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Calculus of Variations

Intrinsic regular submanifolds in Heisenberg groups are differentiable graphs

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Abstract We characterize intrinsic regular submanifolds in the Heisenberg group as intrinsic differentiable graphs.

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Contents

1 Introduction

The notion of rectifiable set is a central one in calculus of variations and in geometric measure theory. To develop a theory of rectifiable sets inside Carnot groups has been the object of much research in the last 10 years (see e.g. [2,3,6,7,10,12,13,19–21,24]).

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Rectifiable sets, in Euclidean spaces, are generalizations of C^1 or of Lipschitz submanifolds, hence, understanding the objects that, inside Carnot groups, naturally take the role of C^1 or of Lipschitz submanifolds is a preliminary task in developing a satisfactory theory of rectifiable sets inside Carnot groups.

In this paper we address this problem considering functions acting between *complementary subgroups* of a (Carnot) group $\mathbb{G}$ (Definition 3.1) and introducing, for these functions, the notions of intrinsic Lipschitz continuity or of intrinsic differentiability. Then, Lipschitz or C^1 submanifolds will be objects that, locally, are *intrinsic graphs* of these functions (Definition 3.3).

Intrinsic graphs came out naturally while studying non critical level sets of differentiable functions from $\mathbb{H}^n$ to $\mathbb{R}$ (see [10] and also [6]); indeed these level sets can always be locally described as intrinsic graphs, (see Proposition 3.12 of [13]). The simple idea of intrinsic graph is the following one: let $\mathbb{G}_1$, $\mathbb{G}_2$ be *complementary subgroups* of a group $\mathbb{G}$, then the intrinsic (left) graph of $f : \mathbb{G}_1 \to \mathbb{G}_2$ is the set

$$\text{graph}\,(f) = \{g \cdot f(g) : g \in \mathbb{G}_1\}.$$

More generally, we say that a subset S of a Carnot group $\mathbb{G}$, is a *(left) intrinsic graph*, in direction of a homogeneous subgroup $\mathbb{H}$, if S intersects each left coset of $\mathbb{H}$ in at most a single point.

Intrinsic Lipschitz continuity and *intrinsic differentiability* are defined respecting the geometry of the ambient space $\mathbb{G}$.

$f : \mathbb{G}_1 \to \mathbb{G}_2$ is *intrinsic Lipschitz* (Definition 3.8) if, at each point $p \in \text{graph}\,(f)$, there is an intrinsic cone (Definition 3.7), with vertex p, axis $\mathbb{G}_2$ and fixed opening, intersecting graph (f) only at p.

$f : \mathbb{G}_1 \to \mathbb{G}_2$ is *intrinsic differentiable at* $g \in \mathbb{G}_1$ if there is a homogeneous subgroup $\mathbb{H}$ of $\mathbb{G}$ such that, in $p = g \cdot f(g) \in \text{graph}\,(f)$, the left coset $p \cdot \mathbb{H}$ is the *tangent plane* to graph (f) in p, that is if $p \cdot \mathbb{H}$ is the limit of group dilations of graph (f) centered in p (Definition 3.13).

In this paper we focus our attention on intrinsic differentiable functions and on their relation with C^1 submanifolds. On the contrary, intrinsic Lipschitz functions are studied here only as far as they are useful for this topic. A more advanced analysis of intrinsic Lipschitz functions can be found in [14].

Let us come now to C^1 surfaces in groups. A class of surfaces that, in $\mathbb{H}^n$ and, sometimes, in more general Carnot groups, are a good generalization, to the group setting, of C^1 submanifolds are the so called H-regular submanifolds (see Definition 4.1 and the Ref. [2,6,13,20,24,26]).

Since the theory of H-regular surfaces seems, up to now, approximately complete only on $\mathbb{H}^n$, we will limit ourselves to describe this case. In $\mathbb{H}^n$, H-regular submanifolds are differently defined according to their topological dimension k. Precisely, if $k \leq n$, a k-dimensional or *low dimensional* H-regular submanifold is, locally, the image in $\mathbb{H}^n$ of an open set of $\mathbb{R}^k$, through an injective, Pansu differentiable function; while a k-codimensional, or *low codimensional*, H-regular submanifold is, locally, the non critical level set of a Pansu differentiable function $\mathbb{H}^n \to \mathbb{R}^k$.

These surfaces are different from Euclidean C^1 surfaces and are very different from each other. Indeed k-dimensional H-regular submanifolds are a subclass of k-dimensional Euclidean C^1 submanifolds of $\mathbb{R}^{2n+1}$ (see [13], Theorem 3.5). On the contrary, k-codimensional H-regular submanifolds can be very irregular, even fractals, from an Euclidean point of view (see [13,18]). Nevertheless H-regular submanifolds can, very reasonably, be considered as C^1 submanifolds because (i) they have a tangent plane at each point and the tangent planes are cosets of subgroups of $\mathbb{H}^n$ that are also blow-up limits of the surface, (ii) the tangent planes depend continuously on the point, (iii) H-regular submanifolds have integer (Heisenberg) dimension, locally finite (Heisenberg) Hausdorff measure and this measure can be obtained, by integration, through an area type formula (see [13]).

Our main result here (Theorem 4.2), is a characterization of H-regular surfaces in $\mathbb{H}^n$, both low dimensional and low codimensional, as *uniformly intrinsic differentiable graphs* of functions between *complementary subgroups* of $\mathbb{H}^n$.

Describing H-regular submanifolds as (intrinsic differentiable) graphs is more general and flexible than using parametrizations or level sets. Indeed, differently from $\mathbb{R}^n$—where d-dimensional C^1 embedded submanifolds are *equivalently* defined as non-critical level sets of differentiable functions $\mathbb{R}^n \to \mathbb{R}^{n-d}$ or as images of injective differentiable maps $\mathbb{R}^d \to \mathbb{R}^n$ (or as *graphs of* C^1 *functions* $\mathbb{R}^d \to \mathbb{R}^{n-d}$)—in $\mathbb{H}^n$, low dimensional H-regular surfaces cannot be seen as non critical level sets and low codimensional ones cannot be seen as (bilipschitz) images of open sets. The reasons for this are rooted in the algebraic structure of $\mathbb{H}^n$. From one side, low dimensional horizontal subgroups of $\mathbb{H}^n$ are not normal subgroups, hence they are not kernels of homogeneous homomorphisms, while they are tangent spaces to low dimensional submanifolds; on the other side, injective homogeneous homomorphisms $\mathbb{R}^d \to \mathbb{H}^n$ do not exist, if $d \geq n+1$ (see [2,19]).

We hope that our approach, here limited to $\mathbb{H}^n$, might prove itself useful to define intrinsic C^1 and Lipschitz submanifolds in more general Carnot groups. With this aim, we made the effort of writing many statements and proofs in a coordinate free fashion. We hope this will show how many concepts, here discussed, find their natural setting in more general Carnot groups than the Heisenberg groups.

Finally, we recall that the class of functions, called here *uniformly intrinsic differentiable*, is not a new one. It was already introduced and studied, with different names and approaches, by many authors, at least in relation with 1-codimensional graphs (see [1,6,26]).

Finally we would like to thank Francesco Serra Cassano and Davide Vittone for their interest and many useful talks.

2 Notations and preliminaries

For a general review on Carnot and Heisenberg groups see [8,9,16,17,25] and the recent ones [4,5]. Here we limit ourselves to fix some notations.

2.1 Carnot groups

A *graded group* of step k is a connected, simply connected Lie group $\mathbb{G}$ whose Lie algebra $\mathfrak{g}$, of dimension n, is the direct sum of k subspaces $\mathfrak{g}_i$, $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, such that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j},$$

for $1 \leq i, j \leq k$ and $\mathfrak{g}_i = 0$ for $i > k$.

A *Carnot group* $\mathbb{G}$ of step k is a graded group of step k, where $\mathfrak{g}_1$ generates all $\mathfrak{g}$. That is $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$, for $i = 1, \dots, k$.

The exponential map is a one to one diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$. Let $X_1, \dots, X_n$ be a base for $\mathfrak{g}$ such that $X_1, \dots, X_{m_1}$ is a base for $\mathfrak{g}_1$ and, for $1 < j \leq k$, $X_{m_{j-1}+1}, \dots, X_{m_j}$ is a base for $\mathfrak{g}_j$. Then any $p \in \mathbb{G}$ can be written, in a unique way, as $p = \exp(p_1X_1 + \cdots + p_nX_n)$ and we can identify p with the n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$. The explicit expression of the group operation $\cdot$, determined by the Campbell-Hausdorff formula (see [4] or [8]), has the form

$$x \cdot y = x + y + \mathcal{Q}(x, y), \qquad \forall x, y \in \mathbb{R}^n \tag{1}$$

where $\mathcal{Q}(x, y) = (\mathcal{Q}_1(x, y), \dots, \mathcal{Q}_n(x, y)) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Here, $Q_i(x, y) = 0$, for $i = 1, \dots, m_1$ and, for $1 < j \le k$ and $m_{j-1} + 1 \le i \le m_j$, we have, $Q_i(x, y) = Q_i\left(x_1, \dots, x_{m_{j-1}}, y_1, \dots, y_{m_{j-1}}\right)$.

If $p \in \mathbb{G}$, $p^{-1} = (-p_1, \dots, -p_n)$ is the inverse of p and $e = (0, \dots, 0)$ is the identity of $\mathbb{G}$.

If $\mathbb{G}$ is a graded group, for all $\lambda > 0$, the *(non isotropic) dilations* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ are automorphisms of $\mathbb{G}$ defined as $\delta_\lambda(x_1, \dots, x_n) = (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \dots, \lambda^{\alpha_n} x_n)$, where $\alpha_i = j$, if $m_{j-1} < i \le m_j$.

Given any homogeneous norm $\|\cdot\|$, the distance in $\mathbb{G}$ is defined as

$$d(x, y) = d(y^{-1} \cdot x, 0) = \left\| y^{-1} \cdot x \right\|, \quad \text{for all } x, y \in \mathbb{G}. \tag{2}$$

A possible homogeneous norm is the following one, if $p = (p^1, \dots, p^k) \in \mathbb{R}^n = \mathbb{G}$, with $p^j \in \mathbb{R}^{m_j - m_{j-1}}$, for $j = 1, \dots, k$, then

$$\|p\| = \max_{j=1,\dots,k} \left\{ \varepsilon_j \left\| p^j \right\|_{\mathbb{R}^{m_j - m_{j-1}}}^{1/\alpha_j} \right\}, \tag{3}$$

where $\varepsilon_1 = 1$, and $\varepsilon_2, \dots \varepsilon_k \in (0, 1]$ are suitable positive constants depending on $\mathbb{G}$ (see Theorem 5.1 of [12]).

The distance d is comparable with the Carnot Carathèodory distance of $\mathbb{G}$ and is well behaved with respect to left translations and dilations, that is

$$d(z \cdot x, z \cdot y) = d(x, y) \quad , \quad d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y)$$

for $x, y, z \in \mathbb{G}$ and $\lambda > 0$. For $r > 0$ and $p \in \mathbb{G}$, we denote by $B(p, r)$ the open ball associated with d.

A *homogeneous subgroup* of a Carnot group $\mathbb{G}$ (see [25] 5.2.4) is a subgroup $\mathbb{H}$ such that $\delta_\lambda g \in \mathbb{H}$, for all $g \in \mathbb{H}$ and for all $\lambda > 0$.

The *(linear) dimension* of a (sub)group is the dimension of its Lie algebra. The *metric dimension* of a subgroup, or of a subset, is its Hausdorff dimension, where Hausdorff measures are constructed from the distance in (2).

2.2 Heisenberg groups

The n-dimensional Heisenberg group $\mathbb{H}^n$ is identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted as $p = (p_1, \dots, p_{2n}, p_{2n+1}) \in \mathbb{R}^{2n+1}$.
For $p, q \in \mathbb{H}^n$, the group operation is

$$p \cdot q = \left(p_1 + q_1, \dots, p_{2n} + q_{2n}, p_{2n+1} + q_{2n+1} + \sum_{i=1}^{n} (p_i q_{i+n} - p_{i+n} q_i)/2 \right).$$

For $\lambda > 0$, dilations $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ are defined as

$$\delta_\lambda p := (\lambda p_1, \dots, \lambda p_{2n}, \lambda^2 p_{2n+1}), \quad \text{for all } p \in \mathbb{H}^n.$$

The standard base of the Lie algebra $\mathfrak{h}$ of $\mathbb{H}^n$ is given by the left invariant vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n, T$, where, for $i = 1, \dots, n$,

$$X_i(p) := \partial_i - \frac{1}{2} p_{i+n} \partial_{2n+1}, \quad Y_i(p) := \partial_{i+n} + \frac{1}{2} p_i \partial_{2n+1}, \quad T(p) := \partial_{2n+1}.$$

The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n, Y_1, \dots, Y_n$. Denoting by $\mathfrak{h}_2$ the linear span of T, then $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ and $[\mathfrak{h}_1, \mathfrak{h}_1] = \mathfrak{h}_2$.

The Lie algebra $\mathfrak{h}$ is also endowed with a scalar product $\langle \cdot, \cdot \rangle$ making the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n, T$ orthonormal.

The centre of $\mathbb{H}^n$ is the subgroup $\mathbb{T} := \exp(\mathfrak{h}_2) = \{(0, \dots, 0, p_{2n+1})\}$.

The *horizontal bundle* $H\mathbb{H}^n$ is the subbundle of the tangent bundle $T\mathbb{H}^n$ whose fibers $H\mathbb{H}^n_p$ are spanned by the horizontal vectors $X_1(p), \cdots, Y_n(p)$.

If $p \in \mathbb{H}^n$, the homogeneous norm in (3) becomes

$$\|p\| := \max\{\|(p_1, \dots, p_{2n})\|_{\mathbb{R}^{2n}}, |p_{2n+1}|^{1/2}\}$$

while the distance d is defined as in (2).

Let $\pi : \mathbb{H}^n \to \mathbb{R}^{2n}$ be $\pi(p) = \pi(p_1, \dots, p_{2n}, p_{2n+1}) := (p_1, \dots, p_{2n})$. Notice that any $p \in \mathbb{H}^n$ can be uniquely written as $p = (\pi(p), p_{2n+1}) = (\pi(p), 0) \cdot p_{\mathbb{T}}$, where $p_{\mathbb{T}} = (0, \cdots, 0, p_{2n+1}) \in \mathbb{T}$ and $(\pi(p), 0) \in H\mathbb{H}^n_e$.

2.3 Calculus

The notion of *P-differentiability* for functions acting between graded groups, was introduced by Pansu in [23].

Definition 2.1 Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be graded groups, with homogeneous norms $\|\cdot\|_1$, $\|\cdot\|_2$ and dilations δ^1_λ, δ^2_λ, then $L : \mathbb{G}_1 \to \mathbb{G}_2$ is said to be *H-linear* (see [19]), if L is a *homogeneous homomorphism*, that is if L is a group homomorphism from $\mathbb{G}_1$ to $\mathbb{G}_2$ and if,

$$L(\delta^1_\lambda g) = \delta^2_\lambda L(g), \quad \text{for all } g \in \mathbb{G}_1 \text{ and } \lambda > 0.$$

The norm of L is $\|L\| = \sup\{\|L(g)\|_2 : \|g\|_1 \le 1\}$.

Definition 2.2 Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be graded groups with homogeneous norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Then $f : \mathcal{A} \subset \mathbb{G}_1 \to \mathbb{G}_2$ is *P-differentiable* in $g_0 \in \mathcal{A}$ if there is a H-linear function $df_{g_0} : \mathbb{G}_1 \to \mathbb{G}_2$ such that

$$\left\| \left(df_{g_0}(g_0^{-1} \cdot g)\right)^{-1} \cdot f(g_0)^{-1} \cdot f(g) \right\|_2 = o\left(\|g_0^{-1} \cdot g\|_1\right), \quad \text{as } \|g_0^{-1} \cdot g\|_1 \to 0.$$

The H-linear function df_{g_0} is the *P-differential* of f in g_0. We say that f is *continuously P-differentiable* in $\mathcal{A}$, $f \in C^1_H(\mathcal{A}, \mathbb{G}_2)$, if f is P-differentiable in every $g \in \mathcal{A}$ and if df_g depends continuously on g.

We recall the following inequality proved in [9].

Theorem 2.3 *Let $\mathcal{U}$ be open in $\mathbb{H}^n$ and $f \in C^1_H(\mathcal{U}, \mathbb{R}^k)$. Then there are $c > 1$, $C > 0$ such that, if $B(p_0, cr) \subset \mathcal{U}$ and $p, q \in B(p_0, r)$, then*

$$\left\| f(q) - f(p) - df_p(p^{-1} \cdot q) \right\|_{\mathbb{R}^k} \le C \sup_{x \in B(p_0, cr)} \|df_x - df_p\| \cdot d(p, q).$$

3 Intrinsic graphs

3.1 Complementary subgroups and graphs

Definition 3.1 Two homogeneous subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$ of a Carnot group $\mathbb{G}$ are *complementary subgroups* in $\mathbb{G}$, and we write

$$\mathbb{G} = \mathbb{G}_1 \cdot \mathbb{G}_2,$$

if $\mathbb{G}_1 \cap \mathbb{G}_2 = \{e\}$ and, for each $g \in \mathbb{G}$, there are $g_{\mathbb{G}_1} \in \mathbb{G}_1$ and $g_{\mathbb{G}_2} \in \mathbb{G}_2$ such that $g = g_{\mathbb{G}_1} \cdot g_{\mathbb{G}_2}$. If $\mathbb{G}_1$, $\mathbb{G}_2$ are complementary subgroups in $\mathbb{G}$ and one of them is a normal subgroup then $\mathbb{G}$ is said to be a *semidirect product* of $\mathbb{G}_1$ and $\mathbb{G}_2$.

Proposition 3.2 *If $\mathbb{G} = \mathbb{G}_1 \cdot \mathbb{G}_2$ as in Definition 3.1, each $g \in \mathbb{G}$ has unique* components *$g_{\mathbb{G}_1} \in \mathbb{G}_1$, $g_{\mathbb{G}_2} \in \mathbb{G}_2$, such that $g = g_{\mathbb{G}_1} \cdot g_{\mathbb{G}_2}$. The maps*

$$g \to g_{\mathbb{G}_1} \text{ and } g \to g_{\mathbb{G}_2}$$

are continuous and there is a constant $c = c(\mathbb{G}_1, \mathbb{G}_2) > 0$ such that

$$c\left(\|g_{\mathbb{G}_1}\| + \|g_{\mathbb{G}_2}\|\right) \le \|g\| \le \|g_{\mathbb{G}_1}\| + \|g_{\mathbb{G}_2}\|. \tag{4}$$

Proof Uniqueness is trivial. The continuity of the maps $g \to g_{\mathbb{G}_1}$ and $g \to g_{\mathbb{G}_2}$ follows, e.g., by direct considerations on the form of the product in $\mathbb{G}$ (see (1)). Indeed, the m_1 components in the first layer of $g_{\mathbb{G}_1}$ and $g_{\mathbb{G}_2}$ are the components of the euclidean projections of the first m_1 components of g, hence they depend continuously on g. Now, the values of the polynomials from $Q_{m_1+1}(g_{\mathbb{G}_1}, g_{\mathbb{G}_2})$ to $Q_{m_2}(g_{\mathbb{G}_1}, g_{\mathbb{G}_2})$ are determined and depend continuously on g. In turn, the components of the second layer of $g_{\mathbb{G}_1}$ and $g_{\mathbb{G}_2}$ are given by the projections of the components from $(g_{m_1+1} - Q_{m_1+1}(g_{\mathbb{G}_1}, g_{\mathbb{G}_2}))$ to $(g_{m_2} - Q_{m_2}(g_{\mathbb{G}_1}, g_{\mathbb{G}_2}))$. Then the procedure repeats.
By homogeneity, it is enough to prove the left hand side of (4) when $\|g\| = 1$, and in this case (4) follows by a compactness argument. The right hand side of (4) is just triangular inequality. □

Definition 3.3 If $\mathbb{G}_1$, $\mathbb{G}_2$ are complementary subgroups of $\mathbb{G}$ we say that $S \subset \mathbb{G}$ is a *(left) graph from* $\mathbb{G}_1$ *to* $\mathbb{G}_2$ (or over $\mathbb{G}_1$ along $\mathbb{G}_2$) if

$$S \cap (\xi \cdot \mathbb{G}_2) \text{ contains at most one point,}$$

for all $\xi \in \mathbb{G}_1$. Equivalently, if there is $\varphi : \mathcal{E} \subset \mathbb{G}_1 \to \mathbb{G}_2$ such that

$$S = \{\xi \cdot \varphi(\xi) : \xi \in \mathcal{E}\}$$

and we say that *S is the graph of φ*, $S = \text{graph}(\varphi)$.

Remark 3.4 A more general definition of graph inside $\mathbb{G}$ can be considered. Assume that $\mathbb{H}$ is a homogeneous subgroup of $\mathbb{G}$. Even if no complementary subgroup of $\mathbb{H}$ exists in $\mathbb{G}$, we can say that a set $S \subset \mathbb{G}$ is a $\mathbb{H}$-*graph* (or a graph along $\mathbb{H}$) if S intersect each coset of $\mathbb{H}$ in at most one point. Such a notion has been used many times in the literature, mainly inside $\mathbb{H}^n$. Many authors e.g. considered sets as $S = \{(x_1, \dots, y_n, \varphi(x_1, \dots, y_n))\} \subset \mathbb{H}^n$, were φ is a real valued function. These sets are $\mathbb{T}$-graphs. We recall that $\mathbb{T}$ has no complementary subgroup in $\mathbb{H}^n$ (see Proposition 3.18).

If $S = \text{graph}(\varphi)$ with $\varphi : \mathbb{G}_1 \to \mathbb{G}_2$ then both $\delta_\lambda S$ and $q \cdot S$ are graphs from $\mathbb{G}_1$ to $\mathbb{G}_2$; if $\mathbb{G}$ is also the semidirect product of $\mathbb{G}_1$ and $\mathbb{G}_2$ then the algebraic form of the translated graph can be explicitly given (see also [20]).

Proposition 3.5 *Let $\mathbb{G}_1$, $\mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$, $\varphi : \mathcal{E} \subset \mathbb{G}_1 \to \mathbb{G}_2$ and $S = \{\xi \cdot \varphi(\xi) : \xi \in \mathcal{E}\} = \text{graph}(\varphi)$. Then, for all $\lambda > 0$, the dilated set $\delta_\lambda S$ is a graph, precisely*

$$\delta_\lambda S = graph(\varphi_\lambda),$$

with $\varphi_\lambda := \delta_\lambda \circ \varphi \circ \delta_{1/\lambda} : \delta_\lambda \mathcal{E} \subset \mathbb{G}_1 \to \mathbb{G}_2$.

Proof Just observe that $\delta_\lambda S = \delta_\lambda(\xi \cdot \varphi(\xi)) = \delta_\lambda \xi \cdot \delta_\lambda(\varphi(\xi))$. □

Proposition 3.6 *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$, $\varphi : \mathcal{E} \subset \mathbb{G}_1 \to \mathbb{G}_2$ and $S = \text{graph}(\varphi)$. Then, for any $q \in \mathbb{G}$, there are $\mathcal{E}_q \subset \mathbb{G}_1$ and $\varphi_q : \mathcal{E}_q \subset \mathbb{G}_1 \to \mathbb{G}_2$, such that*

$$\text{graph}(\varphi_q) := q \cdot S = \{\eta \cdot \varphi_q(\eta) : \ \eta \in \mathcal{E}_q\}.$$

where φ_q is as in (5). The statement can be made more explicit if we assume that $\mathbb{G}$ is the semidirect product of $\mathbb{G}_1$ and $\mathbb{G}_2$. In this case we have,

(i) *If $\mathbb{G}_1$ is normal in $\mathbb{G}$ then $\mathcal{E}_q := q \cdot \mathcal{E} \cdot (q_{\mathbb{G}_2})^{-1} \subset \mathbb{G}_1$, and, for $y \in \mathcal{E}_q$,*

$$\varphi_q(y) = q_{\mathbb{G}_2} \cdot \varphi(q_{\mathbb{G}_2}^{-1} \cdot q_{\mathbb{G}_1}^{-1} \cdot y \cdot q_{\mathbb{G}_2}).$$

(ii) *If $\mathbb{G}_2$ is normal in $\mathbb{G}$ then $\mathcal{E}_q := q_{\mathbb{G}_1} \cdot \mathcal{E} \subset \mathbb{G}_1$ and, for $y \in \mathcal{E}_q$,*

$$\varphi_q(y) = y^{-1} \cdot q_{\mathbb{G}_1} \cdot q_{\mathbb{G}_2} \cdot q_{\mathbb{G}_1}^{-1} \cdot y \cdot \varphi(q_{\mathbb{G}_1}^{-1} \cdot y).$$

Proof Observe that the map $\tau_q : \mathbb{G}_1 \to \mathbb{G}_1$ defined as $\tau_q(x) := (q \cdot x)_{\mathbb{G}_1}$ is injective. Indeed, from

$$q \cdot x = (q \cdot x)_{\mathbb{G}_1} \cdot (q \cdot x)_{\mathbb{G}_2}, \quad q \cdot x' = (q \cdot x')_{\mathbb{G}_1} \cdot (q \cdot x')_{\mathbb{G}_2}, \quad (q \cdot x)_{\mathbb{G}_1} = (q \cdot x')_{\mathbb{G}_1}$$

we get $q \cdot x \cdot (q \cdot x)_{\mathbb{G}_2}^{-1} = q \cdot x' \cdot (q \cdot x')_{\mathbb{G}_2}^{-1}$. Hence $x \cdot (q \cdot x)_{\mathbb{G}_2}^{-1} = x' \cdot (q \cdot x')_{\mathbb{G}_2}^{-1}$ and finally $x = x'$, because of the uniqueness of the components (see Proposition 3.2). Hence,

$$q \cdot S = \{(q \cdot x)_{\mathbb{G}_1} \cdot (q \cdot x)_{\mathbb{G}_2} \cdot \varphi(x) : \ x \in \mathcal{E}\} = \{y \cdot \varphi_q(y) : \ y \in \mathcal{E}_q\},$$

where, $\mathcal{E}_q = \{(q \cdot x)_{\mathbb{G}_1} : x \in \mathcal{E}\}$ and

$$\varphi_q(y) = (q \cdot \tau_q(y)^{-1})_{\mathbb{G}_2} \cdot \varphi(\tau_q(y)^{-1}) \tag{5}$$

for $y = (q \cdot x)_{\mathbb{G}_1} \in \mathcal{E}_q$. This concludes the proof of the first part.
Case (*i*): Assume $\mathbb{G}_1$ is a normal subgroup. Because $q \cdot x = q_{\mathbb{G}_1} \cdot q_{\mathbb{G}_2} \cdot x = q_{\mathbb{G}_1} \cdot q_{\mathbb{G}_2} \cdot x \cdot q_{\mathbb{G}_2}^{-1} \cdot q_{\mathbb{G}_2}$ then $(q \cdot x)_{\mathbb{G}_1} = q \cdot x \cdot q_{\mathbb{G}_2}^{-1}$. It follows that

$$\mathcal{E}_q = \{q \cdot x \cdot q_{\mathbb{G}_2}^{-1} : x \in \mathcal{E}\},$$

and that $\tau_q(y)^{-1} = q^{-1} \cdot y \cdot q_{\mathbb{G}_2}$ for $y \in \mathcal{E}_q$. Hence, for $y \in \mathcal{E}_q$, we have $\varphi_q(y) = (q_{\mathbb{G}_2} \cdot q^{-1} \cdot y \cdot q_{\mathbb{G}_2})_{\mathbb{G}_2} \cdot \varphi(q^{-1} \cdot y \cdot q_{\mathbb{G}_2}) = (q_{\mathbb{G}_1}^{-1} \cdot y \cdot q_{\mathbb{G}_2})_{\mathbb{G}_2} \cdot \varphi(q^{-1} \cdot y \cdot q_{\mathbb{G}_2}) = q_{\mathbb{G}_2} \cdot \varphi(q^{-1} \cdot y \cdot q_{\mathbb{G}_2})$.
Case (*ii*): Assume $\mathbb{G}_2$ is a normal subgroup. Then $(q \cdot x)_{\mathbb{G}_1} = (q_{\mathbb{G}_1} \cdot x \cdot x^{-1} \cdot q_{\mathbb{G}_2} \cdot x)_{\mathbb{G}_1} = q_{\mathbb{G}_1} \cdot x$. It follows that

$$\mathcal{E}_q = \{q_{\mathbb{G}_1} \cdot x : \ x \in \mathcal{E}\} = q_{\mathbb{G}_1} \cdot \mathcal{E}$$

and that $\tau_q(y)^{-1} = q_{\mathbb{G}_1}^{-1} \cdot y$ for $y \in \mathcal{E}_q$. Hence, for $y \in \mathcal{E}_q$, we have $\varphi_q(y) = (q \cdot q_{\mathbb{G}_1}^{-1} \cdot y)_{\mathbb{G}_2} \cdot \varphi(q_{\mathbb{G}_1}^{-1} \cdot y) = (y \cdot y^{-1} \cdot q_{\mathbb{G}_1} \cdot q_{\mathbb{G}_2} \cdot q_{\mathbb{G}_1}^{-1} \cdot y)_{\mathbb{G}_2} \cdot \varphi(q_{\mathbb{G}_1}^{-1} \cdot y) = y^{-1} \cdot q_{\mathbb{G}_1} \cdot q_{\mathbb{G}_2} \cdot q_{\mathbb{G}_1}^{-1} \cdot y \cdot \varphi(q_{\mathbb{G}_1}^{-1} \cdot y)$. □

3.2 Intrinsic Lipschitz graphs

The notion of *intrinsic Lipschitz continuity*, for functions acting between complementary subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$ of $\mathbb{G}$, was originally suggested by Corollary 3.17 of [13] together with the fact that a H-regular surface keeps being H-regular after a (left) translation, (see the Definition given in [15]).

We propose here a geometric definition. We say that $f : \mathbb{G}_1 \to \mathbb{G}_2$, is intrinsic Lipschitz continuous if, at each $p \in \text{graph}\,(f)$, there is an (intrinsic) closed cone with vertex p, axis $\mathbb{G}_2$ and fixed opening, intersecting graph (f) only in p. The equivalence of this definition and other ones, more algebraic, is the content of Propositions 3.10 and 3.20.

Notice also that $\mathbb{G}_1$ and $\mathbb{G}_2$ are metric spaces, being subsets of $\mathbb{G}$, hence it makes sense to speak also of *metric Lipschitz continuous* functions from $\mathbb{G}_1$ to $\mathbb{G}_2$. As usual, $f : \mathbb{G}_1 \to \mathbb{G}_2$ is said (metric) Lipschitz if there is $L > 0$ such that, for all $g, g' \in \mathbb{G}_1$,

$$\left\| f(g)^{-1} \cdot f(g') \right\| = d\left(f(g), f(g')\right) \leq Ld(g, g') = L \left\| g^{-1} \cdot g' \right\|. \tag{6}$$

The notions of intrinsic Lipschitz continuity and of Lipschitz continuity are different ones (see Example 3.21) and we will try to convince the reader, that intrinsic Lipschitz continuity seems more useful in the context of functions acting between subgroups of a Carnot group.

Let us come to the basic definitions. By *intrinsic (closed) cone* we mean

Definition 3.7 Let $\mathbb{G}_1, \mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$, $q \in \mathbb{G}$ and $\alpha > 0$. The closed *cone* $C_{\mathbb{G}_1,\mathbb{G}_2}(q, \alpha)$ with *base* $\mathbb{G}_1$*, axis* $\mathbb{G}_2$*, vertex* q*, opening* α is

$$C_{\mathbb{G}_1,\mathbb{G}_2}(q, \alpha) := q \cdot C_{\mathbb{G}_1,\mathbb{G}_2}(e, \alpha)$$

where

$$C_{\mathbb{G}_1,\mathbb{G}_2}(e, \alpha) := \left\{p \in \mathbb{G} : \left\| p_{\mathbb{G}_1} \right\| \leq \alpha \left\| p_{\mathbb{G}_2} \right\|\right\}.$$

If $0 < \alpha < \beta$, $C_{\mathbb{G}_1,\mathbb{G}_2}(q, \alpha) \subset C_{\mathbb{G}_1,\mathbb{G}_2}(q, \beta)$ and $C_{\mathbb{G}_1,\mathbb{G}_2}(e, 0) = \mathbb{G}_2$, moreover, for all $t > 0$,

$$\delta_t\left(C_{\mathbb{G}_1,\mathbb{G}_2}(e, \alpha)\right) = C_{\mathbb{G}_1,\mathbb{G}_2}(e, \alpha).$$

Definition 3.8 Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$. We say that $f : \mathcal{A} \subset \mathbb{G}_1 \to \mathbb{G}_2$ is *intrinsic Lipschitz (continuous)* in $\mathcal{A}$, if there is $M > 0$ such that, for all $q \in \text{graph}\,(f)$,

$$C_{\mathbb{G}_1,\mathbb{G}_2}(q, 1/M) \cap \text{graph}\,(f) = \{q\}. \tag{7}$$

The Lipschitz constant of f in $\mathcal{A}$ is the infimum of the numbers M such that (7) holds. An intrinsic Lipschitz function, with Lipschitz constant not exceeding $L > 0$, is called a L *Lipschitz function*.

Remark 3.9 Intrinsic Lipschitz continuity is invariant under left translations of the graph. That is, for all $q \in \mathbb{G}$,

$$f : \mathbb{G}_1 \to \mathbb{G}_2 \text{ is } L \text{ Lipschitz, if and only if } f_q : \mathbb{G}_1 \to \mathbb{G}_2 \text{ is } L \text{ Lipschitz.}$$

We give now algebraic characterizations of intrinsic Lipschitz functions.

Proposition 3.10 *Let $\mathbb{G}_1$, $\mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$. Then $f : \mathcal{E} \subset \mathbb{G}_1 \to \mathbb{G}_2$ is intrinsic Lipschitz in $\mathcal{E}$, if and only if there is $L > 0$ such that, for all $q \in$ graph (f) and for all $x \in \mathcal{A}_{q^{-1}}$,*

$$\left\| f_{q^{-1}}(x) \right\| \leq L \, \|x\| \,. \tag{8}$$

Proof The equivalence of (8) and (7) follows from Definition 3.7 and from (ii) of Proposition 3.6 observing that, if $q \in$ graph (f), then $C_{\mathbb{G}_1,\mathbb{G}_2}(q, 1/L) \cap$graph $(f) = \{q\}$ is equivalent with $C_{\mathbb{G}_1,\mathbb{G}_2}(e, 1/L) \cap$ graph $(f_{q^{-1}}) = \{e\}$. □

Remark 3.11 If $f : \mathcal{E} : \mathbb{G}_1 \to \mathbb{G}_2$ is intrinsic Lipschitz, then it is continuous. Indeed, if $f(e) = e$ then, by (8), f is continuous in e. To prove the continuity in $x \in \mathcal{E}$, observe that $f_{q^{-1}}$ is continuous in e, where $q = x \cdot f(x)$.

3.3 Intrinsic differentiable graphs

We come now to the definition of *intrinsic differentiability* for functions acting between complementary subgroups of $\mathbb{G}$. As usual differentiability amounts to the existence of approximating linear functions. *Intrinsic linear* functions, acting between complementary subgroups, are functions whose graphs are homogeneous subgroups.

Definition 3.12 Let $\mathbb{G}_1, \mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$. We say that $L : \mathbb{G}_1 \to \mathbb{G}_2$ is an *intrinsic linear function* if graph $(L) = \{g \cdot L(g) : g \in \mathbb{G}_1\}$ is an homogeneous subgroup of $\mathbb{G}$.

If $f(e) = e$ we say that $f : \mathbb{G}_1 \to \mathbb{G}_2$, is *intrinsic differentiable in e* if there is an intrinsic linear map $L : \mathbb{G}_1 \to \mathbb{G}_2$ such that, for all $g \in \mathbb{G}_1$,

$$\left\| L(g)^{-1} \cdot f(g) \right\| = o(\|g\|), \quad \text{as } \|g\| \to 0, \tag{9}$$

where $o(t)/t \to 0$ as $t \to 0^+$.

Up to this point the definition of intrinsic differentiability is the same as the definition of P-differentiability. The differences appear (see Definition 3.14), when we extend the previous notion to any point $\bar{g}$ of $\mathbb{G}_1$ using (9) in a *translation invariant* way. That is, given $\bar{g} \in \mathbb{G}_1$ we consider $\bar{p} = \bar{g} \cdot f(\bar{g})$ and the translated function $f_{\bar{p}^{-1}}$ that, by definition, satisfies $f_{\bar{p}^{-1}}(e) = e$. Now we say that f is intrinsic differentiable in $\bar{g}$ if and only if $f_{\bar{p}^{-1}}$ satisfies (9) (see Definition 3.13). We give also a uniform version of Definition 3.13 in Definition 3.16 and algebraic characterizations of both, when $\mathbb{G} = \mathbb{H}^n$, in Propositions 3.25 and 3.26.

Definition 3.13 Let $\mathbb{G}_1, \mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$ and $f : \mathcal{A} \subset \mathbb{G}_1 \to \mathbb{G}_2$ with $\mathcal{A}$ relatively open in $\mathbb{G}_1$. For $\bar{p} := \bar{g} \cdot f(\bar{g}) \in$ graph (f) we consider the translated function $f_{\bar{p}^{-1}}$ defined in the neighborhood $\mathcal{A}_{\bar{p}^{-1}}$ of e in $\mathbb{G}_1$, (see Proposition 3.5). We say that f is *intrinsic differentiable in* $\bar{g} \in \mathcal{A}$ if there is an intrinsic linear map $df_{\bar{g}} : \mathbb{G}_1 \to \mathbb{G}_2$ such that, for all $g \in \mathcal{A}_{\bar{p}^{-1}}$,

$$\left\| df_{\bar{g}}(g)^{-1} \cdot f_{\bar{p}^{-1}}(g) \right\| = o(\|g\|), \quad \text{as } \|g\| \to 0. \tag{10}$$

The map $df_{\bar{g}}$ is called the *intrinsic differential* of f.

Remark 3.14 P-differentiability and intrinsic differentiability are *different*.
Indeed, assume $\mathbb{G} = \mathbb{H}^n$, $\mathbb{G}_1 = \mathbb{W} = \{w = (0, p_2, \dots, p_{2n+1})\}$ and $\mathbb{G}_2 = \mathbb{V} = \{v =$

$(p_1, 0, \dots, 0))\}$. Then, by (ii) of Proposition 3.25, we know that $f : \mathbb{W} \to \mathbb{V}$ is intrinsic differentiable in $w \in \mathbb{W}$ if, for all $w' \in \mathbb{W}$,

$$\left\| df_w(w^{-1} \cdot w')^{-1} \cdot f(w)^{-1} \cdot f(w') \right\| = o\left(\left\| f(w)^{-1} \cdot w^{-1} \cdot w' \cdot f(w) \right\| \right);$$

while f is P-differentiable in $w \in \mathbb{W}$ if, for all $w' \in \mathbb{W}$,

$$\left\| df_w(w^{-1} \cdot w')^{-1} \cdot f(w)^{-1} \cdot f(w') \right\| = o\left(\left\| w^{-1} \cdot w' \right\| \right).$$

On the contrary, if $\mathbb{G} := \mathbb{G}_1 \times \mathbb{G}_2$, it is easy to convince oneself that

$$f : \mathbb{G}_1 \to \mathbb{G}_2 \text{ is P-differentiable}$$

if and only if

$$f : \mathbb{G}_1 \to \mathbb{G}_2 \text{ is intrinsic differentiable.}$$

Hence, intrinsic differentiability is a generalization of P-differentiability.

Remark 3.15 Intrinsic differentiability is invariant by left translations of the graph. Indeed, let $q_1 = g_1 \cdot f(g_1)$ and $q_2 = g_2 \cdot f(g_2) \in \text{graph}\,(f)$; then f is intrinsic differentiable in $g_1 \in \mathbb{G}_1$ if and only if $f_{q_1^{-1}}$ is intrinsic differentiable in e. Consequently, f is intrinsic differentiable in g_1 if and only if $f_{q_2 \cdot q_1^{-1}} \equiv (f_{q_1^{-1}})_{q_2}$ is intrinsic differentiable in g_2.

Definition 3.16 Let $\mathbb{G}_1, \mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$. We say that $f : \mathcal{A} \subset \mathbb{G}_1 \to \mathbb{G}_2$ is *uniformly intrinsic differentiable* in $\mathcal{A}$ if

(i) f is intrinsic differentiable at each $\bar{g} \in \mathcal{A}$;
(ii) $df_{\bar{g}} : \mathbb{G}_1 \to \mathbb{G}_2$ depends continuously on $\bar{g}$, that is, for each compact $\mathcal{K} \subset \mathcal{A}$, there is $\eta = \eta_{\mathcal{K}} : \mathbb{R}^+ \to \mathbb{R}^+$, with $\eta(t) \to 0$ as $t \downarrow 0$ such that

$$\sup_{g \in \mathcal{K}} \left\| df_{\bar{g}_1}(g)^{-1} \cdot df_{\bar{g}_2}(g) \right\| \le \eta\left(\left\| g_1^{-1} \cdot g_2 \right\| \right); \tag{11}$$

(iii) for each compact $\mathcal{K} \subset \mathcal{A}$, there is $\varepsilon = \varepsilon_{\mathcal{A},\mathcal{K}} : \mathbb{R}^+ \to \mathbb{R}^+$, with $\varepsilon(t) \to 0$ as $t \downarrow 0$, and such that, for all $g \in \mathcal{K}_{\bar{p}^{-1}}$ and for all $\bar{g} \in \mathcal{K}$,

$$\left\| df_{\bar{g}}(g)^{-1} \cdot f_{\bar{p}^{-1}}(g) \right\| \le \varepsilon(\|g\|)\, \|g\|. \tag{12}$$

Intrinsic differentiability implies local intrinsic Lipschitz continuity.

Proposition 3.17 *Let $\mathbb{G}_1$, $\mathbb{G}_2$ be complementary subgroups in $\mathbb{G}$ and $f : \mathcal{A} \subset \mathbb{G}_1 \to \mathbb{G}_2$ be uniformly intrinsic differentiable in $\mathcal{A}$. Then, for all $p \in \mathcal{A}$ there is $r > 0$ such that f is intrinsic Lipschitz in $\mathcal{A} \cap B(p, r)$.*

The proof is elementary.

3.4 Graphs in Heisenberg groups

In Heisenberg groups $\mathbb{H}^n$, all the notions, described in the preceding sections for general Carnot groups, can be made more explicit; the key point is the structure of complementary subgroups of $\mathbb{H}^n$, (see also [20]).

Proposition 3.18 *All homogeneous subgroups of $\mathbb{H}^n$ are either horizontal, that is contained in the horizontal fiber $H\mathbb{H}^n_e$, or vertical, that is containing the subgroup $\mathbb{T}$. A horizontal subgroup has linear dimension and metric dimension k, with $1 \le k \le n$ and it is algebraically isomorphic and isometric to $\mathbb{R}^k$. A vertical subgroup can have any dimension d, with $1 \le d \le 2n+1$, its metric dimension is $d+1$ and it is a normal subgroup.*
All couples $\mathbb{V}$, $\mathbb{W}$, of complementary subgroups of $\mathbb{H}^n$ are of the type

(i) *$\mathbb{V}$ horizontal of dimension k, $1 \le k \le n$,*
(ii) *$\mathbb{W}$ vertical and normal of dimension $2n+1-k$.*

Proof Observe that, $\mathbb{V} \subset \mathbb{H}^n$ is a homogeneous subgroup of $\mathbb{H}^n$, if and only if, $\mathbb{V} = \exp \mathfrak{v}$, where $\mathfrak{v}$ is a homogeneous subalgebra of $\mathfrak{h}$. Then, there exist linearly independent $v_1, \dots, v_k \in \mathfrak{h}$, with $1 \le k \le 2n+1$, such that $\mathfrak{v} := \text{span}\,(v_1, \dots, v_k)$ and it must be $[v_i, v_j] \in \mathfrak{v}$, for each $i, j = 1, \dots, k$. It follows that, if $\mathbb{V}$ is horizontal, that is, if $v_i \in \mathfrak{h}_1$, for each $i = 1, \dots, k$, then necessarily we have $[v_i, v_j] = 0$ for each $i, j = 1, \dots, k$ and it must be $k \le n$. Otherwise, suppose there exists $v \in \mathfrak{h}_1$, such that $v + T \in \mathfrak{v}$. Then both $\lambda v + \lambda T \in \mathfrak{v}$ and $\lambda v + \lambda^2 T \in \mathfrak{v}$, yielding that $T \in \mathfrak{v}$. Finally, observe that, if $\mathbb{V}$ is a horizontal subgroup with $\dim \mathfrak{v} = k$, then it is isomorphic and also isometric to $\mathbb{R}^k$, for, in this case, if $x, y \in \mathbb{V}$, the points $x \cdot \delta_\lambda(x^{-1} \cdot y) \in \mathbb{V}$ for each $0 \le \lambda \le 1$, form an horizontal segment connecting them. On the contrary, if $\mathbb{W}$ is a vertical subgroup with $\dim \mathfrak{w} = k$, then, in general, $\mathbb{W}$ is not isomorphic to $\mathbb{R}^k$ and it is never isometric to $\mathbb{R}^k$, having metric dimension equal to $k+1$ (see [22], Theorem 2).
The second part follows readily because, if $\mathbb{H}^n = \mathbb{G}_1 \cdot \mathbb{G}_2$, then $\mathbb{G}_1, \mathbb{G}_2$ cannot be both vertical subgroups or horizontal subgroups. □

The following characterizations of H-linear functions follows from Theorem 3.1.12 in [19] (see also [10,13]).

Proposition 3.19 *Let $1 \le k \le n$, and let $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ be the $2n \times 2n$ symplectic matrix. Then*

(i) *$L : \mathbb{R}^k \to \mathbb{H}^n$ is H-linear if and only if there is a $2n \times k$ matrix A with $A^T J A = 0$ such that, for all $x \in \mathbb{R}^k$, $L(x) = (Ax, 0)$.*
(ii) *$L : \mathbb{H}^n \to \mathbb{R}^k$ is H-linear, if and only if there is a $k \times 2n$ matrix A, such that for all $p \in \mathbb{H}^n$ $L(p) = A\pi(p)^t$.*

Proposition 3.20 *Assume $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Proposition 3.18. Then*

(i) *$f : \mathcal{A} \subset \mathbb{V} \to \mathbb{W}$ is* intrinsic Lipschitz continuous*, if and only if the map $\Phi_f : \mathcal{A} \to \mathbb{H}^n$, defined as $\Phi_f(v) := v \cdot f(v)$, is metric Lipschitz in $\mathcal{A}$, that is if and only if there is $\tilde{L} > 0$ such that, for all $v, \bar{v} \in \mathcal{A}$,*

$$\left\| \Phi_f(\bar{v})^{-1} \cdot \Phi_f(v) \right\| \le \tilde{L} \left\| \bar{v}^{-1} \cdot v \right\|, \tag{13}$$

(ii) *$f : \mathcal{A} \subset \mathbb{W} \to \mathbb{V}$ is* intrinsic Lipschitz continuous in *$\mathcal{A}$, if and only if there is $L > 0$ such that, for all $w, w' \in \mathcal{A}$,*

$$\left\| f(w)^{-1} \cdot f(w') \right\| \le L \left\| f(w)^{-1} \cdot w^{-1} \cdot w' \cdot f(w) \right\|. \tag{14}$$

Proof To prove (i) recall (ii) of Proposition 3.6. If $q = x \cdot f(x) \in \text{graph}\,(f)$ then, for all $\eta \in \mathcal{A}_{q^{-1}}$, $f_{q^{-1}}(\eta) = \eta^{-1} \cdot f(x)^{-1} \cdot \eta \cdot f(x \cdot \eta)$. Hence, from (13), setting $\eta = x^{-1} \cdot v$, we have

$$\begin{aligned}\left\| f_{q^{-1}}(\eta)\right\| &= \left\| v^{-1}\cdot x\cdot f(x)^{-1}\cdot x^{-1}\cdot v\cdot f(v)\right\|\\ &\le \left\| v^{-1}\cdot x\right\| + \left\| f(x)^{-1}\cdot x^{-1}\cdot v\cdot f(v)\right\|\\ &= \left\| v^{-1}\cdot x\right\| + \left\| \Phi_f(x)^{-1}\cdot \Phi_f(v)\right\| \le (1+\tilde{L})\left\|\eta\right\|.\end{aligned}$$

On the other side,

$$\begin{aligned}&\Phi_f(v)^{-1}\cdot\Phi_f(\bar v) = f(v)^{-1}\cdot v^{-1}\cdot \bar v\cdot f(\bar v) = f(v)^{-1}\cdot v^{-1}\cdot x\cdot v\cdot f(x\cdot v)\\ &\quad = x\cdot x^{-1}\cdot f(v)^{-1}\cdot x\cdot f(x\cdot v),\end{aligned}$$

where $x = v^{-1}\cdot\bar v$. Now from (8) we get (13).

To prove (ii) observe that, from (8) and (i) of Proposition 3.6, for any $\bar x\in\mathcal{A}$, and for any y in the domain of $f_{q^{-1}}$,

$$\left\| f_{q^{-1}}(y)\right\| = \left\| f(\bar x)^{-1}\cdot f\left(\bar x\cdot f(\bar x)\cdot y\cdot f(\bar x)^{-1}\right)\right\| \le L\left\|y\right\|.$$

Setting $x = \bar x\cdot f(\bar x)\cdot y\cdot f(\bar x)^{-1}$, that is $y = f(\bar x)^{-1}\cdot \bar x^{-1}\cdot x\cdot f(\bar x)$, then, for all $x,\bar x\in\mathcal{A}$, $\left\| f(\bar x)^{-1}\cdot f(x)\right\| \le L\left\| f(\bar x)^{-1}\cdot(\bar x^{-1}\cdot x)\cdot f(\bar x)\right\|$.
This completes the proof of (ii). □

Example 3.21 Here we show that condition (6) is not invariant under left translations of the graph. It follows that neither intrinsic Lipschitz continuity implies (metric) Lipschitz continuity nor the opposite.
Let $\mathbb{H}^1 = \mathbb{W}\cdot\mathbb{V}$ where $\mathbb{V} = \{v = (v_1,0,0)\}$ and $\mathbb{W} = \{w = (0,w_2,w_3)\}$; $\|w\| = \max\{|w_2|, |w_3|^{1/2}\}$ and $\|v\| = |v_1|$.

(1) $\varphi:\mathbb{W}\to\mathbb{V}$, defined as $\varphi(w) := \left(1+|w_3|^{1/2},0,0\right)$, satisfies (6) with $L=1$, hence φ is Lipschitz. On the contrary φ is not intrinsic Lipschitz. Indeed, let $p := (1,0,0)\in\text{graph}(\varphi)$, from Proposition 3.6 we have $\varphi_{p^{-1}}(w) = \left(|w_2+w_3|^{1/2},0,0\right)$. For $\varphi_{p^{-1}}$, (8) does not hold. This shows also that condition (6) is not invariant under graph translations.
(2) $\psi:\mathbb{W}\to\mathbb{V}$, defined as $\psi(w) := \left(1+|w_3-w_2|^{1/2},0,0\right)$, is intrinsic Lipschitz; indeed, with $p = (1,0,0)$ and $\varphi(w) := \left(|w_3|^{1/2},0,0\right)$ we have $\psi(w) = \varphi_p(w)$, so that ψ is intrinsic Lipschitz because φ is intrinsic Lipschitz. On the contrary ψ is not Lipschitz, in the sense of (6), as can be easily observed.

Analogously,

(1) the constant function $\varphi:\mathbb{V}\to\mathbb{W}$ defined as $\varphi(v) := (0,1,0)$, for all $v\in\mathbb{V}$, is Lipschitz but it is not intrinsic Lipschitz;
(2) $\psi:\mathbb{V}\to\mathbb{W}$ defined as $\psi(v) := (0,1,-v_1)$ for all $v\in\mathbb{V}$, is intrinsic Lipschitz continuous but it is not Lipschitz.

The following result is related with Proposition 3.1 of [1]. It states that, for each intrinsic Lipschitz $f:\mathbb{W}\to\mathbb{V}$, there is a distance d_f on the domain $\mathbb{W}$—the d_f distance of two points of $\mathbb{W}$ being the distance in $\mathbb{H}^n$ of the corresponding points on graph (f)—such that $f:(\mathbb{W},d_f)\to\mathbb{V}$ is Lipschitz.

Proposition 3.22 *Let $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$, as in Proposition 3.18, $f:\mathbb{W}\to\mathbb{V}$ and $\Phi_f:\mathbb{W}\to\mathbb{H}^n$ defined as $\Phi_f(w) := w\cdot f(w)$. Define, for all $w,w'\in\mathbb{W}$,*

$$\tau_f(w,w') := \|f(w)^{-1}\cdot w^{-1}\cdot w'\cdot f(w)\|.$$

If f is intrinsic L Lipschitz, then

$$c\,\tau_f(w,w') \le \|\Phi_f(w)^{-1}\cdot\Phi_f(w')\| \le (1+L)\tau_f(w,w'), \tag{15}$$

where $c = c(\mathbb{W},\mathbb{V})\in(0,1)$ is the constant in (4).

Proof From (ii) of Proposition 3.20, it follows

$$\begin{aligned}\left\|\Phi_f(w)^{-1}\cdot\Phi_f(w')\right\| &= \left\|f(w)^{-1}\cdot w^{-1}\cdot w'\cdot f(w)\cdot f(w)^{-1}\cdot f(w')\right\|\\ &\le \tau_f(w,w')+\left\|f(w)^{-1}\cdot f(w')\right\|\le(1+L)\tau_f(w,w').\end{aligned}$$

Moreover, $f(w)^{-1}\cdot w^{-1}\cdot w'\cdot f(w)=\left(\Phi_f(w)^{-1}\cdot\Phi_f(w')\right)_{\mathbb{W}}$. Hence, by (4),

$$c\,\tau_f(w,w')=c\,\left\|\left(\Phi_f(w)^{-1}\cdot\Phi_f(w')\right)_{\mathbb{W}}\right\|\le\left\|\Phi_f(w)^{-1}\cdot\Phi_f(w')\right\|.$$

□

Intrinsic linear functions in $\mathbb{H}^n$ can be characterized in terms of H-linear functions.

Proposition 3.23 *Let $\mathbb{H}^n=\mathbb{W}\cdot\mathbb{V}$ as in Proposition 3.18. Then*

(i) *$L:\mathbb{V}\to\mathbb{W}$ is intrinsic linear if and only if $\Phi_L:\mathbb{V}\to\mathbb{H}^n$, defined as $\Phi_L(v):=v\cdot L(v)$, is H-linear.*
(ii) *$L:\mathbb{W}\to\mathbb{V}$ is intrinsic linear if and only if L is H-linear.*

Proof Part (i): If Φ_L is H-linear then it is elementary to check that graph (L) is a homogeneous subgroup.
Conversely, if graph (L) is a homogeneous subgroup, then for each $v\in\mathbb{V}$ and $\lambda>0$ there is $\bar v\in\mathbb{V}$ such that $\delta_\lambda(v\cdot L(v))=\bar v\cdot L(\bar v)$. Hence $\delta_\lambda v\cdot\delta_\lambda(L(v))=\bar v\cdot L(\bar v)$. By Proposition 3.2, it follows $\delta_\lambda v=\bar v$ so that $L(\delta_\lambda v)=L(\bar v)=\delta_\lambda(L(v))$. Hence L is homogeneous and Φ_L is homogeneous too.
For all $v,v'\in\mathbb{V}$ there is $\bar v\in\mathbb{V}$ such that $v\cdot L(v)\cdot v'\cdot L(v')=\bar v\cdot L(\bar v)$ hence $v\cdot v'\cdot v'^{-1}\cdot L(v)\cdot v'\cdot L(v')=\bar v\cdot L(\bar v)$. Because $\mathbb{W}$ is normal in $\mathbb{H}^n$, then $v\cdot v'=\bar v$ and, consequently, $v'^{-1}\cdot L(v)\cdot v'\cdot L(v')=\bar v\cdot L(\bar v)$. Hence Φ_L is additive, indeed, for all $v,v'\in\mathbb{V}$,

$$\Phi_L(v\cdot v')=v\cdot v'\cdot v'^{-1}\cdot L(v)\cdot v'\cdot L(v')=\Phi_L(v)\cdot\Phi_L(v'),$$

Part (ii): Assume that $L:\mathbb{W}\to\mathbb{V}$ is H-linear. Then, as before, graph (L) is homogeneous. Because $\mathbb{W}$ is normal and $\mathbb{V}$ is commutative, then, for all $g\in\mathbb{H}^n$ and $w\in\mathbb{W}$,

$$L(g^{-1}\cdot w\cdot g)=L(w). \tag{16}$$

Indeed, (see Theorem 3.1.12 of [19]) $L(w)$ does not depend on the $(2n+1)^{\text{th}}$ component of w, while the first $2n$ components of $g^{-1}\cdot w\cdot g$ and of w coincide.
From (16), for all $w,w'\in\mathbb{W}$ we have

$$\begin{aligned}w\cdot L(w)\cdot w'\cdot L(w') &= \underbrace{w\cdot L(w)\cdot w'\cdot(L(w))^{-1}}_{=\bar w\in\mathbb{W}}\cdot\underbrace{L(w)\cdot L(w')}_{\in\mathbb{V}}\\ &=\bar w\cdot L(w)\cdot L\left(L(w)\cdot w'\cdot L(w)^{-1}\right)=\bar w\cdot L(\bar w).\end{aligned}$$

This proves that graph (L) is a homogeneous group.
Conversely, assume that graph (L) is a homogeneous group. Arguing as in Part (i), we show that L is homogeneous.
Then, for all $w,w'\in\mathbb{W}$, there is $\bar w\in\mathbb{W}$ such that $w\cdot L(w)\cdot w'\cdot L(w')=\bar w\cdot L(\bar w)$. Hence, $\bar w=w\cdot L(w)\cdot w'\cdot L(w)^{-1}$ and $L(\bar w)=L(w)\cdot L(w')$. Let $\tilde w:=L(w)\cdot w'\cdot L(w)^{-1}$, then, for all $w,\tilde w\in\mathbb{W}$,

$$L(w\cdot\tilde w)=L(w)\cdot L\left(L(w)^{-1}\cdot\tilde w\cdot L(w)\right). \tag{17}$$

Now observe that for all $w_{\mathbb{T}}\in\mathbb{W}\cap\mathbb{T}$ we have $w_{\mathbb{T}}\cdot w_{\mathbb{T}}=\delta_{\sqrt2}(w_{\mathbb{T}})$. Moreover, because $w_{\mathbb{T}}$ is in the centre of $\mathbb{H}^n$, (17) gives that $L(w_{\mathbb{T}}\cdot w_{\mathbb{T}})=L(w_{\mathbb{T}})\cdot L(w_{\mathbb{T}})$ and, in turn

$L(w_{\mathbb{T}}) \cdot L(w_{\mathbb{T}}) = \delta_2(L(w_{\mathbb{T}}))$ because $L(w_{\mathbb{T}})$ belongs to the horizontal subgroup $\mathbb{V}$. Hence, by the homogeneity of L we get that

$$\delta_{\sqrt{2}}(L(w_{\mathbb{T}})) = L(w_{\mathbb{T}} \cdot w_{\mathbb{T}}) = L(w_{\mathbb{T}}) \cdot L(w_{\mathbb{T}}) = \delta_2(L(w_{\mathbb{T}}))$$

that eventually gives

$$L(w_{\mathbb{T}}) = e. \tag{18}$$

Recall that any $w \in \mathbb{W}$ can be written in a unique way as $w = \pi(g) \cdot w_{\mathbb{T}}$, with $w_{\mathbb{T}} \in \mathbb{T}$ and $\pi(w) \in \mathbb{W} \cap H\mathbb{H}^n_e$. Hence, from (17) and (18),

$$L(w) = L(\pi(w) \cdot w_{\mathbb{T}}) = L(\pi(w)) \cdot L(w_{\mathbb{T}}) = L(\pi(w)) \tag{19}$$

for all $w \in \mathbb{W}$. So that, because $\pi(g^{-1} \cdot w \cdot g) = \pi(w)$, for all $w \in \mathbb{W}$ and $g \in \mathbb{H}^n$, from (17) and (19) and for all $w, \tilde{w} \in \mathbb{W}$, we get $L(w \cdot \tilde{w}) = L(w) \cdot L\left(\pi((L(w)^{-1} \cdot \tilde{w} \cdot L(w))\right) = L(w) \cdot L(\pi(\tilde{w})) = L(w) \cdot L(\tilde{w})$. This proves the additivity of L and concludes the Proposition. □

Example 3.24 Each condition in Proposition 3.23, cannot characterize all intrinsic linear functions. Let $\mathbb{H}^1 = \mathbb{W} \cdot \mathbb{V}$, with $\mathbb{V} = \{v = (v_1, 0, 0)\}$ and $\mathbb{W} = \{w = (0, w_2, w_3)\}$.

(1) For any fixed $a \in \mathbb{R}$, the function $L : \mathbb{V} \to \mathbb{W}$ defined as $L(v) = (0, av_1, -av_1^2/2)$ is intrinsic linear because graph $(L) = \{(t, at, 0) : t \in \mathbb{R}\}$ is a horizontal 1-dimensional subgroup of $\mathbb{H}^1$. But L is not a group homomorphism from $\mathbb{V}$ to $\mathbb{W}$.
(2) For any fixed $a \in \mathbb{R}$, the function $L : \mathbb{W} \to \mathbb{V}$ defined as $L(w) = (aw_2, 0, 0)$ is intrinsic linear because graph $(L) = \{(at, t, s) : t, s \in \mathbb{R}\}$ is a vertical 2-dimensional subgroup of $\mathbb{H}^1$. The parametric function $\Phi_L : \mathbb{W} \to \mathbb{H}^1$ acts as $\Phi_L(w) = (aw_2, w_2, w_3 - aw_2^2/2)$ and, consequently, Φ_L is not a group homomorphism from $\mathbb{V}$ to $\mathbb{H}^n$.

Proposition 3.25 *Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Proposition 3.18. Then,*

(i) *$f : \mathcal{A} \subset \mathbb{V} \to \mathbb{W}$ is intrinsic differentiable in $\bar{g} \in \mathcal{A}$ if and only if the parameterization map $\Phi_f : \mathcal{A} \to \mathbb{H}^n$, $\Phi_f(g) := g \cdot f(g)$, is P-differentiable in $\bar{g}$ and, for all $g \in \mathbb{V}$,*

$$(d\Phi_f)_{\bar{g}}(g) = g \cdot df_{\bar{g}}(g). \tag{20}$$

(ii) *$f : \mathcal{A} \subset \mathbb{W} \to \mathbb{V}$ is intrinsic differentiable in $\bar{g} \in \mathcal{A}$ if and only if there is an intrinsic linear map $df_{\bar{g}} : \mathbb{W} \to \mathbb{V}$, such that*

$$\left\|df_{\bar{g}}(\bar{g}^{-1} \cdot g)^{-1} \cdot f(\bar{g})^{-1} \cdot f(g)\right\| = o\left(\left\|f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g})\right\|\right)$$

for $g \in \mathcal{A}$ and $\left\|f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(\bar{g})\right\| \to 0$.

Proof Case (*i*): If f is intrinsic differentiable in $\bar{g}$ with intrinsic differential $df_{\bar{g}}$, by Proposition 3.23, the map $g \mapsto g \cdot df_{\bar{g}}(g)$ is H-linear from $\mathbb{V}$ to $\mathbb{H}^n$. We define

$$(d\Phi_f)_{\bar{g}} : \mathbb{V} \to \mathbb{H}^n \text{ as } (d\Phi_f)_{\bar{g}}(g) := g \cdot df_{\bar{g}}(g).$$

Observe that, from Proposition 3.6, if $\bar{p} = \bar{g} \cdot f(\bar{g})$ then

$$df_{\bar{g}}(\eta)^{-1} \cdot f_{\bar{p}^{-1}}(\eta) = df_{\bar{g}}(\eta)^{-1} \cdot \eta^{-1} \cdot f(\bar{g})^{-1} \cdot \eta \cdot f(\bar{g} \cdot \eta)$$

and, defining $g := \bar{g} \cdot \eta$,

$$\begin{aligned} &df_{\bar{g}}(\bar{g}^{-1} \cdot g)^{-1} \cdot f_{\bar{p}^{-1}}(\bar{g}^{-1} \cdot g) \\ &= df_{\bar{g}}(\bar{g}^{-1} \cdot g)^{-1} \cdot (\bar{g}^{-1} \cdot g)^{-1} \cdot f(\bar{g})^{-1} \cdot \bar{g}^{-1} \cdot g \cdot f(g) \\ &= (d\Phi_f)_{\bar{g}}(\bar{g}^{-1} \cdot g)^{-1} \cdot \Phi_f(\bar{g})^{-1} \cdot \Phi_f(g). \end{aligned} \tag{21}$$

Hence (10) yields

$$\left\|(d\Phi_f)_{\bar g}(\bar g^{-1}\cdot g)^{-1}\cdot \Phi_f(\bar g)^{-1}\cdot \Phi_f(g)\right\| = o\left(\|\bar g^{-1}\cdot g\|\right), \tag{22}$$

as $\|\bar g^{-1}\cdot g\|\to 0$, that is Φ_f is P-differentiable in $\bar g$.

Conversely, if Φ_f is P-differentiable in $\bar g$ then its P-differential $(d\Phi_f)_{\bar g}:\mathbb{V}\to\mathbb{H}^n$ is H-linear and (22) holds. From (4), both the $\mathbb{W}$ component and the $\mathbb{V}$ component of the left hand side of (22) have to be $o\left(\|\bar g^{-1}\cdot g\|\right)$. Looking at the $\mathbb{V}$ component we get

$$\left\|((d\Phi_f)_{\bar g}(\bar g^{-1}\cdot g))_{\mathbb{V}}^{-1}\cdot \bar g^{-1}\cdot g\right\| = o\left(\|\bar g^{-1}\cdot g\|\right). \tag{23}$$

Notice that, by (i) of Proposition 3.19, we get that the restriction $((d\Phi_f)_{\bar g})_{|\mathbb{V}}$ of $(d\Phi_f)_{\bar g}$ to $\mathbb{V}$ is a linear map from $\mathbb{V}\equiv\mathbb{R}^k$ to itself. Hence from (23) $((d\Phi_f)_{\bar g})_{|\mathbb{V}}$ is the identity in $\mathbb{V}$, and, in turn,

$$(d\Phi_f)_{\bar g}(g) = g\cdot L_{f,\bar g}(g), \tag{24}$$

with $L_{f,\bar g} = ((d\Phi_f)_{\bar g})_{|\mathbb{W}}:\mathbb{V}\to\mathbb{W}$.
From (24) and Proposition 3.23 we have that $L_{f,\bar g}$ is an intrinsic linear map from $\mathbb{V}$ to $\mathbb{W}$. We define the P-differential of f at $\bar g$ as

$$df_{\bar g} := L_{f,\bar g}.$$

Now, (22) and (21) yield the intrinsic differentiability of f in $\bar g$.
Case (*ii*): given (i) of Proposition 3.6, for all $\eta\in\mathbb{W}$,

$$df_{\bar g}(\eta)^{-1}\cdot f_{\bar p^{-1}}(\eta) = df_{\bar g}(\eta)^{-1}\cdot f(\bar g)^{-1}\cdot f\left(\bar g\cdot f(\bar g)\cdot\eta\cdot f(\bar g)^{-1}\right)$$

and defining g such that $\eta = f(\bar g)^{-1}\cdot\bar g^{-1}\cdot g\cdot f(\bar g)$

$$\begin{aligned}&= df_{\bar g}\left(f(\bar g)^{-1}\cdot\bar g^{-1}\cdot g\cdot f(\bar g)\right)^{-1}\cdot f(\bar g)^{-1}\cdot f(g)\\ &= df_{\bar g}\left(\bar g^{-1}\cdot g\right)^{-1}\cdot f(\bar g)^{-1}\cdot f(g).\end{aligned}$$

Now the equivalence between Definition 3.13 and (ii) is clear. □

Proposition 3.26 *Let* $\mathbb{H}^n=\mathbb{W}\cdot\mathbb{V}$, *as in Proposition* 3.18. *Then*

(i) $f:\mathcal{A}\subset\mathbb{V}\to\mathbb{W}$ *is uniformly intrinsic differentiable in* $\mathcal{A}$ *if and only if the parameterization map* $\Phi_f:\mathcal{A}\to\mathbb{H}^n$, *is continuously P-differentiable in* $\mathcal{A}$.
(ii) $f:\mathcal{A}\subset\mathbb{W}\to\mathbb{V}$ *is uniformly intrinsic differentiable in* $\mathcal{A}$ *if and only if it is intrinsic differentiable at each* $g\in\mathcal{A}$ *with differential* df_g *continuously dependent on* g *and if, for each compact* $\mathcal{K}\subset\mathcal{A}$,

$$\sup_{\substack{\bar g,g\in\mathcal{K}\\ 0<\|\bar g^{-1}\cdot g\|<\delta}}\frac{\|df_{\bar g}(\bar g^{-1}\cdot g)^{-1}\cdot f(\bar g)^{-1}\cdot f(g)\|}{\|f(\bar g)^{-1}\cdot\bar g^{-1}\cdot g\cdot f(\bar g)\|}\to 0 \text{ as } \quad \delta\to 0. \tag{25}$$

Proof *Case* (*i*): the equivalence between uniformly intrinsic differentiability of f and continuous P-differentiability of Φ_f follows immediately from (20) and applying Theorem 4.6 in [20].
Case (*ii*): because f is continuous in $\mathcal{A}$, then f is bounded in each compact $\mathcal{K}\subset\mathcal{A}$; hence, $\|f(\bar g)^{-1}\cdot\bar g^{-1}\cdot g\cdot f(\bar g)\|$ is comparable with $\|\bar g^{-1}\cdot g\|$ in $\mathcal{K}$. Now the equivalence between (25) and the two conditions (ii) and (iii) of Definition 3.16, follows from the same steps used in the proof of Case (ii) of Proposition 3.25. □

4 H-regular submanifolds are intrinsic differentiable graphs

This section contains our main theorem. We prove that $S \subset \mathbb{H}^n$ is a H-regular submanifold if and only if S is, locally, a uniformly intrinsic differentiable graph.

We recall the definitions of H-regular submanifolds, of dimension k or of codimension k (see [13] and also [10] or [26]).

Definition 4.1 Let k be an integer, $1 \le k \le n$.

(i) A subset $S \subset \mathbb{H}^n$ is a *k-dimensional* H-regular submanifold if for each $p \in S$ there are an open $\mathcal{U} \subset \mathbb{H}^n$ with $p \in \mathcal{U}$, an open $\mathcal{A} \subset \mathbb{R}^k$ and an injective, continuously Pansu differentiable $f : \mathcal{A} \to \mathcal{U}$, with injective Pansu differential, such that

$$S \cap \mathcal{U} = f(\mathcal{A}).$$

(ii) A subset $S \subset \mathbb{H}^n$ is a *k-codimensional* H-regular submanifold if for each $p \in S$ there are an open $\mathcal{U} \subset \mathbb{H}^n$, with $p \in \mathcal{U}$, and $f : \mathcal{U} \to \mathbb{R}^k$, $f \in C^1_H(\mathcal{U}; \mathbb{R}^k)$ with surjective Pansu differential, such that

$$S \cap \mathcal{U} = \{x \in \mathcal{U} : f(x) = 0\}.$$

Theorem 4.2 *The following statements are equivalent*

(1) *$S \subset \mathbb{H}^n$ is a H-regular submanifold.*
(2) *For all $p \in S$ there is an open $\mathcal{U} \subset \mathbb{H}^n$ such that $p \in \mathcal{U}$ and $S \cap \mathcal{U}$ is the graph of a uniformly intrinsic differentiable function φ acting between complementary subgroups of $\mathbb{H}^n$.*

More precisely, with $1 \le k \le n$, if S is k-dimensional H-regular then φ is defined on a k-dimensional horizontal subgroup and if S is k-codimensional H-regular then φ is defined on a $(2n+1-k)$-dimensional normal subgroup.

Proof We divide the proof in two parts: first we deal with a k-dimensional S, then with a k-codimensional S.

First part:

(1) $\Longrightarrow$ (2). In Theorem 3.5 of [13] it is proved that any k-dimensional H-regular surface S is an Euclidean C^1, k-dimensional, submanifold of $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ and that, at each $p \in S$, there is a k-dimensional horizontal subgroup $\mathbb{V}$ such that its coset $p \cdot \mathbb{V}$ is the Euclidean tangent k-plane T_pS.
Fix $p \in S$ and let $\mathbb{V} := p^{-1} \cdot T_pS$, $\mathbb{W} := \mathbb{V}^\perp$. If $\mathcal{V}$ is a small open neighborhood of the origin, then $(p^{-1} \cdot S) \cap \mathcal{V}$ is an Euclidean C^1 graph from $\mathbb{V}$ to $\mathbb{W}$, and there are an open $\mathcal{O} \subset \mathbb{V}$ and a function $\varphi : \mathcal{O} \to \mathbb{W}$, continuously differentiable in $\mathcal{O}$, such that

$$(p^{-1} \cdot S) \cap \mathcal{V} = \{v + \varphi(v) : v \in \mathcal{O}\}$$

The map $\Phi : \mathcal{O} \to \mathbb{H}^n$ defined as $\Phi(v) := v + \varphi(v)$ is Euclidean C^1; once more by Theorem 3.5 of [13], the image of the Euclidean differential $d_{euc}\Phi_v$ is an horizontal k-dimensional subgroup of $\mathbb{H}^n$, for all $v \in \mathcal{O}$, hence, by Theorem 1.1 of [20], Φ is continuously P-differentiable in $\mathcal{O}$.

Finally, $(p^{-1}\cdot S)\cap\mathcal{V}=\{v+\varphi(v): v\in\mathcal{O}\}=\{v\cdot\psi(v): v\in\mathcal{O}\}$, where $\psi:\mathcal{O}\to\mathbb{W}$ is given by

$$\psi(v)=\left(\varphi_1(v),\cdots,\varphi_{2n}(v),\varphi_{2n+1}(v)-\frac{1}{2}\sum_{i=1}^{n}(v_i\varphi_{n+i}(v)-v_{n+i}\varphi_i(v))\right).$$

By Proposition 3.26, the function ψ is uniformly intrinsic differentiable in $\mathcal{O}$ because the associated parametric map Φ_ψ is continuously P-differentiable, being $\Phi_\psi\equiv\Phi$.
Hence $(p^{-1}\cdot S)\cap\mathcal{V}=\text{graph}(\psi)$ and $S\cap(p\cdot\mathcal{V})=\text{graph}(\psi_p)$.
(2) $\Longrightarrow$ (1). Let $\mathbb{H}^n=\mathbb{W}\cdot\mathbb{V}$, as in Proposition 3.18, and $f:\mathcal{A}\subset\mathbb{V}\to\mathbb{W}$ be uniformly intrinsic differentiable in $\mathcal{A}$. Then $\Phi_f:\mathcal{A}\to\mathbb{H}^n$ is uniformly P-differentiable in $\mathcal{A}$. Hence, graph $(f)=\Phi_f(\mathcal{A})$ is a k-dimensional H-regular submanifold.

Second part:

(1) $\Longrightarrow$ (2). Let $S\subset\mathbb{H}^n$ be a H-regular surface of codimension k. By Proposition 3.12 of [13], for each $p\in S$ there are $\mathcal{U}$, open in $\mathbb{H}^n$, and $f\in C^1_H(\mathcal{U},\mathbb{R}^k)$ such that $p\in\mathcal{U}$, $S\cap\mathcal{U}=\{x\in\mathcal{U}: f(x)=0\}$, and there are complementary subgroups $\mathbb{V}$ and $\mathbb{W}$, such that $df_{x|\mathbb{V}}:\mathbb{V}\to\mathbb{R}^k$ is one to one, for all $x\in\mathcal{U}$.
Finally, there are a relatively open $\mathcal{A}\subset\mathbb{W}$ and a continuous $\varphi:\mathcal{A}\to\mathbb{V}$ such that $p_{\mathbb{W}}\in\mathcal{A}$ and $S\cap\mathcal{U}=\text{graph}(\varphi)=\{w\cdot\varphi(w): w\in\mathcal{A}\}$. We will prove that φ is uniformly intrinsic differentiable in $\mathcal{A}$.

Let $w\in\mathcal{A}$, $x=w\cdot\varphi(w)$ and define $d\varphi_w:=-\left(df_{x|\mathbb{V}}\right)^{-1}\circ df_{x|\mathbb{W}}$.
Notice that $d\varphi_w:\mathbb{W}\to\mathbb{V}$ is H-linear, hence, by (ii) of Proposition 3.23, it is intrinsic linear. Moreover, $d\varphi_w$ depends continuously on w, because $f\in C^1_H(\mathcal{U},\mathbb{R}^k)$.
We prove that, for any sufficiently small compact $\mathcal{K}\subset\mathcal{A}$,

$$\sup_{\substack{w,\eta\in\mathcal{K}\\0<\|w^{-1}\cdot\eta\|<\delta}}\frac{\|(d\varphi_w(w^{-1}\cdot\eta))^{-1}\cdot\varphi(w)^{-1}\cdot\varphi(\eta)\|}{\|\varphi(w)^{-1}\cdot w^{-1}\cdot\eta\cdot\varphi(w)\|}\to 0,\quad\text{as }\delta\to 0. \tag{26}$$

By (ii) of Proposition 3.26, (26) completes the proof.

Notice that, for all $\eta\in\mathbb{W}$ and $v\in\mathbb{V}$,

$$\begin{aligned}\left((df_{x|\mathbb{V}})^{-1}\circ df_x\right)(\eta\cdot v)&=\left(df_{x|\mathbb{V}}\right)^{-1}\left(df_{x|\mathbb{W}}(\eta)\cdot df_{x|\mathbb{V}}(v)\right)\\&=\left((df_{x|\mathbb{V}})^{-1}\circ df_{x|\mathbb{W}}\right)(\eta)\cdot v\end{aligned}\tag{27}$$

By (27) we have

$$\begin{aligned}&\left\|(d\varphi_w(w^{-1}\cdot\eta))^{-1}\cdot\varphi(w)^{-1}\cdot\varphi(\eta)\right\|\\&\quad=\left\|\left((df_{x|\mathbb{V}})^{-1}\circ df_{x|\mathbb{W}}\right)(w^{-1}\cdot\eta)\cdot\varphi(w)^{-1}\cdot\varphi(\eta)\right\|\\&\quad=\left\|(df_{x|\mathbb{V}})^{-1}\left(df_x(\Phi_\varphi(w)^{-1}\cdot\Phi_\varphi(\eta))\right)\right\|\end{aligned}$$

where $\Phi_\varphi(w):=w\cdot\varphi(w)$; by Theorem 2.3, there is $\delta>0$ such that

$$\begin{aligned}&\le\left\|(df_{x|\mathbb{V}})^{-1}\right\|\left\|f(\Phi_\varphi(\eta))-f(\Phi_\varphi(w))-df_x(\Phi_\varphi(w)^{-1}\cdot\Phi_\varphi(\eta))\right\|_{\mathbb{R}^k}\\&\le C\sup_{w\in\mathcal{K}}\left\|(df_{x|\mathbb{V}})^{-1}\right\|\sup_{\substack{w,\eta\in\mathcal{K}\\\|w^{-1}\cdot\eta\|<\delta}}\left\|df_{\Phi_\varphi(w)}-df_{\Phi_\varphi(\eta)}\right\|\ \|\Phi_\varphi(w)^{-1}\cdot\Phi_\varphi(\eta)\|.\end{aligned}$$

From Proposition 3.17, φ is intrinsic Lipschitz in $\mathcal{K}$, hence (15) gives

$$\|\Phi_\varphi(w)^{-1}\cdot\Phi_\varphi(\eta)\| \le (1+L)\tau_\varphi(w,\eta) = (1+L)\|\varphi(w)^{-1}\cdot w^{-1}\cdot\eta\cdot\varphi(w)\|$$

where $L = L(\mathcal{K})$ is the Lipschitz constant of φ.
Finally, use that df is uniformly continuous in $\tilde{\mathcal{K}} := \{w\cdot\varphi(w) : w\in\mathcal{K}\}$ and that $\|w^{-1}\cdot\eta\| < \delta$ implies $\|\Phi_\varphi(w)^{-1}\cdot\Phi_\varphi(\eta)\| < c(\mathcal{K},\delta)$, with $c(\mathcal{K},\delta)\to 0^+$ as $\delta\to 0$, to obtain (26).

(2) $\Longrightarrow$ (1). Let $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$ as in Proposition 3.18 and $\mathcal{A}$ be open in $\mathbb{W}$. We have to prove that if $\varphi : \mathcal{A}\to\mathbb{V}$ is uniformly intrinsic differentiable in $\mathcal{A}$, then $S = \{w\cdot\varphi(w),\ w\in\mathcal{A}\}$ is a k-co-dimensional H-regular submanifold. That is, we have to prove that, given $\bar p = \bar w\cdot\varphi(\bar w)\in S$, there are an open neighborhood $\mathcal{U}$ of $\bar p$ and a function $f\in C^1_H(\mathcal{U},\mathbb{R}^k)$, such that,

$$S\cap\mathcal{U} = \{x\in\mathcal{U} : f(x) = 0\} \tag{28}$$

and, for all $x\in\mathcal{U}$,

$$df_x : \mathbb{H}^n\to\mathbb{R}^k \text{ is surjective.} \tag{29}$$

Without loss of generality, we can assume that $\mathcal{A}$ is relatively compact and we define

$$\mathcal{F} := \{w\cdot\varphi(w) :\ w\in\bar{\mathcal{A}}\}.$$

For $x = w\cdot\varphi(w)\in\mathcal{F}$ and for all $p\in\mathbb{H}^n$ let

$$h_x(p) := (d\varphi_w(p_{\mathbb{W}}))^{-1}\cdot p_{\mathbb{V}}, \tag{30}$$

Notice that, for $x\in\mathcal{F}$, h_x is a H-linear function from $\mathbb{W}$ to $\mathbb{V}$.
Indeed, h_x is homogenous and, for all $p, q\in\mathbb{H}^n$, we have $h_x(p\cdot q) = (d\varphi_w[(p\cdot q)_{\mathbb{W}}])^{-1}\cdot(p\cdot q)_{\mathbb{V}} = \left(d\varphi_w(p_{\mathbb{W}}\cdot p_{\mathbb{V}}\cdot q_{\mathbb{W}}\cdot p_{\mathbb{V}}^{-1})\right)^{-1}\cdot p_{\mathbb{V}}\cdot q_{\mathbb{V}} = (d\varphi_w(p_{\mathbb{W}}))^{-1}\cdot(d\varphi_w(q_{\mathbb{W}}))^{-1}\cdot p_{\mathbb{V}}\cdot q_{\mathbb{V}} = h_x(p)\cdot h_x(q)$.

The map $x\mapsto h_x$, from $\mathcal{F}$ to the set of H-linear functions from $\mathbb{H}^n$ to $\mathbb{R}^k$, is continuous. Indeed, because of intrinsic uniform differentiability of φ in $\mathcal{A}$, $d\varphi_w$ is continuous from $\mathcal{A}$ to the set of H-linear functions from $\mathbb{W}$ to $\mathbb{V}$.
Hence, if we associate, as in Proposition 3.19, to each h_x a matrix $Q_x\in\mathbb{R}^{k,2n}$, then the map $Q:\mathcal{F}\to\mathbb{R}^{k,2n}$, sending $x\in\mathcal{F}$ to Q_x, is continuous.

Now define, for $x, y\in\mathcal{F}$, $x\neq y$,

$$R(x,y) := -\frac{h_x(x^{-1}\cdot y)}{d(x,y)}.$$

If $\mathcal{K}$ is a compact subset of $\mathcal{F}$, then

$$\sup\{\|R(x,y)\| : x, y\in\mathcal{K},\ 0 < d(x,y) < \delta\}\to 0 \text{ as } \delta\to 0. \tag{31}$$

Indeed, from (15), there exists $c = c(\mathbb{W},\mathbb{V}) > 0$ such that, for all $x = w\cdot\varphi(w)$ and $y = \eta\cdot\varphi(\eta)$ in $\mathcal{K}$, we have: $c\left\|\varphi(w)^{-1}\cdot w^{-1}\cdot\eta\cdot\varphi(w)\right\| \equiv c\tau_\varphi(w,\eta)\le d(x,y)$. Hence,

$$\|R(x,y)\| \le \frac{1}{c}\frac{\|(d\varphi_w(w^{-1}\cdot\eta))^{-1}\cdot\varphi(w)^{-1}\cdot\varphi(\eta)\|}{\left\|\varphi(w)^{-1}\cdot w^{-1}\cdot\eta\cdot\varphi(w)\right\|}\to 0.$$

Now, because φ is uniform intrinsic differentiable, (31) follows from (25).

We can now apply Whitney's theorem (see Theorem 2.10 of [11]) to the functions

$$g:\mathcal{F}\to\mathbb{R},\quad Q:\mathcal{F}\to\mathbb{R}^{k,2n},$$

where $g(x) = 0$ for all $x \in \mathcal{F}$. We get a function $f \in C^1_H(\mathbb{H}^n, \mathbb{R}^k)$, vanishing on $\mathcal{F}$ and with a surjective differential at all points of $\mathcal{F}$.

Indeed, from (30) and for all $x \in \mathcal{F}$, $h_{x|\mathbb{V}} : \mathbb{V} \to \mathbb{R}^k$ is one to one.

To conclude our proof we have to provide an open neighborhood $\mathcal{U}$ of $\bar{p}$ satisfying (28) and (29). Then, fix $r > 0$ and define $\mathcal{U}$ as

$$\mathcal{U} = \{w \cdot v \in \mathbb{H}^n : \; w \in \mathcal{I}_r \subset \mathbb{W}, \; v \in \mathbb{V} \cap B(\varphi(\bar{w}), r)\} \tag{32}$$

where $\mathcal{I}_r \subset \mathcal{A}$ is a neighborhood of $\bar{w}$, $\mathcal{I}_r$ is open in $\mathbb{W}$ and such that, $\varphi(\mathcal{I}_r) \subset \mathbb{V} \cap B(\varphi(\bar{w}), r)$. By definition $\bar{p} \in \mathcal{U}$ and, if we choose r small enough, by continuity of df on $\mathbb{H}^n$, df_x is surjective for all $x \in \mathcal{U}$, hence (29) holds.

Moreover, by continuity, for r small, $df_x : \mathbb{V} \to \mathbb{R}^k$ is one to one, for all $x \in \mathcal{U}$. Hence, for each $\tilde{w} \in \mathcal{I}_r$, the map $v \mapsto f(\tilde{w} \cdot v)$ is one to one in $\mathbb{V} \cap B(\varphi(\bar{w}), r)$. Hence, if $x = w \cdot v \in \mathcal{U}$ and if $f(x) = 0$, then $x = w \cdot \varphi(w) \in \mathcal{F}$.

So also (28) holds and the proof is completed. □

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