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Calculus of Variations

The limit of $W^{1,1}$ homeomorphisms with finite distortion

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Abstract We show that the limit f of a weakly convergent sequence of $W^{1,1}$ homeomorphisms f_j with finite distortion has finite distortion as well, provided that it is a homeomorphism. Moreover, the lower semicontinuity of the distortions is deduced both in case of outer and inner distortion.

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1 Introduction

In this paper, we study the convergence of a sequence of homeomorphisms $f_j : \Omega \mapsto \Omega'$ of Sobolev class $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ with finite distortion, where Ω and Ω' are bounded open sets in $\mathbb{R}^n$, $n \geq 2$.

Recall that a mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is said to be of *finite distortion* if its Jacobian $J_f \in L'_{\mathrm{doc}}(\Omega)$ and is strictly positive almost everywhere on the set where $Df \neq 0$. For such a mapping the *distortion* K_f is defined as

$$K_f(x) = \begin{cases} \dfrac{|Df(x)|^n}{J_f(x)} & \text{if } J_f(x) > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{1.1}$$

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Notice that $K_f(x)$ is the smallest function greater than or equal to 1 and such that

$$|Df(x)|^n \le K_f(x) J_f(x) \quad \text{for a.e. } x \in \Omega. \tag{1.2}$$

Our first result deals with the convergence of the inverse mappings f_j^{-1} of a sequence f_j of homeomorphisms of finite distortion. In fact, a recent result proved in [5] (see also [11,12], for the case $n = 2$ and [3,13,14]) states that if $f \in W^{1,n-1}(\Omega, \Omega')$ is a homeomorphism of finite distortion, then the inverse map f^{-1} belongs to $W^{1,1}(\Omega', \Omega)$ and has finite distortion too.

In particular our Theorem 3.2 shows that if f_j is a sequence of homeomorphisms of finite distortion, satisfying reasonable equi-boundedness assumptions, then the inverse mappings f_j^{-1} converge weakly in $W^{1,1}$.

In the literature the study of a sequence of mappings of finite distortion has been also considered from a different point of view, namely to find under which conditions weak limits are also maps of finite distortion. To this aim, we recall the following result, proved in [10], where the maps f_j are assumed to converge weakly in $W^{1,n}$ to f and the corresponding distortions K_{f_j} converge in the biting sense to some function K.

Theorem 1.1 *Suppose that $f_j : \Omega \mapsto \mathbb{R}^n$ is a sequence of mappings of finite distortion which converge weakly in $W^{1,n}(\Omega, \mathbb{R}^n)$ to f and suppose that the functions K_{f_j} converge in the biting sense to K. Then f has finite distortion and*

$$K_f(x) \le K(x) < \infty \quad \textit{for a.e. } x \in \Omega.$$

A more general version of this result has been proved in [15] in the context of Orlicz–Sobolev spaces.

An important tool in the proof of Theorem 1.1 is the continuity of the Jacobian operator

$$f \in W^{1,n}(\Omega, \mathbb{R}^n) \mapsto J_f \in L^1(\Omega)$$

with respect to weak convergence in $W^{1,n}$ of mappings of finite distortion and weak convergence in L^1 of Jacobians. Notice that such a continuity is not guaranteed, even in dimension $n = 2$, when we assume that mappings f_j belong only to $W^{1,1}$ and converge weakly in $W^{1,1}$. On the other hand this result pertains to mappings of finite distortion which are not necessarily one-to-one, though they are continuous, as a consequence of the required summability of their gradients.

In this paper, we present a different kind of result. On one side, we assume more on the maps f_j and f by requiring that they are both homeomorphisms, on the other side, we weaken significantly the integrability assumptions on the gradients by requiring only that $Df_j, Df \in L^1$. Denoting by $\mathrm{Hom}(\Omega, \Omega')$ the set of all homeomorphisms between Ω and Ω', our main result reads as follows.

Theorem 1.2 *Let $f_j, f \in W^{1,1}(\Omega, \mathbb{R}^n) \cap \mathrm{Hom}(\Omega, \Omega')$, with $f_j \rightharpoonup f$ weakly in $W^{1,1}(\Omega, \mathbb{R}^n)$. Assume that*

$$|Df_j(x)|^n \leqslant K_j(x) J_{f_j}(x) \quad \textit{for a.e. } x \in \Omega, \tag{1.3}$$

where $K_j : \Omega \to [1, \infty)$ is a Borel function for all j and K_j converges in the biting sense to K. Then f is a map of finite distortion and $K_f(x) \le K(x)$ for a.e. $x \in \Omega$.

Finally, we observe that in Theorem 1.2 the finite distortion assumption (1.3) can be replaced by a similar one involving inner distortion (see Theorem 4.1).

2 Preliminary results

In the sequel it will be convenient to work with a pointwise definition of a gradient of a Sobolev map. To this aim let us consider a function $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. We say that a point x is a point of *approximate continuity* if there exists $z \in \mathbb{R}^N$ such that

$$\lim_{r\to 0} \fint_{B_r(x)} |f(y) - z|\, dy = 0.$$

The vector z for which the equality above holds is called the *approximate limit* of f at x and is denoted by $f^*(x)$.

Let x be a point of approximate continuity for f. We say that f is *approximately differentiable* at x if there exists a $N \times n$ matrix, denoted by $Df(x)$, such that

$$\lim_{r\to 0} \fint_{B_r(x)} \frac{|f(y) - f^*(x) - Df(x)(y-x)|}{r}\, dy = 0. \tag{2.1}$$

The *approximate gradient* $Df(x)$ is uniquely determined by equality (2.1) and it can be easily checked that the set

$$\mathcal{D}_f = \{x \in \Omega : f \text{ is approximately differentiable at } x\}$$

is a Borel set and the map $Df : \mathcal{D}_f \mapsto \mathbb{R}^{nN}$ is a Borel map ([2, Proposition 3.71]).

In the sequel by Df we shall always denote the approximate gradient defined above. Note that if f is differentiable in the classical sense at x the approximate gradient $Df(x)$ coincides with the usual gradient. Moreover, if $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, then f is approximately differentiable almost everywhere in Ω and its approximate differential gradient coincides almost everywhere with the distributional gradient ([2, Proposition 3.83]).

Another feature of the definition (2.1) is its local nature. In fact, if $f, g \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, then ([2, Proposition 3.73])

$$Df(x) = Dg(x) \quad \text{for a.e. } x \in \mathcal{D}_f \cap \mathcal{D}_g \cap \{f = g\}. \tag{2.2}$$

Finally, we remark that definition (2.1) of approximate gradient is slightly stronger than the one introduced in [9]. However, for a Sobolev map the two definitions agree, up to a set of measure zero.

Next lemma is a technical result that will be useful in the sequel.

Lemma 2.1 *Let $f : \Omega \mapsto \Omega'$ be a one-to-one map such that $f \in W^{1,1}(\Omega, \Omega')$ and $f^{-1} \in W^{1,1}(\Omega', \Omega)$. Set $E = \{y \in \mathcal{D}_{f^{-1}} : |J_{f^{-1}}(y)| > 0\}$. Then, there exists a Borel set $A \subset E$, with $|E \setminus A| = 0$ such that $f^{-1}(A) \subset \{x \in \mathcal{D}_f : |J_f(x)| > 0\}$, with the property that*

$$Df^{-1}(y) = \left[Df\left(f^{-1}(y)\right)\right]^{-1} \quad \textit{for all } y \in A.$$

Proof Fix $\varepsilon > 0$. By a well known approximation result there exist a Lipschitz map $h : \mathbb{R}^n \mapsto \mathbb{R}^n$ and a measurable set $F_\varepsilon \subset E$, with $|E \setminus F_\varepsilon| < \varepsilon$, such that $f^{-1}(y) = h(y)$ for all $y \in F_\varepsilon$. As a consequence, recalling (2.2), we have that $Df^{-1}(y) = Dh(y)$ for a.e. $y \in F_\varepsilon$, hence $|J_h(y)| > 0$ for a.e. $y \in F_\varepsilon$.

Thus, by the Lipschitz linearization lemma of Federer ([2, Lemma 2.74] or [9, Lemma 3.2.2]), F_ε can be decomposed, up to a set of zero measure, into the union of countably many, pairwise disjoint, compact sets H_i such that for all i, the map $h_{|H_i}$ is invertible, $\left(h_{|H_i}\right)^{-1}$ is Lipschitz, h is differentiable, $|J_h(y)| > 0$ and $Df^{-1}(y) = Dh(y)$ for all $y \in H_i$. Finally, let us denote by $g_i : \mathbb{R}^n \mapsto \mathbb{R}^n$ a Lipschitz function such $g_i(x) = \left(h_{|H_i}\right)^{-1}(x)$ for all $x \in h(H_i)$. Since $h(g_i(x)) = x$ for all $x \in h(H_i)$ and $g_i(h(y)) = y$ for all $y \in H_i$, using the a.e. differentiability of Lipschitz functions and (2.2) again we easily get that for all i

$$Dh(g_i(x)) = [Dg_i(x)]^{-1} \quad \text{for a.e. } x \in h(H_i).$$

Since $g_i(x) = f(x)$ for every $x \in h(H_i)$, from the equality above we deduce that for all i there exists a null Borel set $M_i \subset h(H_i) = f^{-1}(H_i)$ such that f is approximately differentiable at every point $x \in f^{-1}(H_i) \setminus M_i$, and

$$Dh(f(x)) = [Df(x)]^{-1} \quad \text{for any } x \in f^{-1}(H_i)\backslash M_i,$$

i.e., $Dh(y) = \left[Df\left(f^{-1}(y)\right)\right]^{-1}$ for all $y \in H_i \backslash f(M_i)$. Notice that $f(M_i) = g_i(M_i)$ and thus, since g_i is a Lipschitz map, we may deduce that $f(M_i)$ is a Borel set of zero Lebesgue measure. In conclusion, recalling that $Df^{-1}(y) = Dh(y)$ for all $y \in \cup_i H_i$, we have proved that the approximate gradient $Df(x)$ exists for all $x \in \cup_i(f^{-1}(H_i)\backslash M_i)$ and

$$Df^{-1}(y) = \left[Df\left(f^{-1}(y)\right)\right]^{-1} \quad \text{for all } y \in \cup_i(H_i \backslash f(M_i)),$$

where $\cup_i(H_i \backslash f(M_i))$ is a Borel subset of F_ε of full measure. From this equality, the assertion easily follows. □

Remark 2.2 **[Validity of the Area formula]** In the sequel we are going to use the *area formula* for maps in $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n) \cap \mathrm{Hom}(\Omega, \Omega')$. To this aim, we recall that if f is such a map, and $\mathcal{D}_f$ is the set of points in Ω where f is approximately differentiable, then the area formula holds in $\mathcal{D}_f$, i.e.,

$$\int_{\mathcal{D}_f} \varphi(f(x))|J_f(x)|\,dx = \int_{f(\mathcal{D}_f)} \varphi(y)\,dy \tag{2.3}$$

for every nonnegative Borel function φ in $\mathbb{R}^n$. Equality (2.3) is proved by covering $\mathcal{D}_f$ with a countable family of measurable sets such that the restriction of f to each member of the family is a Lipschitz map ([9, Theorem 3.1.8]) and by applying the usual area formula for Lipschitz maps. In particular, denoting by $\mathcal{J}^0_f \subset \mathcal{D}_f$, the set of points where J_f is zero, we have that $|f(\mathcal{J}^0_f)| = 0$. This result can be viewed as a *weak version of the classical Sard theorem*.

Notice that, as a consequence of (2.3), we have that for any Borel set $E \subset \Omega$ and any nonnegative Borel function φ in $\mathbb{R}^n$ the following inequality holds

$$\int_E \varphi(f(x))|J_f(x)|\,dx \le \int_{f(E)} \varphi(y)\,dy. \tag{2.4}$$

However, if f satisfies the (N) Lusin condition, inequality (2.4) clearly holds as an equality.

Next theorem is a slight variant of the result proved in [5], with the only difference that the (outer) distortion K_f defined in (1.1) is replaced by the inner distortion. To this aim, let us recall that a mapping $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ is said to be of *finite inner distortion* if its Jacobian $J_f \in L^l_{\mathrm{loc}}(\Omega)$ and is strictly positive almost everywhere on the set where $\mathrm{Adj}\, Df \neq 0$. Here, if A is a $n \times n$ matrix, $\mathrm{Adj}\, A$ denotes the transpose of the cofactor matrix of A. If f is a map of finite inner distortion, similarly to (1.2), we call *inner distortion* of f the smallest function $K^I_f \ge 1$ such that

$$|\mathrm{Adj}\, Df(x)|^n \le K^I_f(x) J_f(x)^{n-1} \quad \text{for a.e. } x \in \Omega. \tag{2.5}$$

Notice that in (2.5) and in the rest of the paper, by $|A|$ we denote the operator norm of the $n \times n$ matrix A, i.e., $|A| = \sup\{|A\xi| : \xi \in \mathbb{R}^n,\ |\xi| = 1\}$.

Clearly, a map of finite (outer) distortion is also of finite inner distortion and in dimension $n = 2$ the two notions coincide. In general, as a consequence of the Hadamard inequality $|\mathrm{Adj}\, A| \le |A|^{n-1}$, we have immediately that if f has finite distortion, then $K_f^I(x) \le (K_f(x))^{n-1}$ for all x and the inequality can be strict if $n \ge 3$.

Theorem 2.3 *Let $f \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap \mathrm{Hom}(\Omega, \Omega')$ be a map such that*

$$|\mathrm{Adj}\, Df(x)|^n \le K(x) J_f(x)^{n-1} \quad \textit{a.e. in } \Omega, \tag{2.6}$$

for some Borel function $K\colon \Omega \to [1, \infty)$. Then, f^{-1} is a $W^{1,1}(\Omega'; \mathbb{R}^n)$ map of finite distortion. Moreover,

$$|Df^{-1}(y)|^n \le K(f^{-1}(y)) J_{f^{-1}}(y) \quad \textit{a.e. in } \Omega' \tag{2.7}$$

and

$$\int_{\Omega'} |Df^{-1}(y)|\, dy = \int_{\Omega} |\mathrm{Adj}\, Df(x)|\, dx. \tag{2.8}$$

Proof The proof that f^{-1} is a $W^{1,1}(\Omega'; \mathbb{R}^n)$ map goes exactly as the proof of Theorem 1.2 in [5], where the finite distortion assumption on f was used only to derive Eq. (4.6). However, one can easily check that in the proof only the weaker assumption (2.6) is actually needed. Then, the fact that f^{-1} has finite distortion follows directly from Theorem 4.5 in [5]. Thus, we are reduced to show only (2.7) and (2.8).

Notice that, since f^{-1} is a map of finite distortion, in order to prove (2.7) it is enough to restrict ourselves to the points $y \in A$, where A is the Borel set provided by Lemma 2.1. To this aim, let us denote by $F \subset \Omega$ a Borel set, with $|F| = 0$ such that (2.6) holds for all $x \in \Omega \setminus F$. Then, for any $y \in A \setminus f(F)$, from Lemma 2.1 and from (2.6) we have

$$|Df^{-1}(y)|^n = \frac{\left|\mathrm{Adj}\, Df\left(f^{-1}(y)\right)\right|^n}{J_f(f^{-1}(y))^n} \le \frac{K(f^{-1}(y))}{J_f(f^{-1}(y))} = K(f^{-1}(y)) J_{f^{-1}}(y). \tag{2.9}$$

Then, (2.7) follows, since from area formula (2.3) we get

$$\int_{A \cap f(F)} J_{f^{-1}}(y)\, dy = |f^{-1}(A) \cap F| = 0,$$

hence $|A \cap f(F)| = 0$. Using Lemma 2.1 and recalling that f^{-1} is a map of finite distortion, from (2.9) and the area formula we have

$$\begin{aligned} \int_{\Omega'} |Df^{-1}(y)|\, dy &= \int_{A} |Df^{-1}(y)|\, dy = \int_{A} \frac{\left|\mathrm{Adj}\, Df\left(f^{-1}(y)\right)\right|}{J_f(f^{-1}(y))}\, dy \\ &= \int_{A} \left|\mathrm{Adj}\, Df\left(f^{-1}(y)\right)\right| J_{f^{-1}}(y)\, dy \le \int_{\Omega} \left|\mathrm{Adj}\, Df(x)\right| dx. \end{aligned}$$

To show the opposite inequality, let us apply Lemma 2.1 again, thus getting a Borel set $\widetilde{A} \subset \widetilde{E} = \{x \in \mathcal{D}_f : J_f(x) > 0\}$, such that $|\widetilde{E} \setminus \widetilde{A}| = 0$ and $Df(x) = \left[Df^{-1}(f(x))\right]^{-1}$ for all $x \in \widetilde{A}$. Then, from the assumption (2.6) and the area formula we obtain

$$\int_{\Omega} \left|\text{Adj}\, Df(x)\right| dx = \int_{\widetilde{A}} \left|\text{Adj}\, Df(x)\right| dx = \int_{\widetilde{A}} \left|Df^{-1}(f(x))\right| J_f(x)\, dx \leq \int_{\Omega'} |Df^{-1}(y)|\, dy,$$

thus proving (2.8). □

3 Weak convergence of the inverse mappings

Let us start with the following

Lemma 3.1 *Let $f_j, f \in \text{Hom}(\Omega, \Omega')$ be such that $f_j \to f$ uniformly in Ω. Then, $f_j^{-1} \to f^{-1}$ locally uniformly in Ω'.*

Proof Fix a compact subset H of Ω'. We argue by contradiction. If f_j^{-1} does not converge uniformly to f^{-1} in H, we can find an increasing sequence $\{j_r\}$ and a corresponding sequence of points $y_{j_r} \in H$ such that $y_{j_r} \to y \in H$ and

$$\liminf_{r\to\infty} |f_{j_r}^{-1}(y_{j_r}) - f^{-1}(y_{j_r})| > 0. \tag{3.1}$$

On the other hand, by the uniform convergence of f_j to f we have that $f(f_{j_r}^{-1}(y_{j_r})) - y_{j_r} = f(f_{j_r}^{-1}(y_{j_r})) - f_{j_r}(f_{j_r}^{-1}(y_{j_r})) \to 0$ as $r \to \infty$. From this, recalling that $y_{j_r} \to y$, we deduce that $f(f_{j_r}^{-1}(y_{j_r})) \to y$, and in turn, by the continuity of f^{-1}, that

$$\lim_{r\to\infty} \left(f_{j_r}^{-1}(y_{j_r}) - f^{-1}(y_{j_r}) \right) = 0$$

which contradicts (3.1). Hence, the result follows. □

We recall that if $K : \mathbb{R}^n \to [0, \infty)$ is a measurable function with compact support, the *spherically decreasing rearrangement* K^* of K is defined by setting, for every $x \in \mathbb{R}^n$,

$$K^*(x) = \sup\left\{t \geq 0 : \left|\{K > t\}\right| > \omega_n |x|^n\right\},$$

where ω_n denotes the measure of the unit ball. Notice that from this definition one easily gets that for all $t > 0$

$$\left|\{K^* > t\}\right| = \left|\{K > t\}\right|. \tag{3.2}$$

Theorem 3.2 *Let $f_j \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap \text{Hom}(\Omega, \Omega')$ be a sequence of maps of finite inner distortions K_j. Assume that $K_j^* \leq K$ for all j, for some Borel function $K : \mathbb{R}^n \to [0, \infty)$, and that the sequence* Adj Df_j *is equi-integrable in Ω. Then f_j^{-1} is locally weakly compact in $W^{1,1}(\Omega'; \mathbb{R}^n)$.*

Moreover, if $f_j \to f \in \text{Hom}(\Omega, \Omega')$ uniformly in Ω, the maps f_j^{-1} converge weakly in $W^{1,1}(\Omega'; \mathbb{R}^n)$ and locally uniformly in Ω' to f^{-1}.

Proof From Theorem 2.3, the maps f_j^{-1} belong to $W^{1,1}(\Omega'; \mathbb{R}^n)$ and have finite distortion. Moreover, (2.8) implies that the sequence f_j^{-1} is bounded in $W^{1,1}(\Omega', \mathbb{R}^n)$, hence is locally compact in $L^1(\Omega', \mathbb{R}^n)$.

Therefore, we need only to show that the sequence Df_j^{-1} is equi-integrable. To this aim, let us set, for $j, h \in \mathbb{N}$, $F_{jh} = \{x \in \Omega : K_j(x) > h\}$. For any Borel set $E \subset \Omega'$, we have

$$\int_E |Df_j^{-1}(y)|\,dy = \int_{E\setminus f_j(F_{jh})} |Df_j^{-1}(y)|\,dy + \int_{E\cap f_j(F_{jh})} |Df_j^{-1}(y)|\,dy = I_1 + I_2. \quad (3.3)$$

If $y \in E\setminus f_j(F_{jh})$, then $K_j\left(f_j^{-1}(y)\right) \le h$, hence, by applying (2.7) to each f_j and using Hölder inequality and inequality (2.4),

$$I_1 \le h^{1/n} \int_E J_{f_j^{-1}}(y)^{1/n}\,dy \le (h|\Omega|)^{1/n}|E|^{(n-1)/n}. \quad (3.4)$$

To estimate I_2, define for $j, h \in \mathbb{N}$

$$E_{jh} = E \cap f_j(F_{jh}) \cap A_j$$

where, for all j, A_j is the set relative to f_j^{-1} provided by Lemma 2.1. Recalling that each f_j^{-1} is a map of finite distortion, from area formula (2.3) and Lemma 2.1, we get

$$\begin{aligned} I_2 &= \int_{E_{jh}} \frac{|Df_j^{-1}(y)|}{J_{f_j^{-1}}(y)} J_{f_j^{-1}}(y)\,dy = \int_{f_j^{-1}(E_{jh})} \frac{|Df_j^{-1}(f_j(x))|}{J_{f_j^{-1}}(f_j(x))}\,dx \\ &= \int_{f_j^{-1}(E_{jh})} \left|\operatorname{Adj} Df_j(x)\right| dx \le \int_{F_{jh}} \left|\operatorname{Adj} Df_j(x)\right| dx. \end{aligned}$$

From this inequality, (3.3) and (3.4), we conclude that for any measurable set $E \subset \Omega'$ and for any $j, h \in \mathbb{N}$

$$\int_E |Df_j^{-1}(y)|\,dy \le \int_{F_{jh}} \left|\operatorname{Adj} Df_j(x)\right| dx + (h|\Omega|)^{1/n}|E|^{(n-1)/n}.$$

Notice that $|F_{jh}| \to 0$ as $h \to \infty$, uniformly with respect to j, since from (3.2) we have $|F_{jh}| = \left|\{K_j^* > h\}\right| \le \left|\{K > h\}\right|$ and $K(x) < \infty$ a.e. in Ω. Therefore, from the equi-integrability of the sequence $\operatorname{Adj} Df_j$ we deduce that, given any $\varepsilon > 0$, there exists h_ε such that

$$\sup_{j\in\mathbb{N}} \int_{F_{jh_\varepsilon}} \left|\operatorname{Adj} Df_j(x)\right| dx < \varepsilon.$$

Therefore, if $|E| < \dfrac{\varepsilon^{n/(n-1)}}{(h_\varepsilon|\Omega|)^{1/(n-1)}}$, we get that for all j

$$\int_E |Df_j^{-1}(y)|\,dy < 2\varepsilon,$$

thus proving the equi-integrability of the sequence Df_j^{-1}.

If we assume in addition that $f_j \to f \in \operatorname{Hom}(\Omega, \Omega')$ uniformly in Ω the local uniform convergence of f_j^{-1} to f^{-1} follows from Lemma 3.1. Moreover, since Ω and Ω' are both bounded, we have also that f_j^{-1} to f^{-1} in $L^1(\Omega', \mathbb{R}^n)$. Hence, the weak convergence in $W^{1,1}(\Omega', \mathbb{R}^n)$ easily follows from the equi-integrability of the sequence Df_j^{-1}. □

4 Lower semicontinuity of the distortion

In this section, we establish the lower semicontinuity of the distortions of a sequence of homeomorphisms converging weakly in $W^{1,1}$ (see Corollary 4.2 below). This property is an immediate consequence of Theorem 1.2 whose proof is also given here.

To this aim, let us recall that a sequence of measurable functions $h_j : \Omega \to \mathbb{R}$ is said to converge in the *biting sense* in Ω to a measurable function $h : \Omega \to \mathbb{R}$ if there exists an increasing sequence of measurable sets $E_k \subset \Omega$, with $\cup_k E_k = \Omega$, such that $h_j, h \in L^1(E_k)$ for all j, k and $h_j \rightharpoonup h$ weakly in $L^1(E_k)$ for all k.

An important feature of this convergence is the property that if h_j is a sequence bounded in $L^1(\Omega)$, then there exists a subsequence h_{j_r} converging in the biting sense in Ω (see [4] or [1, Lemma 1.6]).

Proof of Theorem 1.2 Since f_j converges to f weakly in $W^{1,1}(\Omega, \mathbb{R}^n)$, passing possibly to a subsequence, we may assume without loss of generality that $f_j(x) \to f(x)$ a.e. in Ω.

For any $\sigma > 0$ denote by $\Omega_\sigma \subset \Omega$ a measurable set such that $f_j \to f$ uniformly in Ω_σ, $K_j \rightharpoonup K$ weakly in $L^1(\Omega_\sigma)$ and $|\Omega \backslash \Omega_\sigma| < \sigma$. For all $M > 1$ we set

$$L_M = \left\{x \in \mathcal{D}_f : K(x) + |Df(x)| \leq M\right\} \backslash f^{-1}\left(\mathcal{J}^0_{f^{-1}}\right),$$

where $\mathcal{D}_f$ is the set of points where f is approximately differentiable and $\mathcal{J}^0_{f^{-1}}$ is the set of points in $\mathcal{D}_{f^{-1}}$, where $J_{f^{-1}} = 0$. We are going to show that

$$\int_H |Df(x)|^n \, dx \leq \int_H K(x) J_f(x) \, dx \quad \text{for all compact sets } H \subset L_M \cap \Omega_\sigma. \tag{4.1}$$

In fact, once this inequality is proved, since $\mathcal{D}_f$ has full measure in Ω and by the weak Sard theorem $|f^{-1}\left(\mathcal{J}^0_{f^{-1}}\right)| = 0$, from the arbitrariety of H, M and σ we easily conclude that $|Df(x)|^n \leq K(x)J_f(x)$ for a.e. $x \in \Omega$.

So, let us fix a compact subset H of $L_M \cap \Omega_\sigma$. Given a nonnegative function $\varphi \in C_0(\Omega)$, from the assumption (1.3) and the weak convergence of f_j to f in $W^{1,1}(\Omega, \mathbb{R}^n)$, we immediately get

$$\int_H |Df(x)|\varphi(x) \, dx \leq \liminf_{j\to\infty} \int_H |Df_j(x)|\varphi(x) \, dx \leq \liminf_{j\to\infty} \int_H \left(K_{f_j}(x) J_{f_j}(x)\right)^{1/n} \varphi(x) \, dx. \tag{4.2}$$

Let us now denote by ψ a bounded, strictly positive, continuous function in Ω. By applying Hölder inequality (once if $n = 2$ and twice if $n \geq 3$) and inequality (2.4) for f_j, we get

$$\begin{aligned}
\int_H \left(K_j J_{f_j}\right)^{1/n} \varphi\, dx &\leq \left(\int_H (K_j\psi)^{\frac{1}{n-1}} \varphi^{\frac{n(n-2)}{(n-1)^2}}\, dx\right)^{\frac{n-1}{n}} \left(\int_H \frac{\varphi^{\frac{n}{n-1}}(x) J_{f_j}(x)}{\psi(x)}\, dx\right)^{\frac{1}{n}} \\
&\leq \left(\int_H K_j(x)\psi(x)\, dx\right)^{\frac{1}{n}} \left(\int_H \varphi^{\frac{n}{n-1}}(x)\, dx\right)^{\frac{n-2}{n}} \left(\int_H \frac{\varphi^{\frac{n}{n-1}}(x) J_{f_j}(x)}{\psi(x)}\, dx\right)^{\frac{1}{n}} \qquad (4.3)\\
&\leq \left(\int_H K_j(x)\psi(x)\, dx\right)^{\frac{1}{n}} \left(\int_H \varphi^{\frac{n}{n-1}}(x)\, dx\right)^{\frac{n-2}{n}} \left(\int_{\Omega'} \frac{\varphi^{\frac{n}{n-1}}(f_j^{-1}(y))}{\psi(f_j^{-1}(y))} \chi_H(f_j^{-1}(y))\, dy\right)^{\frac{1}{n}}.
\end{aligned}$$

Fix $y \in \Omega'$. Notice that if there exists a subsequence f_{j_r} of f_j such that $f_{j_r}^{-1}(y) \in H$, the same argument used in the proof of Lemma 3.1 gives immediately that $f_{j_r}^{-1}(y) \to f^{-1}(y)$. As a consequence, we get that

$$\limsup_{j\to\infty} \frac{\varphi^{\frac{n}{n-1}}(f_j^{-1}(y))}{\psi(f_j^{-1}(y))} \chi_H(f_j^{-1}(y)) \leq \frac{\varphi^{\frac{n}{n-1}}(f^{-1}(y))}{\psi(f^{-1}(y))} \chi_H(f^{-1}(y)) \quad \text{for all } y \in \Omega'.$$

Thus, combining (4.2) and (4.3), and passing to the limit as $j \to \infty$, by Fatou Lemma and the weak convergence of K_j in H, we get

$$\begin{aligned}
&\int_H |Df(x)|\varphi(x)\, dx \\
&\quad\leq \limsup_{j\to\infty} \left(\int_H K_j\psi\, dx\right)^{\frac{1}{n}} \left(\int_H \varphi^{\frac{n}{n-1}}(x)\, dx\right)^{\frac{n-2}{n}} \left(\int_{\Omega'} \frac{\varphi^{\frac{n}{n-1}}(f_j^{-1}(y))}{\psi(f_j^{-1}(y))} \chi_H(f_j^{-1}(y))\, dy\right)^{\frac{1}{n}} \\
&\quad\leq \left(\int_H K\psi\, dx\right)^{\frac{1}{n}} \left(\int_H \varphi^{\frac{n}{n-1}}(x)\, dx\right)^{\frac{n-2}{n}} \left(\int_{f(H)} \frac{\varphi^{\frac{n}{n-1}}(f^{-1}(y))}{\psi(f^{-1}(y))}\, dy\right)^{\frac{1}{n}}. \qquad (4.4)
\end{aligned}$$

Now, let us fix $m \in \mathbb{N}$ and set $E_m = \left\{x \in \mathcal{D}_f : 1/m \leq J_f(x) \leq m\right\}$. Given $\varepsilon > 0$, we denote by ψ_h a sequence of continuous, equibounded functions such that $\psi_h(x) \geq \varepsilon$ for all $x \in \Omega$ such that

$$\psi_h(x) \to J_f(x)\chi_{E_m}(x) + \varepsilon \quad \text{for a.e. } x \in \Omega.$$

Recall that $H \subset \mathcal{D}_f$ and that , by (2.3), $f_{|\mathcal{D}_f}$ satisfies the Lusin (N) property. Thus,

$$\psi_h(f^{-1}(y)) \to J_f(f^{-1}(y))\chi_{E_m}(f^{-1}(y)) + \varepsilon \quad \text{for a.e. } y \in f(H).$$

Thus, inserting ψ_h in place of ψ in (4.4) and passing to the limit, first as $h \to \infty$ and then as $m \to \infty$, we get

$$\int_H |Df|\varphi dx \le \left(\int_H K(J_f(x)+\varepsilon)dx\right)^{\frac{1}{n}} \left(\int_H \varphi^{\frac{n}{n-1}} dx\right)^{\frac{n-2}{n}} \times \left(\int_{f(H)} \frac{\varphi^{\frac{n}{n-1}}(f^{-1}(y))}{J_f(f^{-1}(y))\chi_E(f^{-1}(y))+\varepsilon} dy\right)^{\frac{1}{n}},$$

where $E = \{x \in \mathcal{D}_f : J_f(x) > 0\}$.

Recalling that $|f(\mathcal{J}_f^0)| = 0$, we have that $|f(H \setminus E)| = 0$, hence $\chi_E(f^{-1}(y)) = 1$ for a.e. $y \in f(H)$. Thus, letting $\varepsilon \to 0$ in the inequality above, we get

$$\int_H |Df(x)|\varphi(x)dx \le \left(\int_H K(x)J_f(x)dx\right)^{\frac{1}{n}} \times \left(\int_H \varphi^{\frac{n}{n-1}}(x)dx\right)^{\frac{n-2}{n}} \left(\int_{f(H\cap E)} \frac{\varphi^{\frac{n}{n-1}}(f^{-1}(y))}{J_f(f^{-1}(y))} dy\right)^{\frac{1}{n}}.$$

By the definition of L_M, it follows that $f(H \cap E) \cap \mathcal{J}_{f^{-1}}^0 = \emptyset$. Therefore, since $\Omega' \setminus \mathcal{D}_{f^{-1}}$ is a null set, from Lemma 2.1 we have that $J_{f^{-1}}(y) = 1/J_f(f^{-1}(y))$ for a.e. $y \in f(H \cap E)$ and thus, using (2.4), we get

$$\begin{aligned}
\int_H |Df|\varphi\, dx &\le \left(\int_H K(x)J_f(x)\, dx\right)^{\frac{1}{n}} \\
&\quad \times \left(\int_H \varphi^{\frac{n}{n-1}}(x)\, dx\right)^{\frac{n-2}{n}} \left(\int_{f(H\cap E)} \varphi^{\frac{n}{n-1}}(f^{-1}(y))J_{f^{-1}}(y)\, dy\right)^{\frac{1}{n}} \\
&\le \left(\int_H K(x)J_f(x)\, dx\right)^{\frac{1}{n}} \left(\int_H \varphi^{\frac{n}{n-1}}(x)\, dx\right)^{\frac{n-2}{n}} \left(\int_{H\cap E} \varphi^{\frac{n}{n-1}}(x)\, dx\right)^{\frac{1}{n}} \\
&\le \left(\int_H K(x)J_f(x)\, dx\right)^{\frac{1}{n}} \left(\int_H \varphi^{\frac{n}{n-1}}(x)\, dx\right)^{\frac{n-1}{n}}.
\end{aligned}$$

Finally, let us replace φ in this inequality by φ_h, where $\varphi_h \in C_0(\Omega)$, $0 \le \varphi_h(x) \le M^{n-1}$ for all $h \in \mathbb{N}$ and any $x \in \Omega$ and

$$\varphi_h(x) \to |Df(x)|^{n-1} \quad \text{for a.e. } x \in L_M.$$

Then, letting $h \to \infty$, we get

$$\int_H |Df(x)|^n \, dx \leq \left(\int_H K(x) J_f(x) \, dx \right)^{\frac{1}{n}} \left(\int_H |Df(x)|^n \, dx \right)^{\frac{n-1}{n}}, \tag{4.5}$$

hence (4.1) follows. This concludes the proof. □

A slightly different result is obtained with a simple variant of the argument used in the proof of Theorem 1.2.

Theorem 4.1 *Let $f_j, f \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap \mathrm{Hom}(\Omega, \Omega')$ with $f_j \rightharpoonup f$ weakly in $W^{1,n-1}(\Omega, \mathbb{R}^n)$. Assume that*

$$\left|\mathrm{Adj}\, Df_j(x)\right|^n \leq K_j(x) J_{f_j}^{n-1} \quad \textit{for a.e. } x \in \Omega, \tag{4.6}$$

where $K_j : \Omega \to [1, \infty)$ is a Borel function and K_j converges to K in the biting sense. Then f has finite inner distortion $K_f^I \leq K(x)$ for a.e. $x \in \Omega$.

Proof Assume $n \geq 3$, since for $n = 2$ the assertion reduces to Theorem 1.2.

As in the proof of Theorem 1.2, we start by observing that $f_j(x) \to f(x)$ a.e. in Ω and that for any $\sigma > 0$ there exists a measurable set $\Omega_\sigma \subset \Omega$ such that $f_j \to f$ uniformly in Ω_σ, $K_j \rightharpoonup K$ in $L^1(\Omega_\sigma)$, with $|\Omega \backslash \Omega_\sigma| < \sigma$.

For $M > 1$ we set

$$L_M = \{x \in \mathcal{D}_f : K(x) + \left|\mathrm{Adj}\, Df_j(x)\right| \leq M\} \backslash f^{-1}\left(\mathcal{J}^0_{f^{-1}}\right).$$

Our aim is to show that for every compact set $H \subset L_M \cap \Omega_\sigma$ we have

$$\int_H \left|\mathrm{Adj}\, Df_j(x)\right|^n dx \leq \int_H K(x) J_f(x)^{n-1} \, dx. \tag{4.7}$$

Indeed, as before, establishing this inequality will conclude the proof.

Thus, let us fix a compact set H and a nonnegative function $\varphi \in C_0(\Omega)$. Setting for all $(x, A) \in \Omega \times \mathbb{R}^{n^2}$

$$F(x, A) = \chi_H(x)\varphi(x)\left|\mathrm{Adj}\, A\right|,$$

F turns out to be a polyconvex integrand with growth $(n-1)$. Therefore, using the lower semicontinuity theorem by Acerbi–Fusco ([1]), from (4.6) we have

$$\begin{aligned} \int_H \left|\mathrm{Adj}\, Df(x)\right| \varphi(x) dx &\leq \liminf_{j \to \infty} \int_H \left|\mathrm{Adj}\, Df_j(x)\right| \varphi(x) dx \\ &\leq \liminf_{j \to \infty} \int_H \left(K_j(x) J_{f_j}(x)^{n-1}\right)^{\frac{1}{n}} \varphi(x) dx. \end{aligned}$$

Let us denote by ψ a strictly positive and bounded continuous function in Ω. By Hölder's inequality and (2.4) we get

$$\int_H \left(K_j(x)J_{f_j}(x)^{n-1}\right)^{\frac{1}{n}}\varphi(x)\,dx \le \left(\int_H K_j(x)\psi^{n-1}(x)\,dx\right)^{\frac{1}{n}}\left(\int_H \frac{J_{f_j}(x)\varphi^{\frac{n}{n-1}}(x)}{\psi(x)}\,dx\right)^{\frac{n-1}{n}}$$
$$\le \left(\int_H K_j(x)\psi^{n-1}(x)\,dx\right)^{\frac{1}{n}}\left(\int_{\Omega'} \frac{\varphi^{\frac{n}{n-1}}(f_j^{-1}(y))}{\psi(f_j^{-1}(y))}\chi_H(f_j^{-1}(y))\,dy\right)^{\frac{n-1}{n}}$$

and then, arguing exactly as in the proof of Theorem 1.2, we deduce first the inequality

$$\int_H \left|\mathrm{Adj}\, Df(x)\right|\varphi(x)\,dx \le \left(\int_H K(x)\psi^{n-1}(x)\,dx\right)^{\frac{1}{n}}\left(\int_{f(H)} \frac{\varphi^{\frac{n}{n-1}}(f^{-1}(y))}{\psi(f^{-1}(y))}\,dy\right)^{\frac{n-1}{n}}$$

and then

$$\int_H \left|\mathrm{Adj}\, Df(x)\right|\varphi(x)\,dx \le \left(\int_H K(x)J_f(x)^{n-1}\,dx\right)^{\frac{1}{n}}\left(\int_H \varphi^{\frac{n}{n-1}}(x)\,dx\right)^{\frac{n-1}{n}}.$$

Finally, replace in this inequality φ by $\varphi_h \in C_0(\Omega)$, $0 \le \varphi_h \le M^{n-1}$, such that

$$\varphi_h(x) \to \left|\mathrm{Adj}\, Df(x)\right|^{n-1} \qquad \text{for a.e. } x \in L_M$$

and let $h \to \infty$ to obtain

$$\int_H \left|\mathrm{Adj}\, Df(x)\right|^n dx \le \left(\int_H K(x)J_f^{n-1}\,dx\right)^{\frac{1}{n}}\left(\int_H \left|\mathrm{Adj}\, Df(x)\right|^n dx\right)^{\frac{n-1}{n}}.$$

From this inequality (4.7) follows, thus concluding the proof. □

Corollary 4.2 *Let $f_j, f \in W^{1,1}(\Omega,\mathbb{R}^n)\cap \mathrm{Hom}(\Omega,\Omega')$, with $f_j \rightharpoonup f$ weakly in $W^{1,1}(\Omega,\mathbb{R}^n)$. Assume that the maps f_j have all finite distortions K_{f_j} and that the sequence K_{f_j} is bounded in $L^1(\Omega)$. Then f is a map with finite distortion and*

$$\int_\Omega K_f(x)\,dx \le \liminf_{j\to\infty}\int_\Omega K_{f_j}(x)\,dx. \tag{4.8}$$

Proof In order to prove (4.8) we may assume without loss of generality that the lim inf on the right hand side is a limit. If this is the case, there exists a subsequence $K_{f_{j_r}}$ converging in the biting sense to a measurable function $\widetilde{K}$. Thus, from Theorem 1.2, we have

$$\int_\Omega K_f(x)\,dx \le \int_\Omega \widetilde{K}(x)\,dx \le \lim_{r\to\infty}\int_\Omega K_{f_{j_r}}(x)\,dx$$

and the assertion follows.

A similar result clearly holds for the inner distortions if in Corollary 4.2 we assume that the maps f_j, f satisfy the asssumptions of Theorem 4.1.

Example 4.3 Let $\varphi : \mathbb{R} \to [c, +\infty)$, $c > 0$, be a 1-periodic function, strictly increasing in $(0, 1)$ and such that $\int_0^1 \varphi dt = 1$, but $\varphi \notin L^p((0, 1))$ for all $p > 1$. Set, for all $j \in \mathbb{N}$, $(x, y) \in \Omega = (0, 1) \times (0, 1)$,

$$f_j(x, y) = \left(\int_0^x \varphi(jt)dt, \int_0^y \varphi(jt)dt \right).$$

Then, f_j is a sequence of homeomorphisms from Ω onto Ω weakly converging to the identity map f in $W^{1,1}(\Omega, \Omega)$. All maps f_j are of finite distortion and for a.e. $(x, y) \in \Omega$

$$K_{f_j}(x, y) = \frac{\max\{\varphi^2(jx), \varphi^2(jy)\}}{\varphi(jx)\varphi(jy)}.$$

Thus, the functions K_{f_j} converge weakly in $L^1(\Omega)$ to the constant function

$$K \equiv \int_0^1 \int_0^1 \frac{\max\{\varphi^2(s), \varphi^2(t)\}}{\varphi(s)\varphi(t)} dsdt.$$

Recalling that φ is strictly increasing in $(0, 1)$, we easily get that

$$K \equiv 2 \int_0^1 \varphi(s)\, ds \int_0^s \frac{1}{\varphi(t)} dt > 2 \int_0^1 \varphi(s) \frac{s}{\varphi(s)} ds = 1 \equiv K_f,$$

thus showing that the inequality $K_f \leq K$ provided by Theorem 1.2 can be everywhere strict even in very simple situations. Notice also that since $f_j \notin W^{1,2}$ for all j, Theorem 1.1 does not apply to this example.

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