.

If $f \in W^{1,p}(\mathbb{R}^n)$ for some $p \ge 1$, then the classical Pólya–Szegö principle (see [8,28,52]) asserts that then f^* is locally absolutely continuous, $f^\star \in W^{1,p}(\mathbb{R}^n)$, and

$$\|\nabla f^\star\|_p \le \|\nabla f\|_p. \tag{2.8}$$

In fact, a full analogue of (2.8) for the affine energy $\mathcal{E}_p$ turns out to hold. It was shown in [38] that if $1 \leq p < n$, then

$$\mathcal{E}_p(f^\star) \leq \mathcal{E}_p(f) \tag{2.9}$$

for every $f \in W^{1,p}(\mathbb{R}^n)$. An inspection of the proof given in [38] shows that the same argument will work for $p > n$ as well. Instead, the proof given in [38] breaks down when $p = n$, exactly the case of interest for applications to Theorem 1.1. In this borderline situation, a limiting argument could be used to establish (2.9). However, a main aim of this note is to present in Theorem 2.1 below a direct unified approach to inequality (2.9), which yields (2.9) simultaneously for all $p \geq 1$. Apart from its own interest, such an approach makes the proof of Theorem 1.1 self-contained, and, by avoiding limiting arguments in p, is useful in analyzing the equality cases, an issue of possible interest for future developments. (In this connection, see e.g. [8,18,22,29], where the cases of equality in (2.8) and in related inequalities are characterized.)

Theorem 2.1 *Suppose $n > 1$ and $p \geq 1$. If $f \in W^{1,p}(\mathbb{R}^n)$, then $f^\star \in W^{1,p}(\mathbb{R}^n)$, and*

$$\mathcal{E}_p(f^\star) \leq \mathcal{E}_p(f). \tag{2.10}$$

Note that the left-hand sides of (2.8) and (2.9) agree, since

$$\mathcal{E}_p(f^\star) = \left(\int_0^{|\mathrm{sprt}\, f|} \left(n\omega_n^{1/n} s^{1/n'} (-f^{*'}(s)) \right)^p ds \right)^{1/p} = \|\nabla f^\star\|_p, \tag{2.11}$$

if $f \in W^{1,p}(\mathbb{R}^n)$. Thus, in view of (1.6), Theorem 2.1 provides a strengthened version of the standard Pólya–Szegö principle.

3 Elements of the L^p Brunn–Minkowski theory

The proof of Theorem 2.1 relies on tools from the rapidly evolving L^p Brunn–Minkowski theory of convex bodies (see, e.g., [12,14,27,31–41,44,48–50]). In particular, on the affine L^p isoperimetric inequality [37] (see [9] for an alternate proof) and on the solution of the normalized L^p Minkowski problem [40]. We recall some basic facts from this theory that will be needed in what follows.

A convex body is a compact convex set in $\mathbb{R}^n$ with nonempty interior, which, throughout this paper, will be assumed to contain the origin in its interior. Each convex body K is uniquely determined by its support function $h_K : \mathbb{R}^n \to [0, \infty)$ defined by

$$h_K(u) = \max\{u \cdot x : x \in K\} \quad \text{for } u \in \mathbb{R}^n.$$

For real $p \geq 1$, real $\varepsilon > 0$ and convex bodies K and L, the Minkowski–Firey L^p combination $K +_p \varepsilon L$ is the convex body whose support function obeys

$$h_{K +_p \varepsilon L}(\cdot)^p = h_K(\cdot)^p + \varepsilon^p h_L(\cdot)^p. \tag{3.1}$$

When $p = 1$, the subscript in "$+_p$" can be suppressed without causing any ambiguity — see the observation following equation (3.5) below.

The L^p-mixed volume $V_p(K, L)$ of K and L is defined by

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{|K +_p \varepsilon^{\frac{1}{p}} L| - |K|}{\varepsilon}.$$

The existence of this limit was proved in [33]. In particular,

$$V_p(K, K) = |K|, \tag{3.2}$$

for every convex body K. The L^p extension of the classical Minkowski inequality (established in [33] for $p > 1$) states that

$$V_p(K, L)^n \geq |K|^{n-p}|L|^p, \tag{3.3}$$

with equality, for $p > 1$, if and only if $K = \varepsilon L$ for some $\varepsilon > 0$, and equality, for $p = 1$, if and only if $K = x + \varepsilon L$ for some $x \in \mathbb{R}^n$ and $\varepsilon > 0$. In [33] it is also shown that

$$V_p(K, L) = \frac{1}{n} \int_{\mathbf{S}^{n-1}} h_L(v)^p h_K(v)^{1-p} dS_K(v) \tag{3.4}$$

for all convex bodies K and L, where S_K is a Borel measure on $\mathbf{S}^{n-1}$ called the surface area measure of K (see, e.g. [51]).

The mixed volume $V_1(E, K)$ of a compact set E and a convex body K in $\mathbb{R}^n$ is defined as

$$V_1(E, K) = \frac{1}{n} \liminf_{\varepsilon \to 0^+} \frac{|E + \varepsilon K| - |E|}{\varepsilon}. \tag{3.5}$$

Here $E + \varepsilon K = \{x + \varepsilon y : x \in E \text{ and } y \in K\}$. Notice that, if E is a convex body, the definition of $E + \varepsilon K$ coincides with Definition (3.1) for $p = 1$. The Brunn–Minkowski inequality (see e.g., [51] or [25]) states that

$$|E + K|^{\frac{1}{n}} \geq |E|^{\frac{1}{n}} + |K|^{\frac{1}{n}}. \tag{3.6}$$

Definition (3.5) and the Brunn–Minkowski inequality immediately give the Minkowski inequality

$$V_1(E, K) \geq |E|^{\frac{1}{n'}} |K|^{\frac{1}{n}}. \tag{3.7}$$

If E has a C^1 boundary, then the following integral representation for $V_1(E, K)$ holds:

$$V_1(E, K) = \frac{1}{n} \int_{\partial E} h_K(\nu(x)) d\mathcal{H}^{n-1}(x), \tag{3.8}$$

where $\nu(x)$ denotes the outward unit normal vector to ∂E at x, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure—see [56].

The L^p-projection function of a convex body K is denoted by $v_p(K, \cdot) : \mathbb{R}^n \to (0, \infty)$ and is defined by

$$v_p(K, u)^p = \frac{1}{2} \int_{\mathbf{S}^{n-1}} |u \cdot v|^p h_K(v)^{1-p} \, dS_K(v), \quad \text{for } u \in \mathbb{R}^n. \tag{3.9}$$

The L^p-projection body $\Pi_p K$ of K is the convex body whose support function is $v_p(K, \cdot)$. The polar $\Pi_p^* K$ of the convex body $\Pi_p K$ is defined as

$$\Pi_p^* K = \left\{x \in \mathbb{R}^n : |x \cdot y| \leq 1 \text{ for all } y \in \Pi_p K\right\}.$$

On observing that

$$\partial \Pi_p^* K = \left\{v_p(K, u)^{-1} u : u \in \mathbf{S}^{n-1}\right\},$$

and making use of polar coordinates, one sees that

$$|\Pi_p^* K| = \frac{1}{n} \int\limits_{\mathbf{S}^{n-1}} v_p(K,u)^{-n} du. \tag{3.10}$$

The L^p Petty projection inequality [37] (see [9] for an alternate approach) is an affine isoperimetric inequality that states that for every convex body K in $\mathbb{R}^n$,

$$|K|^{(n-p)/n} |\Pi_p^* K|^{p/n} \leq \omega_n \omega_{p-1}/\omega_{n+p-2}, \tag{3.11}$$

with equality for $p = 1$ if and only if K is an ellipsoid and with equality for $p > 1$ if and only if K is an ellipsoid centered at the origin.

The solution to the even normalized L^p Minkowski problem [40] will play a key role in our proof of Theorem 2.1. It states that with each $p \geq 1$ and with each even Borel measure λ on $\mathbf{S}^{n-1}$, whose support does not lie in the intersection of $\mathbf{S}^{n-1}$ with a proper subspace, there is uniquely associated an origin-symmetric convex body K, such that

$$\lambda = \frac{1}{|K| h_K^{p-1}} S_K. \tag{3.12}$$

4 Proof of Theorem 2.1

The proof follows roughly the same rearrangement argument used to prove the Euclidean Sobolev inequality. In the Euclidean case, the proof reduces to applying the classical Euclidean isoperimetric inequality to the level sets. In the affine case, we would like to apply the L^p affine isoperimetric inequality (3.11) instead. This cannot be done directly, because the L^p affine isoperimetric inequality applies only to convex bodies, and the level sets are not necessarily convex. Even when the level sets are convex, the L^p affine isoperimetric inequality alone is not enough. Instead, it is necessary to write the L^p gradient integrals over level sets in terms of L^p mixed volumes of convex bodies. This is done by solving a family of L^p Minkowski problems. Estimates for the L^p gradient integrals then reduce to estimates for L^p mixed volumes of convex bodies. The L^p affine isoperimetric inequality is crucial for this.

Assume that $p \geq 1$, and $n > 1$. Suppose that $f \in C_0^\infty(\mathbb{R}^n)$, the space of infinitely differentiable functions having compact support in $\mathbb{R}^n$. By Sard's Lemma, for a.e. $t > 0$,

$$\{|f| > t\} \text{ is a bounded open set with a } C^1 \text{ boundary,} \tag{4.1}$$

$$\partial\{|f| > t\} = \{|f| = t\}, \tag{4.2}$$

and

$$\nabla f(x) \neq 0, \quad \text{for } x \in \{|f| = t\}. \tag{4.3}$$

For every positive t satisfying (4.1)–(4.3), define $\mathrm{v}_p(f,t,\cdot) : \mathbb{R}^n \to (0,\infty)$ by

$$\mathrm{v}_p(f,t,u)^p = \frac{1}{2} \int\limits_{\{|f|=t\}} |u \cdot \nabla f(x)|^p |\nabla f(x)|^{-1} d\mathcal{H}^{n-1}(x), \quad \text{for } u \in \mathbb{R}^n, \tag{4.4}$$

and, for each convex body Q, define $\mathrm{V}_p(f,t,Q)$ by

$$\mathrm{V}_p(f,t,Q)=\frac{1}{n}\int\limits_{\{|f|=t\}} h_Q(\nu(x))^p|\nabla f(x)|^{p-1}d\mathcal{H}^{n-1}(x). \tag{4.5}$$

Observe that, for each convex body Q,

$$\mathrm{V}_1(f,t,Q)=\mathrm{V}_1(\{|f|\geq t\},Q) \quad \text{for a.e. } t>0. \tag{4.6}$$

Lemma 4.1 *If $f\in C_0^\infty(\mathbb{R}^n)$, then for a.e. $t>0$, there exists a unique origin-symmetric convex body $K_t=K_t(f,p)$, such that*

$$\mathrm{v}_p(f,t,u)^p=\frac{v_p(K_t,u)^p}{|K_t|}, \quad \textit{for } u\in\mathbf{S}^{n-1}, \tag{4.7}$$

and

$$\mathrm{V}_p(f,t,Q)=\frac{V_p(K_t,Q)}{|K_t|}, \tag{4.8}$$

for every origin-symmetric convex body Q in $\mathbb{R}^n$.

Proof Suppose $t>0$ is such that (4.1)–(4.3) are fulfilled. Let λ_t be the even positive Borel measure on $\mathbf{S}^{n-1}$ satisfying

$$\int\limits_{\mathbf{S}^{n-1}} g(v)d\lambda_t(v)=\int\limits_{\{|f|=t\}} g(\nu(x))|\nabla f(x)|^{p-1}d\mathcal{H}^{n-1}(x) \tag{4.9}$$

for every even Borel function $g:\mathbf{S}^{n-1}\to\mathbb{R}$. Since, for fixed $u\in\mathbf{S}^{n-1}$,

$$\mathcal{H}^{n-1}\left(\{x:|f(x)|=t \text{ and } u\cdot\nu(x)\neq 0\}\right)>0,$$

and (4.3) holds, one has

$$\int\limits_{\{|f|=t\}} |u\cdot\nu(x)||\nabla f(x)|^{p-1}d\mathcal{H}^{n-1}(x)>0. \tag{4.10}$$

Hence, by (4.9),

$$\int\limits_{\mathbf{S}^{n-1}} |u\cdot v|\,d\lambda_t(v)>0, \quad \text{for } u\in\mathbf{S}^{n-1}. \tag{4.11}$$

Consequently, the measure λ_t is not supported in the intersection of $\mathbf{S}^{n-1}$ with any subspace. By the solution to the even normalized L^p Minkowski problem (3.12), there exists a unique origin-symmetric convex body K_t such that

$$\lambda_t=\frac{1}{|K_t|h_{K_t}^{p-1}}S_{K_t}. \tag{4.12}$$

Equation (4.7) follows from the chain

$$\begin{aligned} \mathrm{v}_p(f,t,u)^p &= \frac{1}{2}\int\limits_{\{|f|=t\}} \frac{|u\cdot\nabla f(x)|^p}{|\nabla f(x)|}\,d\mathcal{H}^{n-1}(x) \\ &= \frac{1}{2}\int\limits_{\{|f|=t\}} |u\cdot\nu(x)|^p|\nabla f(x)|^{p-1}d\mathcal{H}^{n-1}(x) \\ &= \frac{1}{2}\int\limits_{\mathbf{S}^{n-1}} |u\cdot v|^p d\lambda_t(v) \\ &= \frac{1}{2|K_t|}\int\limits_{\mathbf{S}^{n-1}} |u\cdot v|^p h_{K_t}(v)^{1-p}dS_{K_t}(v) \\ &= \frac{v_p(K_t,u)^p}{|K_t|}, \end{aligned} \tag{4.13}$$

for $u\in\mathbf{S}^{n-1}$, where the second equality holds since $\nu(x)=\nabla f(x)/|\nabla f(x)|$ for $x\in\{|f|=t\}$, the third equality is a consequence of (4.9), the fourth equality is due to (4.12), and the final identity is (3.9).

As for (4.8), note that by (4.5), (4.9), (4.12), and (3.4),

$$\begin{aligned} \mathrm{V}_p(f,t,Q) &= \frac{1}{n}\int\limits_{\{|f|=t\}} h_Q(\nu(x))^p|\nabla f(x)|^{p-1}d\mathcal{H}^{n-1}(x) \\ &= \frac{1}{n}\int\limits_{\mathbf{S}^{n-1}} h_Q(v)^p d\lambda_t(v) \\ &= \frac{1}{n|K_t|}\int\limits_{\mathbf{S}^{n-1}} h_Q(v)^p h_{K_t}(v)^{1-p}dS_{K_t}(v) \\ &= \frac{V_p(K_t,Q)}{|K_t|}. \end{aligned} \tag{4.14}$$

To see that K_t is unique, suppose that K_t' is an origin-symmetric convex body that also satisfies (4.8). Then

$$\frac{V_p(K_t,Q)}{|K_t|} = \frac{V_p(K_t',Q)}{|K_t'|}, \tag{4.15}$$

for every origin-symmetric convex body Q in $\mathbb{R}^n$. Choosing $Q=K_t$ in (4.15), and making use of (3.2) and (3.3) entail that $|K_t'|\ge|K_t|$. The same argument applied with $Q=K_t'$ in (4.15) shows that $|K_t|\ge|K_t'|$. Thus, $|K_t'|=|K_t|$, and the equality condition in (3.3), with $K=K_t$ and $L=K_t'$, implies that necessarily $K_t=K_t'$. □

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1 We may assume that f does not vanish identically, otherwise the statement holds trivially. The fact that $f^\star\in W^{1,p}(\mathbb{R}^n)$ whenever $f\in W^{1,p}(\mathbb{R}^n)$ is classical—see e.g. Brothers and Ziemer [8]. To establish (2.10), let us first assume that $f\in C_0^\infty(\mathbb{R}^n)$. From the coarea formula and definition (4.4),

$$\|D_v f\|_p^p = \int_{\mathbb{R}^n} |v\cdot\nabla f(x)|^p dx$$
$$= \int_0^\infty \int_{\{|f|=t\}} \frac{|v\cdot\nabla f(x)|^p}{|\nabla f(x)|} d\mathcal{H}^{n-1}(x)\, dt$$
$$= 2\int_0^\infty \mathrm{v}_p(f,t,v)^p dt, \tag{4.16}$$

for $v \in \mathbf{S}^{n-1}$. Minkowski's inequality for integrals tells us that

$$\left(\int_{\mathbf{S}^{n-1}} \left(\int_0^\infty \mathrm{v}_p(f,t,v)^p dt\right)^{-n/p} dv\right)^{-p/n} \geq \int_0^\infty \left(\int_{\mathbf{S}^{n-1}} \frac{1}{\mathrm{v}_p(f,t,v)^n} dv\right)^{-p/n} dt. \tag{4.17}$$

Let $K_t = K_t(f,p)$ be the unique origin-symmetric convex body guaranteed by Lemma 4.1 for a.e. $t > 0$. Owing to (4.7), (3.10), and (3.11), we have

$$\left(\frac{1}{n}\int_{\mathbf{S}^{n-1}} \frac{1}{\mathrm{v}_p(f,t,v)^n} dv\right)^{-p/n} = \left(\frac{1}{n}\int_{\mathbf{S}^{n-1}} \frac{|K_t|^{\frac{n}{p}}}{v_p(K_t,v)^n} dv\right)^{-p/n}$$
$$= \frac{1}{|K_t|\,|\Pi_p^* K_t|^{\frac{p}{n}}}$$
$$\geq \frac{\omega_{n+p-2}}{\omega_n \omega_{p-1}} \frac{1}{|K_t|^{\frac{p}{n}}} \tag{4.18}$$

for a.e. $t > 0$. Thanks to (4.8) applied with $Q = K_t$, and (3.2), one gets

$$\mathrm{V}_p(f,t,K_t) = \frac{V_p(K_t,K_t)}{|K_t|} = 1, \quad \text{for a.e. } t > 0. \tag{4.19}$$

Assume, for the time being, that $p > 1$. From Eq. (4.19), Hölder's inequality, Eq. (3.8) and inequality (3.7), we deduce that, for a.e. $t > 0$,

$$n^{1/p}\left(\int_{\{|f|=t\}} \frac{1}{|\nabla f(x)|} d\mathcal{H}^{n-1}(x)\right)^{\frac{1}{p'}}$$
$$= \big(n\mathrm{V}_p(f,t,K_t)\big)^{\frac{1}{p}} \left(\int_{\{|f|=t\}} \frac{1}{|\nabla f(x)|} d\mathcal{H}^{n-1}(x)\right)^{\frac{1}{p'}}$$
$$= \left(\int_{\{|f|=t\}} h_{K_t}(\nu(x))^p |\nabla f(x)|^{p-1} d\mathcal{H}^{n-1}(x)\right)^{\frac{1}{p}} \left(\int_{\{|f|=t\}} \frac{1}{|\nabla f(x)|} d\mathcal{H}^{n-1}(x)\right)^{\frac{1}{p'}}$$

$$
\begin{aligned}
&\geq \int_{\{|f|=t\}} h_{K_t}(\nu(x))d\mathcal{H}^{n-1}(x) \\
&= nV_1(\{|f|>t\}, K_t) \\
&\geq n|\{|f|>t\}|^{\frac{1}{n'}}|K_t|^{\frac{1}{n}} \\
&= n\mu_f(t)^{\frac{1}{n'}}|K_t|^{\frac{1}{n}}. \qquad (4.20)
\end{aligned}
$$

By the coarea formula again,

$$
\mu_f(t) = |\{|f|>t\} \cap \{\nabla f = 0\}| + \int_t^\infty \int_{\{|f|=\tau\}} \frac{1}{|\nabla f(x)|} d\mathcal{H}^{n-1}(x)\, d\tau, \quad \text{for } t > 0,
$$

(see, e.g. [8]), and since the non-increasing function μ_f is the sum of two nonincreasing functions, we have

$$
-\mu_f'(t) \geq \int_{\{|f|=t\}} \frac{1}{|\nabla f(x)|} d\mathcal{H}^{n-1}(x) \quad \text{for a.e. } t > 0. \qquad (4.21)
$$

Combining (4.20) and (4.21) yields

$$
\frac{1}{|K_t|^{\frac{p}{n}}} \geq \frac{\mu_f(t)^{p-\frac{p}{n}}}{(-\mu_f'(t)/n)^{p-1}} \quad \text{for a.e. } t > 0. \qquad (4.22)
$$

Inequality (4.22) also holds for $p = 1$, since, by (4.19) and (4.6),

$$
\begin{aligned}
1 &= \frac{1}{n} \int_{\{|f|=t\}} h_{K_t}(\nu(x))d\mathcal{H}^{n-1}(x) = V_1(\{|f| \geq t\}, K_t) \geq |\{|f|>t\}|^{\frac{1}{n'}}|K_t|^{\frac{1}{n}} \\
&= \mu_f(t)^{\frac{1}{n'}}|K_t|^{\frac{1}{n}}.
\end{aligned}
$$

From (1.3), (4.16), (4.17), (4.18) and (4.22) we obtain

$$
\begin{aligned}
\mathcal{E}_p(f)^p &= c_{n,p}^p \left(\int_{\mathbf{S}^{n-1}} \|D_v f\|_p^{-n} dv \right)^{-p/n} \\
&= c_{n,p}^p \left(\int_{\mathbf{S}^{n-1}} \left(2\int_0^\infty \mathrm{v}_p(f,t,v)^p dt \right)^{-n/p} dv \right)^{-p/n} \\
&\geq \frac{2}{n^{p/n}} c_{n,p}^p \int_0^\infty \left(\frac{1}{n} \int_{\mathbf{S}^{n-1}} \frac{1}{\mathrm{v}_p(f,t,v)^n} dv \right)^{-p/n} dt \\
&\geq n\omega_n^{p/n} \int_0^\infty \frac{1}{|K_t|^{\frac{p}{n}}} dt \\
&\geq n^p \omega_n^{p/n} \int_0^\infty \frac{\mu_f(t)^{p-\frac{p}{n}}}{(-\mu_f'(t))^{p-1}} dt. \qquad (4.23)
\end{aligned}
$$

An inspection of the proof of (4.23) shows that, when $f = f^\star$ and $K_t = K_t(f^\star, p)$, all the inequalities turn into equalities, since $K_t(f^\star, p)$ is a ball centered at the origin for a.e. $t > 0$ (in particular, equality holds in (4.21) if $f = f^\star$—see, e.g. [17, Lemmas 2.4 and 2.6]). Consequently, since $\mu_f = \mu_{f^\star}$, we deduce that

$$\mathcal{E}_p(f^\star)^p = n^p \omega_n^{p/n} \int_0^\infty \frac{\mu_f(t)^{p-\frac{p}{n}}}{(-\mu'_f(t))^{p-1}}\, dt. \tag{4.24}$$

The desired inequality (2.10) for $f \in C_0^\infty(\mathbb{R}^n)$ now follows from (4.23) and (4.24).

To establish inequality (2.10) for an arbitrary $f \in W^{1,p}(\mathbb{R}^n)$, consider a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ such that $f_k \in C_0^\infty(\mathbb{R}^n)$ for $k \in \mathbb{N}$, and $f_k \to f$ in $W^{1,p}(\mathbb{R}^n)$. We already know that

$$\mathcal{E}_p(f_k^\star) \le \mathcal{E}_p(f_k) \quad \text{for } k \in \mathbb{N}. \tag{4.25}$$

It is easily seen that $\|D_v f_k\|_p \to \|D_v f\|_p$ uniformly for $v \in \mathbf{S}^{n-1}$. Moreover, the function $v \mapsto \|D_v f\|_p$ is strictly positive and (Lipschitz) continuous on $\mathbf{S}^{n-1}$, and hence attains a positive minimum on $\mathbf{S}^{n-1}$. Consequently, $1/\|D_v f_k\|_p^n \to 1/\|D_v f\|_p^n$ uniformly for $v \in \mathbf{S}^{n-1}$, whence

$$\lim_{k\to\infty} \mathcal{E}_p(f_k) = \mathcal{E}_p(f). \tag{4.26}$$

On the other hand, $f_k^\star \to f^\star$ in $L^p(\mathbb{R}^n)$, thanks to the contractivity of the spherically symmetric rearrangement in $L^p(\mathbb{R}^n)$ (see, e.g. [13]). Hence, one can infer that $f_k^\star \rightharpoonup f^\star$ weakly in $W^{1,p}(\mathbb{R}^n)$. Since $\mathcal{E}_p(f_k^\star) = \|\nabla f_k^\star\|_p$ and $\mathcal{E}_p(f^\star) = \|\nabla f^\star\|_p$, and since the $L^p(\mathbb{R}^n)$ norm of the gradient is lower semicontinuous with respect to weak convergence in $W^{1,p}(\mathbb{R}^n)$,

$$\liminf_{k\to\infty} \mathcal{E}_p(f_k^\star) \ge \mathcal{E}_p(f^\star). \tag{4.27}$$

Inequality (2.10) follows from (4.25)–(4.27). □

5 Proof of Theorems 1.1 and 1.2

The finiteness of the quantity m_n, as defined in (1.8), is the content of the following result by Moser.

Lemma 5.1 *Let $p \in (1, \infty)$, and let*

$$m_p = \sup_\phi \int_0^\infty e^{\phi(s)^{p'} - s} ds, \tag{5.1}$$

where ϕ ranges among all non-decreasing, locally absolutely continuous functions in $[0, \infty)$ fulfilling $\phi(0) = 0$ and $\int_0^\infty \phi'(s)^p ds \le 1$. Then

$$m_p < \infty. \tag{5.2}$$

Incidentally, observe that the quantity m_p given by (5.1) remains unchanged if the class of trial functions is enlarged to include also not necessarily (positive and) monotone functions ϕ, provided that $\phi(s)^{p'}$ is replaced by $|\phi(s)|^{p'}$ and $\phi'(s)^p$ by $|\phi'(s)|^p$. This can be easily seen on replacing $\phi(s)$ by $\int_0^s |\phi'(s)| ds$.

The original proof of Lemma 5.1 for $p \geq 2$ contained in Moser [45] is rather involved. A simplified approach, exploiting a technique of Garsia (see [2, Sect. 3.8]), is presented in Adams [1], where an even more general result is established; a very similar proof, for $p = 2$, can be found in Marshall [43], where the possibility of an extension to the case where $p \neq 2$ is also mentioned. For completeness, we reproduce a proof of Lemma 5.1 along the same lines as those of Adams [1] and Marshall [43].

Proof of Lemma 5.1 For each $t \in \mathbb{R}$, define

$$E_t = \left\{ s \geq 0 : s - \phi(s)^{p'} \leq t \right\},$$

and let

$$c_p = \frac{7}{1 - (1 + 2^{1-p})^{\frac{1}{1-p}}}.$$

From the fact that $\phi(0) = 0$, the Hölder inequality and the fact that $\int_0^\infty \phi'(s)^p ds \leq 1$, we have that

$$\phi(s)^{p'} = \left(\int_0^s \phi'(r) dr \right)^{p'} \leq s \quad \text{for } s > 0.$$

Hence,

$$E_t = \emptyset, \quad \text{for } t < 0. \tag{5.3}$$

We now show that

$$|E_t| \leq (c_p + 2)t, \quad \text{for } t > 0. \tag{5.4}$$

Inequality (5.4) trivially holds if $E_t \subset [0, 2t]$. If this is not the case, then inequality (5.4) will follow if we prove that

$$s_2 - s_1 \leq c_p t \tag{5.5}$$

for every $s_1, s_2 \in E_t$ satisfying $2t \leq s_1 < s_2$. From the definition of E_t and the Hölder inequality,

$$s_1 - t \leq \left(\int_0^{s_1} \phi'(r) dr \right)^{p'} \leq s_1 \left(\int_0^{s_1} \phi'(r)^p dr \right)^{\frac{1}{p-1}} \leq s_1 \left(1 - \int_{s_1}^{\infty} \phi'(r)^p dr \right)^{\frac{1}{p-1}}.$$

Thus,

$$\int_{s_1}^{\infty} \phi'(r)^p dr \leq 1 - \left(1 - \frac{t}{s_1} \right)^{p-1}. \tag{5.6}$$

From the definition of E_t, the Hölder inequality again, and (5.6),

$$s_2 - t \le \left(\int_0^{s_1} \phi'(r)dr + \int_{s_1}^{s_2} \phi'(r)dr \right)^{p'}$$

$$\le \left[s_1^{\frac{1}{p'}} \left(\int_0^{s_1} \phi'(r)^p dr \right)^{\frac{1}{p}} + (s_2 - s_1)^{\frac{1}{p'}} \left(\int_{s_1}^{s_2} \phi'(r)^p dr \right)^{\frac{1}{p}} \right]^{p'}$$

$$\le \left[s_1^{\frac{1}{p'}} + (s_2 - s_1)^{\frac{1}{p'}} \left(\int_{s_1}^{\infty} \phi'(r)^p dr \right)^{\frac{1}{p}} \right]^{p'}$$

$$\le \left[s_1^{\frac{1}{p'}} + (s_2 - s_1)^{\frac{1}{p'}} \left(1 - \left(1 - \frac{t}{s_1} \right)^{p-1} \right)^{\frac{1}{p}} \right]^{p'}.$$

Therefore,

$$\frac{s_2}{s_1} - \frac{t}{s_1} \le \left[1 + \left(\frac{s_2}{s_1} - 1 \right)^{\frac{1}{p'}} \left(1 - \left(1 - \frac{t}{s_1} \right)^{p-1} \right)^{\frac{1}{p}} \right]^{p'}. \tag{5.7}$$

Set $M = \frac{s_2 - s_1}{t}$ and $z = 1 - \frac{t}{s_1}$. Then, $\frac{1}{2} \le z < 1$ and (5.7) can be rewritten as

$$M(1-z) + z \le \left(1 + M^{\frac{1}{p'}} (1-z)^{\frac{1}{p'}} (1 - z^{p-1})^{\frac{1}{p}} \right)^{p'}. \tag{5.8}$$

The convexity of the function $\tau \mapsto \tau^{p'}$ entails that, for $\lambda \in [0, 1]$,

$$\left(1 + M^{\frac{1}{p'}} (1-z)^{\frac{1}{p'}} (1 - z^{p-1})^{\frac{1}{p}} \right)^{p'} \le (1-\lambda)^{1-p'} + \lambda^{1-p'} M(1-z)(1-z^{p-1})^{\frac{1}{p-1}}. \tag{5.9}$$

Choosing $\lambda = 1 - z^{2(p-1)}$ and combining (5.8) and (5.9) yield

$$M \le \frac{(1-\lambda)^{1-p'} - z}{(1-z)\left(1 - \lambda^{1-p'} (1 - z^{p-1})^{\frac{1}{p-1}} \right)} = \frac{z^{-2} + z^{-1} + 1}{1 - (1 + z^{p-1})^{\frac{1}{1-p}}} \le c_p,$$

whence (5.5) follows.

The Layer Cake principle and a simple change of variables imply that

$$\int_0^{\infty} e^{-g(s)} ds = \int_{-\infty}^{\infty} |\{s > 0 : g(s) \le t\}| \, e^{-t} \, dt$$

for every measurable function $g : (0, \infty) \to \mathbb{R}$. Thus,

$$\int_0^\infty e^{\phi(s)^{p'}-s} ds = \int_{-\infty}^\infty |E_t| e^{-t} dt \le c_p + 2,$$

where the inequality is a consequence of (5.3) and (5.4). □

Proof of Theorem 1.1 By (2.7) and (2.5), we have

$$\int_{\mathbb{R}^n} e^{(n\omega_n^{1/n}|f(x)|)^{n'}} dx = \int_0^{|\text{sprt } f|} e^{(n\omega_n^{1/n} f^*(s))^{n'}} ds. \tag{5.10}$$

On the other hand, (2.11) and (2.10) tell us that

$$\mathcal{E}_n(f) \ge \mathcal{E}_n(f^\star) = \left(\int_0^{|\text{sprt } f|} \left(n\omega_n^{1/n} s^{1/n'} (-f^{*'}(s)) \right)^n ds \right)^{1/n}. \tag{5.11}$$

Owing to (5.10) and (5.11), the affine Moser–Trudinger inequality (1.9) will follow if we show that, for each $a > 0$,

$$\sup_\psi \frac{1}{a} \int_0^a e^{(n\omega_n^{1/n}\psi(s))^{n'}} ds = m_n, \tag{5.12}$$

as ψ ranges among all non-increasing locally absolutely continuous function $\psi : (0, a] \to [0, \infty)$ such that $\psi(a) = 0$ and $\int_0^a (n\omega_n^{1/n} s^{1/n'}(-\psi'(s)))^n ds \le 1$. Given such a ψ, define the non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ by

$$\phi(t) = n\omega_n^{1/n} \psi(ae^{-t}), \quad \text{for } t > 0. \tag{5.13}$$

Note that $\phi(0) = 0$. The change of variable

$$s = ae^{-t} \tag{5.14}$$

gives

$$\int_0^\infty \phi'(t)^n \, dt = \int_0^a \left(n\omega_n^{1/n} s^{1/n'} (-\psi'(s)) \right)^n ds \ \le \ 1,$$

and

$$\int_0^\infty e^{\phi(t)^{n'}} e^{-t} dt = \frac{1}{a} \int_0^a e^{(n\omega_n^{1/n}\psi(s))^{n'}} ds.$$

Hence, Eq. (5.12) follows from Lemma 5.1, since, for each fixed a, the class of functions appearing in definition (5.1) agrees with the class of functions ϕ given by (5.13) with ψ as above.

The sharpness of the constant $n\omega_n^{1/n}$ in (1.9) can be verified on testing the inequality on the same sequence $\{f_k\}_{k\in\mathbb{N}}$ of (radially decreasing) functions as in Moser [45], namely

$$f_k(x) = \begin{cases} \frac{k^{1/q'}}{n\omega_n^{1/n}} & \text{if } |x| \le e^{-k/n} \\ \frac{k^{-1/q}}{\omega_n^{1/n}} \log\left(\frac{1}{|x|}\right) & \text{if } e^{-k/n} < |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the sequence $\{f_k(A(x - x_0))\}$ is also extremal in (1.9), for any $x_0 \in \mathbb{R}^n$ and any $A \in GL(n)$, since inequality (1.9) is invariant under affine transformations in $\mathbb{R}^n$. □

We conclude with the proof of Theorem 1.2.

Proof of Theorem 1.2 Similarly to the proof of Theorem 1.1, a crucial ingredient here is the symmetrization inequality (2.9) for $p > n$. This inequality, together with (2.11), reads

$$\mathcal{E}_p(f) \ge \mathcal{E}_p(f^\star) = \left(\int_0^{|\mathrm{sprt}\, f|} \left(n\omega_n^{1/n} s^{1/n'} (-f^{*'}(s)) \right)^p ds \right)^{1/p} \tag{5.15}$$

for every $f \in W^{1,p}(\mathbb{R}^n)$ with $|\mathrm{sprt}\, f| < \infty$. By the (local) absolute continuity of f^*, and by (2.6), (2.5) and the Hölder inequality,

$$\begin{aligned} \|f\|_{L^\infty(\mathbb{R}^n)} &= f^*(0) = \int_0^{|\mathrm{sprt}\, f|} (-f^{*'}(s)) ds \\ &\le \left(\int_0^{|\mathrm{sprt}\, f|} \left(-f^{*'}(s) s^{1/n'}\right)^p ds \right)^{1/p} \left(\int_0^{|\mathrm{sprt}\, f|} s^{-p'/n'} ds \right)^{1/p'} \\ &= n^{1/p'} \left(\frac{p-1}{p-n} \right)^{1/p'} |\mathrm{sprt}\, f|^{\frac{1}{n} - \frac{1}{p}} \left(\int_0^{|\mathrm{sprt}\, f|} \left(-f^{*'}(s) s^{1/n'}\right)^p ds \right)^{1/p}. \end{aligned} \tag{5.16}$$

Inequality (1.12) follows from (5.15) and (5.16).

Equality holds in (1.12) for any function having the form (1.13) with $x_0 = 0$ and $A = I$; actually, any such function is spherically symmetric, so that equality holds in (5.15), and renders the inequality in (5.16) an equality. Equality continues to hold in (1.12) even if $x_0 \ne 0$ and $A \ne I$ in (1.13), owing to the invariance of (1.12) under affine transformations. □

References

1. Adams, D.R.: A sharp inequality of J. Moser for higher order derivatives. Ann. Math. **128**, 385–398 (1988)
2. Adams, D.R., Hedberg, L.I.: Function Spaces and Potential Theory. Springer, Berlin (1996)
3. Alberico, A.: Moser type inequalities for higher order derivatives in Lorentz spaces. Potential Anal. **28**, 389–400 (2008)
4. Alvino, A., Ferone, V., Trombetti, G.: Moser-type inequalities in Lorentz spaces. Potential Anal. **5**, 273–299 (1996)

5. Aubin, T.: Problèmes isopérimetriques et espaces de Sobolev. J. Differ. Geom. **11**, 573–598 (1976)
6. Balogh, Z.M., Manfredi, J.J., Tyson, J.T.: Fundamental solution for the Q-Laplacian and sharp Moser–Trudinger inequality in Carnot groups. J. Funct. Anal. **204**, 35–49 (2003)
7. Beckner, W.: Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality. Ann. Math. **138**, 213–242 (1993)
8. Brothers, J.E., Ziemer, W.P.: Minimal rearrangements of Sobolev functions. J. Reine Angew. Math. **384**, 153–179 (1988)
9. Campi, S., Gronchi, P.: The L^p-Busemann–Petty centroid inequality. Adv. Math. **167**, 128–141 (2002)
10. Carleson, L., Chang, S.Y.A.: On the existence of an extremal function for an inequality of J. Moser. Bull. Sci. Math. **110**, 113–127 (1986)
11. Chang, S.-Y.A., Yang, P.C.: Conformal deformation of metrics on S^2. J. Differ. Geom. **27**, 259–296 (1988)
12. Chen, W.: L_p Minkowski problem with not necessarily positive data. Adv. Math. **201**, 77–89 (2006)
13. Chiti, G.: Rearrangements of functions and convergence in Orlicz spaces. Appl. Anal. **9**, 23–27 (1979)
14. Chou, K.-S., Wang, X.-J.: The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry. Adv. Math. **205**, 33–83 (2006)
15. Cianchi, A.: Moser–Trudinger inequalities without boundary conditions and isoperimetric problems. Indiana Univ. Math. J. **54**, 669–705 (2005)
16. Cianchi, A.: Moser–Trudinger trace inequalities. Adv. Math. **217**, 2005–2044 (2008)
17. Cianchi, A., Fusco, N.: Functions of bounded variation and rearrangements. Arch. Ration. Mech. Anal. **165**, 1–40 (2002)
18. Cianchi, A., Fusco, N.: Steiner symmetric extremals in Pólya–Szegö type inequalities. Adv. Math. **203**, 673–728 (2006)
19. Cohn, W.S., Lu, G.: Best constants for Moser-Trudinger inequalities on the Heisenberg group. Indiana Univ. Math. J. **50**, 1567–1591 (2001)
20. de Figuereido, D.G., do, Ó, Marcos, J., Ruf, B.: On an inequality by N.Trudinger and J.Moser and related elliptic equations. Comm. Pure Appl. Math. **55**, 135–152 (2002)
21. Federer, H., Fleming, W.: Normal and integral currents. Ann. Math. **72**, 458–520 (1960)
22. Ferone, A., Volpicelli, R.: Convex rearrangement: equality cases in the Pólya–Szegö inequality. Calc. Var. Partial Differ. Equ. **21**, 259–272 (2004)
23. Flucher, M.: Extremal functions for Trudinger–Moser inequality in 2 dimensions. Comment. Math. Helvetici **67**, 471–497 (1992)
24. Fontana, L.: Sharp borderline Sobolev inequalities on compact Riemannian manifolds. Comment. Math. Helvetici **68**, 415–454 (1993)
25. Gardner, R.J.: The Brunn–Minkowski inequality. Bull. Am. Math. Soc. **39**, 355–405 (2002)
26. Hang, F.: On the higher order conformal covariant operators on the sphere. Commun. Contemp. Math. **9**, 279–299 (2007)
27. Hu, C., Ma, X.-N., Shen, C.: On the Christoffel–Minkowski problem of Firey's p-sum. Calc. Var. Partial Differ. Equ. **21**, 137–155 (2004)
28. Kawohl, B.: Rearrangements and convexity of level sets in PDE. Lecture Notes in Math., vol. 1150. Springer, Berlin (1985)
29. Kawohl, B.: On the isoperimetric nature of a rearrangement inequality and its consequences for some variational problems. Arch. Ration. Mech. Anal. **94**, 227–243 (1986)
30. Lin, K.C.: Extremal functions for Moser's inequality. Trans. Am. Math. Soc. **348**, 2663–2671 (1996)
31. Ludwig, M.: Ellipsoids and matrix-valued valuations. Duke Math. J. **119**, 159–188 (2003)
32. Ludwig, M.: Minkowski valuations. Trans. Am. Math. Soc. **357**, 4191–4213 (2005)
33. Lutwak, E.: The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem. J. Differ. Geom. **38**, 131–150 (1993)
34. Lutwak, E.: The Brunn–Minkowski–Firey theory. II. Affine and geominimal surface areas. Adv. Math. **118**, 244–294 (1996)
35. Lutwak, E., Oliker, V.: On the regularity of solutions to a generalization of the Minkowski problem. J. Differ. Geom. **41**, 227–246 (1995)
36. Lutwak, E., Yang, D., Zhang, G.: A new ellipsoid associated with convex bodies. Duke Math. J. **104**, 375–390 (2000)
37. Lutwak, E., Yang, D., Zhang, G.: On L_p affine isoperimetric inequalities. J. Differ. Geom. **56**, 111–132 (2000)
38. Lutwak, E., Yang, D., Zhang, G.: Sharp affine L_p Sobolev inequalities. J. Differ. Geom. **62**, 17–38 (2002)
39. Lutwak, E., Yang, D., Zhang, G.: The Cramer–Rao inequality for star bodies. Duke Math. J. **112**, 59–81 (2002)
40. Lutwak, E., Yang, D., Zhang, G.: On the L_p Minkowski problem. Trans. Am. Math. Soc. **356**, 4359–4370 (2004)

41. Lutwak, E., Yang, D., Zhang, G.: Optimal Sobolev Norms and the L^p Minkowski Problem. Int. Math. Res. Not. (2006) Art. ID 62987, 21 pp
42. Maz'ya, V.M.: Classes of domains and imbedding theorems for function spaces, Dokl. Akad. Nauk. SSSR **133** (1960), 527–530 (Russian); English translation: *Soviet Math. Dokl.* **1** (1960), 882–885
43. Marshall, D.E.: A new proof of a sharp inequality concerning the Dirichlet integral. Ark. Math. **27**, 131–137 (1989)
44. Meyer, M., Werner, E.: On the p-affine surface area. Adv. Math. **152**, 288–313 (2000)
45. Moser, J.: A sharp form of an inequality by Trudinger. Indiana Univ. Math. J. **20**, 1077–1092 (1970/71)
46. Pohozhaev, S.I.: On the imbedding Sobolev theorem for $pl = n$. Doklady Conference, Section Math. Moscow Power Inst, pp. 158–170 (1965) (Russian)
47. Ruf, B.: A sharp Trudinger–Moser type inequality for unbounded domains in $\mathbb{R}^2$. J. Funct. Anal. **219**, 340–367 (2005)
48. Ryabogin, D., Zvavitch, A.: The Fourier transform and Firey projections of convex bodies. Indiana Univ. Math. J. **53**, 667–682 (2004)
49. Schütt, C., Werner, E.: Surface bodies and p-affine surface area. Adv. Math. **187**, 98–145 (2004)
50. Stancu, A.: The discrete planar L_0-Minkowski problem. Adv. Math. **167**, 160–174 (2002)
51. Schneider, R.: Convex Bodies: The Brunn–Minkowski Theory. Cambridge University Press, Cambridge (1993)
52. Talenti, G.: Best constant in Sobolev inequality. Ann. Mat. Pura Appl. **110**, 353–372 (1976)
53. Talenti, G.: Inequalities in rearrangement invariant function spaces. In: Krbec, M., Kufner, A., Opic, B., Rákosnik, J. (eds.) Nonlinear Analysis, Function Spaces and Applications, vol. 5, pp. 177–230. Prometheus, Prague (1994)
54. Trudinger, N.S.: On imbeddings into Orlicz spaces and some applications. J. Math. Mech. **17**, 473–483 (1967)
55. Yudovic, V.I.: Some estimates connected with integral operators and with solutions of elliptic equations. Dokl. Akad.Nauk SSSR, vol. 138, pp. 805–808 (1961), English translation in Soviet Math. Doklady 2 (1961), 746–749
56. Zhang, G.: The affine Sobolev inequality. J. Differ. Geom. **53**, 183–202 (1999)