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Calculus of Variations

Harmonic maps from manifolds of L^∞-Riemannian metrics

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Abstract For a bounded domain $\Omega \subset \mathbf{R}^n$ endowed with L^∞-metric g, and a C^5-Riemannian manifold $(N, h) \subset \mathbf{R}^k$ without boundary, let $u \in W^{1,2}(\Omega, N)$ be a weakly harmonic map, we prove that (1) $u \in C^\alpha(\Omega, N)$ for $n = 2$, and (2) for $n \geq 3$, if, in additions, $g \in \text{VMO}(\Omega)$ and u satisfies the quasi-monotonicity inequality (1.5), then there exists a closed set $\Sigma \subset \Omega$, with $H^{n-2}(\Sigma) = 0$, such that $u \in C^\alpha(\Omega\backslash\Sigma, N)$ for some $\alpha \in (0, 1)$.

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1 Introduction

For $n \geq 2$, let Ω be a bounded domain in $\mathbf{R}^n$. Throughout this paper, let g be a bounded (or L^∞), measurable Riemannian metric on $\mathbf{R}^n$, namely, there exists $\Lambda > 0$ such that $g = \sum_{\alpha,\beta=1}^n g_{\alpha\beta}\, dx_\alpha\, dx_\beta$ satisfies:

$$\Lambda^{-1} I_n \leq (g_{\alpha\beta})(x) \leq \Lambda I_n, \quad \forall x \in \mathbf{R}^n. \tag{1.1}$$

Let $(N, h) \subset \mathbf{R}^k$ be a compact, at least C^5-Riemannian manifold without boundary, isometrically embedded into an Euclidean space $\mathbf{R}^k$. For $1 < p < \infty$, define the Sobolev space $W^{1,p}(\Omega, N)$ by

$$W^{1,p}(\Omega, N) := \{u : \Omega \to \mathbf{R}^k \mid E_p(u) < +\infty, \quad u(x) \in N \quad \text{for a.e. } x \in \Omega\}$$

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where

$$E_p(u) = \int_\Omega \left(\sum_{i=1}^k |\nabla u^i|_g^2\right)^{\frac{p}{2}} dv_g$$

is the pth Dirichlet energy of u with respect to g,

$$|\nabla u^i|_g^2 = \sum_{\alpha,\beta=1}^n g^{\alpha\beta} \frac{\partial u^i}{\partial x_\alpha} \frac{\partial u^i}{\partial x_\beta}, \quad 1 \le i \le k,$$

where $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, and $dv_g = \sqrt{g}\,dx = \sqrt{\det(g_{\alpha\beta})}\,dx$ is the volume element of (Ω, g).

Let $d_g(x, y)$ and $d_0(x, y) \equiv |x - y|$ be the distance functions with respect to g and g_0 (the Euclidean metric), respectively. Since g is L^∞-Riemannian metric on $\mathbf{R}^n$, it is easy to see that there exists $0 < C_\Lambda < +\infty$ such that

$$C_\Lambda^{-1} d_0(x, y) \le d_g(x, y) \le C_\Lambda d_0(x, y), \quad \forall x,\ y \in \mathbf{R}^n. \tag{1.2}$$

In particular, $f \in C^\alpha(\Omega, N)$ with respect to g iff $f \in C^\alpha(\Omega, N)$ with respect to g_0, and for any open set $U \subset \mathbf{R}^m$ and $1 \le p < +\infty$,

$$C_\Lambda^{-1} \int_U |h|_g^p\, dv_g \le \int_U |h|^p\, dx \le C_\Lambda \int_U |h|_g^p\, dv_g \tag{1.3}$$

holds for any vector field $h \in L^p(U, \mathbf{R}^n)$, here $|h| = (\sum_{i=1}^n h_i^2)^{\frac{1}{2}}$ and dx is the volume element of g_0.

Definition 1 A map $u \in W^{1,2}(\Omega, N)$ is a weakly harmonic map, if it is a critical point of $E_2(\cdot)$.

It is readily seen that any weakly harmonic map $u \in W^{1,2}(\Omega, N)$ satisfies the harmonic map equation:

$$\Delta_g u + A_g(u)(\nabla u, \nabla u) = 0, \quad \text{in } \mathcal{D}'(\Omega) \tag{1.4}$$

where $\Delta_g = \frac{1}{\sqrt{g}} \sum_{\alpha,\beta=1}^n \frac{\partial}{\partial x_\alpha}(\sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x_\beta})$ is the Laplace-Beltrami operator of (Ω, g), and $A(y)(\cdot,\cdot) : T_y N \times T_y N \to (T_y N)^\perp,\ y \in N$ is the second fundamental form of $N \subset \mathbf{R}^k$, and

$$A_g(u)(\nabla u, \nabla u) = \sum_{\alpha,\beta=1}^n g^{\alpha\beta} A(u) \left(\frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta}\right).$$

Regularity of harmonic maps from manifolds with C^∞-Riemannian metrics g has been extensively studied by many people. Schoen-Uhlenbeck [21], Giaquinta-Guisti [9] independently proved that any minimizing harmonic map is smooth off a closed set whose Hausdorff dimension is at most $(n - 3)$. Hélein [12,13] proved that any weakly harmonic map from a Riemannian surface is smooth. Evans [6] and Bethuel [1] proved that any stationary harmonic map in dimensions at least three is smooth off a closed set of zero $(n-2)$-dimensional Hausdorff measure.

In this paper, we are mainly interested in seeking the minimal regularity assumption on Riemannian metrics g such that any weakly harmonic map $u \in W^{1,2}(\Omega, N)$ enjoys (partial) Hölder continuity.

In this context, our first theorem is

Theorem A *For $n = 2$ and a L^∞-Riemannian metric g on $\mathbf{R}^n$, let $u \in W^{1,2}(\Omega, N)$ be a weakly harmonic map. Then $u \in C^\alpha(\Omega, N)$ for some $\alpha \in (0, 1)$.*

Remark 1 For $n \geq 2$, if, in addition, $g \in C^{m,\beta}(\Omega)$ for some $m \geq 0$ and $\beta \in (0, 1)$ and $N \in C^{m+5}$, then theorem A and the theory of higher regularity of harmonic maps (cf. Giaquinta [8]) imply that if $u \in C^\alpha(\Omega, N)$, then $u \in C^{m+1,\delta}(\Omega, N)$ for $\delta = \min\{\alpha, \beta\}$.

For $n \geq 3$, Riviére [19] constructed an example of weakly harmonic map from B^3 to S^2 that is singular everywhere. It turns out that the stationarity or suitable energy monotonicity inequality plays a crucial role for the partial regularity of weakly harmonic maps. To this end, we introduce

Definition 2 (quasi-monotonicity inequality) A map $u \in W^{1,2}(\Omega, N)$ enjoys the quasi-monotonicity inequality property, if there exist $K = K(n, g) > 0$ and $r_0 = r_0(n, g) > 0$ such that for any $x \in \Omega$ and $0 < r \leq R < \min\{r_0, \mathrm{dist}(x, \partial\Omega)\}$, we have

$$r^{2-n} \int_{B_r(x)} |\nabla u|^2\, dx \leq K R^{2-n} \int_{B_R(x)} |\nabla u|^2\, dx. \tag{1.5}$$

Remark 2 (a) For $n = 2$, (1.5) holds automatically for $u \in W^{1,2}(\Omega, N)$ with $K = 1$.

(b) For $n \geq 3$ and $g \in C^2(\Omega)$, it is well-known that both minimizing harmonic maps and stationary (or C^2)-harmonic maps enjoy the quasi-monotonicity inequality property (cf. [21], Preiss [18], and Schoen [20]).

(c) In proposition 5.1 and 5.2 below, we verify that for $n \geq 3$, both minimizing harmonic maps with respect to Dini continuous g and stationary harmonic maps with respect to Lipschitz continuous g enjoy the quasi-monotonicity inequality property.

It is also well-known that certain regularity of the coefficients is necessary for the regularity of second order elliptic systems (cf. [8]). To this end, we recall

Definition 3 (a) For any open set $U \subset \mathbf{R}^n$, a function $f \in \mathrm{BMO}(U)$, if $f \in L^1_{\mathrm{loc}}(U)$ and

$$[f]_{\mathrm{BMO}(U)} := \sup \left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} |f - f_{x,r}| \mid B_r(x) \subset U \right\} < \infty$$

where $f_{x,r} = \frac{1}{|B_r(x)|} \int_{B_r(x)} f$.

(b) For any open set $U \subset \mathbf{R}^n$, a function $f \in \mathrm{VMO}(U)$, if $f \in \mathrm{BMO}(U)$ and

$$\lim_{r \to 0} \sup_{x \in U} [f]_{\mathrm{BMO}(U \cap B_r(x))} = 0.$$

Now we are ready to state our second theorem.

Theorem B *For $n \geq 3$ and $g \in VMO(\Omega)$, suppose that $u \in W^{1,2}(\Omega, N)$ is a weakly harmonic map satisfying the quasi-monotonicity inequality (1.5). Then there exist a closed set $\Sigma \subset \Omega$, with $H^{n-2}(\Sigma) = 0$, and $\alpha \in (0, 1)$ such that $u \in C^\alpha(\Omega \backslash \Sigma, N)$. Here H^{n-2} denotes the $(n-2)$-dimensional Hausdorff measure with respect to g_0.*

We would like to mention that Shi [22] proved the partial regularity theorem, similar to theorem B, for minimizing harmonic maps from manifolds with L^∞-Riemannian metrics. However, the argument in [22] relies heavily on the minimality property. Our method is of PDE nature and partly motivated by the techniques developed by [1,6,12,13].

The paper is written as follows. In Sect. 2, for any C^5-Riemannian manifold N, we outline the Coulomb gauge frame construction by Hélein [12,13] on $u^*TN|_\Omega$ with respect to g. In Sect. 3, we utilize the $W_0^{1,p}$-solvablity theorem on $\nabla \cdot (A\nabla u) = \nabla \cdot F$ by Meyers [17] ($n = 2$) and Di Fazio [5] ($n \geq 3$) for bounded measurable elliptic matrix A to obtain the Div-Curl decomposition theorem on (Ω, g). In Sect. 4, we establish the decay Lemma on the $M^{p,n-p}$ norm of u, $\|u\|_{M^{p,n-p}(\cdot)}$, under the smallness condition of $\|\nabla u\|_{M^{2,n-2}(\cdot)}$. In Sect. 5, we provide two examples in which the quasi-monotonicity inequality (1.5) holds. In Sect. 6, we make some final remarks.

2 Construction of Coulomb gauge frame

In this section, we sketch the Coulomb gauge frame construction on u^*TN by Hélein [12,13] to (Ω, g) for any C^5-Riemannian manifold N and L^∞-Riemannian metric g on $\mathbf{R}^n$.

Let $l = \dim(N)$. For any ball $B \subset \Omega$, $\{e_i\}_{i=1}^l \subset W^{1,2}(B, \mathbf{R}^k)$ is called to be a frame of u^*TN on B, if $\{e_i(x)\}_{i=1}^l$ forms an orthonormal base of $T_{u(x)}N$ for a.e. $x \in B$.

For a vector field $V = (V_1, \ldots, V_n) : \Omega \to \mathbf{R}^n$, define the divergence of V with respect to g by

$$\mathrm{div}_g(V) = \sum_{\alpha,\beta=1}^n \frac{\partial}{\partial x_\alpha}(\sqrt{g} g^{\alpha\beta} V_\beta).$$

First we have

Lemma 2.1 *Assume that there exist a C^5-Riemannian manifold $\hat{N} \subset \mathbf{R}^k$ and a totally geodesic, isometric embedding $i : N \to \hat{N}$. If $u \in W^{1,2}(\Omega, N)$ solves (1.4), then $\hat{u} = i \circ u \in W^{1,2}(\Omega, \hat{N})$ also solves (1.4).*

Proof Straightforward calculations (cf. Jost [14]) imply that

$$\begin{aligned}\Delta_g \hat{u} &= \nabla i(u)(\Delta_g u) + \sum_{\alpha,\beta=1}^n g^{\alpha\beta} (\nabla^2 i)(u) \left(\frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta}\right) \\ &= \nabla i(u)(A_g(u)(\nabla u, \nabla u)) \\ &= \hat{A}_g(\hat{u})(\nabla \hat{u}, \nabla \hat{u})\end{aligned}$$

where $\hat{A}$ denotes the second fundamental form of $\hat{N}$ in $\mathbf{R}^k$. □

With help of Lemma 2.1 and the enlargement construction by Hélein [12,13], we may assume that N is parallelizable so that we have

Proposition 2.2 *Assume that $N \in C^5$ is parallelizable and g is L^∞-Riemannian metric on $\mathbf{R}^n$. Let $\Omega \subset \mathbf{R}^n$ be a bounded domian and $B \subset \Omega$ be a ball. If $u \in W^{1,2}(B, N)$, then there exists a Coulomb gauge frame $\{e_i\}_{i=1}^l \subset W^{1,2}(B, \mathbf{R}^k)$ of u^*TN on B, i.e.*

$$div_g(\langle \nabla e_i, e_j \rangle) = 0 \quad in\ B, \quad 1 \leq i,\ j \leq l \tag{2.1}$$

$$\sum_{\alpha,\beta=1}^n g^{\alpha\beta} \left\langle \frac{\partial e_i}{\partial x_\beta}, e_j \right\rangle x_\beta = 0 \quad on\ \partial B, \quad 1 \leq i, j \leq l, \tag{2.2}$$

and

$$\sum_{i=1}^{l}\int_B |\nabla e_i|^2\,dx \le C\int_B |\nabla u|^2\,dx. \tag{2.3}$$

Proof As N is parallelizable, there exists a smooth orthonormal frame $\{\hat{e}_i(y)\}_{i=1}^l$ of TN. For $1\le i\le l$, define $\bar{e}_i(x)=\hat{e}_i(u(x))$ for a.e. $x\in B$. Then $\{\bar{e}_i\}_{i=1}^l$ forms a frame of u^*TN on B. Denote $\mathrm{SO}(l)$ as the special orthonormal group of order l, consider the minimization problem:

$$\inf\left\{\sum_{i,j=1}^{l}\int_B |\nabla(R_{ij}\bar{e}_j)|_g^2\,dv_g : R=(R_{ij})\in W^{1,2}(B,\mathrm{SO}(l))\right\}. \tag{2.4}$$

By the direct method, there is $R^0\in W^{1,2}(B,SO(l))$ such that $e_\alpha(x)=\sum_{\beta=1}^l R^0_{\alpha\beta}(x)\bar{e}_\beta(x)$, $1\le\alpha\le l$, satisfies

$$\sum_{\alpha=1}^{l}\int_B |\nabla e_\alpha|_g^2\,dv_g \le \sum_{\alpha,\beta=1}^{l}\int_B |\nabla(R_{\alpha\beta}\bar{e}_\beta)|_g^2\,dv_g,\quad \forall R\in W^{1,2}(B,\mathrm{SO}(l)). \tag{2.5}$$

In particular, we have

$$\sum_{\alpha=1}^{l}\int_B |\nabla e_\alpha|_g^2\,dv_g \le \sum_{\alpha,\beta=1}^{l}\int_B |\nabla(\delta_{\alpha\beta}\bar{e}_\beta)|_g^2\,dv_g \le C\int_B |\nabla u|_g^2\,dv_g. \tag{2.6}$$

This, combined with (1.3), implies (2.3). Moreover, the first variation similar to [12,13] implies that $\langle\nabla e_i, e_j\rangle$, $1\le i,j\le l$, satisfies the Euler–Lagrange equation (2.1) and the Neumann condition (2.2). Hence the proof is complete. □

3 Div–curl decomposition

In this section, we prove that if the metric g is either L^∞ for $n=2$ or in $\mathrm{VMO}(\Omega)$ for $n\ge 3$, then the div–curl decomposition holds, namely, *any $F\in L^p(\Omega,\mathbf{R}^n)$ can be decomposed into the sum of ∇G, with $G\in W_0^{1,p}(\Omega)$, and a div_g-free $H\in L^p(\Omega,\mathbf{R}^n)$, for p sufficiently close to $\frac{n}{n-1}$*. The key ingredients are $W_0^{1,p}$-solvability results by Meyers [17] for $n=2$, and Di Fazio [5] for $n\ge 3$.

More precisely, we have

Theorem 3.1 *Let g be L^∞-Riemannian metric on $\mathbf{R}^n$ and $B\subset\Omega\subset\mathbf{R}^n$ be a ball. If, in addition, $g\in VMO(\Omega)$ for $n\ge 3$, then there exists $\delta_0=\delta(n,g)>0$ such that for $p\in(\frac{n}{n-1}-\delta_0,\frac{n}{n-1}+\delta_0)$ and any $F\in L^p(B,\mathbf{R}^n)$ there exist $G\in W_0^{1,p}(B)$ and $H\in L^p(B,\mathbf{R}^n)$, with $\mathrm{div}_g(H)=0$ in Ω, such that*

$$F=\nabla G+H \quad in \quad B, \tag{3.1}$$

and

$$\|\nabla G\|_{L^p(B)}+\|H\|_{L^p(B)}\le C(p,g)\|F\|_{L^p(B)} \tag{3.2}$$

where $L^p(B)$ is L^p-space with respect to g_0.

The proof of Theorem 3.1 relies on the following $W_0^{1,p}$-solvability result.

Proposition 3.2 [17] *For $n \geq 2$ and any ball $B \subset \Omega$, assume that $A = (a_{ij}) \in L^\infty(B, \mathbf{R}^{n\times n})$ is symmetric and uniformly elliptic, then there exists $\delta_0 = \delta_0(n) > 0$ such that, for any $p \in (2-\delta_0, 2+\delta_0)$ and $F \in L^p(B, \mathbf{R}^n)$, there exists a unique solution $u \in W_0^{1,p}(B)$ to the Dirichlet problem*:

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}, \quad \text{in } B,$$
$$u = 0, \quad \text{on } \partial B. \tag{3.3}$$

Moreover,

$$\|\nabla u\|_{L^p(B)} \leq C(p, A)\|F\|_{L^p(B)}. \tag{3.4}$$

Proposition 3.3 [5] *For $n \geq 3$ and ball $B \subset \Omega$, assume that $A = (a_{ij}) \in L^\infty \cap VMO(B, \mathbf{R}^{n\times n})$ is symmetric and uniformly elliptic, then for any $p \in (1, +\infty)$ and $F \in L^p(B, \mathbf{R}^n)$, there exists a unique solution $u \in W_0^{1,p}(B)$ to (3.3) satisfying (3.4).*

Proof of Theorem 3.1 Consider the Dirichlet problem:

$$\operatorname{div}_g(\nabla G) = \operatorname{div}_g(F), \ \text{ in } B$$
$$G = 0, \ \text{ on } \partial B. \tag{3.5}$$

Observe that (3.5) is equivalent to

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial G}{\partial x_j}\right) = \sum_{i=1}^n \frac{\partial \hat{F}_i}{\partial x_i}, \ \text{ in } B$$
$$G = 0, \quad \text{on } \partial B \tag{3.6}$$

where $a_{ij} = \sqrt{g}g^{ij}$ and $\hat{F}_i = \sum_{j=1}^n \sqrt{g}g^{ij}F_j$. Since g satisfies (1.1), it is easy to see that $(a_{ij}) \in L^\infty(B, \mathbf{R}^{n\times n})$ is symmetric and uniformly elliptic. Moreover, we have $\|\hat{F}\|_{L^p(B)} \leq \|F\|_{L^p(B)}$. For $n = 2$, Proposition 3.2 implies that there exists $\delta_0 > 0$ such that (3.5) is uniquely solvable in $W_0^{1,p}(B)$ for any $p \in (2-\delta_0, 2+\delta_0)$. For $n \geq 3$, since $g \in \text{VMO}(B)$ implies $(a_{ij}) \in \text{VMO}(B)$, Proposition 3.3 implies (3.5) is uniquely solvable in $W_0^{1,p}(B)$ for any $1 < p < \infty$. Set $H = F - \nabla G$, (3.5) implies $\operatorname{div}_g(H) = 0$ in B. Moreover, for any $p \in (\frac{n}{n-1} - \delta_0, \frac{n}{n-1} + \delta_0)$, (3.4) yields

$$\|H\|_{L^p(B)} \leq \|F\|_{L^p(B)} + \|\nabla G\|_{L^p(B)} \leq C\|F\|_{L^p(B)}. \tag{3.7}$$

The completes the proof of Theorem 3.1. □

4 Decay estimate in Morrey spaces

In this section, we prove both Theorems A and B. The crucial step is to establish that under the smallness condition of $\|\nabla u\|_{M^{2,n-2}(B)}$, $\|u\|_{M^{p,n-p}(B_r)}$ decays as r^α for some $\alpha \in (0, 1)$. The ideas are suitable modifications of techniques developed by Hélein [12,13], Evans [6], and Bethuel [1]. In order to achieve it, we need two new ingredients: (1) the div–curl decomposition Proposition 3.1, and (2) a new approach to estimate the L^p norm of div_g-free vector fields.

First we define Morrey spaces.

Definition 4.1 For $1 \le p \le n$ and any open set $U \subset \mathbf{R}^n$, the Morrey space $M^{p,n-p}(U)$ is defined by

$$M^{p,n-p}(U) = \left\{ f \in L^p(U) \mid \|f\|^p_{M^{p,n-p}(U)} \equiv \sup_{B_r(x)\subset U} \left\{ r^{p-n} \int\limits_{B_r(x)} |f|^p \, dx \right\} < +\infty \right\}.$$

Now we have

Lemma 4.1 (ϵ_0-decay estimate) *For any bounded domain $\Omega \subset \mathbf{R}^n$ and L^∞-Riemannian metric g on $\mathbf{R}^n$. If, in addition, $g \in VMO(\Omega)$ for $n \ge 3$, then there exist $\delta_n > 0$, $\epsilon_0 = \epsilon_0(g, N) > 0$, and $\theta_0 = \theta_0(g, N) \in (0, \frac{1}{2})$ such that if $u \in W^{1,2}(\Omega, N)$ is a weakly harmonic map satisfying the quasi-monotonicity inequality* (1.5), *and for $B_r(x) \subset \Omega$,*

$$r^{2-n} \int\limits_{B_r(x)} |\nabla u|_g^2 \, dv_g \le \epsilon_0^2 \tag{4.1}$$

then, for any $p \in \left(\frac{n}{n-1} - \delta_n, \frac{n}{n-1}\right)$,

$$\|\nabla u\|_{M^{p,n-p}(B_{\theta_0 r}(x))} \le \frac{1}{2} \|\nabla u\|_{M^{p,n-p}(B_r(x))}. \tag{4.2}$$

Proof of Lemma 4.1 By Lemma 2.1, assume that N is parallelizable. For $x \in \Omega$ and $r > 0$, let $g_{x,r}(y) = g(x + ry)$ and $u_{x,r}(y) = u(x + ry)$ for $y \in B$. Observe that $g_{x,r}$ is L^∞-Riemannian metric on B and $u_{x,r} \in W^{1,2}(B, N)$ is a weakly harmonic map with respect to $g_{x,r}$, satisfies the quasi-monotonicity inequality (1.5), and

$$\int\limits_B |\nabla u|^2_{g_{x,r}} \, dv_{g_{x,r}} = r^{2-n} \int\limits_{B_r(x)} |\nabla u|_g^2 \, dv_g \le \epsilon_0^2. \tag{4.3}$$

Hence, without loss of generality, assume $x = 0$ and $r = 1$. It follows from (1.5) that there exists $K > 0$ such that

$$\|\nabla u\|_{M^{2,n-2}(B_{\frac{1}{2}})} \le K \|\nabla u\|_{L^2(B)} \le K\epsilon_0^2. \tag{4.4}$$

For any $\theta \in (0, \frac{1}{2})$, let $B_{2\theta} \subset B_{\frac{1}{2}}$ be an arbitrary ball of radius 2θ and $\eta \in C_0^\infty(B)$ be such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B_θ, $\eta = 0$ outside $B_{2\theta}$, and $|\nabla \eta| \le 2\theta^{-1}$. Denote the average of u over $B_{2\theta}$ by $u_{2\theta} = \frac{1}{|B_{2\theta}|} \int_{B_{2\theta}} u \, dv_g$, and $|B_{2\theta}|$ is the volume of $B_{2\theta}$ with respect to g.

Let $\{e_\alpha\}_{\alpha=1}^l \in W^{1,2}(B_{2\theta}, \mathbf{R}^k)$ be the Coulomb gauge frame of u^*TN on $B_{2\theta}$ given by Proposition 2.2.

Let

$$\langle p, q\rangle = \sum_{i=1}^n p_i q_i, \ \langle p, g\rangle_g = \sum_{i,j=1}^n g^{ij} p_i q_j, \quad p = (p_1, \dots, p_n), \ q = (q_1, \dots, q_n) \in \mathbf{R}^n$$

denote the inner products with respect to g_0 and g on $\mathbf{R}^n$, respectively.

By Theorem 3.1, there exists $\delta_n > 0$ such that for any $p \in (\frac{n}{n-1} - \delta_n, \frac{n}{n-1})$, there are $\phi_\alpha \in W_0^{1,p}(B_{2\theta})$ and $\psi_\alpha \in L^p(B_{2\theta})$ such that

$$\langle \nabla((u - u_{2\theta})\eta), e_\alpha\rangle = \nabla\phi_\alpha + \psi_\alpha, \quad \mathrm{div}_g(\psi_\alpha) = 0, \quad \text{in } B_{2\theta}, \tag{4.5}$$

and

$$\|\nabla\phi_\alpha\|_{L^p(B_{2\theta})} + \|\psi_\alpha\|_{L^p(B_{2\theta})} \le C\|\nabla((u-u_{2\theta})\eta)\|_{L^p(B_{2\theta})} \le C\|\nabla u\|_{L^p(B_{2\theta})} \tag{4.6}$$

where we have used the Poincaré inequality in the last inequality of (4.6).

Using the Coulomb gauge frame $\{e_\alpha\}_{\alpha=1}^l$, (1.4) can be written as:

$$\mathrm{div}_g(\langle \nabla u, e_\alpha\rangle) = \sum_{\beta=1}^{l}\sum_{i,j=1}^{n} g^{ij}\left\langle \frac{\partial u}{\partial x_i}, \left\langle \frac{\partial e_\alpha}{\partial x_j}, e_\beta\right\rangle\right\rangle e_\beta \quad \text{in } B_{2\theta}. \tag{4.7}$$

We estimate ϕ_α, ψ_α as follows. Let $\phi_\alpha^{(1)} \in W^{1,2}(B_\theta)$ be the weak solution of

$$\sum_{i,,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial \phi_\alpha^{(1)}}{\partial x_j}\right) = 0, \quad \text{in } B_\theta \tag{4.8}$$

$$\phi_\alpha^{(1)} = \phi_\alpha, \quad \text{on } \partial B_\theta. \tag{4.9}$$

where $a_{ij} = \sqrt{g}g^{ij}$, $1 \le i, j \le n$. Let $\phi_\alpha^{(2)} = \phi_\alpha - \phi_\alpha^{(1)}$, then $\phi_\alpha^{(2)}$ satisfies

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial \phi_\alpha^{(2)}}{\partial x_j}\right) = \sum_{\beta=1}^{l}\sum_{i,j=1}^{n} g^{ij}\left\langle \frac{\partial u}{\partial x_i}, \left\langle \frac{\partial e_\alpha}{\partial x_j}, e_\beta\right\rangle\right\rangle e_\beta, \text{ in } B_\theta, \tag{4.10}$$

$$\phi_\alpha^{(2)} = 0, \quad \text{on } \partial B_\theta. \tag{4.11}$$

Step I(a) Estimation of $\nabla\phi_\alpha^{(1)}$.

It is well-known (cf. [11]) that there exists $\delta \in (0,1)$ such that $\phi_\alpha^{(1)} \in C^\delta(B_\theta)$, and for any $0 < r \le \frac{\theta}{2}$ and $p > 1$,

$$[\phi_\alpha^{(1)}]^p_{C^\delta(B_r)} \le C\theta^{p-n}\int_{B_\theta} |\nabla\phi_\alpha^{(1)}|^p\,dx, \quad 0 < r \le \frac{\theta}{2}.$$

On the other hand, since $\phi_\alpha^{(2)} \in W_0^{1,2}(B_\theta)$ satisfies

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial \phi_\alpha^{(2)}}{\partial x_j}\right) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial \phi_\alpha}{\partial x_j}\right), \quad \text{in } B_\theta,$$

Theorem 3.1 implies that there exists $\delta_n > 0$ such that, for $p \in (\frac{n}{n-1} - \delta_n, \frac{n}{n-1})$,

$$\|\nabla\phi_\alpha^{(2)}\|_{L^p(B_\theta)} \le C\|\nabla\phi_\alpha\|_{L^p(B_\theta)} \le C\|\nabla u\|_{L^p(B_{2\theta})}.$$

In particular, we have

$$\|\nabla\phi_\alpha^{(1)}\|_{L^p(B_\theta)} \le \|\nabla\phi_\alpha\|_{L^p(B_\theta)} + \|\nabla\phi_\alpha^{(2)}\|_{L^p(B_\theta)} \le C\|\nabla u\|_{L^p(B_{2\theta})},$$

and, for $0 < r \le \frac{\theta}{2}$ and $p \in (\frac{n}{n-1} - \delta_n, \frac{n}{n-1})$,

$$[\phi_\alpha^{(1)}]^p_{C^\delta(B_r)} \le C\theta^{p-n}\int_{B_{2\theta}} |\nabla u|^p\,dx.$$

This, combined with the Cacciopolli inequality, implies that for any $\tau \in (0, \frac{1}{4})$ and $p \in \left(\frac{n}{n-1} - \delta_n, \frac{n}{n-1}\right)$, we have

$$
\begin{aligned}
(\tau\theta)^{p-n} \int_{B_{\tau\theta}} |\nabla \phi_\alpha^{(1)}|^p \, dx &\leq C[\phi_\alpha^{(1)}]^p_{C^\delta(B_{2\tau\theta})} \\
&\leq C\tau^{p\delta}\theta^{p-n} \int_{B_{2\theta}} |\nabla u|^p \, dx \\
&\leq C\tau^{p\delta} \|\nabla u\|_{M^{p,n-p}(B_1)}.
\end{aligned} \tag{4.12}
$$

Step I(b) Estimation of $\nabla \phi_\alpha^{(2)}$.

First, we claim

There exists $\delta_n > 0$ such that for any $p \in (\frac{n}{n-1} - \delta_n, \frac{n}{n-1})$, if $f \in W_0^{1,p}(B_\theta)$ then

$$
\|\nabla f\|_{L^p(B_\theta)} \leq C \sup \left\{ \int_{B_\theta} \langle \nabla f, \nabla v\rangle_g \, dv_g : v \in W_0^{1,p'}(B_\theta), \|\nabla v\|_{L^{p'}(B_\theta)} = 1 \right\} \tag{4.13}
$$

where $p' = \frac{p}{p-1}$.

To see (4.13), observe that by L^p-duality, there exists $v \in L^{p'}(B_\theta)$, with $\|v\|_{L^{p'}(B_\theta)} = 1$, such that

$$
\|\nabla f\|_{L^p(B_\theta)} \leq C \int_{B_\theta} \langle \nabla f, v\rangle_g \, dv_g. \tag{4.14}
$$

On the other hand, by Theorem 3.1, there exists $\delta_n > 0$ such that if $p \in (\frac{n}{n-1} - \delta_n, \frac{n}{n-1})$, then there exist $v_1 \in W_0^{1,p'}(B_\theta)$ and $v_2 \in L^{p'}(B_\theta, \mathbf{R}^n)$, with $\mathrm{div}_g(v_2) = 0$ in B_θ, such that

$$
v = \nabla v_1 + v_2 \quad \text{in } B_\theta, \; \|\nabla v_1\|_{L^{p'}(B_\theta)} + \|v_2\|_{L^{p'}(B_\theta)} \leq C\|v\|_{L^{p'}(B_\theta)}. \tag{4.15}
$$

This and (4.14) imply

$$
\begin{aligned}
\|\nabla f\|_{L^p(B_\theta)} &\leq C \left(\int_{B_\theta} \langle \nabla f, \nabla v_1\rangle_g \, dv_g + \int_{B_\theta} \langle \nabla f, v_2\rangle_g \, dv_g \right) \\
&= C \int_{B_\theta} \langle \nabla f, \nabla v_1\rangle_g \, dv_g,
\end{aligned}
$$

where we have used $\mathrm{div}_g(v_2) = 0$ in the last step. Hence (4.13) holds.

Applying (4.13) to eqn. (4.7), we have that for $p \in (\frac{n}{n-1} - \delta_n, \frac{n}{n-1})$, there exists $v \in W_0^{1,p'}(B_\theta)$ such that

$$
\begin{aligned}
\|\nabla \phi_\alpha^{(2)}\|_{L^p(B_\theta)} &\leq C \int_{B_\theta} \langle \nabla \phi_\alpha^{(2)}, \nabla v\rangle_g \, dv_g \\
&= -C \sum_{\beta=1}^{l} \sum_{i,j=1}^{n} \int_{B_\theta} \sqrt{g} g^{ij} \left\langle \frac{\partial u}{\partial x_i}, \left\langle \frac{\partial e_\alpha}{\partial x_j}, e_\beta \right\rangle \right\rangle (e_\beta v) \, dx.
\end{aligned} \tag{4.16}
$$

To estimate the right hand side, we need the Hardy-BMO duality theorem (cf. [7]) and the tri-linear estimate (cf. [3,6]).

Proposition 4.2 ([6]) *Suppose that* $f \in W^{1,2}(\mathbf{R}^n)$, $h \in L^2(\mathbf{R}^n, \mathbf{R}^n)$ *with* $div(h) = \sum_{i=1}^n \frac{\partial h_i}{\partial x_i} = 0$, *and* $v \in BMO(\mathbf{R}^n)$. *Then we have*

$$\left| \int_{\mathbf{R}^n} \langle \nabla f, h\rangle v\, dx \right| \le C \|\nabla f\|_{L^2(\mathbf{R}^n)} \|h\|_{L^2(\mathbf{R}^n)} \|v\|_{BMO(\mathbf{R}^n)}. \tag{4.17}$$

Let $\hat{u} : \mathbf{R}^n \to \mathbf{R}^k$ be an extension of u such that

$$\|\nabla \hat{u}\|_{L^2(\mathbf{R}^n)} \le C \|\nabla u\|_{L^2(B_{2\theta})}, \quad [\hat{u}]_{\mathrm{BMO}(\mathbf{R}^n)} \le C[u]_{\mathrm{BMO}(B_{2\theta})}. \tag{4.18}$$

Let $w_\alpha^i = \sum_{\beta=1}^l \sum_{j=1}^n \sqrt{g} g^{ij} \langle \frac{\partial e_\alpha}{\partial x_j}, e_\beta\rangle$, $1 \le i \le n$, and $w_\alpha = (w_\alpha^1, \ldots, w_\alpha^n)$. Then, by (2.1), we have

$$\mathrm{div}(w_\alpha) = \sum_{i=1}^n \frac{\partial w_\alpha^i}{\partial x_i} = \sqrt{g} \sum_{\beta=1}^l \mathrm{div}_g(\langle \nabla e_\alpha, e_\beta\rangle) = 0 \quad \text{on } B_{2\theta}.$$

This, combined with (2.2), implies that there exists an extension $\hat{w}_\alpha \in L^2(\mathbf{R}^n, \mathbf{R}^n)$ of w_α such that

$$\mathrm{div}(\hat{w}_\alpha) = 0 \text{ in } \mathbf{R}^n, \quad \|\hat{w}_\alpha\|_{L^2(\mathbf{R}^n)} \le C\|w_\alpha\|_{L^2(B_{2\theta})} \le C\|\nabla u\|_{L^2(B_{2\theta})}. \tag{4.19}$$

Putting (4.17)–(4.19) into (4.16), we have

$$\begin{aligned} \|\nabla \phi_\alpha^{(2)}\|_{L^p(B_\theta)} &\le -C \int_{\mathbf{R}^n} \langle \nabla u, \hat{\omega}_\alpha\rangle (v e_\alpha)\, dx \\ &= C \int_{\mathbf{R}^n} \langle \hat{u}, \hat{w}_\alpha\rangle \nabla (v e_\alpha)\, dx \\ &\le C[\hat{u}]_{\mathrm{BMO}(\mathbf{R}^n)} \|\hat{w}_\alpha\|_{L^2(\mathbf{R}^n)} \|\nabla(v e_\alpha)\|_{L^2(\mathbf{R}^n)} \\ &\le C\|\nabla u\|_{L^2(B_{2\theta})} [u]_{\mathrm{BMO}(B_{2\theta})} \|\nabla(v e_\alpha)\|_{L^2(B_\theta)}. \end{aligned} \tag{4.20}$$

To estimate $\|\nabla(v e_\alpha)\|_{L^2(B_\theta)}$, note that for $p \in (1, \frac{n}{n-1})$, $p' = \frac{p}{p-1} > n$ and hence the Sobolev embedding theorem implies $v \in W_0^{1,p'}(B_\theta) \subset C_0^{1-\frac{n}{p'}}(B_\theta)$, and

$$\|v\|_{L^\infty(B_\theta)} \le C\theta^{1-\frac{n}{p'}} = C\theta^{1-n+\frac{n}{p}}. \tag{4.21}$$

Moreover, by Hölder inequality, we have

$$\|\nabla v\|_{L^2(B_\theta)} \le C\theta^{\frac{n}{2}-\frac{n}{p'}} \|\nabla v\|_{L^{p'}(B_\theta)} \le C\theta^{\frac{n}{p}-\frac{n}{2}}. \tag{4.22}$$

Therefore we have

$$\begin{aligned} \|\nabla(v e_\alpha)\|_{L^2(B_\theta)} &\le C(\|\nabla v\|_{L^2(B_\theta)} + \|v\|_{L^\infty(B_\theta)} \|\nabla e_\alpha\|_{L^2(B_\theta)}) \\ &\le C\theta^{\frac{n}{p}-\frac{n}{2}}[1 + \theta^{1-\frac{n}{2}} \|\nabla u\|_{L^2(B_{2\theta})}] \\ &\le C\theta^{\frac{n}{p}-\frac{n}{2}}(1 + \|\nabla u\|_{M^{2,n-2}(B_1)}) \\ &\le C\theta^{\frac{n}{p}-\frac{n}{2}}(1 + \epsilon_0) \le C\theta^{\frac{n}{p}-\frac{n}{2}}. \end{aligned} \tag{4.23}$$

Putting (4.23) into (4.20), and combining with (4.12), we have, for any $\tau \in (0, \frac{1}{4})$,

$$\left\{ (\tau\theta)^{p-n} \int_{B_{\tau\theta}} |\nabla \phi_\alpha|^p \, dx \right\}^{\frac{1}{p}} \le C[\tau^\delta + \tau^{1-\frac{n}{p}} \epsilon_0] \|\nabla u\|_{M^{p,n-p}(B_1)} \tag{4.24}$$

where we have used the Poincaré inequality:

$$[u]_{\mathrm{BMO}(B_{2\theta})} \le C\|\nabla u\|_{M^{p,n-p}(B_{2\theta})} \le C\|\nabla u\|_{M^{p,n-p}(B_1)}. \tag{4.25}$$

Step II Estimation of ψ_α.

It follows from (4.5) and Proposition 4.2 that we have

$$\begin{aligned}
\int_{B_\theta} |\psi_\alpha|_g^2 \, dv_g &= \sum_{i,j=1}^n \int_{B_\theta} a_{ij} \psi_\alpha^i \psi_\alpha^j \, dx \\
&= \sum_{i,j=1}^n \int_{B_\theta} a_{ij} \psi_\alpha^i \left\langle \frac{\partial((u-u_{2\theta})\eta)}{\partial x_j}, e_\alpha \right\rangle dx \\
&\quad - \sum_{i,j=1}^n \int_{B_\theta} a_{ij} \psi_\alpha^i \frac{\partial \phi_\alpha}{\partial x_j} \, dx \\
&= - \sum_{i,j=1}^n \int_{B_\theta} a_{ij} \psi_\alpha^i \left\langle (u-u_{2\theta})\eta, \frac{\partial e_\alpha}{\partial x_j} \right\rangle dx \\
&\le C\|\psi_\alpha\|_{L^2(B_\theta)} \|\nabla e_\alpha\|_{L^2(B_\theta)} [(u-u_{2\theta})\eta]_{\mathrm{BMO}(B_\theta)} \\
&\le C\|\psi_\alpha\|_{L^2(B_\theta)} \|\nabla u\|_{L^2(B_\theta)} \|\nabla u\|_{M^{p,n-p}(B_1)}
\end{aligned} \tag{4.26}$$

where we have used the fact $\mathrm{div}_g(\psi_\alpha) = 0$, i.e.

$$\sum_{i,j=1}^n \int_{B_\theta} a_{ij} \psi_\alpha^i \frac{\partial \eta}{\partial x_j} \, dx = 0, \quad \forall \eta \in W_0^{1,2}(B_\theta),$$

and

$$[(u-u_{2\theta})\eta]_{\mathrm{BMO}(B_\theta)} \le C[u]_{\mathrm{BMO}(B_{2\theta})} \le C\|\nabla u\|_{M^{p,n-p}(B_1)}. \tag{4.27}$$

By Hölder inequality, (4.26) yields

$$\left\{ \theta^{p-n} \int_{B_\theta} |\psi_\alpha|^p \, dx \right\}^{\frac{1}{p}} \le C\epsilon_0 \|\nabla u\|_{M^{p,n-p}(B_1)}. \tag{4.28}$$

It follows from (4.5), (4.24), and (4.28) that for any $\tau \in (0, \frac{1}{4})$, any ball $B_{2\theta} \subset B_{\frac{1}{2}}$,

$$\left\{ (\tau\theta)^{p-n} \int_{B_{\tau\theta}} |\nabla u|^p \, dx \right\}^{\frac{1}{p}} \le C(\tau^\delta + \tau^{1-\frac{n}{p}} \epsilon_0) \|\nabla u\|_{M^{p,n-p}(B_1)}. \tag{4.29}$$

Taking superum over all balls $B_{2\theta} \subset B_{\frac{1}{2}}$, we have

$$\|\nabla u\|_{M^{p,n-p}(B_{\frac{\tau}{2}})} \leq C(\tau^{\delta} + \tau^{1-\frac{n}{p}}\epsilon_0)\|\nabla u\|_{M^{p,n-p}(B_1)}. \tag{4.30}$$

Therefore, by choosing $\tau = \tau_1 = 4C^{\frac{-1}{\delta}}$ and $\epsilon_0 = \frac{1}{4C}\tau_0^{\frac{n}{p}-1}$ sufficiently small, we have, for $\tau_0 = \frac{\tau_1}{2} > 0$,

$$\|\nabla u\|_{M^{p,n-p}(B_{\tau_0})} \leq \frac{1}{2}\|\nabla u\|_{M^{p,n-p}(B_1)}. \tag{4.31}$$

This completes the proof of Lemma 4.1. □

Proof of Theorem A For $n = 2$, the absolute continuity of $\int |\nabla u|^2$ implies that there exists $r_0 > 0$ such that

$$\int_{B_r(x)} |\nabla u|^2 \, dx \leq \epsilon_0^2, \quad \forall r \leq r_0, \ x \in \Omega. \tag{4.32}$$

Hence, applying Lemma 4.1 repeatedly, we have that for some $p \in (1, 2)$ and $\tau_0 \in (0, \frac{1}{2})$,

$$(\tau_0^m r_0)^{p-2} \int_{B_{\tau_0^m r_0}(x)} |\nabla u|^p \leq 2^{-pm}\epsilon_0^p, \quad \forall m \geq 1, \quad \forall x \in \Omega. \tag{4.33}$$

This implies that there exists $\alpha_0 \in (0, 1)$ such that

$$r^{p-2} \int_{B_r(x)} |\nabla u|^p \leq C(\epsilon_0, p) r^{\alpha}, \quad \forall r \in (0, r_0), \ x \in \Omega. \tag{4.34}$$

Hence, by Morrey's Lemma (cf. [8]), we conclude $u \in C^{\alpha}(\Omega, N)$. This completes the proof of Theorem A. □

Proof of Theorem B Define

$$\Sigma = \left\{ x \in \Omega : \lim_{r \downarrow 0} r^{2-n} \int_{B_r(x)} |\nabla u|^2 \geq \epsilon_0^2 \right\}.$$

It is well-known (cf. [21]) that $H^{n-2}(\Sigma) = 0$. Moreover, by Lemma 4.1, $\Sigma \subset \Omega$ is a closed set. For any $x_0 \in \Omega \setminus \Sigma$, there exists $r_0 > 0$ such that $B_{2r_0}(x_0) \cap \Sigma = \emptyset$, and

$$r^{2-n} \int_{B_r(x)} |\nabla u|^2 \leq \epsilon_0^2, \quad \forall x \in B_{r_0}(x_0), \ r \leq r_0.$$

Therefore, by Lemma 4.1, we have that for some $p \in (1, \frac{n}{n-1})$ and $\tau_0 \in (0, 1)$,

$$(\tau_0^m r_0)^{p-n} \int_{B_{\tau_0^m r_0}(x)} |\nabla u|^p \leq 2^{-pm}\epsilon_0^p, \quad \forall m \geq 1, \quad \forall x \in B_{r_0}(x_0). \tag{4.35}$$

This implies that there is $\alpha \in (0, 1)$ such that

$$r^{p-n} \int_{B_r(x)} |\nabla u|^p \leq C(\epsilon_0, p) r^{p\alpha}, \quad \forall x \in B_{r_0}(x_0), \quad \forall r \in (0, r_0). \tag{4.36}$$

Hence, by Morrey's Lemma, we conclude $u \in C^{\alpha}(B_{r_0}(x_0), N)$ and $u \in C^{\alpha}(\Omega \backslash \Sigma, N)$. □

5 Quasi-monotonicity inequality

In this section, we derive the quasi-monotonicity inequality (1.5) for two classes of harmonic maps in dimensions $n \geq 3$: (1) minimizing harmonic maps with respect to Dini-continuous metrics g, and (2) stationary harmonic maps with respect to Lipschitz continuous metrics g.

Definition 5.1 A map $u \in W^{1,2}(\Omega, N)$ is a minimizing harmonic map, if

$$\int_{\Omega} |\nabla u|_g^2 \, dv_g \leq \int_{\Omega} |\nabla v|_g^2 \, dv_g, \quad \forall v \in W^{1,2}(\Omega, N) \quad \text{with } v|_{\partial\Omega} = u|_{\partial\Omega}. \tag{5.1}$$

Recall that $f : \Omega \to \mathbf{R}^{n \times n}$ is Dini-continuous, if there exist $r_0 > 0$ and a monotonically non-decreasing $\omega : [0, r_0] \to \mathbf{R}_+$, with $\omega(0) = 0$ and $\int_0^{r_0} \frac{\omega(t)}{t} \, dt < \infty$, such that

$$|f(x) - f(y)| \leq \omega(|x - y|), \quad \forall x, y \in \Omega, \ |x - y| \leq r_0. \tag{5.2}$$

Proposition 5.1 *For $n \geq 3$, suppose that g is a Dini-continuous metric on Ω and $u \in W^{1,2}(\Omega, N)$ is a minimizing harmonic map. Then u satisfies the quasi-monotonicity inequality* (1.5).

Proof It suffices to prove (1.5) for $x = 0 \in \Omega$. Assume $g_0 = g(0)$ is the Euclidean metric on $\mathbf{R}^n$. For $0 < r < \min\{r_0, \text{dist}(0, \partial\Omega)\}$, define

$$\begin{aligned} v(x) &= u\left(\frac{rx}{|x|}\right), \quad x \in B_r \\ &= u(x), \quad x \in \Omega \backslash B_r. \end{aligned}$$

Then the minimality of u implies

$$\int_{B_r} |\nabla u|_g^2 \, dv_g \leq \int_{B_r} |\nabla v|_g^2 \, dv_g. \tag{5.3}$$

It follows from the Dini-continuity of g that

$$\max_{x \in B_r} |g(x) - g_0| \leq \omega(r), \quad \forall 0 < r \leq \min\{r_0, \text{dist}(0, \partial\Omega)\},$$

where ω is the modular of continuity of g. This and (5.3) imply that there exists $C_0 > 0$ such that

$$(1 - C_0 \omega(r)) \int_{B_r} |\nabla u|^2 \, dx \leq \int_{B_r} |\nabla v|^2 \, dx, \quad \forall 0 < r \leq \min\{r_0, \text{dist}(0, \partial\Omega)\}. \tag{5.4}$$

Direct calculations imply

$$\int_{B_r} |\nabla v|^2 \, dx = \frac{r}{n-2} \int_{\partial B_r} \left(|\nabla u|^2 - |\frac{\partial u}{\partial r}|^2 \right) dH^{n-1}.$$

Therefore we have, for $0 < r \le \min\{r_0, \mathrm{dist}(0, \partial\Omega)\}$,

$$(n-2)(1 - C_0\omega(r))r^{1-n} \int_{B_r} |\nabla u|^2\, dx \le r^{2-n} \int_{\partial B_r} |\nabla u|^2\, dH^{n-1} - r^{2-n} \int_{\partial B_r} |\frac{\partial u}{\partial r}|^2\, dH^{n-1}. \tag{5.5}$$

This yields, for $0 < r \le \min\{r_0, \mathrm{dist}(0, \partial\Omega)\}$,

$$\begin{aligned} \frac{d}{dr}&\left\{ e^{\{(n-2)C_0 \int_0^r \frac{\omega(t)}{t}\, dt\}} r^{2-n} \int_{B_r} |\nabla u|^2\, dx \right\} \\ &\ge e^{\{(n-2)C_0 \int_0^r \frac{\omega(t)}{t}\, dt\}} r^{2-n} \int_{\partial B_r} |\frac{\partial u}{\partial r}|^2\, dH^{n-1} \\ &\ge r^{2-n} \int_{\partial B_r} |\frac{\partial u}{\partial r}|^2\, dH^{n-1}. \end{aligned} \tag{5.6}$$

Integrating (5.6), we have, for $0 < r \le R \le \min\{r_0, \mathrm{dist}(0, \partial\Omega)\}$,

$$\begin{aligned} \int_{B_R \setminus B_r} &|x|^{2-n} |\frac{\partial u}{\partial r}|^2\, dx + r^{2-n} \int_{B_r} |\nabla u|^2\, dx \\ &\le e^{\{(n-2)C_0 \int_0^R \frac{\omega(t)}{t}\, dt\}} R^{2-n} \int_{B_R} |\nabla u|^2\, dx. \end{aligned} \tag{5.7}$$

This implies (1.5) holds for $K = e^{\{(n-2)C_0 \int_0^{r_0} \frac{\omega(t)}{t}\, dt\}}$. □

Next we consider stationary harmonic maps.

Definition 5.2 A weakly harmonic map $u \in W^{1,2}(\Omega, N)$ is a stationary harmonic map, if it is a critical point of E_2 with respect to the domain variations:

$$\frac{d}{dt}|_{t=0} \int_\Omega |\nabla u(x + tX(x))|_g^2\, dv_g = 0, \quad \forall X \in C_0^1(\Omega, \mathbf{R}^n). \tag{5.8}$$

We have

Proposition 5.2 *For $n \ge 3$, let g be a Lipschitz continuous Riemannian metric on Ω. Then any stationary map $u \in W^{1,2}(\Omega, N)$ satisfies (1.5) for some $K = K(n, g) > 0$.*

Proof For simplicity, assume $x = 0 \in \Omega$ and $g(0) = g_0$. Define the energy-stress tensor

$$S_{\alpha\beta} = \frac{1}{2}|\nabla u|_g^2 g_{\alpha\beta} - \left\langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta} \right\rangle, \quad 1 \le \alpha, \beta \le n.$$

Then it is well-known (cf. [13]) that the stationarity (5.8) implies

$$\sum_{\alpha,\beta=1}^n \int_\Omega (L_X g^{\alpha\beta}) S_{\alpha\beta}\, dv_g = 0 \tag{5.9}$$

where

$$L_X g^{\alpha\beta} = \sum_{\gamma=1}^{n} \left[X_\gamma \frac{\partial g^{\alpha\beta}}{\partial x_\gamma} - \frac{\partial X_\alpha}{\partial x_\gamma} g^{\gamma\beta} - \frac{\partial X_\beta}{\partial x_\gamma} g^{\gamma\alpha} \right]$$

is the Lie derivative of $(g^{\alpha\beta})$ with respect to X.

For $B_r \subset \Omega$, and $\eta(x) = \eta(|x|) \in C_0^1(B_r)$ with $0 \le \eta \le 1$, let $X(x) = x\eta(|x|)$. Then we have

$$\frac{\partial X_\alpha}{\partial x_\gamma} = \delta_{\alpha\gamma}\eta(|x|) + \eta'(|x|)\frac{x_\alpha x_\gamma}{|x|}, \quad 1 \le \alpha, \gamma \le n,$$

and

$$L_X g^{\alpha\beta} = \eta(|x|) \sum_{\gamma=1}^{n} x_\gamma \frac{\partial g^{\alpha\beta}}{\partial x_\gamma} - 2\eta(|x|)g^{\alpha\beta} - 2\eta'(|x|) \sum_{\gamma=1}^{n} \frac{x_\beta x_\gamma}{|x|} g^{\alpha\gamma}.$$

Since g is Lipschitz continuous, there exist $r_0 > 0$ and $C_0 > 0$ depending on Lip(g) such that

$$\|\nabla g^{\alpha\beta}\|_{L^\infty(B_r)} \le C_0 \text{Lip}(g), \quad \forall 0 < r \le r_0. \tag{5.10}$$

Let $\text{I} \equiv \sum_{\alpha,\beta,\gamma=1}^{n} \int_{B_r} x_\gamma \eta(|x|) \frac{\partial g^{\alpha\beta}}{\partial x_\gamma} S_{\alpha\beta}\, dv_g$. Then we have

$$\begin{aligned} |\text{I}| &\le \sum_{\alpha,\beta,\gamma=1}^{n} \int_{B_r} |x_\gamma| \left|\frac{\partial g^{\alpha\beta}}{\partial x_\gamma}\right| |S_{\alpha\beta}|\, dv_g \\ &\le r\|\nabla g^{\alpha\beta}\|_{L^\infty(B_r)} \sum_{\alpha,\beta=1}^{n} \int_{B_r} |S_{\alpha\beta}|\, dv_g \le Cr \int_{B_r} |\nabla u|_g^2\, dv_g \end{aligned}$$

for $C = C_0 \text{Lip}(g)$.

Set $\text{II} \equiv -2\sum_{\alpha,\beta=1}^{n} \int_{B_r} \eta(|x|) g^{\alpha\beta} S_{\alpha\beta}\, dv_g$. Then we have

$$\begin{aligned} \text{II} &= -2 \sum_{\alpha,\beta=1}^{n} \int_{B_r} \eta(|x|) g^{\alpha\beta} \left(\frac{1}{2}|\nabla u|_g^2 g_{\alpha\beta} - \left\langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta} \right\rangle \right) dv_g \\ &= (2-n) \int_{B_r} \eta(|x|)|\nabla u|_g^2\, dv_g. \end{aligned}$$

For $\text{III} \equiv -2\sum_{\alpha,\beta,\gamma=1}^{n} \int_{B_r} \eta'(|x|) \frac{x_\beta x_\gamma}{|x|} g^{\alpha\gamma} S_{\alpha\beta}\, dv_g$, we have

$$\begin{aligned} \text{III} &= -2 \sum_{\alpha,\beta,\gamma=1}^{n} \int_{B_r} \eta'(|x|) \frac{x_\beta x_\gamma}{|x|} g^{\alpha\gamma} \left(\frac{1}{2}|\nabla u|_g^2 g_{\alpha\beta} - \left\langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta} \right\rangle \right) dv_g \\ &= -\int_{B_r} \eta'(|x|)|x||\nabla u|_g^2\, dv_g \\ &\quad +2 \sum_{\alpha,\beta,\gamma=1}^{n} \int_{B_r} \eta'(|x|) \frac{x_\beta x_\gamma}{|x|} g^{\alpha\gamma} \left\langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta} \right\rangle dv_g \\ &= \text{IV} + \text{V}. \end{aligned}$$

Observe that (5.10) implies, for $0 < r \le r_0$,

$$g^{\alpha\gamma}(x) = \delta_{\alpha\gamma} + h_{\alpha\gamma}(x), \quad |h_{\alpha\gamma}|(x) \le C_0 \mathrm{Lip}(g)|x|, \quad \forall x \in B_r, \ \forall 1 \le \alpha, \gamma \le n.$$

Hence we have

$$V = 2\int_{B_r} |x|\eta'(|x|) \left|\frac{\partial u}{\partial r}\right|^2 dv_g + 2\sum_{\alpha,\gamma=1}^{n} \int_{B_r} \eta'(|x|) x_\gamma h_{\alpha\gamma} \left\langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial r}\right\rangle dv_g. \quad (5.11)$$

As

$$0 = \sum_{\alpha,\beta=1}^{n} \int_{\Omega} (L_X g^{\alpha\beta}) S_{\alpha\beta}\, dv_g = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

we have

$$\begin{aligned}(2-n)\int_{B_r} \eta(|x|)|\nabla u|_g^2\, dv_g - \int_{B_r} |x|\eta'(|x|)\left(|\nabla u|_g^2 - 2|\frac{\partial u}{\partial r}|^2\right) dv_g \\ \ge -Cr\int_{B_r} |\nabla u|_g^2\, dv_g - 2\sum_{\alpha,\gamma=1}^{n}\int_{B_r} \eta'(|x|)x_\gamma h_{\alpha\gamma}\left\langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial r}\right\rangle dv_g. \quad (5.12)\end{aligned}$$

For small $\epsilon > 0$, let $\eta = \eta_\epsilon(|x|) \in C_0^{0,1}(B_r)$ be such that $\eta_\epsilon(t) = 1$ for $0 \le t \le r - \epsilon$, $\eta_\epsilon(t) = 0$ for $t \ge r$, and $\eta'_\epsilon(t) = -\frac{1}{\epsilon}$ for $r - \epsilon \le t \le r$. Putting η into (5.12) and sending ϵ to zero, we obtain

$$\begin{aligned}(2-n)\int_{B_r} |\nabla u|_g^2\, dv_g + r\int_{\partial B_r} |\nabla u|_g^2\, dH_g^{n-1} \\ \ge 2r\int_{\partial B_r} |\frac{\partial u}{\partial r}|^2\, dH_g^{n-1} - Cr\int_{B_r} |\nabla u|_g^2\, dv_g \\ +2\sum_{\alpha,\gamma=1}^{n}\int_{\partial B_r} x_\gamma h_{\alpha\gamma}\langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial r}\rangle\, dH_g^{n-1} \\ \ge 2r\int_{\partial B_r} |\frac{\partial u}{\partial r}|^2\, dH_g^{n-1} - Cr\int_{B_r} |\nabla u|_g^2\, dv_g \\ -Cr^3\int_{\partial B_r} |\nabla u|_g^2\, dH_g^{n-1} \quad (5.13)\end{aligned}$$

where dH_g^{n-1} is the $(n-1)$-dimensional Hausdorff measure with respect to g, and we have used the Hölder inequality in the last step:

$$\begin{aligned}
&2\sum_{\alpha,\gamma=1}^{n} \Big|\int_{\partial B_r} x_\gamma h^{\alpha\gamma}\left\langle \frac{\partial u}{\partial \alpha}, \frac{\partial u}{\partial r}\right\rangle dH_g^{n-1}\Big| \\
&\le r\int_{\partial B_r} \Big|\frac{\partial u}{\partial r}\Big|^2 dH_g^{n-1} + r\left(\sum_{\alpha,\gamma=1}^{n} \max_{B_r} |h^{\alpha\gamma}|^2\right)\int_{\partial B_r} |\nabla u|_g^2\, dH_g^{n-1} \\
&\le r\int_{\partial B_r} \Big|\frac{\partial u}{\partial r}\Big|^2 dH_g^{n-1} + Cr^3\int_{\partial B_r} |\nabla u|_g^2\, dH_g^{n-1}.
\end{aligned}$$

Let $f(r) = \int_{B_r} |\nabla u|_g^2\, dv_g$, we have $f'(r) = \int_{\partial B_r} |\nabla u|_g^2\, dH^{n-1}$ for a.e. $r > 0$. Hence (5.13) yields

$$(2-n+Cr)f(r) + r(1+Cr)f'(r) \ge r\int_{\partial B_r} \left|\frac{\partial u}{\partial r}\right|^2 dH_g^{n-1}.$$

In particular, there exists a small $r_0 > 0$ depending on g such that for $0 < r \le r_0$,

$$(2-n+O(r))f(r) + rf'(r) \ge \frac{r}{2}\int_{\partial B_r} \left|\frac{\partial u}{\partial r}\right|^2 dH_g^{n-1} \tag{5.14}$$

where $C^{-1}r \le O(r) \le Cr$. Therefore we have, $0 < r \le r_0$,

$$\frac{d}{dr}\left(e^{O(r)} r^{2-n} f(r)\right) \ge \frac{1}{2} e^{O(r)} r^{2-n}\int_{\partial B_r} \left|\frac{\partial u}{\partial r}\right|^2 dH_g^{n-1}. \tag{5.15}$$

Integrating (5.15) over $0 < r \le R \le r_0$, we have

$$e^{O(R)} R^{2-n} f(R) \ge r^{2-n} f(r) + \frac{1}{2}\int_{B_R\setminus B_r} |x|^{2-n}\left|\frac{\partial u}{\partial r}\right|^2 dv_g. \tag{5.16}$$

This, combined with (1.3), implies (1.5) with $K = e^{O(r_0)}$. □

Remark 5.1 The monotonicity inequality (5.15) has been derived by Garofalo–Lin [10] for second order elliptic equations with divergence structure by a different method.

6 Final remarks

This section is devoted to some further discussions on Theorems A and B. The first remark asserts that for $n \ge 3$, $g \in \mathrm{VMO}(\Omega)$ can be weaken. The second remark concerns the optimal Hausdorff dimension estimate on minimizing harmonic map from domains with Dini continuous metrics. The third remark concerns the blow-up analysis of stationary harmonic maps from domians with Lipschitz continuous Riemannian metrics.

Theorem 6.1 *For $n \ge 3$, there exists $\delta_0 > 0$ such that if g is a L^∞-Riemannian metric on Ω with $[g]_{BMO(\Omega)} \le \delta_0$ and $u \in W^{1,2}(\Omega, N)$ is a weakly harmonic map satisfying the* *quasi-monotonicity inequality* (1.5), *then there are* $\alpha \in (0, 1)$ *and closed subset* $\Sigma \subset \Omega$, *with* $H^{n-2}(\Sigma) = 0$, *such that* $u \in C^\alpha(\Omega \backslash \Sigma, N)$.

Proof It follows from the same arguments as in Theorem B, except that we need to replace Proposition 3.3 by the following proposition, due to Byun–Wang [2] (see also Caffarelli–Peral [4]).

Lemma 6.2 *For* $n \geq 3$ *and ball* $B \subset \Omega$, *assume that* $A = (a_{ij}) \in L^\infty(B, \mathbf{R}^{n\times n})$ *is symmetric, and uniformly elliptic with ellipticity constant* $\Lambda > 0$. *For any* $p \in (1, +\infty)$ *and* $F \in L^p(B, \mathbf{R}^n)$, *there exists* $\delta_p > 0$ *such that if* $[g]_{BMO(B)} \leq \delta_p$, *then there exists a unique solution* $G \in W_0^{1,p}(B)$ *to the Dirichlet problem*:

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial G}{\partial x_j}\right) = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}, \quad in\ B \tag{6.1}$$

$$G = 0, \quad on\ \partial B. \tag{6.2}$$

Moreover,

$$\|\nabla G\|_{L^p(B)} \leq C([A]_{BMO(B)}, n, \Lambda)\|F\|_{L^p(B)}. \tag{6.3}$$

□

Theorem 6.2 *For* $n \geq 3$ *and a Dini-continuous Riemannian metric* g *in* $\Omega \subset \mathbf{R}^n$, *if* $u \in W^{1,2}(\Omega, N)$ *is a minimizing harmonic map, then there exist* $\alpha \in (0, 1)$ *and closed subset* $\Omega \subset \Omega$, *which is discrete for* $n = 3$ *and has Hausdorff dimension at most* $(n-3)$ *for* $n \geq 4$, *such that* $u \in C^\alpha(\Omega \backslash \Sigma, N)$.

Proof Note that the Dini-continuity of g implies $g \in \mathrm{VMO}(\Omega)$. Since u is a minimizing harmonic map, Proposition 5.1 implies that u satisfies the monotonicity inequality (5.7). Define

$$\Sigma = \left\{ x \in \Omega \mid \Theta(u, x) \equiv \lim_{r\downarrow 0} r^{2-n} \int_{B_r(x)} |\nabla u|^2 \geq \epsilon_0^2 \right\} \tag{6.4}$$

where ϵ_0 is given by Lemma 4.1. Then, by theorem B, we have that $u \in C^\alpha(\Omega\backslash\Sigma, N)$ for some $\alpha \in (0, 1)$.

To prove the Hausdorff dimension estimate of Σ, define the rescalled map $u_{x_0,r_i}(x) = u(x_0 + r_i x) : B_2 \to N$ for any $x_0 \in \Sigma$ and $r_i \downarrow 0$. It is easy to see that u_{x_0,r_i} is minimizing harmonic map with respect to $g_i(x) = g(x_0 + r_i x)$. Since g is Dini-continuous, we know $g_i \to g_0$, the Euclidean metric, uniformly on B_2.

It follows from Luckhaus' extension Lemma (see [16]) and the minimality of u that there exists a minimizing harmonic map $\phi \in W^{1,2}(B_2, N)$ with respect to g_0 such that after taking possible subsequences, $u_{x_0,r_i}(x) \equiv u(x_0 + r_i x) \to \phi$ strongly in $W^{1,2}(B_2, N)$. Moreover, the monotonicity inequality (5.7) yields $\frac{\partial\phi}{\partial r} = 0$ a.e. in B_2 and $\phi(x) = \phi(\frac{x}{|x|})$ for a.e. $x \in B_2$. Now we can apply Federer's dimension reduction argument (cf. [21]) to conclude that Σ is discrete for $n = 3$, and has Hausdorff dimension at most $(n-3)$ for $n \geq 4$. □

Theorem 6.3 *For* $n \geq 3$ *and a Lipschitz continuous metric* g *on* $\Omega \subset \mathbf{R}^n$. *Assume that* N *doesn't support nonconstant harmonic maps from* S^2. *If* $u \in W^{1,2}(\Omega, N)$ *is a stationary harmonic map, then there exist* $\alpha \in (0, 1)$ *and closed subset* $\Sigma \subset \Omega$, *which is discrete for* $n = 4$, *and has Hausdorff dimension at most* $(n-4)$ *for* $n \geq 5$, *such that* $u \in C^\alpha(\Omega\backslash\Sigma, N)$.

Proof Note that the Lipschitz continuity of g implies $g \in \text{VMO}(\Omega)$. It follows from the stationarity and Proposition 5.2 that u satisfies the monotonicity inequality (5.16). Therefore, Theorem B implies $u \in C^\alpha(\Omega\backslash\Sigma, N)$ for some $\alpha \in (0, 1)$, with Σ given by (6.4).

For any $x_0 \in \Sigma$ and $r_i \downarrow 0$, $u_{x_0,r_i} \in W^{1,2}(B_2, N)$ are stationary harmonic maps with respect to g_i. It follows from (5.16) that there is a harmonic map $\phi \in W^{1,2}(B_2, N)$ with respect to g_0, which is homogeneous of degree zero, such that after passing to subsequences, $u_{x_0,r_i}(x) \equiv u(x_0 + r_i x) \to \phi$ weakly in $W^{1,2}(B_2, N)$. One can check the blow-up analysis by Lin [15] applies to stationary harmonic maps with respect to Lipschitz continuous metrics g as long as we have theorem B, (5.16), and N doesn't support harmonic S^2's. In particular, $u_{x_0,r_i} \to \phi$ strongly in $W^{1,2}(B_2, N)$. With this strong convergence, one can show Σ is discrete for $n = 4$, and has Hausdorff dimension at most $(n - 4)$ for $n \geq 5$. □

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