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Calculus of Variations

Γ-limits and relaxations for rate-independent evolutionary problems

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Abstract This work uses the energetic formulation of rate-independent systems that is based on the stored-energy functionals $\mathcal{E}$ and the dissipation distance $\mathcal{D}$. For sequences $(\mathcal{E}_k)_{k\in\mathbb{N}}$ and $(\mathcal{D}_k)_{k\in\mathbb{N}}$ we address the question under which conditions the limits q_∞ of solutions $q_k : [0, T] \to \mathcal{Q}$ satisfy a suitable limit problem with limit functionals $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$, which are the corresponding Γ-limits. We derive a sufficient condition, called *conditional upper semi-continuity of the stable sets*, which is essential to guarantee that q_∞ solves the limit problem. In particular, this condition holds if certain *joint recovery sequences* exist. Moreover, we show that time-incremental minimization problems can be used to approximate

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the solutions. A first example involves the numerical approximation of functionals using finite-element spaces. A second example shows that the stop and the play operator converge if the yield sets converge in the sense of Mosco. The third example deals with a problem developing microstructure in the limit $k \to \infty$, which in the limit can be described by an effective macroscopic model.

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1 Introduction

Rate-independent models for material behavior are useful in many contexts. Elastoplasticity is the most prominent application, but recently also damage, fracture, hysteretic behavior in magnetic, magnetostrictive and ferroelectric materials, and phase transformations in shape-memory alloys have been described via such models, see [25,26] and the references there.

Here, we want to contribute to the abstract mathematical foundations for such models. While a quite flexible existence theory has been developed over the last years (cf. [14,25,27, 32,34]), there is still a need to develop a theory for parameter dependence and for numerical approximation properties. The first part of this work will address these questions in the framework of Γ-convergence. In the second part, we are concerned with the question of relaxation of rate-independent evolutionary systems. This topic is important for the understanding of evolution of microstructures in materials, see [4,9,19,24,28,35]. While the static questions of Γ-convergence or relaxation are well studied, the related questions for evolutionary systems are treated less systematically, see e.g., [7,8,37]. Only recently, a systematic study for gradient flows was initialized in [36,38–41].

To present our main ideas we introduce the main notions. The state space of our system is denoted by $\mathcal{Q}$ and the stored-energy functional $\mathcal{E} : [0, T]\times\mathcal{Q} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ is assumed to depend on the (process) time through a time-dependent loading. Additionally, there is given a dissipation distance $\mathcal{D} : \mathcal{Q}\times\mathcal{Q} \to [0, \infty]$, which is assumed to satisfy the triangle inequality but may be unsymmetric. Here, $\mathcal{D}(q_0, q_1)$ measures the minimal amount of energy that is dissipated when the state is changed from q_0 into q_1. In rate-independent systems the dissipation depends only on the path but not on the velocity.

A process $q : [0, T] \to \mathcal{Q}$ is called an *energetic solution* of the rate-independent process associated with the functionals $\mathcal{E}$ and $\mathcal{D}$, if it satisfies the *stability condition* (S) and the *energy balance* (E) for all $t \in [0, T]$:

$$\begin{aligned}
&\text{(S)} \quad \forall \widetilde{q} \in \mathcal{Q}: \ \mathcal{E}(t, q(t)) \leq \mathcal{E}(t, \widetilde{q}) + \mathcal{D}(q(t), \widetilde{q}),\\
&\text{(E)} \quad \mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{D}}(q; [0, t]) = \mathcal{E}(0, q(0)) + \int_0^t \partial_s \mathcal{E}(s, q(s))\,\mathrm{d}s.
\end{aligned} \tag{1.1}$$

Here, the dissipation $\mathrm{Diss}_{\mathcal{D}}(q; [r, s])$ along a part of the curve is defined as a total variation with respect to the "metric" $\mathcal{D}$. In this case, we also say that q solves the *energetic formulation* (S)&(E). If $\mathcal{E}$ and $\mathcal{D}$ are replaced by $\mathcal{E}_k$ and $\mathcal{D}_k$, we call this the energetic formulation $(\mathrm{S})_k\&(\mathrm{E})_k$.

Under the assumption that $\mathcal{Q}$ is a Banach space, that $\mathcal{D}$ is translation invariant, i.e. $\mathcal{D}(q_0, q_1) = \mathcal{R}(q_1 - q_0)$, and that $\mathcal{E}(t, \cdot)$ is convex, the energetic formulation (S)&(E) is equivalent to the doubly nonlinear differential inclusion

$$0 \in \partial\mathcal{R}(\dot{q}(t)) + \partial\mathcal{E}(t, q(t)) \ \subset \mathcal{Q}^* \text{ (dual space)}$$

cf. [25,32]. The advantage of the energetic formulation (S)&(E) is that it is totally derivative free and hence can be formulated on an abstract topological space $\mathcal{Q}$, see [27]. The stability is a purely static concept and the evolutionary concept is brought into bearing solely by the scalar energy balance.

In Sects. 2 and 3 we study the situation that a sequence of pairs $(\mathcal{E}_k, \mathcal{D}_k)$ is given as well as limit functionals $(\mathcal{E}_\infty, \mathcal{D}_\infty)$. Assume that $q_k : [0, T] \to \mathcal{Q}$ is an energetic solution associated with $\mathcal{E}_k$ and $\mathcal{D}_k$. We study the question in what sense $(\mathcal{E}_k, \mathcal{D}_k)$ has to converge to $(\mathcal{E}_\infty, \mathcal{D}_\infty)$ such that a limit process $q(t) = \lim_{k\to\infty} q_k(t)$ solves the energetic formulation $(\mathrm{S})_\infty$&$(\mathrm{E})_\infty$. It turns out that the right notion of convergence is related to Γ-convergence. However, it is easy to see that

$$\mathcal{E}_\infty = \Gamma\text{–}\lim_{k\to\infty} \mathcal{E}_k \;\text{ and }\; \mathcal{D}_\infty = \Gamma\text{–}\lim_{k\to\infty} \mathcal{D}_k \tag{1.2}$$

is not sufficient. See (2.14) for the definition of Γ-convergence and Example 3.2 for a simple system where (1.2) is not sufficient for convergence of solutions. Note also, that the Γ-limit $\mathcal{D}_\infty$ may no longer satisfy the triangle inequality, so this will be an extra assumption.

Central objects are the set of stable states and stable sequences. The *sets of stable states* $\mathcal{S}_k(t)$ depend on $t \in [0, T]$ and $k \in \mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ and are defined via

$$\mathcal{S}_k(t) := \{\, q \in \mathcal{Q}\,;\; \mathcal{E}_k(t, q) < \infty,\ \forall \widetilde{q} \in \mathcal{Q} :\ \mathcal{E}_k(t, q) \le \mathcal{E}_k(t, \widetilde{q}) + \mathcal{D}_k(q, \widetilde{q})\,\}. \tag{1.3}$$

A sequence $(t_l, q_{k_l})_{l\in\mathbb{N}}$ is called a *stable sequence* if

$$q_{k_l} \in \mathcal{S}_{k_l}(t_l) \;\text{ and }\; \sup_{l\in\mathbb{N}} \mathcal{E}_{k_l}(t_l, q_{k_l}) < \infty. \tag{1.4}$$

Here we always assume that $(k_l)_{l\in\mathbb{N}}$ denotes a subsequence, i.e., $k_l < k_{l+1} \to \infty$. The crucial conditions for the desired convergence result are now

(a) $\mathcal{E}_\infty(t, q) \le \inf\{\, \liminf_{l\to\infty} \mathcal{E}_{k_l}(t_l, q_{k_l})\,;\; (t_l, q_{k_l}) \text{ is stable and } (t_l, q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\to} (t, q)\,\}$,

(b) $\mathcal{D}_\infty(q, \widetilde{q}) \le \inf\{\, \liminf_{l\to\infty} \mathcal{D}_{k_l}(q_{k_l}, \widetilde{q}_{k_l})\,;\; (t_l, q_{k_l}),\ (\widetilde{t}_l, \widetilde{q}_{k_l}) \text{ are stable}, (t_l, q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\to} (t, q),\ (\widetilde{t}_l, \widetilde{q}_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\to} (\widetilde{t}, \widetilde{q})\,\}$,

(c) $\forall$ stable sequences $(t_l, q_{k_l})_{l\in\mathbb{N}} :\ \ (t_l, q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\to} (t, q) \implies q \in \mathcal{S}_\infty(t)$.

While the conditions (a) and (b) are usually satisfied by assuming (1.2), the condition (c) is genuinely new and concerns the interplay between the two sequences $(\mathcal{E}_k)_{k\in\mathbb{N}}$ and $(\mathcal{D}_k)_{k\in\mathbb{N}}$. In Sect. 2 we provide several sufficient conditions for the implication (c), which can be understood as conditioned upper semi-continuity of the stable sets. The strongest of these conditions is that $\mathcal{E}_\infty = \Gamma\text{–}\lim_{k\to\infty} \mathcal{E}_k$ and that $\mathcal{D}_k$ continuously converges to $\mathcal{D}_\infty$. Note that (a) and (b) only ask for a lower estimate, however our theorems will prove that, along the approximate solutions, the lower limits $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$ are attained, see assertions (i) and (ii) in the Theorems 3.1, 3.4, and 4.1.

Having in mind numerical approximation we also combine this result with time discretizations. The most effective way to study energetic formulations is based on the incremental minimization problems

$$(\mathrm{IP})_k \qquad q_j^k \in \operatorname{Arg\,min}\{\, \mathcal{E}_k(t_j^k, \widetilde{q}) + \mathcal{D}_k(q_{j-1}^k, \widetilde{q})\,;\; \widetilde{q} \in \mathcal{Q}\,\},$$

where $\Pi_k = \left\{0 = t_0^k < t_1^k < \cdots < t_{N_k}^k = T\right\}$ is an arbitrary partition of $[0, T]$. Using the same conditions as for the above convergence result together with suitable uniform compactness results, we show that the piecewise constant interpolants $q_k : [0, T] \to \mathcal{Q}$ associated with solutions of $(\mathrm{IP})_k$ contain a subsequence that converges to a solution of $(\mathrm{S})_\infty\&(\mathrm{E})_\infty$, see Theorem 3.4.

In Sect. 4 we consider the situation that the sequences $(\mathcal{E}_k)_{k\in\mathbb{N}}$ and $(\mathcal{D}_k)_{k\in\mathbb{N}}$ are constant, i.e. $\mathcal{E}_k = \mathcal{E}_1$ and $\mathcal{D}_k = \mathcal{D}_1$. However, we do not assume that $\mathcal{E}_1$ and $\mathcal{D}_1$ are lower semi-continuous. Hence, $(\mathrm{IP})_k$ may not be solvable and we replace it by an approximate incremental problem $(\mathrm{AIP})_k$ where we only need to reach the infimum up to an accuracy $\varepsilon_k(t_j^k - t_{j-1}^k)$. Of course, $(\mathrm{AIP})_k$ is solvable and we study the sequence $q_k : [0, T] \to \mathcal{Q}$ of piecewise constant interpolants. Using a slightly strengthened version of the upper semi-continuity of the stable sets we show that the sequence $(q_k)_{k\in\mathbb{N}}$ again contains a convergent subsequence the limit of which solves $(\mathrm{S})_\infty\&(\mathrm{E})_\infty$. The construction of subsequences relies on an abstract version of Helly's selection principle that is due to [27] and that we prove in a slightly more general form in Appendix A.

In the final Sect. 5 we illustrate the two main results by three relatively simple examples. In Sect. 5.1 we deal with a quadratic energy functional $\mathcal{E}_\infty$ on a Hilbert space $H = \mathcal{Q}$ and a weakly continuous and translationally invariant dissipation distance $\mathcal{D}_\infty$. Defining a sequence H_k of finite-dimensional subspaces of H with $\cup_{k=1}^\infty H_k$ dense in H, we define $\mathcal{E}_k$ equal to $\mathcal{E}_\infty$ on H_k and $+\infty$ else. Letting $\mathcal{D}_k = \mathcal{D}_\infty$ it is easy to check the abstract conditions and, thus, a convergence result for space-time discretizations is established. The idea of using Γ-convergence for treating numerical approximations was first investigated in [19]; see also [15,16] for independent work in the context of fracture. As a particular application, this provides the convergence result in elastoplasticity derived first in [17]. Further applications that use the full strength of the theory developed here, are found in [30]. Stronger convergence results of numerical methods, also giving specific convergence rates are discussed in [2,18].

In Sect. 5.2 we address the question of the continuity of the *play* and the *stop operator* with respect to the yield or characteristic set $\mathcal{C}_k$. This question was studied in [20, Theorem 3.12] and [40, Corollary 4.6] and we show that our abstract result recovers the known results.

The example in Sect. 5.3 deals with $\mathcal{Q} = \mathrm{H}^1((0, 1))$ equipped with the weak topology, with the dissipation $\mathcal{D}_k(q, \widetilde{q}) = \|\widetilde{q} - q\|_{\mathrm{L}^1}$ and with the energy functional

$$\mathcal{E}_k(t, q) = \int_0^1 W(q'(x)) + q(x)^2 - f(t, x)q(x)\,\mathrm{d}x,$$

where $W : \mathbb{R} \to \mathbb{R}$ is a coercive, nonconvex double-well potential. The Γ-limits in the weak topology of $\mathrm{H}^1((0, 1))$ of the constant sequences $\mathcal{D}_k = \mathcal{D}_1$ and $\mathcal{E}_k = \mathcal{E}_1$ are $\mathcal{D}_\infty = \mathcal{D}_1$ and $\mathcal{E}_\infty = \mathrm{conv}\mathcal{E}_1$, which has the same form as $\mathcal{E}_k$ but W is replaced by its convexification W^{**}. Using the results of Sect. 4 we show that the solutions of $(\mathrm{AIP})_k$, which develop microstructure, converge weakly to an energetic solution associated with the relaxed functionals $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$. The question of relaxations of this type was already addressed in [24,28,34]. However, rigorous results were only obtained in [9,43]. The analogous is obtained by regularizing $\mathcal{E}_1$ in the form $\mathcal{E}_k(t, z) = \mathcal{E}_1(t, z) + \frac{1}{k}\int_0^1 (z''(x))^2\,\mathrm{d}x$.

Another application of the theory presented here is given in [15], where the Γ-convergence of families of crack problems is studied. There the notion of "stability of the unilateral minimality property" is used for what we call upper semi-continuity of the stable sets.

2 Assumptions and preliminary results

Throughout this work we assume that the state space $\mathcal{Q}$ is a product $\mathcal{Q} = \mathcal{F}\times\mathcal{Z}$, where each of the factors is a Hausdorff topological space. All our notions concerning (lower semi−) continuity, closedness and compactness are in fact meant "sequentially". (The typical applications we have in mind are the weak topologies in separable, reflexive Banach spaces, possibly restricted to a weakly closed subset.) We will denote the convergence in these spaces by $\overset{\mathcal{Q}}{\to}$, $\overset{\mathcal{F}}{\to}$, and $\overset{\mathcal{Z}}{\to}$ respectively. For sequences $(t_k, q_k)_{k\in\mathbb{N}}$ we write $(t_k, q_k) \overset{[0,T]\times\mathcal{Q}}{\to} (t, q)$ if $t_k \to t$ in $\mathbb{R}$ and $q_k \overset{\mathcal{Q}}{\to} q$.

On the state space $\mathcal{Q}$ a sequence of time-dependent energy functionals $\mathcal{E}_k : [0, T]\times\mathcal{Q} \to \mathbb{R}_\infty$ as well as a limit $\mathcal{E}_\infty : [0, T]\times\mathcal{Q} \to \mathbb{R}_\infty$ are given. Moreover, we have a sequence of dissipation distances $\mathcal{D}_k : \mathcal{Z}\times\mathcal{Z} \to [0, \infty]$ and a limit $\mathcal{D}_\infty : \mathcal{Z}\times\mathcal{Z} \to [0, \infty]$. Note that our dissipation distances are not assumed to be symmetric, i.e. $\mathcal{D}_k(z_1, z_2) \neq \mathcal{D}_k(z_2, z_1)$ is possible. Moreover, we allow for the value $+\infty$, which is often needed in continuum mechanical models. We use the notation $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ which enables us to address the sequence as well as the limits together.

Throughout we will switch between the two equivalent notations $q \in \mathcal{Q}$ and $(\varphi, z) \in \mathcal{F}\times\mathcal{Z}$ as it is most appropriate in the given context. In particular, we also consider $\mathcal{D}_k$, $k \in \mathbb{N}_\infty$, as functions on $\mathcal{Q} \times \mathcal{Q}$ and write $\mathcal{D}_k(q_1, q_2)$ instead of $\mathcal{D}_k(z_1, z_2)$, where $q_j = (\varphi_j, z_j) \in \mathcal{F}\times\mathcal{Z} = \mathcal{Q}$ is taken for granted.

To formulate our assumptions we recall the definition of the stable sets $\mathcal{S}_k(t)$ from (1.3) and call a sequence $(t_l, q_{k_l})_{l\in\mathbb{N}}$ a *stable sequence* (abbreviated as "stab.seq." further on), if

$$q_{k_l} \in \mathcal{S}_{k_l}(t_l) \text{ for all } l \in \mathbb{N} \quad \text{and} \quad \sup_{l\in\mathbb{N}} \mathcal{E}_{k_l}(t_l, q_{k_l}) < \infty. \tag{2.1}$$

Note that $(q_{k_l})_{l\in\mathbb{N}}$ denotes a subsequence to indicate the index k_l for which we have stability. We now state our assumptions in one list and comment on it afterwards.

$$\begin{aligned}&\textit{Pseudo-distance}: \quad \forall\, k \in \mathbb{N}_\infty\ \forall\, z_1, z_2, z_3 \in \mathcal{Z}:\\ &\mathcal{D}_k(z_1, z_1) = 0 \ \text{ and } \ \mathcal{D}_k(z_1, z_3) \le \mathcal{D}_k(z_1, z_2) + \mathcal{D}_k(z_2, z_3).\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\textit{Lower semi-continuity of } \mathcal{D}_k:\\ &\forall\, k \in \mathbb{N}_\infty: \quad \mathcal{D}_k : \mathcal{Z}\times\mathcal{Z} \to [0, \infty] \text{ is lower semi-continuous.}\end{aligned} \tag{2.3}$$

$$\begin{aligned}&\textit{Positivity of } \mathcal{D}_\infty: \quad \text{For all compact } \mathcal{K} \subset \mathcal{Z}:\\ &\text{If } z_k \in \mathcal{K} \text{ and } \min\{\mathcal{D}_\infty(z_k, z), \mathcal{D}_\infty(z, z_k)\} \to 0, \text{ then } z_k \overset{\mathcal{Z}}{\to} z.\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\textit{Lower } \Gamma\textit{-limit for } \mathcal{D}_k:\\ &\forall \text{ stab.seq. } (t_l, q_{k_l}),\ (\widetilde{t}_l, \widetilde{q}_{k_l}) \text{ with } (t_l, q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\to} (t, q),\ (\widetilde{t}_l, \widetilde{q}_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\to} (\widetilde{t}, \widetilde{q}):\\ &\mathcal{D}_\infty(q, \widetilde{q}) \le \liminf_{l\to\infty} \mathcal{D}_{k_l}(q_{k_l}, \widetilde{q}_{k_l}).\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\textit{Compactness of energy sublevels}:\\ &\text{For all } t \in [0, T] \text{ and all } E \in \mathbb{R} \text{ we have}\\ &\text{(i)} \quad \forall\, k \in \mathbb{N}_\infty: \quad \{\, q \in \mathcal{Q};\ \mathcal{E}_k(t, q) \le E \,\} \text{ is compact;}\\ &\text{(ii)} \quad \textstyle\bigcup_{k=1}^{\infty}\{\, q \in \mathcal{Q};\ \mathcal{E}_k(t, q) \le E \,\} \text{ is relatively compact.}\end{aligned} \tag{2.6}$$

Here (with our agreement about "sequential" notions) relative compactness of $A \subset \mathcal{Q}$ means that every sequence in A has a convergent subsequence.

$$\begin{aligned}&\textit{Uniform control of the power } \partial_t\mathcal{E}_k:\\ &\exists\, c_0^E \in \mathbb{R}\ \exists\, c_1^E > 0\ \forall\, k \in \mathbb{N}_\infty\ \forall\, t \in [0, T]\ \forall\, q \in \mathcal{Q}:\\ &\text{If } \mathcal{E}_k(t, q) < \infty, \text{ then } \mathcal{E}_k(\cdot, q) \in \mathrm{C}^1([0, T]) \text{ and}\\ &|\partial_t\mathcal{E}_k(s, q)| \le c_1^E(c_0^E + \mathcal{E}_k(s, q)) \text{ for all } s \in [0, T].\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\textit{Uniform time-continuity of the power } \partial_t\mathcal{E}_\infty :\\&\forall\, \varepsilon>0\ \forall\, E\in\mathbb{R}\ \exists\, \delta>0:\\&\mathcal{E}_\infty(0,q)\le E \text{ and } |t_1-t_2|<\delta \implies |\partial_t\mathcal{E}_\infty(t_1,q)-\partial_t\mathcal{E}_\infty(t_2,q)|<\varepsilon.\end{aligned} \tag{2.8}$$

$$\begin{aligned}&\textit{Conditioned continuous convergence of the power}:\\&\forall \text{ stab.seq. } (t_l,q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\to} (t,q): \quad \partial_t\mathcal{E}_{k_l}(t_l,q_{k_l})\to\partial_t\mathcal{E}_\infty(t,q)\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\textit{Lower } \Gamma\textit{-limit for } \mathcal{E}_k :\\&\forall \text{ stab.seq. } (t_l,q_{k_l}) \text{ with } (t_l,q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\to} (t,q):\ \mathcal{E}_\infty(t,q)\le\liminf_{l\to\infty}\mathcal{E}_{k_l}(t_l,q_{k_l}).\end{aligned} \tag{2.10}$$

$$\begin{aligned}&\textit{Conditioned upper semi-continuity of stable sets}:\\&\forall \text{ stab.seq. } (t_l,q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\to} (t,q): \quad q\in\mathcal{S}_\infty(t).\end{aligned} \tag{2.11}$$

Assumptions (2.2)–(2.5) mainly concern the dissipation distances, whereas assumptions (2.6)–(2.10) are mainly on the stored-energy functionals. Conditions (2.5), (2.9)–(2.11) are based on the stable sets, which involve the interplay of $\mathcal{E}_k$ and $\mathcal{D}_k$.

For a given function $z:\ [0,T]\to\mathcal{Z}$ (defined everywhere!) we define the dissipation associated with $\mathcal{D}_k,\ k\in\mathbb{N}_\infty$, on the subinterval $[r,s]$, via

$$\mathrm{Diss}_k(z;[r,s])=\sup\Big\{\sum_{j=1}^{N}\mathcal{D}_k(z(t_{j-1}),z(t_j))\ ;\ N\in\mathbb{N},\ r\le t_0<t_1<\cdots<t_N\le s\Big\}.$$

The lower Γ-limit condition (2.5) for $\mathcal{D}_k$ implies that, if $z_k:[0,T]\to\mathcal{Z}$ converges pointwise to $z:[0,T]\to\mathcal{Z}$ and if $(t,q_k(t))$ is stable for all $t\in[0,T]$, then

$$\mathrm{Diss}_\infty(z;[r,s])\le\liminf_{k\to\infty}\mathrm{Diss}_k(z_k;[r,s]). \tag{2.12}$$

The positivity condition (2.4) for $\mathcal{D}_\infty$ implies that a function z with $\mathrm{Diss}_\infty(z;[0,T])<\infty$ is continuous on $[0,T]$ except for at most countably many points, namely the jump points of $t\mapsto\mathrm{Diss}_\infty(z;[0,t])$.

The major compactness result is a generalization of Helly's selection principle, which is proved in Appendix A. Using (2.2), (2.4) and (2.5) it is shown that every sequence of functions $z_k:[0,T]\to\mathcal{Z}$ for which $\mathrm{Diss}_k(z_k;[0,T])$ is bounded has a pointwise convergent subsequence.

The compactness condition (2.6) on the energy functionals implies lower semi-continuity of each $\mathcal{E}_k(t,\cdot):\mathcal{Q}\to\mathbb{R}_\infty$ and is essential for constructing solutions for incremental minimization problems.

For a given $q\in\mathcal{Q}$ the mapping $t\mapsto\mathcal{E}_k(t,q)$ maps $[0,T]$ into $\mathbb{R}_\infty$. Hence the partial derivative $\partial_t\mathcal{E}(t,q)$ makes sense even though $\mathcal{Q}$ does not have a manifold structure. Moreover, it has the physical dimension of a power, namely energy divided by time. In [29] $\int_0^t\partial_s\mathcal{E}(s,q(s))\,\mathrm{d}s$ is called the reduced work of the external forces, since it relates to the "work of the external forces", as used in the mechanics literature. In the simple case $\mathcal{E}(t,\varphi,z)=\mathcal{U}(\varphi,z)-\langle\ell(t),\varphi\rangle$ the former has the form $-\int_0^t\langle\dot\ell(s),\varphi(s)\rangle\,\mathrm{d}s$ while the latter one reads $\int_0^t\langle\ell(s),\partial_s\varphi(s)\rangle\,\mathrm{d}s$. From our energy balance (E) in (1.1) it is clear that $\partial_t\mathcal{E}(t,q(t))$ is the power associated with the changing external forces. For simplicity, we continue to call this term simply *power*.

Condition (2.7) gives a uniform energetic control on the power $\partial_t\mathcal{E}_k(t,q)$. Using a simple Gronwall argument yields the estimate

$$\mathcal{E}_k(t_1,q)+c_0^E\le\mathrm{e}^{c_1^E|t_1-t_2|}\Big(\mathcal{E}_k(t_2,q)+c_0^E\Big), \tag{2.13}$$

which provides simple a priori estimates for the energy and the dissipation along solutions, see Step 1 in the proof of Theorem 3.4.

The continuity condition (2.9) for the power $\partial_t\mathcal{E}_k$ is weaker than the so-called continuous convergence of $\partial_t\mathcal{E}_k$ to $\partial_t\mathcal{E}_\infty$, viz., $(t_l, q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\to} (t,q) \implies \partial_t\mathcal{E}_{k_l}(t_l, q_{k_l}) \to \partial_t\mathcal{E}_\infty(t,q)$. In fact, we only need to know the convergence of the power along converging stable sequences. We will see that, under some additional assumptions, the convergence of stable sequences leads to improved convergence, e.g., to convergence of the energies $\mathcal{E}_{k_l}(t_l, q_{k_l}) \to \mathcal{E}_\infty(t,q)$, see Proposition 2.2(A) below. In the Banach space context this may be used to convert a weak convergence into a strong one. Moreover, the abstract Proposition 3.3 in [14] shows that this energy convergence together with the lower semi-continuity (2.10) of $(\mathcal{E}_k)_{k\in\mathbb{N}_\infty}$ and (2.8) implies the conditioned continuous convergence (2.9) of the power.

The two conditions (2.5) and (2.10) on the lower Γ-limits of $\mathcal{D}_k$ and $\mathcal{E}_k$, respectively, are formulated in a general setting involving the stable sequences. However, in all the applications in this paper we will use the major results under the stronger assumption that $\mathcal{D}_\infty$ and $\mathcal{E}_\infty$ are the Γ-limits in the usual sense:

$$\mathcal{I}_\infty = \Gamma\text{-}\lim_{k\to\infty} \mathcal{I}_k \overset{\text{def}}{\iff} \begin{cases} \text{(i) } q_k \overset{\mathcal{Q}}{\to} q \implies \mathcal{I}_\infty(q) \le \liminf_{k\to\infty}\mathcal{I}_k(q_k), \\ \text{(ii) } \forall q\in\mathcal{Q}\ \exists\,(\widehat{q}_k)_{k\in\mathbb{N}} \text{ with } \widehat{q}_k \overset{\mathcal{Q}}{\to} q: \\ \qquad\qquad \mathcal{I}_\infty(q) \ge \limsup_{k\to\infty}\mathcal{I}_k(\widehat{q}_k). \end{cases} \tag{2.14}$$

Here the sequence $(\widehat{q}_k)_{k\in\mathbb{N}}$ is called a *recovery sequence* for the limit q. Clearly (i) and (ii) gives $\mathcal{I}_k(\widehat{q}_k) \to \mathcal{I}_\infty(q)$. Our weaker assumptions (2.5) and (2.10) can be useful in certain more involved applications since the additional stability and energy boundedness for the converging sequences might be helpful in establishing the desired lower bound. However, our main results in Sections 3 and 4 imply that along our solution sequences q_k we will have convergence of the energies, see the statements (i) in the Theorems 3.1, 3.4, and 4.1.

The major condition that makes the whole theory working is (2.11). This condition couples the potentials $\mathcal{E}_k$ and $\mathcal{D}_k$ and provides a kind of upper Γ-limit estimate for $\mathcal{E}_k$ and $\mathcal{D}_k$ simultaneously. In [15] a similar condition is derived to study the Γ-convergence of the solutions in families of crack problems. There our notion of stability is called "unilateral minimality property" and our notion of upper semi-continuity of the stable sets is called "stability of the unilateral minimality property". In that paper the Theorems 7.2 and 8.3 provide what we call condition (2.11).

Lemma 2.1 *The upper semi-continuity condition* (2.11) *is equivalent to*

$$\begin{aligned} &\forall\ \textit{stab.seq.}\ (t_l, q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\to} (t,q)\ \forall \widetilde{q}\in\mathcal{Q}\ \exists\,(\widetilde{q}_{k_l})_{l\in\mathbb{N}}: \\ &\limsup_{l\to\infty}\big(\mathcal{E}_{k_l}(t_l,\widetilde{q}_{k_l}) + \mathcal{D}_{k_l}(q_{k_l},\widetilde{q}_{k_l}) - \mathcal{E}_{k_l}(t_l,q_{k_l})\big) \le \mathcal{E}_\infty(t,\widetilde{q}) + \mathcal{D}_\infty(q,\widetilde{q}) - \mathcal{E}_\infty(t,q). \end{aligned} \tag{2.15}$$

Proof For abbreviation we set $\mathcal{H}_k(t,q,\widetilde{q}) = \mathcal{E}_k(t,\widetilde{q}) + \mathcal{D}_k(q,\widetilde{q}) - \mathcal{E}_k(t,q)$. Then, $q\in\mathcal{S}_k(t)$ is equivalent to $\mathcal{H}_k(t,q,\widetilde{q}) \ge 0$ for all $\widetilde{q}\in\mathcal{Q}$.

The implication (2.11) ⇒ (2.15) follows immediately by taking the sequence $\widetilde{q}_{k_l} = q_{k_l}$. Then, (2.15) holds, since $\mathcal{H}_{k_l}(t_l, q_{k_l}, \widetilde{q}_{k_l}) = 0$ and (2.11) implies $\mathcal{H}_\infty(t,q,\widetilde{q}) \ge 0$.

The opposite implication (2.15) ⇒ (2.11) is seen as follows. For arbitrary $\widetilde{q}$ we choose a sequence $(\widetilde{q}_{k_l})_{l\in\mathbb{N}}$ according to (2.15). Using $q_{k_l}\in\mathcal{S}_{k_l}(t_l)$ we have $\mathcal{H}_{k_l}(t_l,q_{k_l},\widetilde{q}_{k_l}) \ge 0$. Taking the $\limsup_{l\to\infty}$ and employing (2.15) we conclude $\mathcal{H}_\infty(t,q,\widetilde{q}) \ge 0$. Since $\widetilde{q}\in\mathcal{Q}$ was arbitrary, this gives $q\in\mathcal{S}_\infty(t)$. □

Note that condition (2.15) does not ask for $\widetilde{q}_{k_l} \overset{\mathcal{Q}}{\to} \widetilde{q}$, hence $(\widetilde{q}_{k_l})_{l\in\mathbb{N}}$ is not a recovery sequence in the sense of (2.14). In fact, the inequality in (2.15) has the property that the right-hand side depends on $\widetilde{q}$ but not on $(\widetilde{q}_{k_l})_{l\in\mathbb{N}}$, while the left-hand side is independent of $\widetilde{q}$. Nevertheless, the condition is useful when choosing a suitable sequence $(\widetilde{q}_{k_l})_{l\in\mathbb{N}}$ with $\widetilde{q}_{k_l} \overset{\mathcal{Q}}{\to} \widetilde{q}$ such that $\mathcal{E}_{k_l}(t_l,\widetilde{q}_{k_l})+\mathcal{D}_{k_l}(q_{k_l},\widetilde{q}_{k_l})-\mathcal{E}_{k_l}(t_l,q_{k_l}) \to \mathcal{E}_\infty(t,\widetilde{q})+\mathcal{D}_\infty(q,\widetilde{q})-\mathcal{E}_\infty(t,q)$. For later use we display this slight strengthening of (2.15) for finding a *joint recovery sequence* $(\widetilde{q}_{k_l})_{l\in\mathbb{N}}$:

$$\begin{aligned}&\forall \text{ stab.seq. } (t_l,q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\to} (t,q)\ \forall \widetilde{q}\in\mathcal{Q}\ \exists \widetilde{q}_{k_l} \overset{\mathcal{Q}}{\to} \widetilde{q}:\\ &\limsup_{l\to\infty}\big(\mathcal{E}_{k_l}(t_l,\widetilde{q}_{k_l})+\mathcal{D}_{k_l}(q_{k_l},\widetilde{q}_{k_l})-\mathcal{E}_{k_l}(t_l,q_{k_l})\big) \le \mathcal{E}_\infty(t,\widetilde{q})+\mathcal{D}_\infty(q,\widetilde{q})-\mathcal{E}_\infty(t,q).\end{aligned} \tag{2.16}$$

We provide two more conditions which are stronger than (2.16) and, hence, can be used to establish the crucial upper semi-continuity (2.11) of the stable sets. The weaker of these two conditions is based on the existence of a joint recovery sequence and reads

$$\begin{aligned}&\forall \text{ stab.seq. } (t_l,q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\to} (t,q)\ \forall \widetilde{q}\in\mathcal{Q}\ \exists \widetilde{q}_{k_l} \overset{\mathcal{Q}}{\to} \widetilde{q}:\\ &\limsup_{l\to\infty}\big(\mathcal{E}_{k_l}(t_l,\widetilde{q}_{k_l})+\mathcal{D}_{k_l}(q_{k_l},\widetilde{q}_{k_l})\big) \le \mathcal{E}_\infty(t,\widetilde{q})+\mathcal{D}_\infty(q,\widetilde{q}).\end{aligned} \tag{2.17}$$

The stronger of these two conditions consists on two separate convergence results for the energy functionals and for the dissipation distances: $\mathcal{E}_\infty$ is the Γ-limit of $\mathcal{E}_k$, i.e.,

$$\begin{aligned}&(2.10) \text{ holds and } \ \forall t\in[0,T]\ \forall \widehat{q}\in\mathcal{Q}\\ &\exists (\widehat{q}_k)_{k\in\mathbb{N}} \text{ with } \widehat{q}_k \overset{\mathcal{Q}}{\to} \widehat{q}: \quad \mathcal{E}_\infty(t,\widehat{q}) \ge \limsup_{k\to\infty} \mathcal{E}_k(t,\widehat{q}_k),\end{aligned} \tag{2.18}$$

and $\mathcal{D}_k$ continuously converges to $\mathcal{D}_\infty$ conditioned by bounded energy, i.e.,

$$\left.\begin{aligned}q_k \overset{\mathcal{Q}}{\to} q \text{ and } \widetilde{q}_k \overset{\mathcal{Q}}{\to} \widetilde{q}\\ \sup_{k\in\mathbb{N}}\big(\mathcal{E}_k(t,q_k)+\mathcal{E}_k(t,\widetilde{q}_k)\big)<\infty\end{aligned}\right\} \implies \mathcal{D}_k(q_k,\widetilde{q}_k) \to \mathcal{D}_\infty(q,\widetilde{q}). \tag{2.19}$$

Proposition 2.2 *Assume that* (2.10) *holds.*
(*A*) *If for each stable sequence* (t_l,q_{k_l}) *that converges to* (t,q) *there exists a sequence* $(\widetilde{q}_l)_{l\in\mathbb{N}}$ *such that* $\limsup_{l\to\infty}\mathcal{E}_{k_l}(t_l,\widetilde{q}_l)+\mathcal{D}_{k_l}(q_{k_l},\widetilde{q}_l) \le \mathcal{E}_\infty(t,q)$, *then the energy converges along the stable sequences, i.e.,*

$$\forall \textit{ stab.seq. } (t_l,q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\to} (t,q): \quad \mathcal{E}_{k_l}(t_l,q_{k_l}) \to \mathcal{E}_\infty(t,q). \tag{2.20}$$

In particular, we have (2.17) $\implies$ (2.20).
(*B*) *We have the following implications*:

$$\big((2.18) \textit{ and } (2.19)\big) \implies (2.17) \implies (2.16) \implies (2.15) \iff (2.11).$$

Proof ad (A). By (2.10) we have $\mathcal{E}_\infty(t,q) \le \liminf_{l\to\infty}\mathcal{E}_{k_l}(t_l,q_{k_l})$. Using $\mathcal{D}_{k_l}(q_{k_l},\widetilde{q}_l)\ge 0$ we immediately obtain $\limsup_{l\to\infty}\mathcal{E}_{k_l}(t_l,q_{k_l}) \le \mathcal{E}_\infty(t,q)$. This proves (2.20). Since (2.17) includes the assumption by specifying $\widetilde{q}=q$, the final implication holds.
ad (B). For the first implication we start from a converging stable sequence $(t_l,q_{k_l}) \to (t,q)$ and from a general $\widetilde{q}$. We choose $\widetilde{q}_l$ via the recovery sequence $\widehat{q}_k$ from (2.18), namely $\widetilde{q}_l = \widehat{q}_{k_l}$. Employing (2.19) we then obtain $\limsup_{l\to\infty}\mathcal{E}_{k_l}(t_l,\widetilde{q}_l)+\mathcal{D}_{k_l}(q_{k_l},\widetilde{q}_l) \le \mathcal{E}_\infty(t,\widetilde{q})+\mathcal{D}_\infty(q,\widetilde{q})$, which is the desired result (2.17).

For "(2.17) ⇒ (2.16)" note that (2.10) implies $\limsup_{l\to\infty}\big(-\mathcal{E}_{k_l}(t_l,q_{k_l})\big)\le-\mathcal{E}_\infty(t,q)$, whenever $(t_l,q_{k_l})\overset{[0,T]\times\mathcal{Q}}{\to}(t,q)$. Adding this to (2.17) we easily find the desired result (2.16).

The next implication follows directly from the definition as the requirement $\widetilde{q}_{k_l}\overset{\mathcal{Q}}{\to}\widetilde{q}$ is dropped. The final equivalence is the content of Lemma 2.1. □

The following examples show that the above implications cannot be reversed. It is easy to provide such examples taking $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$ strictly lower than the corresponding Γ-limits. Our examples below are chosen such that equality between $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$ and the corresponding Γ-limits hold. In particular, this means that (2.10) and (2.18) hold. For simplicity, we drop the dependence on the time $t\in[0,T]$, as the main emphasis of condition (2.11) is on the convergence of q_k. Using the assumptions (2.7)–(2.9) it is then easy to obtain the more general version including $t_k\to t$.

Example 2.3
(I) "(2.16) $\not\Rightarrow$ (2.17)". Consider $\mathcal{Q}=\mathrm{L}^2(\Omega)$ equipped with its weak topology. The sequences $\mathcal{E}_k$ and $\mathcal{D}_k$ are assumed to be constant, namely $\mathcal{E}_k(t,q)=\int_\Omega\frac12 q(x)^2-f(t,x)q(x)\,\mathrm{d}x$ with $f\in\mathrm{C}^1([0,T],\mathrm{L}^2(\Omega))$ and $\mathcal{D}_k(q_0,q_1)=\|q_1-q_0\|_{\mathrm{L}^1}$. Obviously, we have $\mathcal{S}_k(t)=\{\,q\in\mathrm{L}^2(\Omega)\,;\ \|q-f(t,\cdot)\|_{\mathrm{L}^\infty}\le1\,\}$ and it is easy to see that (2.11) holds. However, even without this knowledge, we may establish (2.16) directly. We choose the recovery sequence $\widetilde{q}_{k_l}=\widetilde{q}-q+q_{k_l}$, hence $\widetilde{q}_{k_l}\rightharpoonup\widetilde{q}$. Moreover, $\mathcal{D}_{k_l}(q_{k_l},\widetilde{q}_{k_l})=\|\widetilde{q}-q\|_{\mathrm{L}^1}=\mathcal{D}_\infty(q,\widetilde{q})$ and

$$\begin{aligned}\mathcal{E}_{k_l}(t_l,\widetilde{q}_{k_l})-\mathcal{E}_{k_l}(t_l,q_{k_l})&=\big\langle\tfrac12(\widetilde{q}-q)+q_{k_l}-f(t_l,\cdot),\widetilde{q}-q\big\rangle_{\mathrm{L}^2}\\ &\to\big\langle\tfrac12(\widetilde{q}+q)-f(t_l,\cdot),\widetilde{q}-q\big\rangle_{\mathrm{L}^2}=\mathcal{E}_\infty(t,\widetilde{q})-\mathcal{E}_\infty(t,q),\end{aligned}$$

which proves (2.16) with equality.

To show that (2.17) does not hold we consider $t_l=0$ and the stable sequence q_l with $|q_l-f(0,\cdot)|\equiv1$ but $q_l\rightharpoonup q=f(0,\cdot)$. Moreover, let $\widetilde{q}=q$, such that the right-hand side in (2.17) takes the value $-\frac12\|q\|^2_{\mathrm{L}^2}$. Writing the joint recovery sequence $\widetilde{q}_l$ in the form $\widetilde{q}_l=q_l+w_l$ we must have $w_l\rightharpoonup0$ and the left-hand side in (2.17) gives

$$\begin{aligned}\mathcal{E}(0,\widetilde{q}_l)+\mathcal{D}(q_l,\widetilde{q}_l)&=\int_\Omega\frac12\big(q_l+w_l-q\big)^2-\frac12|q|^2+|w_l|\,\mathrm{d}x\\ &\ge\int_\Omega\frac12-\frac12|q|^2\,\mathrm{d}x\ >\ -\frac12\|q\|^2_{\mathrm{L}^2}=\mathcal{E}(0,q)+\mathcal{D}(q,q),\end{aligned}$$

where we used $|q_l-q|\equiv1$ and minimized with respect to w_l. Thus, we have shown that (2.17) cannot hold.

This example is relevant to the classical linearized elastoplasticity with hardening. An application of (2.16) in the framework of two-scale homogenization is given in [33].
(II) "(2.16) $\not\Rightarrow$ (2.17) $\not\Rightarrow$ (2.19)". We consider $\mathcal{Q}=\mathbb{R}$, $\mathcal{E}_k(q)=\frac12(k^\alpha q)^2$, and $\mathcal{D}_k(q,\widetilde{q})=k^\beta|\widetilde{q}-q|$. Here, $\alpha,\beta\ge0$ are parameters. The corresponding stable sets are $\mathcal{S}_k=[-k^{\beta-\alpha},k^{\beta-\alpha}]$. The Γ-limits are easily obtained, namely $\mathcal{E}_\infty=\mathcal{E}_1$ if $\alpha=0$ and $\mathcal{E}_\infty=I_{\{0\}}$ else and $\mathcal{D}_\infty(q,\widetilde{q})=|\widetilde{q}-q|$ if $\beta=0$ and $\mathcal{D}_\infty(q,\widetilde{q})=I_{\{0\}}(\widetilde{q}-q)$ else.

The different conditions can be checked easily. In particular, (2.19) holds if and only if $\alpha>\beta\ge0$ or if $\alpha=\beta=0$. Condition (2.17) holds if and only if $\alpha>\beta\ge0$ or if $\alpha=0$, which is a strictly bigger set. Note that for $0<\alpha\le\beta$ the property (2.20) does not hold and hence, by Proposition 2.2(A), condition (2.17) must be violated. Finally, condition (2.16) holds in all cases by choosing $\widetilde{q}_{k_l}=q_{k_l}+\widetilde{q}-q$.

(III) "(2.11) $\Leftrightarrow$ (2.15) $\not\Rightarrow$ (2.16)". We let $\mathcal{E}_k(q) = \mathcal{E}(q) = \frac{1}{2}q^2$ for $k \in \mathbb{N}_\infty$ and choose $\mathcal{D}_k$ via $\mathcal{D}_k(q, \widetilde{q}) = \big| \int_{\widetilde{q}}^{q} m_k(p)\,\mathrm{d}p \big|$ with $m_k(p) = 1$ for $p \geq 0$ and k otherwise. The Γ-limit $\mathcal{D}_\infty$ reads $\mathcal{D}_\infty(q, \widetilde{q}) = |\widetilde{q}-q|$ for $q, \widetilde{q} \geq 0$, $\mathcal{D}_\infty(q, \widetilde{q}) = 0$ for $\widetilde{q} = q < 0$, and $+\infty$ otherwise. Some computations give $\mathcal{S}_k = [-k, 1]$ and $\mathcal{S}_\infty = (-\infty, 1]$, and thus (2.11) holds. The sequence $q_k = -1/k$ is a stable sequence converging to $q = 0$. For $\widetilde{q} = 1$, any sequence $(\widetilde{q}_k)_{k\in\mathbb{N}}$ with $\widetilde{q}_k \to \widetilde{q} = 1$ satisfies $\mathcal{D}_k(q_k, \widetilde{q}_k) \to 2 < \mathcal{D}_\infty(q, \widetilde{q}) = \mathcal{D}_\infty(0, 1) = 1$. Hence, since $\mathcal{E}$ is continuous, (2.16) cannot hold.

The next result states that the stability condition (S) in (1.1) implies a lower energy estimate. This observation was first done in [34] and is proved more generally in [25, Proposition 5.7].

Proposition 2.4 *Let the condition* (2.7) *for* $k = \infty$ *and* (2.8) *hold. If* $q : [0, T] \to \mathcal{Q}$ *satisfies* $(\mathrm{S})_\infty$, *if* $\mathcal{E}_\infty(\cdot, q(\cdot)) \in \mathrm{BV}([0, T])$ *and if* $\partial_t\mathcal{E}_\infty(\cdot, q(\cdot)) \in \mathrm{L}^1([0, \mathrm{T}])$, *then for all* $r, s \in [0, T]$ *with* $r < s$ *we have the lower energy estimate*

$$\mathcal{E}_\infty(s, q(s)) + \mathrm{Diss}_\infty(q; [r, s]) \geq \mathcal{E}_\infty(r, q(r)) + \int_r^s \partial_t\mathcal{E}_\infty(t, q(t))\,\mathrm{d}t.$$

Proof Take an arbitrary partition $r = \tau_0 < \tau_1 < \cdots < \tau_N = s$ of $[r, s]$. Testing stability of $q(\tau_{j-1})$ with $q(\tau_j)$ we find

$$\begin{aligned}\mathcal{E}_\infty(\tau_{j-1}, q(\tau_{j-1})) &\leq \mathcal{E}_\infty(\tau_{j-1}, q(\tau_j)) + \mathcal{D}_\infty(q(\tau_{j-1}), q(\tau_j))\\ &= \mathcal{E}_\infty(\tau_j, q(\tau_j)) - \int_{\tau_{j-1}}^{\tau_j} \partial_s\mathcal{E}_\infty(s, q(\tau_j))\,\mathrm{d}s + \mathcal{D}_\infty(q(\tau_{j-1}), q(\tau_j)).\end{aligned}$$

Rearranging this inequality and summation over $j = 1, \ldots, N$ gives

$$\mathcal{E}_\infty(s, q(s)) + \mathrm{Diss}_\infty(q; [r, s]) \geq \mathcal{E}_\infty(s, q(s)) + \sum_{j=1}^{N} \mathcal{D}_\infty(q(\tau_{j-1}), q(\tau_j))$$

$$\geq \mathcal{E}_\infty(r, q(r)) + \sum_{j=1}^{N} \int_{\tau_{j-1}}^{\tau_j} \partial_t\mathcal{E}_\infty(t, q(\tau_j))\,\mathrm{d}t$$

$$= \mathcal{E}_\infty(r, q(r)) + \int_r^s \partial_t\mathcal{E}_\infty(t, q(t))\,\mathrm{d}t \tag{2.21a}$$

$$+ \sum_{j=1}^{N} \partial_s\mathcal{E}_\infty(\tau_j, q(\tau_j))(\tau_j-\tau_{j-1}) \; - \int_r^s \partial_t\mathcal{E}_\infty(t, q(t))\,\mathrm{d}t \tag{2.21b}$$

$$+ \sum_{j=1}^{N} \int_{\tau_{j-1}}^{\tau_j} \big(\partial_t\mathcal{E}_\infty(t, q(\tau_j)) - \partial_t\mathcal{E}_\infty(\tau_j, q(\tau_j))\big)\,\mathrm{d}t \tag{2.21c}$$

Here (2.21a) contains the desired estimate, the term in (2.21b) tends to 0, if we choose a suitable sequence of partitions such that the Riemann sums converge to the L^1 integral, see [13, Lemma 4.12]. The term in (2.21c) tends to 0 because of (2.8). □

Remark 2.5 In fact, the notion of stable sequences could be strengthened slightly by asking also that the dissipation distance remains bounded as well. For this one has to fix a sequence of initial conditions $(q_*^k)_{k\in\mathbb{N}}$ such that the initial conditions q_0^k to be imposed later for the solutions satisfy $D^* = \sup_{k\in\mathbb{N}} \mathcal{D}_k(q_*^k, q_0^k) < \infty$. By the uniform control of power it is shown that all solutions (incremental or continuous) satisfy the a priori bound

$$\mathcal{D}_k\left(q_*^k, q^k(t)\right) + \mathcal{E}_k(t, q^k(t)) \le D^* + 2\mathrm{e}^{c_1^E T}\left(c_0^E + \sup \mathcal{E}_k(0, q_0^k)\right),$$

see (3.10) and (3.11). Hence, we could use the additional condition

$$\sup_{l\in\mathbb{N}} \mathcal{D}_k(q_*^{k_l}, q_{k_l}) \ < \infty \tag{2.22}$$

in the definition (2.1) of stable sequences, which will weaken the crucial condition (2.11) as well as some of the other. Since this does not lead to any substantial improvement in the present analysis, we refrained from using the weakening condition (2.22) in the definition of stable sequences and, thus, keep our text easier readable.

3 Γ-convergence

Our first result concerns the convergence of the solutions $q_k : [0, T] \to \mathcal{Q}$ of the energetic formulations $(\mathrm{S})_k\&(\mathrm{E})_k$ associated with the functionals $\mathcal{E}_k$ and $\mathcal{D}_k$:

$$\begin{aligned}
&(\mathrm{S})_k \quad \forall\, t \in [0, T]: \ q_k(t) \in \mathcal{S}_k(t),\\
&(\mathrm{E})_k \quad \forall\, t \in [0, T]: \ \mathcal{E}_k(t, q_k(t)) + \mathrm{Diss}_k(q_k; [0, t])\\
&\qquad\qquad = \mathcal{E}_k(0, q_k(0)) + \int_0^t \partial_s \mathcal{E}_k(s, q_k(s))\,\mathrm{d}s.
\end{aligned}\tag{3.1}$$

Theorem 3.1 *Let assumptions* (2.5), (2.7)–(2.11) *hold and let* $q_k : [0, T] \to \mathcal{Q}$ *be solutions of* (3.1). *If for all* $t \in [0, T]$ *we have* $q_k(t) \overset{\mathcal{Q}}{\to} q(t)$ *for* $k \to \infty$ *and if* $\mathcal{E}_k(0, q_k(0)) \to \mathcal{E}_\infty(0, q(0))$, *then* $q : [0, T] \to \mathcal{Q}$ *is a solution of* $(\mathrm{S})_\infty\&(\mathrm{E})_\infty$, *i.e., for all* $t \in [0, T]$ *we have*

$$\begin{aligned}
&(\mathrm{S})_\infty \quad q(t) \in \mathcal{S}_\infty(t)\\
&(\mathrm{E})_\infty \quad \mathcal{E}_\infty(t, q(t)) + \mathrm{Diss}_\infty(q; [0, t]) = \mathcal{E}_\infty(0, q(0)) + \int_0^t \partial_s \mathcal{E}_\infty(s, q(s))\,\mathrm{d}s.
\end{aligned}\tag{3.2}$$

Moreover, for all $t \in [0, T]$ *we have*

$$\begin{aligned}
&\text{(i)} \ \ \mathcal{E}_k(t, q_k(t)) \to \mathcal{E}_\infty(t, q(t)),\\
&\text{(ii)} \ \ \mathrm{Diss}_k(q_k; [0, t]) \to \mathrm{Diss}_\infty(q; [0, t]),\\
&\text{(iii)} \ \ \partial_t \mathcal{E}_k(t, q_k(t)) \to \partial_t \mathcal{E}_\infty(t, q(t)).
\end{aligned}\tag{3.3}$$

Proof First we use $\mathcal{E}_k(0, q_k(0)) \to \mathcal{E}_\infty(0, q(0))$ and condition (2.7) to show that $\mathcal{E}_k(t, q_k(t))$ is bounded uniformly in $t \in [0, T]$ and $k \in \mathbb{N}$, see also (2.13). Now, condition (2.11) gives $(\mathrm{S})_\infty$ and condition (2.9) implies the convergence (iii) in (3.3).

Passing to the limit $k \to \infty$ in $(\mathrm{E})_k$ and using (2.12) and (2.10) we find the upper energy estimate

$$\mathcal{E}_\infty(t, q(t)) + \mathrm{Diss}_\infty(q; [0, t]) \le e_*(t) + \delta_*(t) = \mathcal{E}_\infty(0, q(0)) + \int_0^t \partial_s \mathcal{E}_\infty(s, q(s))\, \mathrm{d}s,$$

where $e_*(t) = \liminf_{k\to\infty} \mathcal{E}_k(t, q_k(t))$ and $\delta_*(t) = \liminf_{k\to\infty} \mathrm{Diss}_k(q_k; [0, t])$. Proposition 2.4 shows the opposite estimate and we obtain $e_*(t) = \mathcal{E}_\infty(t, q(t))$ and $\delta_*(t) = \mathrm{Diss}_\infty(q; [0, t])$. Since the limits inferior $e_*(t)$ and $\delta_*(t)$ are identified a priori and do not depend on choosing a subsequence, we conclude that they are true limits such that (i) and (ii) in (3.3) are shown. □

The following counterexample shows that a joint condition on the sequences $(\mathcal{E}_k)_{k\in\mathbb{N}}$ and $(\mathcal{D}_k)_{k\in\mathbb{N}}$ is necessary to obtain the above convergence result. In particular, the above result as well as the conclusion of Theorem 3.4 below may be false if we have merely the following two independent Γ-convergences

$$\mathcal{E}_\infty = \Gamma\text{–}\lim_{k\to\infty} \mathcal{E}_k \text{ and } \mathcal{D}_\infty = \Gamma\text{–}\lim_{k\to\infty} \mathcal{D}_k. \tag{3.4}$$

Example 3.2 Take $\mathcal{Q} = \mathbb{R}^2$ and, for $\alpha > 0$ and $\beta \ge 0$ let

$$\mathcal{E}_k(t, q) = \frac{1}{2}q_1^2 + \frac{k^\alpha}{2}\Big(q_2 - \frac{1}{k}q_1\Big)^2 - tq_1 \quad \text{and} \quad \mathcal{D}_k(q, \widetilde{q}) = |q_1 - \widetilde{q}_1| + k^\beta |q_2 - \widetilde{q}_2|.$$

Under the initial condition $q(0) = 0$, the explicit solution can be obtained from the subdifferential equation

$$0 \in \partial\mathcal{R}_k(\dot{q}) + A_k q - (t, 0)^\top, \quad q(0) = 0,$$

cf. [31,32] for the equivalence to $(\mathrm{S})_k\&(\mathrm{E})_k$ in the convex case. Here

$$A_k = \begin{pmatrix} 1+k^{\alpha-2} & -k^{\alpha-1} \\ -k^{\alpha-1} & k^\alpha \end{pmatrix}, \quad \partial\mathcal{R}_k(v) = \mathrm{Sign}(v_1)\times\big(k^\beta \mathrm{Sign}(v_2)\big) \subset \mathbb{R}^2,$$

where Sign is the multi-valued signum function. With $T(k) = 1 + k^{\beta-1} + k^{\beta+1-\alpha}$ we have the solutions $q_k : [0, \infty) \to \mathbb{R}^2$ with

$$q_k(t) = \begin{cases} (0, 0)^\top & \text{for } t \in [0, 1], \\ \big(\frac{t-1}{k^{\alpha-2}+1}, 0\big)^\top & \text{for } t \in [1, T(k)], \\ \big(t-1-k^{\beta-1}, \frac{t-T(k)}{k}\big)^\top & \text{for } t \ge T(k). \end{cases}$$

For all choices of α and β, the limit $q(t) = \lim_{k\to\infty} q_k(t)$ exists. For $t \in [0, 1]$ we always have $q(t) = 0$, and for $t \ge 1$ we find

$$\lim_{k\to\infty} q_k(t) = \begin{cases} (\max\{0, t-1\}, 0)^\top & \text{for } \beta \in [0, 1) \text{ or } \alpha \in (0, 2), \\ \big(\max\{0, (t-1)/2, t-2\}, 0\big)^\top & \text{for } (\alpha, \beta) = (2, 1), \\ (\max\{0, (t-1)/2\}, 0)^\top & \text{for } \alpha = 2 \text{ and } \beta > 1, \\ \big(\max\{0, t-2\}, 0\big)^\top & \text{for } \alpha > 2 \text{ and } \beta = 1, \\ (0, 0)^\top & \text{for } \alpha > 2 \text{ and } \beta > 1. \end{cases}$$

It is easy to see that we have

$$\mathcal{E}_k(t, \cdot) \xrightarrow{\Gamma} \mathcal{E}_\infty(t, \cdot)\colon q \mapsto \begin{cases} \frac{1}{2}q_1^2 - tq_1 & \text{for } q_2 = 0, \\ \infty & \text{otherwise.} \end{cases}$$

For $\beta = 0$ we have $\mathcal{D}_\infty = \mathcal{D}_k$ and conclude the continuous convergence (2.19). Hence, (2.11) holds. For $\beta > 0$ we have

$$\mathcal{D}_k \xrightarrow{\Gamma} \mathcal{D}_\infty : (q, \widetilde{q}) \mapsto \begin{cases} |q_1 - \widetilde{q}_1| & \text{for } q_2 = \widetilde{q}_2 = 0, \\ \infty & \text{otherwise.} \end{cases}$$

The unique energetic solution associated with $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$ is given by $q(t) = (\max\{0, t-1\}, 0)^\top$. Thus, we conclude that convergence of q_k to the limit solution holds if and only if $\alpha \in (0, 2)$ or $\beta \in [0, 1)$.

It is interesting to see that the crucial conditional upper semi-continuity of (2.11) of the stable sets holds if and only if $\beta \in [0, 1)$. To see this, note $\mathcal{S}_\infty(t) = [t-1, t+1] \times \{0\}$ and that $\mathcal{S}_k(t)$ is the parallelogram defined by the corners $A_k^{-1}(t+\sigma_1, \sigma_2 k^\beta)^\top$ with $\sigma_1, \sigma_2 \in \{-1, 1\}$. Note that the restriction $\sup \mathcal{E}_k(t, q_k) < \infty$ for stable sequences implies $q_k \cdot (0, 1)^\top \to 0$. In fact, the stronger condition of *unconditioned* upper semi-continuity of the stable sets (i.e., (2.11) without the boundedness of the energy in the definition of "stab.seq.") holds if and only if $0 \le \beta < \min\{\alpha, 1\}$.

Remark 3.3 The convergence result of Theorem 3.1 is also recovered in [39, Lemma 8.2] in case $\mathcal{Q}$ is a reflexive Banach space and $\mathcal{E}_k$ are uniformly convex with respect to q. In particular, by assuming $\mathcal{D}_k(z, \tilde{z}) = \mathcal{R}_k(\tilde{z} - z)$ for some non-negative, convex, positively 1-homogeneous, lower semicontinuous, and uniformly linearly bounded functions $\mathcal{R}_k$ and some extra time-regularity for $\mathcal{E}_k$, the convergence of the approximating solutions q_k to an energetic solution q of (3.3) is there obtained by means of a variational approach.

The major result of this section is the construction of solutions of $(\mathrm{S})_\infty \& (\mathrm{E})_\infty$ without first deriving solutions q_k of $(\mathrm{S})_k \& (\mathrm{E})_k$. Instead it is sufficient to have solutions of the time-incremental minimization problems $(\mathrm{IP})_k$. For this we choose a sequence of partitions

$$\Pi_k = \left\{ 0 = \tau_0^k < \tau_1^k < \cdots < \tau_{N_k-1}^k < \tau_{N_k}^k = T \right\}$$

such that the fineness $\phi(\Pi_k) = \max_{j=1,\ldots,N_k} \left(\tau_j^k - \tau_{j-1}^k\right)$ satisfies $\phi(\Pi_k) \to 0$. The time-incremental problem reads as follows:

$$(\mathrm{IP})_k \quad \text{Given } q_0^k \in \mathcal{Q}, \text{ for } j = 1, \ldots, N_k \text{ find} \quad q_j^k \in \operatorname*{Arg\,Min}_{\widetilde{q} \in \mathcal{Q}} \left(\mathcal{E}_k\left(\tau_j^k, \widetilde{q}\right) + \mathcal{D}_k\left(q_{j-1}^k, \widetilde{q}\right) \right).$$

This incremental problem is fully implicit and thus can be called a backward Euler or Rothe scheme. We then define the (backward) piecewise constant interpolants $\overline{q}_k : [0, T] \to \mathcal{Q}$ via

$$\overline{q}_k(t) = q_{j-1}^k \;\text{ for } t \in [\tau_{j-1}^k, \tau_j^k) \quad \text{and } \overline{q}_k(T) = q_{N_k}^k. \tag{3.5}$$

Theorem 3.4 *Let the conditions* (2.2)–(2.11) *hold. Let the sequence of partitions* Π_k, $k \in \mathbb{N}$, *satisfy* $\phi(\Pi_k) \to 0$. *Let* q_0^k, $k \in \mathbb{N}$, *be a sequence of initial conditions satisfying*

$$q_0^k \in \mathcal{S}_k(0), \quad q_0^k \xrightarrow{\mathcal{Q}} q_0 \quad \text{and} \quad \mathcal{E}_k(0, q_0^k) \to \mathcal{E}_\infty(0, q_0) \in \mathbb{R}. \tag{3.6}$$

Then, each $(\mathrm{IP})_k$ *has at least one solution* $\overline{q}_k = (\overline{\varphi}_k, \overline{z}_k) : [0, T] \to \mathcal{Q} = \mathcal{F} \times \mathcal{Z}$ *and there exist a subsequence* $(\overline{q}_{k_j})_{j \in \mathbb{N}}$ *and a solution* $q = (\varphi, z) : [0, T] \to \mathcal{Q} = \mathcal{F} \times \mathcal{Z}$ *of*

$(\mathrm{S})_\infty \& (\mathrm{E})_\infty$ *such that* (i)–(v) *hold* :

$$\begin{aligned}
&\text{(i)}\ \ \forall\, t \in [0,T]: \mathcal{E}_{k_j}(t, \overline{q}_{k_j}(t)) \to \mathcal{E}_\infty(t, q(t)),\\
&\text{(ii)}\ \ \forall\, t \in [0,T]: \mathrm{Diss}_{k_j}(\overline{q}_{k_j}; [0,t]) \to \mathrm{Diss}_\infty(q; [0,t]),\\
&\text{(iii)}\ \ \forall\, t \in [0,T]: \overline{z}_{k_j}(t) \overset{\mathcal{Z}}{\to} z(t),\\
&\text{(iv)}\ \ \partial_t \mathcal{E}_{k_j}(\cdot, \overline{q}_{k_j}(\cdot)) \to \partial_t \mathcal{E}_\infty(\cdot, q(\cdot)) \ \textit{in}\ \mathrm{L}^1([0,\mathrm{T}]),\\
&\text{(v)}\ \ \forall\, t \in [0,T]\ \exists\ \textit{subsequence}\ (K_n^t)_{n\in\mathbb{N}}\ \textit{of}\ (k_j)_{j\in\mathbb{N}}: \quad \overline{\varphi}_{K_n^t}(t) \overset{\mathcal{F}}{\to} \varphi(t).
\end{aligned} \tag{3.7}$$

Moreover, any $\widetilde{q} : [0,T] \to \mathcal{Q}$ *obtained as such a limit is a solution of* $(\mathrm{S})_\infty \& (\mathrm{E})_\infty$.

Finally, if the topology on $\mathcal{F}$ *restricted to compact subsets is separable and metrizable, then the mapping* $\varphi : [0,T] \to \mathcal{F}$ *can be chosen measurable, i.e., for any open subset* $A \subset \mathcal{F}$ *the pre-image* $\varphi^{-1}(A) \subset [0,T]$ *is Lebesgue measurable.*

An alternative way of formulating the convergence in (v) is based on *convergence of nets*, see Remark 3.5 below.

Proof We follow the six steps of the existence proof for rate-independent problems given in [14,25], which rely on techniques introduced in [13]. We add Step 7 to prove the measurability following ideas of [21,22].

Step 1: *A priori estimates*

Using assumptions (2.3) and (2.6) we immediately see that the solution $(q_j^k)_{j\in\{1,\ldots,N_k\}}$ exist by induction on j. Thus, the interpolants $\overline{q}_k : [0,T] \to \mathcal{Q}$ are well defined. Moreover, we have $q_j^k \in \mathcal{S}_k(\tau_j^k)$, since for all $\widetilde{q} \in \mathcal{Q}$ we have

$$\begin{aligned}
\mathcal{E}_k\left(\tau_j^k, q_j^k\right) &\leq_{(\mathrm{IP})_k} \mathcal{E}_k\left(\tau_j^k, \widetilde{q}\right) + \mathcal{D}_k\left(q_{j-1}^k, \widetilde{q}\right) - \mathcal{D}_k\left(q_{j-1}^k, q_j^k\right)\\
&\leq_{(2.2)} \mathcal{E}_k\left(\tau_j^k, \widetilde{q}\right) + \mathcal{D}_k\left(q_j^k, \widetilde{q}\right).
\end{aligned}$$

Letting $e_j^k = \mathcal{E}_k(\tau_j^k, q_j^k)$ and $\delta_j^k = \mathcal{D}_k(q_{j-1}^k, q_j^k)$ and using the minimization property in $(\mathrm{IP})_k$ once again, we derive the upper energy estimate

$$e_j^k + \delta_j^k \leq_{(\mathrm{IP})_k} \mathcal{E}_k\left(\tau_j^k, q_{j-1}^k\right) = e_{j-1}^k + \int_{\tau_{j-1}}^{\tau_j} \partial_s \mathcal{E}_k\left(s, q_{j-1}^k\right) \mathrm{d}s. \tag{3.8}$$

Inserting first (2.7) and then (2.13) into (3.8) we obtain

$$\begin{aligned}
e_j^k + \delta_j^k &\leq e_{j-1}^k + \int_{\tau_{j-1}^k}^{\tau_j^k} c_1^E \left(e_{j-1}^k + c_0^E\right) \mathrm{e}^{c_1^E\left(s-\tau_{j-1}^k\right)}\, \mathrm{d}s\\
&= e_{j-1}^k + \left(e_{j-1}^k + c_0^E\right)\left(\mathrm{e}^{c_1^E\left(\tau_j^k - \tau_{j-1}^k\right)} - 1\right).
\end{aligned} \tag{3.9}$$

Neglecting $\delta_j^k \geq 0$ we obtain by induction $e_j^k + c_0^E \leq (e_0^k + c_0^E)\mathrm{e}^{c_1^E \tau_j^k}$ and using (2.13) and the definition of $\overline{q}_k$ we find, with $E_* = c_0^E + \sup_{k\in\mathbb{N}} \mathcal{E}_k(0, q_0^k)$,

$$\forall\, t \in [0,T]\ \forall\, k \in \mathbb{N}: \quad \mathcal{E}_k(t, \overline{q}_k(t)) + c_0^E \leq E_* \mathrm{e}^{c_1^E t}. \tag{3.10}$$

For this use (2.13) to find $\mathcal{E}_k(t,\overline{q}_k(t))+c_0^E \le \mathrm{e}^{c_1^E(t-\tau_{j-1}^k)}(e_{j-1}^k+c_0^E)$ for $t\in[\tau_{j-1}^k,\tau_j^k)$. Note that $E_*<\infty$ by assumption (3.6). Summing (3.9) over $j\in\{1,\dots,M\}$ we find

$$\begin{aligned}\sum_{j=1}^{M}\delta_j^k &\le e_0^k-e_M^k+\sum_{j=1}^{M}\left(e_{j-1}^k+c_0^E\right)\left(\mathrm{e}^{c_1^E\left(\tau_j^k-\tau_{j-1}^k\right)}-1\right)\\ &\le\left(e_0^k+c_0^E\right)-\left(e_M^k+c_0^E\right)+\left(e_0^k+c_0^E\right)\sum_{j=1}^{M}(\mathrm{e}^{c_1^E\tau_j^k}-\mathrm{e}^{c_1^E\tau_{j-1}^k})\\ &\le\left(e_0^k+c_0^E\right)\mathrm{e}^{c_1^E\tau_M^k}.\end{aligned}$$

Choosing $M=N_k$ and using the definition of $\overline{q}_k$ this provides

$$\mathrm{Diss}_k(\overline{q}_k;[0,T])=\sum\nolimits_{j=1}^{N_k}\delta_j^k\le E_*\mathrm{e}^{c_1^E T}. \tag{3.11}$$

Finally we want to show that the functions $\overline{e}_k:[0,T]\to\mathbb{R}$ with $\overline{e}_k(t)=\mathcal{E}_k(t,\overline{q}_k(t))$ satisfy a BV bound independent of k. For this we test the stability of $q_{j-1}^k\in\mathcal{S}_k(\tau_{j-1}^k)$ by $\widetilde{q}=q_j^k$ and obtain $e_{j-1}^k\le\mathcal{E}_k(\tau_{j-1}^k,q_j^k)+\mathcal{D}_k(q_{j-1}^k,q_j^k)\le e_j^k+\delta_j^k+C(\tau_j^k-\tau_{j-1}^k)$. Together with (3.9) we obtain

$$\left|e_j^k+\delta_j^k-e_{j-1}^k\right|\le C_1\left(\tau_j^k-\tau_{j-1}^k\right), \tag{3.12}$$

where C_1 is independent of k and j. Moreover, for $t\in[\tau_{j-1}^k,\tau_j^k)$ we have $\dot{\overline{e}}_k(t)=\partial_t\mathcal{E}_k(t,q_{j-1}^k)$ and conclude, using (2.7), that $\int_{\tau_{j-1}^k}^{\tau_j^k}|\dot{\overline{e}}_k(t)|\,\mathrm{d}t\le C_2(\tau_j^k-\tau_{j-1}^k)$.

Finally, using (3.12) we estimate the jumps

$$\begin{aligned}\Delta e_j^k &=\lim_{h\searrow 0}\left(\overline{e}_k\left(\tau_j^k\right)-\overline{e}_k\left(\tau_j^k-h\right)\right)=e_j^k-\left(e_{j-1}^k+\int_{\tau_{j-1}^k}^{\tau_j^k}\dot{\overline{e}}_k(t)\,\mathrm{d}t\right)\\ &\le\left|e_j^k-e_{j-1}^k\right|+C_2\left(\tau_j^k-\tau_{j-1}^k\right)\ \le\ \delta_j^k+(C_1+C_2)\left(\tau_j^k-\tau_{j-1}^k\right).\end{aligned}$$

Combining everything we arrive at

$$\begin{aligned}\mathrm{Var}(\overline{e}_k;[0,T]) &=\sum_{j=1}^{N_k}\left(\int_{\tau_{j-1}^k}^{\tau_j^k}|\dot{\overline{e}}_k(t)|\,\mathrm{d}t+\Delta e_j^k\right)\\ &\le\sum_{j=1}^{N_k}\left(\delta_j^k+(C_1+2C_2)\left(\tau_j^k-\tau_{j-1}^k\right)\right)\ \le\ E_*\mathrm{e}^{c_1^E T}+(C_1+2C_2)T.\end{aligned} \tag{3.13}$$

Step 2: *Selection of subsequences*
Estimates (3.10) and (3.11) provides bounds, which are independent of k. The dissipation estimate (3.11) together with the assumptions (2.2),(2.5) and (2.4) allow us to extract a subsequence (not renumbered) and limit functions $z:[0,T]\to\mathcal{Z}$, $e_\infty:[0,T]\to\mathbb{R}$, and $\delta_\infty:[0,T]\to\mathbb{R}$ such that for all $s,t\in[0,T]$ with $s\le t$ we have

$$\begin{aligned}&\mathrm{Diss}_k(\overline{q}_k;[0,t])\to\delta_\infty(t),\quad \overline{e}_k(t)\to e_\infty(t),\\ &\overline{z}_k(t)\overset{\mathcal{Z}}{\to}z(t),\quad \mathrm{Diss}_\infty(z;[s,t])\le\delta_\infty(t)-\delta_\infty(s).\end{aligned}$$

Moreover, the energy boundedness (3.10) together with assumption (2.7) shows that the sequence $p_k : [0,T] \to \mathbb{R}, t \mapsto \partial_t \mathcal{E}_k(t, \overline{q}_k(t))$ is bounded in $\mathrm{L}^\infty([0,T])$. Choosing a further subsequence (not renumbered) we may assume

$$p_k \overset{*}{\rightharpoonup} p_\infty \text{ in } \mathrm{L}^\infty([0,T]).$$

We also define $p^* \in \mathrm{L}^\infty([0,T])$ via

$$p^*(t) = \limsup_{k\to\infty} p_k(t).$$

By Fatou's lemma we know $p_\infty \le p^*$ a.e. on $[0,T]$.

The construction of the limit function $\varphi : [0,T] \to \mathcal{F}$ is more involved and follows [13]. For each $t \in [0,T]$ we define

$$\mathcal{A}(t) = \{\, \widetilde{\varphi} \in \mathcal{F}\,;\; \partial_t \mathcal{E}_\infty(t, \widetilde{\varphi}, z(t)) = p^*(t),\; \exists\,(k_l)_{l\in\mathbb{N}}\colon\; \overline{\varphi}_{k_l}(t) \overset{\mathcal{F}}{\to} \widetilde{\varphi} \,\}.$$

First, we show that $\mathcal{A}(t)$ is nonempty. We are now careful about subsequences, since they now depend on $t \in [0,T]$. First, choose a subsequence $(K_l^t)_{l\in\mathbb{N}}$ such that $p_{K_l^t}(t) \to p^*(t)$ for $l \to \infty$. Next, we use the energy bound (3.10) and the uniform compactness of sublevels postulated in (2.6), which allows us to extract a subsequence $(m_n^t)_{n\in\mathbb{N}}$ from $(K_l^t)_{l\in\mathbb{N}}$ such that $\overline{q}_{m_n^t}(t) \overset{\mathcal{Q}}{\to} q(t) = (\varphi(t), z(t))$ for $n \to \infty$. Let $t_n = \max\{\, \tau \in \Pi_{m_n^t}\,;\; \tau \le t \,\}$, then $\overline{q}_{m_n^t}(t) \in \mathcal{S}_{m_n^t}(t_n)$. Hence, $(t_n, \overline{q}_{m_n^t}(t))$ forms a converging, stable sequence and assumption (2.9) provides

$$\partial_t \mathcal{E}_{m_n^t}(t_n, \overline{q}_{m_n^t}(t)) \to \partial_t \mathcal{E}_\infty(t, q(t)) = p^*(t). \tag{3.14}$$

Thus, $\widetilde{\varphi} = \varphi(t)$ from $q(t) = (\varphi(t), z(t))$ lies in $\mathcal{A}(t)$. This defines the mapping $\varphi : [0,T] \to \mathcal{F}$ with $\varphi(t) \in \mathcal{A}(t)$. (Note that this construction uses the axiom of choice.)

Step 3: *Stability of the limit process*

The limit process $q = (\varphi, z) : [0,T] \to \mathcal{F}\times\mathcal{Z} = \mathcal{Q}$ was defined for each $t \in [0,T]$ such that $\overline{q}_{m_n^t}(t) \to q(t)$ and $\overline{q}_{m_n^t} \in \mathcal{S}_{m_n^t}(t_n)$ with $t_n \to t$. As in Step 2 we have a converging, stable sequence and assumption (2.11) provides $q(t) \in \mathcal{S}_\infty(t)$.

Step 4: *Upper energy estimate*

Recall $\overline{e}_k(t) = \mathcal{E}_k(t, \overline{q}_k(t))$, $\delta_k(t) = \mathrm{Diss}_k(\overline{q}_k; [0,t])$ and the fineness $\phi_k = \phi(\Pi_k) \to 0$. Using the energy bound (3.10) and (2.7) we have $|\overline{e}_k(t) - e_{j-1}^k| \le C\phi_k$ for $t \in \left[\tau_{j-1}^k, \tau_j^k\right)$. Moreover, summing (3.8) over $j \in \{1,\ldots,m\}$ gives $\overline{e}_k(\tau_m^k) + \delta_k(\tau_m^k) \le \overline{e}_k(0) + \int_0^{\tau_m^k} \partial_s \mathcal{E}_k(s, \overline{q}_k(s))\,\mathrm{d}s$. Since $p_k = \partial_s \mathcal{E}_k(\cdot, \overline{q}_k(\cdot))$ is uniformly bounded in $\mathrm{L}^\infty([0,T])$ by C_p, we find

$$\overline{e}_k(t) + \delta_k(t) \le \mathcal{E}_k\left(0, q_0^k\right) + \int_0^t p_k(s)\,\mathrm{d}s + (C + C_p)\phi_k. \tag{3.15}$$

By (2.10) and (2.5) we have $\mathcal{E}_\infty(t, q(t)) \le e_\infty(t) = \lim_{k\to\infty} \overline{e}_k(t)$ and $\mathrm{Diss}_\infty(z; [0,t]) \le \delta_\infty(t) = \lim_{k\to\infty} \delta_k(t)$. Hence, passing to the limit $k \to \infty$ in (3.15) and using the assumption (3.6), we conclude

$$\begin{aligned} \mathcal{E}_\infty(t, q(t)) + \mathrm{Diss}_\infty(q; [0,t]) &\le e_\infty(t) + \delta_\infty(t) \\ &\le \mathcal{E}_\infty(0, q_0) + \int_0^t p_\infty(s)\,\mathrm{d}s \le \mathcal{E}_\infty(0, q_0) + \int_0^t p^*(s)\,\mathrm{d}s. \end{aligned} \tag{3.16}$$

Step 5: *Lower energy estimate*
Since in Step 3 we have found $q(t) \in \mathcal{S}_\infty(t)$ and since (3.14) provides $\partial_t \mathcal{E}_\infty(t, q(t)) = p^*(t)$ with $p^* \in \mathrm{L}^\infty([0, T])$, we can employ Proposition 2.4, which gives the lower energy estimate giving $\mathcal{E}_\infty(t, q(t)) + \mathrm{Diss}_\infty(q; [0, t]) \geq \mathcal{E}_\infty(0, q_0) + \int_0^t p^*(s)\,\mathrm{d}s$.
Step 6: Improved convergence
Combining (3.16) and Step 5 we obtain $\mathcal{E}_\infty(t, q(t)) + \mathrm{Diss}_\infty(q; [0, t]) = e_\infty(t) + \delta_\infty(t)$ for all $t \in [0, T]$ and $p_\infty = p^*$ a.e. in $[0, T]$. Using $\mathcal{E}_\infty(t, q(t)) \leq e_\infty(t)$ and $\mathrm{Diss}_\infty(q; [0, t]) \leq \delta_\infty(t)$ yields $\mathcal{E}_\infty(t, q(t)) = e_\infty(t)$ and $\mathrm{Diss}_\infty(q; [0, t]) = \delta_\infty(t)$ for all $t \in [0, T]$, which establishes the assertions (i) and (ii) in (3.7). Finally, employing Proposition A.2 from [14, Proposition A.2] together with $p_\infty = p^*$ gives (iv) in (3.7).
Step 7: *Measurability of the limit process*
We follow the ideas in [21, Sect. 1.6], see also [22, Sect. 6]. Denote by $\widehat{\mathcal{F}}$ a compact subset of $\mathcal{F}$, which contains all the images of the functions φ_n. By our additional assumption, $\widehat{\mathcal{F}}$ is separable and metrizable. Using the L^1 convergence (3.7.iv) we can choose a further subsequence $(\widehat{k_l})_{l\in\mathbb{N}}$ of $(k_j)_{j\in\mathbb{N}}$ such that for $t \in B$ we have $\partial_t \mathcal{E}_{\widehat{k_l}}(t, \overline{q}_{\widehat{k_l}}(t)) \to \partial_t \mathcal{E}_\infty(t, q(t))$ for $l \to \infty$, where $[0, T]\backslash B$ has measure 0. We now define

$$\mathcal{A}^0(t) = \operatorname*{Limsup}_{l\to\infty}\{\varphi_{\widehat{k_l}}\} = \{\, \widetilde{\varphi} \in \mathcal{F}\ ;\ \exists \text{ subseq. } (l(n))_{n\in\mathbb{N}} :\ \varphi_{\widehat{k}_{l(n)}} \overset{\mathcal{F}}{\to} \widetilde{\varphi}\,\} \subset \widehat{\mathcal{F}}.$$

By construction we have $\mathcal{A}^0(t) \subset \mathcal{A}(t)$ for all $t \in B$. Since all the functions $\varphi_{\widehat{k_l}} : [0, T] \mapsto \mathcal{F}$ are piecewise constant, they are measurable. Hence, the set-valued mapping $t \mapsto \mathcal{A}^0(t) \subset \widehat{\mathcal{F}}$ is measurable with nonempty and closed images, see [1, Theorem 8.2.5]. According to [1, Theorem 8.1.3]) there exits a measurable selection $\varphi : [0, T] \to \widehat{\mathcal{F}}$ with $\varphi(t) \in \mathcal{A}^0(t)$. Without destroying measurability we can modify φ outside the null set $[0, T]\backslash B$ such that we have $\varphi(t) \in \mathcal{A}(t)$ for all $t \in [0, T]$. ◻

Remark 3.5 As in [29,30], the pointwise convergence in (3.7.v) can be formulated alternatively via *convergence on nets*, which is a standard tool of general topology. However, it is not clear whether this net convergence can be combined with the measurable selection constructed above.

To define net convergence, we recall that an index set Ξ is called *directed by an ordering* "$\preceq$", if for any $\xi_1, \xi_2 \in \Xi$ there exists $\xi_3 \in \Xi$ such that both, $\xi_1 \preceq \xi_3$ and $\xi_2 \preceq \xi_3$. Having a directed set $(\Xi, \preceq)$ and another set $\mathcal{B}$, we say that $\{b_\xi\}_{\xi\in\Xi}$ is a *net* in $\mathcal{B}$, if there is a mapping $\Xi \to \mathcal{B} : \xi \mapsto b_\xi$. If $\mathcal{B}$ is a topological space, we write $b = \lim_{\xi\in\Xi} b_\xi$ if, for any neighborhood $\mathcal{N}$ of b there is $\xi_0 \in \Xi$ such that $b_\xi \in \mathcal{N}$ whenever $\xi_0 \preceq \xi$, and then we say that the net $\{b_\xi\}_{\xi\in\Xi}$ *converges* to b (in the so-called Moore-Smith sense).

The notion "net" generalizes that of a "sequence", where Ξ equals $\mathbb{N}$ with the standard ordering. The term "subsequence" is generalized via the notion "finer net". A net $\{\widetilde{x}_{\widetilde{\xi}}\}_{\widetilde{\xi}\in\widetilde{\Xi}}$ in X is called *finer* than the net $\{x_\xi\}_{\xi\in\Xi}$, if there is a mapping $j : \widetilde{\Xi} \to \Xi$ such that $\widetilde{x}_{\widetilde{\xi}} = x_{j(\widetilde{\xi})}$ for all $\widetilde{\xi} \in \widetilde{\Xi}$ and that for any $\xi \in \Xi$ there exists $\widetilde{\xi}_0 \in \widetilde{\Xi}$ such that $j(\widetilde{\xi}) \succeq \xi$ for all $\widetilde{\xi}$ with $\widetilde{\xi} \succeq \widetilde{\xi}_0$. Obviously, a finer net may have an index set $\widetilde{\Xi}$ of strictly greater cardinality than the index set Ξ of the original net.

To reformulate (3.7.v) we use $\Xi \subset \mathbb{N}$ (ordered standardly) to denote the subsequence $(k_j)_{j\in N}$ and $\widetilde{\Xi} \subset \{\text{finite subsets of } [0, T]\}$ to denote pointwise convergence. Here $\widetilde{\Xi}$ is ordered via the usual set inclusion. Then, Theorem 3.4 can be reformulated in such a way that, instead of the mentioned subsequence $\{\bar{q}_{k_j}\}_{j\in\mathbb{N}}$, there exists a net $\{\bar{q}_{k_\xi}\}_{\xi\in\widetilde{\Xi}}$ finer than the subsequence $\{\bar{q}_k\}_{k\in\mathbb{N}}$ and such that $\lim_{\xi\in\widetilde{\Xi}} k_\xi = \infty$, and a process $q : [0, T] \to \mathcal{Q}$ such that, instead of (3.7.v), we have $\lim_{\xi\in\widetilde{\Xi}} \overline{\varphi}_{k_\xi}(t) = \varphi(t)$ for any $t \in [0, T]$.

4 Relaxation

In this section we treat a question that is closely linked to the Γ-convergence considered above. However, this time we consider only one pair of functionals $\mathcal{E}_1$ and $\mathcal{D}_1$ such that the incremental problem (IP) need not have any solution due to missing lower semi-continuity. We provide joint conditions on $\mathcal{E}_1$ and $\mathcal{D}_1$ and suitable relaxations $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$ such that approximate solutions of the incremental problem for $\mathcal{E}_1$ and $\mathcal{D}_1$ converge to energetic solutions associated with $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$. Our assumptions on the stored-energy functionals $\mathcal{E}_j : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$ and dissipation distances $\mathcal{D}_j : \mathcal{Z}\times\mathcal{Z} \to \mathbb{R}_\infty$ need the new notion of the set of α-stable points $\mathcal{S}_j^\alpha(t)$. For $\alpha \geq 0$ we let

$$\mathcal{S}_j^\alpha(t) = \{\, q \in \mathcal{Q}\,;\; \mathcal{E}_j(t,q) < \infty,\ \forall \widetilde{q} \in \mathcal{Q}:\ \mathcal{E}_j(t,q) \leq \alpha + \mathcal{E}_j(t,\widetilde{q}) + \mathcal{D}_j(q,\widetilde{q})\,\}.$$

Note that now j only takes the two values 1 or ∞. Our conditions are the following:

$$\begin{aligned}&\forall\, j \in \{1,\infty\}\ \forall\, z_1,z_2,z_3 \in \mathcal{Z}:\\ &\mathcal{D}_j(z_1,z_1)=0,\ \mathcal{D}_j(z_1,z_3) \leq \mathcal{D}_j(z_1,z_2)+\mathcal{D}_j(z_2,z_3).\end{aligned} \tag{4.1}$$

$$\begin{aligned}&\forall\, q_k \in \mathcal{S}_1^{\alpha_k}(t_k),\ \widetilde{q}_k \in \mathcal{S}_1^{\alpha_k}(\widetilde{t}_k)\ \text{ with }\ \alpha_k \searrow 0\ \text{ and }\ q_k \overset{\mathcal{Q}}{\to} q,\ \widetilde{q}_k \overset{\mathcal{Q}}{\to} \widetilde{q}:\\ &\mathcal{D}_\infty(q,\widetilde{q}) \leq \liminf_{k\to\infty} \mathcal{D}_1(q_k,\widetilde{q}_k).\end{aligned} \tag{4.2}$$

$$\begin{aligned}&\forall \text{ compact } \mathcal{K} \subset \mathcal{Z} \text{ and } z_k \in \mathcal{K}:\\ &\min\{\mathcal{D}_\infty(z_k,z),\mathcal{D}_\infty(z,z_k)\} \to 0 \quad \Longrightarrow \quad z_k \overset{\mathcal{Z}}{\to} z.\end{aligned} \tag{4.3}$$

$$\forall\, t \in [0,T]\ \forall\, E \in \mathbb{R}:\quad \{\, q \in \mathcal{Q}\,;\ \mathcal{E}_1(t,q) \leq E\,\} \text{ is relatively compact.} \tag{4.4}$$

$$\begin{aligned}&\exists\, c_0^E \in \mathbb{R}\ \exists\, c_1^E > 0\ \forall\, t \in [0,T]\ \forall\, j \in \{1,\infty\}:\\ &\text{If } \mathcal{E}_j(t,q) < \infty,\ \text{then } \mathcal{E}_j(\cdot,q) \in \mathrm{C}^1([0,T]) \text{ and}\\ &|\partial_s\mathcal{E}_j(s,q)| \leq c_1^E(\mathcal{E}_j(s,q)+c_0^E) \text{ for all } s \in [0,T].\end{aligned} \tag{4.5}$$

$$\begin{aligned}&\forall\, E \in \mathbb{R}\ \forall\, \varepsilon > 0\ \exists\, \delta > 0:\\ &\mathcal{E}_\infty(0,q) \leq E \text{ and } |t_1-t_2| < \delta \quad \Longrightarrow \quad |\partial_t\mathcal{E}_\infty(t_1,q)-\partial_t\mathcal{E}_\infty(t_2,q)| < \varepsilon.\end{aligned} \tag{4.6}$$

$$\begin{aligned}&(t_k,q_k) \overset{\mathcal{Q}}{\to} (t,q),\ \sup_{k\in\mathbb{N}} \mathcal{E}_1(t_k,q_k) < \infty,\ q_k \in \mathcal{S}_1^{\alpha_k}(t_k) \text{ with } \alpha_k \searrow 0\\ &\quad\Longrightarrow \quad \partial_t\mathcal{E}_1(t_k,q_k) \to \partial_t\mathcal{E}_\infty(t,q).\end{aligned} \tag{4.7}$$

$$q_k \overset{\mathcal{Q}}{\to} q \quad \Longrightarrow \quad \mathcal{E}_\infty(t,q) \leq \liminf_{k\to\infty} \mathcal{E}_1(t,q_k). \tag{4.8}$$

$$\begin{aligned}&q_k \in \mathcal{S}_1^{\alpha_k}(t_k) \text{ with } \alpha_k \searrow 0,\ (t_k,q_k) \overset{[0,T]\times\mathcal{Q}}{\to} (t,q),\ \sup_{k\in\mathbb{N}} \mathcal{E}_1(t_k,q_k) < \infty\\ &\quad\Longrightarrow \quad q \in \mathcal{S}_\infty(t).\end{aligned} \tag{4.9}$$

Like in Sect. 2 the last condition can be established via a hierarchy of several stronger conditions. We only state the simplest one, namely

$$\begin{aligned}&\text{(i) } \mathcal{D}_1 = \mathcal{D}_\infty \text{ and } \mathcal{D}_1 : \mathcal{Z}\times\mathcal{Z} \to [0,\infty) \text{ is continuous,}\\ &\text{(ii) } \mathcal{E}_\infty(t,\cdot) = \Gamma\text{–}\lim_{k\to\infty} \mathcal{E}_1(t,\cdot).\end{aligned} \tag{4.10}$$

Here (i) in (4.10) corresponds to the continuous convergence condition (2.19). The Γ-limit $\mathcal{E}_\infty(t,\cdot)$ of the constant sequence $(\mathcal{E}_1(t,\cdot))_{k\in\mathbb{N}}$ is exactly the lower semi-continuous envelope of $\mathcal{E}_1(t,\cdot)$, see [6,10]. Like in Proposition 2.2 we easily obtain that (4.10) implies (4.9).

The essential difference to the previous section is that the incremental problem (IP) for $\mathcal{E}_1$ and $\mathcal{D}_1$ may not be solvable. We replace it by an approximate incremental problem (AIP). As before we choose an arbitrary sequence $(\Pi_k)_{k\in\mathbb{N}}$ of partitions with fineness $\phi_k := \phi(\Pi_k) \to 0$.

Moreover, the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ with $0 < \varepsilon_k \to 0$ will be used to control the accuracy in the energy minimization.

$$(\mathrm{AIP})_k \quad \begin{cases} \text{Given } q_0^k, \text{ for } j = 1, \ldots, N_k \text{ find iteratively } q_j^k \in \mathcal{Q} \text{ such that} \\ \mathcal{E}_1(\tau_j^k, q_j^k) + \mathcal{D}_1(q_{j-1}^k, q_j^k) \le (\tau_j^k - \tau_{j-1}^k)\varepsilon_k + \inf_{\widetilde{q}\in\mathcal{Q}} \Big(\mathcal{E}_1(\tau_j^k, \widetilde{q}) + \mathcal{D}_1(q_{j-1}^k, \widetilde{q})\Big). \end{cases}$$

Clearly, $(\mathrm{AIP})_k$ has always at least one solution $(q_j^k)_{j=1,\ldots,N_k}$, which leads to piecewise constant interpolants $\overline{q}_k : [0, T] \to \mathcal{Q}$ defined as in (3.5). Our main result is that suitably chosen subsequences converge to a limit process $q : [0, T] \to \mathcal{Q}$, which is an energetic solution associated with $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$.

Theorem 4.1 *Let $(\Pi_k)_{k\in\mathbb{N}}$ be a sequence of partitions of $[0, T]$ with $\phi_k = \phi(\Pi_k) \to 0$ and let $(\varepsilon_k)_{k\in\mathbb{N}}$ satisfy $0 < \varepsilon_k \to 0$. Let $(q_0^k)_{k\in\mathbb{N}}$ be a sequence of initial conditions satisfying*

$$q_0^k \in \mathcal{S}_1^{\varepsilon_k\phi_k}(0), \quad q_0^k \overset{\mathcal{Q}}{\to} q_0, \quad \textit{and} \quad \mathcal{E}_1(0, q_0^k) \to \mathcal{E}_\infty(0, q_0) \in \mathbb{R}. \tag{4.11}$$

Then, for every sequence $(\overline{q}_k)_{k\in\mathbb{N}}$ of piecewise constant interpolants of solutions of $(\mathrm{AIP})_k$ *with initial value q_0^k, there exist a subsequence $(k_l)_{l\in\mathbb{N}}$ and a solution $q = (\varphi, z) : [0, T] \to \mathcal{Q} = \mathcal{F}\times\mathcal{Z}$ of* $(\mathrm{S})_\infty\&(\mathrm{E})_\infty$ *such that* (i)–(v) *hold:*

(i) $\forall\, t \in [0, T] : \ \mathcal{E}_1(t, \overline{q}_{k_l}(t)) \to \mathcal{E}_\infty(t, q(t))$,
(ii) $\forall\, t \in [0, T] : \mathrm{Diss}_1(\overline{q}_{k_l}; [0, t]) \to \mathrm{Diss}_\infty(q; [0, t])$,
(iii) $\forall\, t \in [0, T] : \ z_{k_l}(t) \overset{\mathcal{Z}}{\to} z(t)$,
(iv) $\partial_t\mathcal{E}_1(\cdot, \overline{q}_{k_l}(\cdot)) \to \partial_t\mathcal{E}_\infty(\cdot, q(\cdot))$ *in* $\mathrm{L}^1([0, T])$,
(v) $\forall\, t \in [0, T]\ \exists$ *subsequence* $(K_n^t)_{n\in\mathbb{N}}$ *of* $(k_l)_{n\in\mathbb{N}}$: $\ \overline{\varphi}_{K_n^t}(t) \overset{\mathcal{F}}{\to} \varphi(t)$.

Moreover, any $\widetilde{q} : [0, T] \to \mathcal{Q}$ obtained as such a limit is a solution of $(\mathrm{S})_\infty\&(\mathrm{E})_\infty$.

Finally, if the topology on $\mathcal{F}$ restricted to compact sets is separable and metrizable, then the mapping $\varphi : [0, T] \to \mathcal{F}$ can be chosen measurable.

Proof We follow the proof of Theorem 3.4 and point out the differences only.
Step 1: *A priori estimates*
With $e_j^k = \mathcal{E}_1(\tau_j^k, q_j^k)$ we obtain as in (3.9) the estimate

$$e_j^k + \delta_j^k \le e_{j-1}^k + \varepsilon_k\left(\tau_j^k - \tau_{j-1}^k\right) + \left(e_{j-1}^k + c_0^E\right)\left(\mathrm{e}^{c_1^E\left(\tau_j^k - \tau_{j-1}^k\right)} - 1\right).$$

Introducing the auxiliary variable $E_j^k = e_j^k + c_0^E + \varepsilon_k/c_1^E$ and $E_0^k = e_0^k + c_0^E$ we find

$$E_j^k + \delta_j^k \le \mathrm{e}^{c_1^E\left(\tau_j^k - \tau_{j-1}^k\right)} E_{j-1}^k. \tag{4.12}$$

With $E_* = \sup_{k\in\mathbb{N}}\left(c_0^E + \mathcal{E}_1(0, q_0^k)\right) < \infty$ we find $E_j^k \le \mathrm{e}^{c_1^E\tau_j^k} E_*$ and, hence, the k-independent a priory energy bound $e_j^k \le -c_0^E + E_j^k \le -c_0^E + \mathrm{e}^{c_1^E T} E_*$. Adding (4.12) over $j = 1, \ldots, N_k$ we find

$$\begin{aligned} \sum_{j=1}^{N_k} \delta_j^k &\le E_0^k - E_{N_k}^k + \sum_{j=1}^{N_k} (\mathrm{e}^{c_1^E(\tau_j^k - \tau_{j-1}^k)} - 1) E_{j-1}^k \\ &\le E_0^k + \sum_{j=1}^{N_k} \left(\mathrm{e}^{c_1^E\tau_j^k} E_* - \mathrm{e}^{c_1^E\tau_{j-1}^k} E_*\right) \le \mathrm{e}^{c_1^E T} E_*. \end{aligned}$$

Like in Sect. 3 we define, for the piecewise constant interpolant $\overline{q}_k$, the real-valued functions

$$\delta_k(t) = \text{Diss}_1(\overline{q}_k, [0, t]), \quad \overline{e}_k(t) = \mathcal{E}_1(t, \overline{q}_k(t)), \quad p_k(t) = \partial_t\mathcal{E}_1(t, \overline{q}_k(t)).$$

Like in Step 1 of the proof of Theorem 3.4 we have $|\delta_k(t)+\overline{e}_k(t)-\delta_k(s)-\overline{e}_k(s)| \le C_*|t-s|$ and thus

$$\text{Var}(\delta_k; [0, T]) \le e^{c_1^E T} E_* \text{ and } \text{Var}(\overline{e}_k; [0, T]) \le e^{c_1^E T} E_* + C_* T.$$

Step 2: *Selection of subsequences*
This part is identical to that in Sect. 3. We find one subsequence (k_l) such that

$$\delta_{k_l}(t) \to \delta_\infty(t),\ \overline{e}_{k_l}(t) \to e_\infty(t),\ z_{k_l}(t) \overset{\mathcal{Z}}{\to} z(t),\ p_{k_l} \overset{*}{\rightharpoonup} p_\infty \le p^*.$$

Moreover, for t-dependent subsequences we have $\varphi_{K_n^t}(t) \overset{\mathcal{F}}{\to} \varphi(t)$.
Step 3: *Stability of the limit process*
With $t_k = \min\{\, \tau \in \Pi_k\ ;\ \tau \le t \,\}$ and $\alpha_k = \varepsilon_k\phi_k \ge \varepsilon_k(\tau_j^k - \tau_{j-1}^k)$ we find $\overline{q}_k(t) \in \mathcal{S}_1^{\alpha_k}(t_k)$. Clearly, $(t_k, \overline{q}_k(t)) \overset{[0,T]\times\mathcal{Q}}{\to} (t, q(t))$ and $\mathcal{E}_1(t_k, \overline{q}_k(t)) \le e^{c_1^E T} E_* - c_0^E$. Hence, (4.9) implies the desired result $q(t) \in \mathcal{S}_\infty(t)$.
Step 4: *Upper energy estimate*
Using the approximate minimization property of $q_j^k = \overline{q}_k(\tau_j^k)$ for $j = 1, \ldots, m$ we have, after summation, $\overline{e}_k(\tau_m^k) + \delta_k(\tau_m^k) \le \overline{e}_k(0) + \varepsilon_k\tau_m^k + \int_0^{t_k^k} p_k(s)\,\mathrm{d}s$. As before we obtain the estimate $\overline{e}_k(t)+\delta_k(t) \le \overline{e}_k(0)+\varepsilon_k t+\int_0^t p_k(s)\,\mathrm{d}s+C\phi_k$ for all $t \in [0, T]$. Using $\phi_k, \varepsilon_k \to 0$, $p_k \overset{*}{\rightharpoonup} p_\infty$, $\overline{e}_k(t) \to e_\infty(t)$ and $\delta_k(t) \to \delta_\infty(t)$ we find

$$\begin{aligned}\mathcal{E}_\infty(t, q(t))+\text{Diss}_\infty(q; [0, t]) &\le e_\infty(t)+\delta_\infty(t)\\ &\le \mathcal{E}_\infty(0, q_0)+\int_0^t p_\infty\,\mathrm{d}s \le \mathcal{E}_\infty(0, q_0) + \int_0^t p^*\,\mathrm{d}s.\end{aligned}$$

Step 5: *Lower energy estimate*
Applying Proposition 2.4 to the stable limit process $q : [0, T] \to \mathcal{Q}$ for the limit functionals $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$ results in $\mathcal{E}_\infty(t, q(t)) + \text{Diss}_\infty(q; [0, t]) \ge \mathcal{E}_\infty(0, q_0) + \int_0^t p^*(s)\,\mathrm{d}s$.
Step 6: *Improved convergence*
Exactly as in Step 6 of the proof of Theorem 3.4 we conclude $\text{Diss}_\infty(q; [0, \cdot]) = \delta_\infty$, $\mathcal{E}_\infty(\cdot, q(\cdot)) = e_\infty$, and $p_\infty = p^*$.
Step 7: *Measurability* works exactly as above. □

Remark 4.2 A closely related result concerning relaxations of rate-independent processes is discussed in [28]. There, the case is studied that $\mathcal{Q}$ is a reflexive Banach space and that $\mathcal{D}_1$ is given in the form $\mathcal{D}_1(z, \widetilde{z}) = \mathcal{R}_1(\widetilde{z}-z)$. Besides of the usual technical assumptions, the crucial convergence conditions of the functionals are (4.10), namely (continuous) convergence of $\mathcal{R}_1$ to $\mathcal{R}_\infty$ and Γ-convergence of $\mathcal{E}_1$ to $\mathcal{E}_\infty$. The relaxation of the non-relaxed, in most cases unsolvable rate-independent system $(\text{S})_1\&(\text{E})_1$ is obtained by considering the functional

$$\mathcal{I}_m(q) = \int_0^T e^{-mt}\Big(\mathcal{R}_1(\dot{z}) + m\mathcal{E}_1(t, q(t))\Big)\,\mathrm{d}t.$$

Choosing the minimizers (or suitable approximate minimizers) $q_m : [0, T] \to \mathcal{Q}$ for $\mathcal{I}_m$ under the initial condition $q_m(0) = q_0$ we ask the question how possible accumulation points $q : [0, T] \to \mathcal{Q}$ can be characterized.

The following three features of $\mathcal{I}_m$ strongly depend on the fact that we are dealing with rate-independent systems, i.e., $\mathcal{R}_1$ is 1-homogeneous. First it is shown that for fixed $m \in \mathbb{N}$ the relaxation of $\mathcal{I}_m : \mathrm{L}^1([0, T], \mathcal{Q}) \to \mathbb{R}_\infty$ is given by the same expression but with $\mathcal{R}_1$ and $\mathcal{E}_1$ replaced by $\mathcal{R}_\infty$ and $\mathcal{E}_\infty$. A second result states that every minimizer of $\mathcal{I}_m$ (or of its relaxation) satisfies the energy balance $(\mathrm{E})_j$ for $j \in \{1, \infty\}$, i.e., $\mathcal{E}_j(t, q(t)) + \int_0^t \mathcal{R}_j(\mathrm{d}z) = \mathcal{E}_j(0, q_0) + \int_0^t \partial_s\mathcal{E}_j(s, q(s))\,\mathrm{d}s$. This is surprising since the functional depends on m whereas the energy balance does not. Finally, it is shown that accumulation points q of minimizers q_m of $\mathcal{I}_m$ are solutions of the energetic formulation $(\mathrm{S})_\infty\&(\mathrm{E})_\infty$.

5 Some applications

In this section we provide three examples to illustrate the theory developed above. In the first example we treat the numerical approximation of a standard evolutionary variational inequality with quadratic energy as an application of our Γ-limit theory in Sect. 3. The second example concerns the continuity of the so-called stop and play operators. The third example considers a nonconvex functional $\mathcal{E}_1$ that has a nontrivial lower semi-continuous envelope $\mathcal{E}_\infty$ and thus provides an example of relaxation. For more realistic applications we refer to [19,30], where we also take full advantage of the abstract theory using the weaker conditions (2.15) or (2.17). For similar Γ-convergence results in the context of fracture we refer to [15,16]. In the latter works the "jump transfer" plays the same rôle as (2.15). In the present applications we will rely on the more restrictive assumptions (2.18) and (2.19) for the first application, whereas we exploit directly (2.11) for the second and (4.10) for the third one.

5.1 Approximation via finite-dimensional subspaces

We consider the case that $\mathcal{F}$ and $\mathcal{Z}$ are separable Hilbert spaces $H_\mathcal{F}$ and $H_\mathcal{Z}$, respectively, and set $H = H_\mathcal{F} \times H_\mathcal{Z}$. For the topology we choose the weak topology such that bounded sets are relatively compact. For the energy we assume a quadratic form

$$\mathcal{E}_\infty(t, q) = \frac{1}{2}\langle Aq, q\rangle - \langle \ell(t), q\rangle,$$

where $A = A^* \in \mathcal{L}(H, H^*)$ is a bounded symmetric operator, which is additionally positive definite, i.e., there exists $c > 0$ such that $\langle Aq, q\rangle \geq c\|q\|^2$ for all $q \in H$, where $\|\cdot\|$ stands for the norm in H. The loading satisfies $\ell \in \mathrm{C}^1([0, T], H^*)$.

The dissipation distance is given via a convex, 1-homogeneous functional $\mathcal{R} : H_\mathcal{Z} \to [0, \infty)$, i.e. $\mathcal{R}(\gamma z) = \gamma\mathcal{R}(z)$ for all $\gamma \geq 0$ and $z \in H_\mathcal{Z}$, which satisfies

$$\begin{aligned} &\text{(i)}\ \ z_k \rightharpoonup z \implies \mathcal{R}(z_k) \to \mathcal{R}(z), \\ &\text{(ii)}\ z \neq 0 \implies \mathcal{R}(z) > 0. \end{aligned} \tag{5.1}$$

Now we set $\mathcal{D}_\infty(z_0, z_1) = \mathcal{R}(z_1 - z_0)$.

The sequence of functionals $\mathcal{E}_k$ and $\mathcal{D}_k$ is now obtained by a choosing a nested sequence of finite-dimensional subspaces $H^k_\mathcal{F}$ and $H^k_\mathcal{Z}$, $k \in \mathbb{N}$ such that

$$\begin{aligned} &H^k_\mathcal{F} \subset H^{k+1}_\mathcal{F} \quad \text{and} \quad \textstyle\bigcup_{k=1}^\infty H^k_\mathcal{F} \text{ is dense in } H_\mathcal{F}, \\ &H^k_\mathcal{Z} \subset H^{k+1}_\mathcal{Z} \quad \text{and} \quad \textstyle\bigcup_{k=1}^\infty H^k_\mathcal{Z} \text{ is dense in } H_\mathcal{Z}. \end{aligned} \tag{5.2}$$

We now let $H^k = H^k_{\mathcal{F}} \times H^k_{\mathcal{Z}}$ and define

$$\mathcal{E}_k(t,q) = \begin{cases} \mathcal{E}_\infty(t,q) & \text{for } q \in H^k, \\ \infty & \text{otherwise,} \end{cases} \quad \text{and } \mathcal{D}_k(z_0,z_1) = \begin{cases} \mathcal{R}(z_1 - z_0) & \text{for } z_0, z_1 \in H^k_{\mathcal{Z}}, \\ \infty & \text{otherwise.} \end{cases}$$

We claim that the conditions (2.2)–(2.10) hold and that (2.11) can be deduced via Proposition 2.2 from (2.18) and (2.19).

The triangle inequality (2.2) follows from $\mathcal{R}$ being 1-homogeneous and convex, which gives $\mathcal{R}(z_0+z_1) \leq \mathcal{R}(z_0) + \mathcal{R}(z_1)$. By assumption (5.1)(i) the function $\mathcal{R}$ and hence $\mathcal{D}_\infty : H_{\mathcal{Z}} \times H_{\mathcal{Z}} \to [0,\infty)$ are weakly continuous. The definition of $\mathcal{D}_k$ keeps convexity and strong lower semi-continuity. Thus, all $\mathcal{D}_k$ are weakly lower semi-continuous and (2.3) is established. Using this and $\mathcal{D}_\infty \leq \mathcal{D}_{k+1} \leq D_k$ we immediately obtain the lower Γ-limit condition (2.4). Finally, for sequences $(z_k)_{k\in\mathbb{N}}$ on bounded sets in $H_{\mathcal{Z}}$ the condition $\mathcal{D}_\infty(z_k, z) = \mathcal{R}(z - z_k) \to 0$ implies $z_k \rightharpoonup z$, since z_k has a convergent subsequence, namely $z_{k_l} \rightharpoonup z_*$ for some $z_* \in H_{\mathcal{Z}}$. By (5.1)(i) we have $\mathcal{R}(z - z_*) = \lim_{l\to\infty} \mathcal{R}(z - z_{k_l}) = 0$ and (5.1)(ii) yields $z_* = z$. Hence, the full sequence must converge weakly to z. Thus, all conditions on $\mathcal{D}_k$, $k \in \mathbb{N}$, are satisfied.

For the conditions on $\mathcal{E}_k$, we first consider $\mathcal{E}_\infty$, which satisfies

$$\mathcal{E}_\infty(t,q) \geq \frac{c}{2}\|q\|^2 - \Lambda_0\|q\| \text{ with } \Lambda_0 = \sup_{t\in[0,T]} \|\ell(t)\|_{H^*}.$$

Hence, the sublevels are bounded. By strong continuity and convexity of $\mathcal{E}_\infty$ the sublevels are weakly compact. Since the E-sublevel of $\mathcal{E}_k(t,\cdot)$ is the intersection of H^k with the E-sublevel of $\mathcal{E}_\infty$, the condition (2.6) follows.

With $\Lambda_1 = \sup_{t\in[0,T]} \|\dot{\ell}(t)\|_{H^*}$ and $\partial_t \mathcal{E}_\infty(t,q) = -\langle \dot{\ell}(t), q\rangle$ we obtain $|\partial_t \mathcal{E}_\infty(t,q)| \leq \Lambda_1 \|q\| \leq \frac{\Lambda_1}{\Lambda_0}\big(\frac{2\Lambda_0^2}{c} + \mathcal{E}_\infty(t,q)\big)$. Since $\mathcal{E}_k$ and $\mathcal{E}_\infty$ coincide if $\mathcal{E}_k$ takes finite values, the functionals $\mathcal{E}_k$ satisfy the same estimate. Thus, (2.7) is established. Moreover, by uniform continuity of $\dot{\ell} : [0,T] \to H^*$ we similarly obtain (2.8). Like for $\mathcal{D}_k$, the lower Γ-limit condition follows from $\mathcal{E}_\infty \leq \mathcal{E}_k$ and the weak lower semi-continuity of $\mathcal{E}_\infty$. The convergence of the power is trivial, since $\partial_t \mathcal{E}_k(t,q) = -\langle \dot{\ell}(t), q\rangle$ is linear in q and independent of k.

To prove the crucial upper semi-continuity of the stable sets we use Proposition 2.2 after establishing (2.17). Let (t_l, q_{k_l}) be a stable sequence with limit (t,q). For a given test function $\widetilde{q} \in H$ we choose any sequence $\widetilde{q}_l$ such that $\widetilde{q}_l \in H^{k_l}$ and $\widetilde{q}_l \to \widetilde{q}$. For instance, $\widetilde{q}_l$ may be the orthogonal projection of $\widetilde{q}$ onto H^{k_l}. Hence,

$$\begin{aligned} \mathcal{E}_{k_l}(t_l,\widetilde{q}_l) + \mathcal{D}_{k_l}(q_{k_l},\widetilde{q}_l) &= \mathcal{E}_\infty(t_l,\widetilde{q}_l) + \mathcal{R}(\widetilde{q}_l - q_{k_l}) \\ \to \mathcal{E}_\infty(t,\widetilde{q}) + \mathcal{R}(\widetilde{q} - q) &= \mathcal{E}_\infty(t,\widetilde{q}) + \mathcal{D}_\infty(q,\widetilde{q}), \end{aligned}$$

and (2.17) is established.

As a conclusion, we know that both theorems of Sect. 3 are applicable. In particular, taking finite-dimensional subspaces H^k and choosing time partitions Π_k we are left with a finite number of finite-dimensional minimization problems. If $\phi(\Pi_k) \to 0$ and $(H^k)_{k\in\mathbb{N}}$ exhausts H (i.e., (5.2) holds), then Theorem 3.4 guarantees that there exists subsequences that converge to an energetic solution associated with $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$. In fact, here the solution of $(\mathrm{S})_\infty\&(\mathrm{E})_\infty$ for a given initial value $q_0 \in \mathcal{S}_\infty(0)$ is unique (cf. [25]). This proves that the whole sequence must converge.

We close this subsection by relating our functionals to continuum mechanics. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. We let $H_{\mathcal{F}} = (\mathrm{H}^1_0(\Omega))^d$, which is the space for the displacements $u(t,\cdot) : \Omega \to \mathbb{R}^d$. For some $m \in \mathbb{N}$ we let $H_{\mathcal{Z}} = (\mathrm{H}^1(\Omega))^m$ for the plastic variables, which contain the plastic strain $\varepsilon_{\text{plast}} = Bz$ as well as possible hardening variables.

For the dissipation we choose $\mathcal{R}(z) = \int_\Omega \rho(x, z(x))\,dx$ with $\rho \in C^0(\overline{\Omega}\times\mathbb{R}^m)$ such that $r_1|v| \le \rho(x, v) \le r_2|v|$ for all $(x, v) \in \overline{\Omega}\times\mathbb{R}^m$ with $0 < r_1 \le r_2$ and $\rho(x, \cdot) : \mathbb{R}^m \to [0, \infty)$ is 1-homogeneous and convex. Hence, $\mathcal{R}$ is equivalent to the L^1 norm and (5.1) holds.

The energy functional $\mathcal{E}_\infty$ is usually taken in the form

$$\mathcal{E}_\infty(t, u, z) = \int_\Omega \frac{1}{2}(\varepsilon(u) - Bz):\mathbb{C}(x):(\varepsilon(u) - Bz) + \frac{1}{2}A(x)z \cdot z + \frac{\kappa}{2}\left|\nabla z\right|^2 dx - \int_\Omega f_{\text{ext}}(t) \cdot u\, dx,$$

where $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$, $\kappa > 0$, and $B \in \mathbb{R}^{d\times d\times m}$. Moreover, $\mathbb{C} \in L^\infty(\Omega, \mathrm{Sym}(\mathbb{R}^{d\times d}))$ and $A \in L^\infty(\Omega, \mathrm{Sym}(\mathbb{R}^m))$ are assumed to be uniformly positive definite. Thus, all conditions on $\mathcal{E}_\infty$ are satisfied, if we impose $f_{\text{ext}} \in C^1([0, T], H^{-1}(\Omega)^d)$.

Suitable finite-dimensional approximation spaces are, for instance, finite-element spaces with continuous piecewise affine functions on a triangulation of the domain. The above result provides a simplified and more straightforward convergence proof for elastoplasticity as given in [17]; see also [41] for some related convergence results by variational methods.

Further applications, which use the full strength of the abstract theory developed in the present paper, are found in [30]. Convergence results of numerical methods with explicit convergence rates are discussed in [2,18].

5.2 Continuity of the vector-valued stop and play operator

In a Hilbert space $\mathcal{H}$ with the scalar product $\langle\cdot, \cdot\rangle$ the *play operator* and the *stop operator* of rate-independent hysteresis are defined in terms of the characteristic or yield set $\mathcal{C} \subset \mathcal{H}$, which is non-empty, convex, and closed. The stop operator maps a given input function $\ell \in C^{\mathrm{Lip}}([0, T], \mathcal{H})$ and an initial value $\sigma_0 \in \mathcal{C}$ to the solution $\sigma \in C^{\mathrm{Lip}}([0, T], \mathcal{H})$ of the following evolutionary variational inequality:

$$\sigma(0) = \sigma_0 \text{ and for a.a. } t \in [0, T]\text{: } \sigma(t) \in \mathcal{C} \text{ and } \langle\sigma(t) - \widetilde{\sigma}, \dot{\sigma}(t) - \ell(t)\rangle \le 0 \text{ for all } \widetilde{\sigma} \in \mathcal{C}.$$

The play operator is simply defined via the mapping from (σ_0, ℓ) to $z = \mathcal{P}_\mathcal{C}(\sigma_0, \ell) = \ell - \sigma \in C^{\mathrm{Lip}}([0, T], \mathcal{H})$. These operators can equivalently be defined by the energetic formulation used in this paper. For this we define the quadratic energy functional $\mathcal{E}(t, z) = \frac{1}{2}\langle z, z\rangle - \langle\ell(t), z\rangle$. The dissipation distance is given as $\mathcal{D}(z_0, z_1) = \mathcal{R}(z_1 - z_0)$, where the dissipation potential is the Legendre transform $I^*_\mathcal{C}$ of the indicator function $I_\mathcal{C}$ of the yield set $\mathcal{C}$:

$$\mathcal{R}(v) = I^*_\mathcal{C}(v) = \sup_{\sigma\in\mathcal{H}} \big(\langle\sigma, v\rangle - I_\mathcal{C}(\sigma)\big) = \sup_{\sigma\in\mathcal{C}} \langle\sigma, v\rangle.$$

An important question is now the dependence of the play operator $\mathcal{P}_\mathcal{C}$ on the yields set $\mathcal{C}$. Under the assumptions that all the sets $\mathcal{C}_k$ contain 0, are closed and convex, it is shown in [20] that Hausdorff convergence of $\mathcal{C}_k$ to $\mathcal{C}_\infty$ implies that $\mathcal{P}_{\mathcal{C}_k}(0, \ell)$ converges to $\mathcal{P}_{\mathcal{C}_\infty}(0, \ell)$ in $C^0([0, T], H)$. In [40, Corollary 4.6] this result was generalized to the weaker Mosco convergence:

$$\mathcal{C}_k \xrightarrow{\text{M}} \mathcal{C}_\infty \quad \overset{\text{def}}{\Longleftrightarrow} \quad \begin{cases} \text{(i) } \mathcal{C}_\infty \supset \{\, z \in \mathcal{H}\,;\ z_{k_l} \rightharpoonup z \text{ with } z_{k_l} \in \mathcal{C}_{k_l} \,\}, \\ \text{(ii) } \mathcal{C}_\infty \subset \{\, z \in \mathcal{H}\,;\ \exists\, z_k \in \mathcal{C}_k : z_k \to z \,\}. \end{cases} \tag{5.3}$$

We may now apply our Γ-convergence result from Sect. 3. Since $\mathcal{E}_k$ does not depend on k and is a simple quadratic energy, the sublevels are balls, which are compact with respect to the weak topology. Moreover, the stable sets can be given explicitly in the form

$$\mathcal{S}_k(t) = \{\, z \in \mathcal{H}\,;\ 0 \in \partial\mathcal{R}_k(0) + z - \ell(t) \,\} = \ell(t) - \mathcal{C}_k.$$

The conditioned upper semi-continuity of the stable sets (2.11) now simply means that $z_{k_l} - \ell(t_l) \in \mathcal{C}_{k_l}$, $t_l \to t$ and $z_{k_l} \rightharpoonup z$ imply $z \in \mathcal{C}_\infty$. However, since ℓ is continuous, we easily see that this condition is equivalent to (5.3.i). The remaining condition is the lower Γ-limit (see (2.5)), which now reads

$$v_k \rightharpoonup v \text{ in } \mathcal{H} \quad \Longrightarrow \quad \mathcal{R}_\infty(v) \le \liminf_{k\to\infty} \mathcal{R}_k(v_k). \tag{5.4}$$

It is easily seen that this condition is a consequence of condition (5.3.ii).

In fact, condition (5.3.ii) and (5.4) are actually equivalent in the present situation. Since $0 \in \mathcal{C}_k$ for all k, one can simply follow the first steps in the proof of [3, Theorem 3.11a, p. 282] in order to check that (5.4) yields

$$\forall\, \sigma \in \mathcal{H}: \quad \inf\{\limsup_{k\to\infty} I_{\mathcal{C}_k}(\sigma_k)\ ;\ \sigma_k \to \sigma\} \le I_{\mathcal{C}_\infty}(\sigma),$$

which is clearly equivalent to condition (ii) in (5.3).

Since the limit problem has a unique solution, we additionally conclude that the whole sequence converges and we have thus recovered the result in [39,40] that Mosco convergence of $\mathcal{C}_k$ to $\mathcal{C}_\infty$ implies convergence of the stop operator. In fact, the results in that paper address the more general situation of approximating the data as well.

5.3 An example for relaxation and regularization

This example covers the theory of Sect. 4, where only two pairs of functionals are considered. We choose $\mathcal{Q} = \mathcal{Z} = \mathrm{H}^1((0,1))$ equipped with the weak topology and define the energy functionals

$$\mathcal{E}_1(t,z) = \int_0^1 W(z'(x)) + z(x)^2 - f(t,x)z(x)\,\mathrm{d}x,$$

$$\mathcal{E}_\infty(t,z) = \int_0^1 W^{**}(z'(x)) + z(x)^2 - f(t,x)z(x)\,\mathrm{d}x,$$

where $f \in \mathrm{C}^1([0,T],\mathrm{L}^2((0,1)))$, $W(a) = \min\{(a-1)^2, (a+1)^2\}$ and W^{**} is the convexification of W, i.e., $W^{**}(a) = W(a)$ for $|a| \ge 1$ and $W^{**}(a) = 0$ for $|a| \le 1$. It is a well-known fact that $\mathcal{E}_1$ is not weakly lower semi-continuous on $\mathcal{Z}$ and that $\mathcal{E}_\infty$ is its relaxation on $\mathcal{Z}$. Thus, all conditions on $\mathcal{E}_1$ and $\mathcal{E}_\infty$ are easily proved to hold.

For the dissipation we choose

$$\mathcal{D}_1(z_0,z_1) = \mathcal{D}_\infty(z_0,z_1) = \int_0^1 |z_1(x) - z_0(x)|\,\mathrm{d}x = \|z_1 - z_0\|_{\mathrm{L}^1},$$

which makes it easy to check all the assumptions on $\mathcal{D}_1$ and $\mathcal{D}_\infty$.

The crucial assumption is the upper semi-continuity (4.9) of the stable sets.

Lemma 5.1 *Let $0 < \alpha_l \to 0$, $t_l \to t$, $z_l \rightharpoonup z$ in $\mathcal{Z}$, and $z_l \in \mathcal{S}^{\alpha_l}(t_l)$ (i.e., $\forall\, l \in \mathbb{N}\ \forall\, \widetilde{z} \in \mathcal{Z}$: $\mathcal{E}_1(t_l,z_l) \le \alpha_l + \mathcal{E}_1(t_l,\widetilde{z}) + \mathcal{D}_1(z_l,\widetilde{z})$). Then, $z \in \mathcal{S}_\infty(t)$.*

Proof Choose an arbitrary test function $\widetilde{z} \in \mathcal{Z} = \mathrm{H}^1((0,1))$. Since $\mathcal{E}_\infty$ is the Γ-limit of $(\mathcal{E}_1)_{l\in\mathbb{N}}$, there is a recovery sequence $(\widetilde{z}_l)_{l\in\mathbb{N}}$ such that $\widetilde{z}_l \rightharpoonup \widetilde{z}$ and $\mathcal{E}_1(t_l,\widetilde{z}_l) \to \mathcal{E}_\infty(t,z)$.

Now, we have

$$\mathcal{E}_\infty(t,z) \le \liminf_{l\to\infty} \mathcal{E}_1(t_l,z_l) \le \liminf_{l\to\infty}(\alpha_l+\mathcal{E}_1(t_l,\widetilde{z}_l)+\|\widetilde{z}_l-z_l\|_{\mathrm{L}^1}) = \mathcal{E}_\infty(t,\widetilde{z})+\|\widetilde{z}-z\|_{\mathrm{L}^1},$$

where we have used the weak H^1 continuity of the L^1 norm. Since $\widetilde{z}$ was arbitrary, this proves the assertion. □

Theorem 5.2 *Assume* $0<\varepsilon_k\to 0$ *and* $\phi(\Pi_k)\to 0$ *for a sequence of partitions. Choose* $z_0\in\mathcal{S}_1(0)\subset\mathcal{Z}$ *and define the piecewise constant interpolants* $\overline{z}_k:[0,T]\to\mathcal{Z}$ *associated to some solution of the approximate incremental problem* (*AIP*)$_k$ *with initial value* $z_0^k=z_0$. *Then, there exist a subsequence* $(k_j)_{j\in\mathbb{N}}$ *and a limit function* $z:[0,T]\to\mathcal{Z}$ *such that for all* $t\in[0,T]$ *we have*

$$z_{k_j}(t)\rightharpoonup z(t) \text{ in } \mathrm{H}^1((0,1)),\quad \mathcal{E}_1(t,z_{k_j}(t))\to\mathcal{E}_\infty(t,z(t)),$$

$$\mathrm{Diss}_1(z_{k_j};[0,t])\to\mathrm{Diss}_\infty(z;[0,t])=\int_0^t\|\dot z(t)\|_{\mathrm{L}^1}\,\mathrm{d}t.$$

Moreover, $z:[0,T]\to\mathcal{Z}$ *is an energetic solution associated with* $\mathcal{E}_\infty$ *and* $\mathcal{D}_\infty$ *and satisfies* $z\in\mathrm{L}^\infty([0,T],\mathrm{H}^1((0,1)))\cap\mathrm{C}^{\mathrm{Lip}}([0,T],\mathrm{L}^2((0,1)))$.

The only new part in this result is the time regularity of z, namely $\dot z\in\mathrm{L}^\infty([0,T],\mathrm{L}^2(\Omega))$. This fact is a property of all solutions of $(\mathrm{S})_\infty$&$(\mathrm{E})_\infty$, since $\mathcal{E}_\infty$ is uniformly convex on $\mathrm{L}^2((0,1))$. The proof of this result follows the ideas in [31].

Proposition 5.3 *Every solution* $z:[0,T]\to\mathcal{Z}$ *of* $(\mathrm{S})_\infty$&$(\mathrm{E})_\infty$ *lies in* $\mathrm{C}^{\mathrm{Lip}}([0,T],\mathrm{L}^2((0,1)))$ *and satisfies, for a.e.* $t\in[0,T]$, *the estimate* $\|\dot z(t)\|_{\mathrm{L}^2}\le 2\|\dot f(t)\|_{\mathrm{L}^2}$.

Proof Since $z(s)$ minimizes the functional $\mathcal{E}_\infty(s,\cdot)+\|\cdot-z(s)\|_{\mathrm{L}^1}$, which is uniformly convex in the L^2 norm, we have the obvious estimate

$$\forall\widetilde{z}\in\mathcal{Z}:\quad \mathcal{E}_\infty(s,z(s))+\|\widetilde{z}-z(s)\|_{\mathrm{L}^2}^2\le\mathcal{E}_\infty(s,\widetilde{z})+\|\widetilde{z}-z(s)\|_{\mathrm{L}^1}.$$

Here the left-hand side is a parabola supporting the graph of the functional, which is the right-hand side, in the minimizer $z(s)$. Let $e(r)=\mathcal{E}_\infty(r,z(r))$ for $r\in[0,T]$ and test the above inequality by $\widetilde{z}=z(t)$, then

$$\begin{aligned} e(s)+\|z(t)-z(s)\|_2^2 &\le \mathcal{E}_\infty(s,z(t))+\|z(t)-z(s)\|_{\mathrm{L}^1}\\ &= e(t)-\langle f(s)-f(t),z(t)\rangle+\|z(t)-z(s)\|_{\mathrm{L}^1}.\end{aligned}$$

Assuming $t>s$ and using the energy balance $(\mathrm{E})_\infty$ we have

$$\|z(t)-z(s)\|_{\mathrm{L}^1}\le\mathrm{Diss}(z;[s,t])=e(s)-e(t)-\int_s^t\langle\dot f(\tau),z(\tau)\rangle\,\mathrm{d}\tau.$$

Combining these estimates we arrive at

$$\|z(t)-z(s)\|_2^2\le\int_s^t\langle\dot f(\tau),z(t)-z(\tau)\rangle\,\mathrm{d}\tau\le\sup_{r\in[s,t]}\|\dot f(\tau)\|_2\int_s^t\|z(\tau)-z(t)\|_2\,\mathrm{d}\tau.$$

Now apply [31, Lemma 3.3] to obtain the desired result. □

So far we are not able to prove that solutions associated with microstructure really occur as limits of solutions of $(\mathrm{AIP})_k$. In $(\mathrm{S})_\infty\&(\mathrm{E})_\infty$ this simply means that solutions satisfy $|z'(t,x)| < 1$. However, it is easy to see that $(\mathrm{S})_\infty\&(\mathrm{E})_\infty$ has solutions of this type. Consider the case $f(t,x) = (1-t)x$ and $z_0(x) = x$. Then, the function $z : [0,3] \to \mathrm{H}^1((0,1))$ with

$$z(t,x) = \begin{cases} x & \text{for } x \in [0, 1/(1+t)], \\ \frac{1}{2}\big((1-t)x + 1\big) & \text{for } x \in [1/(1+t), 1]. \end{cases}$$

is a solution. It would be sufficient to show that this solution is unique. Then, all accumulation points of solutions of $(\mathrm{AIP})_k$ would necessarily converge to this unique solution.

Instead of solving the approximate incremental problem we may also treat a regularized problem by using the energies

$$\mathcal{E}_k(t,z) = \int_0^1 \frac{1}{k}\big(z''(x)\big)^2 + W(z'(x)) + z(x)^2 - f(t,x)z(x)\,\mathrm{d}x.$$

We show that for this situation the Γ-convergence result of Sect. 3 is applicable. For this we still keep the underlying space $\mathcal{Q} = \mathcal{Z} = \mathrm{H}^1((0,1))$ equipped with the weak topology. Now each $\mathcal{E}_k$ has compact sublevels as they are closed and bounded in $\mathrm{H}^2((0,1))$, although not uniformly with respect to k, cf. condition (i) in (2.6). In particular, choosing a smooth stable initial value z_0 the standard existence theory for energetic solutions (cf. [14,25,27]) provides for each k energetic solutions z_k, which are solutions of the differential inclusion

$$\begin{aligned} &0 \in \mathrm{Sign}(\partial_t z) + \tfrac{1}{k}\partial_x^4 z - \partial_x\big(\mathrm{D}W(\partial_x z)\big) + 2z - f(t,x) \text{ for a.e. } (t,x) \in [0,T]\times\Omega, \\ &z(0,\cdot) = z_0 \in \mathrm{H}^2((0,1)), \end{aligned}$$

with $z_k \in \mathrm{L}^\infty([0,T], \mathrm{H}^2((0,1))) \cap \mathrm{BV}([0,T], \mathrm{L}^1((0,1)))$. In $\mathrm{L}^\infty([0,T], \mathrm{H}^2((0,1)))$ the norm will tend to ∞ with k, whereas in $\mathrm{L}^\infty([0,T], \mathrm{H}^1((0,1)))$ there is a k-independent bound.

Hence, we may pass to the limit for $k \to \infty$, since it is well-known that $\mathcal{E}_\infty$ is the Γ-limit of $\mathcal{E}_k$, see [6,10]. Theorem 3.1 is applicable and we conclude that convergent subsequences of $(z_k)_{k\in\mathbb{N}}$ exist and that their limit points are energetic solutions associated with the relaxed functionals $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$. Moreover, Theorem 3.4 can be employed to show that the solutions of suitable incremental problems converge to solutions of $(\mathrm{S})_\infty\&(\mathrm{E})_\infty$ as well.

An alternative relaxation is based on so-called Young measures and a continuous extension of W. To be more specific, let

$$\mathcal{Q} := \{\, q = (z,\nu) \in H^1((0,1))\times\mathcal{Y}^2((0,1)) \;;\; \int_{\mathbb{R}} a\,\nu_x(\mathrm{d}a) = z'(x) \text{ for a.a. } x \in (0,1)\,\},$$

where

$$\begin{aligned} \mathcal{Y}^2(0,1) := \Big\{ \nu = (\nu_x)_{x\in(0,1)} \;;\; &\nu_x \text{ is a probability measure on } \mathbb{R}, \\ &\forall\, \psi \in \mathrm{C}_0(\mathbb{R})\colon x \mapsto \int_{\mathbb{R}} \psi(a)\nu_x(\mathrm{d}a) \text{ is measurable}, \\ &\int_{x=0}^{1}\int_{a\in\mathbb{R}} a^2\nu_x(\mathrm{d}a)\,\mathrm{d}x < \infty \Big\} \end{aligned}$$

is the set of the L^2 Young measures. Then it is natural to define

$$\mathcal{E}_1(t,z,\nu) = \begin{cases} \int_0^1 W(z'(x)) + z(x)^2 - f(t,x)z(x)\,\mathrm{d}x & \text{if } \nu_x = \delta_{z'(x)} \text{ a.e. in } (0,1), \\ \infty & \text{else.} \end{cases}$$

while

$$\mathcal{E}_{\mathrm{YM}}(t,z,\nu) = \int_{x=0}^{1} \left(\int_{a\in\mathbb{R}} W(a)\nu_x(\mathrm{d}a) + z(x)^2 - f(t,x)z(x) \right) \mathrm{d}x.$$

The set $\mathcal{Q}$ can be considered as a convex subset of the linear space $\mathrm{H}^1((0,1)) \times \big(\mathrm{C}([0,1]) \otimes \{ a \mapsto \psi(a) + \alpha a^2 \; ; \; \psi \in \mathrm{C}_0(\mathbb{R}), \alpha \in \mathbb{R} \}\big)^*$ under the natural embedding

$$(z,\nu) \mapsto \left(z, \big(g \otimes (\psi + \alpha a^2)\big) \mapsto \int_0^1 g(x) \int_{\mathbb{R}} (\psi(a) + \alpha a^2)\nu_x(\mathrm{d}a)\,\mathrm{d}x \right).$$

This space is standardly topologized by the weak* topology, which makes $\mathcal{E}_{\mathrm{YM}}(t,\cdot)$ the Γ-limit of $\mathcal{E}_1(t,\cdot)$.

Again the theory of Sect. 4 is applicable. This shows that piecewise constant interpolants of the solutions of the approximate incremental problem (AIP) associated with $\mathcal{E}_1$ and $\mathcal{D}_1$ have subsequences, which converge to energetic solutions associated with $\mathcal{E}_{\mathrm{YM}}$ and $\mathcal{D}_\infty$.

In the vectorial, multidimensional case a more sophisticated Young measure relaxation in the rate-independent setting is given in [19]. Related evolutionary systems for Young measures, also in the rate-dependent case, are discussed in [5,11,12,23,24,28,29,42].

A Generalization of Helly's selection principle

The following result is an abstract version of Helly's selection principle which is again a generalization of [27, Theorem 3.2]. Since we are concerned with a sequence $(\mathcal{D}_k)_{k\in\mathbb{N}}$ of dissipation distances rather than with a single one, we give a full independent proof and repeat all assumptions. In particular, we explicitly state that compactness means 'sequential compactness'.

$$\forall\, k \in \mathbb{N}_\infty \; \forall\, z_1, z_2, z_3 \in \mathcal{Z} : \; \mathcal{D}_k(z_1,z_1) = 0, \; \mathcal{D}_k(z_1,z_3) \le \mathcal{D}_k(z_1,z_2) + \mathcal{D}_k(z_2,z_3). \tag{A.1}$$

$$\begin{aligned} &\text{For all sequentially compact } \mathcal{K} \subset \mathcal{Z} \text{ we have :} \\ &\text{If } z_k \in \mathcal{K} \text{ and } \min\{\mathcal{D}_\infty(z_k,z), \mathcal{D}_\infty(z,z_k)\} \to 0, \text{ then } z_k \overset{\mathcal{Z}}{\to} z. \end{aligned} \tag{A.2}$$

$$\big(z_k \to z \text{ and } \widetilde{z}_k \to \widetilde{z} \big) \quad \Longrightarrow \quad \mathcal{D}_\infty(z,\widetilde{z}) \le \liminf_{k\to\infty} \mathcal{D}_k(z_k,\widetilde{z}_k). \tag{A.3}$$

Note that (A.1) and (A.2) are simply recalled from Sect. 2 while (A.3) is stronger than the corresponding assumptions (2.5) and (4.2) (see below).

Additionally, we use that $\mathcal{Z}$ is a Hausdorff topological space, which implies that each converging sequence has a unique limit. For a function $z : [0,T] \to \mathcal{Z}$ and $k \in \mathbb{N}_\infty$ we recall

$$\mathrm{Diss}_k(z;[s,t]) = \sup\left\{\sum_{j=1}^N \mathcal{D}_k(z(t_{j-1}),z(t_j));\, N\in\mathbb{N},\ s\le t_0<t_1<\cdots<t_N\le t\right\}.$$

Of course, we have $\mathcal{D}_k(z(s),z(t)) \le \mathrm{Diss}_k(z;[s,t])$.

Theorem A.1 *Assume that the sequence* $(\mathcal{D}_k)_{k\in\mathbb{N}_\infty}$ *satisfies the conditions* (A.1), (A.2) *and* (A.3). *Moreover, let* $\mathcal{K}$ *be a sequentially compact subset of* $\mathcal{Z}$ *and* $z_k:[0,T]\to\mathcal{Z},\ k\in\mathbb{N}$, *a sequence satisfying*

$$\text{(i) } \forall\, t\in[0,T]\ \forall\, k\in\mathbb{N}:\ z_k(t)\in\mathcal{K} \qquad \text{(ii) } \sup_{k\in\mathbb{N}} \mathrm{Diss}_k(z_k;[0,T])<\infty. \tag{A.4}$$

Then there exist a subsequence $(z_{k_l})_{l\in\mathbb{N}}$ *and limit functions* $z:[0,T]\to\mathcal{Z}$ *and* $\delta:[0,T]\to[0,\infty]$ *with the following properties*:

$$\begin{aligned}
&\text{(a)}\quad \forall\, t\in[0,T]:\ \delta(t)=\lim_{l\to\infty}\mathrm{Diss}_{k_l}(z_{k_l};[0,t])\\
&\text{(b)}\quad \forall\, t\in[0,T]:\ z_{k_l}(t)\overset{\mathcal{Z}}{\to} z(t)\\
&\text{(c)}\quad \forall\, s,t\in[0,T] \text{ with } s<t:\ \mathrm{Diss}_\infty(z;[s,t])\le\delta(t)-\delta(s).
\end{aligned}$$

Proof We define the functions $d_k:[0,T]\to[0,\infty]$ with $d_k(t)=\mathrm{Diss}_k(z_k;[0,t])$ which are nondecreasing by definition and uniformly bounded by (A.4.ii). Hence, the classical Helly's selection principle for real-valued functions provides a subsequence such that $d_{\widetilde{k}_n}(t)\to\delta(t)$ for all $t\in[0,T]$. Hence, $\delta:[0,T]\to[0,\infty]$ is also nondecreasing and bounded. This proves (a).

Denote by $\mathcal{J}\subset[0,T]$ the set of discontinuity points of δ, then $\mathcal{J}$ is countable. Hence, we may choose a countable, dense subset $\mathcal{T}$ of $[0,T]$ with $\mathcal{J}\subset\mathcal{T}$. For each $t\in\mathcal{T}$ any subsequence of $(z_{\widetilde{k}_n}(t))_{n\in\mathbb{N}}$ lies in the sequentially compact set $\mathcal{K}\subset\mathcal{Z}$ and thus contains a convergent subsequence. Hence, using Cantor's diagonal scheme we find a subsequence $(z_{k_l})_{l\in\mathbb{N}}$ of $(z_{\widetilde{k}_n})_{n\in\mathbb{N}}$ such that (a) remains true and additionally we have

$$\forall\, t\in\mathcal{T}:\quad z_{k_l}(t)\overset{\mathcal{Z}}{\to} z(t) \text{ for } l\to\infty.$$

This defines the limit function $z:\ \mathcal{T}\to\mathcal{Z}$.

To show convergence on $[0,T]\backslash\mathcal{T}$ we use the continuity of δ. We fix $t_*\in[0,T]\backslash\mathcal{T}$, then the sequence $(z_{k_l}(t_*))_{l\in\mathbb{N}}$ has a convergent subsequence $z_{\widehat{k}_m}(t_*)\overset{\mathcal{Z}}{\to} z_*$. Moreover, there exists a sequence $t_n\in\mathcal{T}$ with $t_n\to t_*$. Below we will show $z(t_n)\overset{\mathcal{Z}}{\to} z_*$. By the Hausdorff property of $\mathcal{Z}$ we conclude that $(z_{k_l}(t_*))_{l\in\mathbb{N}}$ has exactly one accumulation point and we define $z(t_*)=z_*$.

To show $z(t_n)\overset{\mathcal{Z}}{\to} z_*$ we first assume $t_n<t_*$. Then, using (A.3) we have

$$\mathcal{D}_\infty(z(t_n),z_*)\le\liminf_{m\to\infty}\mathcal{D}_{\widehat{k}_m}(z_{\widehat{k}_m}(t_n),z_{\widehat{k}_m}(t_*))\le\liminf_{m\to\infty}\mathrm{Diss}_{\widehat{k}_m}(z_{\widehat{k}_m};[t_n,t_*])=\delta(t_*)-\delta(t_n).$$

Similarly, for $t_*<t_n$ we obtain $\mathcal{D}_\infty(z_*,z(t_n))\le\delta(t_n)-\delta(t_*)$. Using the continuity of δ in t_* we conclude $\min\{\mathcal{D}_\infty(z(t_n),z_*),\mathcal{D}_\infty(z_*,z(t_n))\}\le|\delta(t_*)-\delta(t_n)|\to 0$ for $n\to\infty$. Employing (A.2) we find $z(t_n)\overset{\mathcal{Z}}{\to} z_*$ as claimed above. Thus, assertion (b) is proved.

The final estimate is obtained using (A.3) again. For any partition of $[s,t]$ we have

$$\sum_{j=1}^N \mathcal{D}_\infty(z(t_{j-1}),z(t_j)) \ \le\ \sum_{j=1}^N \liminf_{l\to\infty}\mathcal{D}_{k_l}(z_{k_l}(t_{j-1}),z_{k_l}(t_j))$$

$$\leq \liminf_{l\to\infty} \sum_{j=1}^{N} \mathcal{D}_{k_l}(z_{k_l}(t_{j-1}), z_{k_l}(t_j)) \leq \liminf_{l\to\infty} \mathrm{Diss}_{k_l}(z_{k_l}; [s,t]) = \delta(t)-\delta(s). \tag{A.5}$$

Thus, $\mathrm{Diss}_\infty(z; [s,t]) \leq \delta(t)-\delta(s)$ and (c) is proved. □

As mentioned above, the latter compactness lemma holds under assumption (A.3), which is stronger than (2.5) and (4.2). In particular, Theorem A.1 is not directly suited for the purposes of checking the compactness of approximating sequences in the proof of Theorems 3.1, 3.4, and 4.1. On the other hand, we actually need to prove compactness for stable sequences only. In particular, by assuming (2.5) [analogously for (4.2)], the sequences $z_k : [0,T] \to \mathcal{Z}$ used in the above proofs are such that the following holds:

$$\begin{aligned} &\forall s_l \to s \text{ and } t_l \to t \text{ with } s_l \leq t_l : \\ &\left(z_{k_l}(s_l) \overset{\mathcal{Z}}{\to} z \text{ and } z_{k_l}(t_l) \overset{\mathcal{Z}}{\to} \widetilde{z}\right) \implies \mathcal{D}_\infty(z,\widetilde{z}) \leq \liminf_{l\to\infty} \mathcal{D}_{k_l}(z_{k_l}(s_l), z_{k_l}(t_l)). \end{aligned} \tag{A.6}$$

It is easily seen that the proof of Theorem A.1 goes through by removing the assumption (A.3) and assuming (A.6) instead. This slight modification of the result is suited for proving the compactness of the sequence of approximating solutions of Theorems 3.1 and 3.4, (and 4.1) under assumption (2.5) [assumption (4.2), resp.] only.

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