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Calculus of Variations

On the Monge–Ampère equation with boundary blow-up: existence, uniqueness and asymptotics

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Abstract We consider the Monge–Ampère equation $\det D^2u = b(x)f(u) > 0$ in Ω, subject to the singular boundary condition $u = \infty$ on $\partial\Omega$. We assume that $b \in C^\infty(\overline{\Omega})$ is positive in Ω and non-negative on $\partial\Omega$. Under suitable conditions on f, we establish the existence of positive strictly convex solutions if Ω is a smooth strictly convex, bounded domain in $\mathbb{R}^N$ with $N \geq 2$. We give asymptotic estimates of the behaviour of such solutions near $\partial\Omega$ and a uniqueness result when the variation of f at ∞ is regular of index q greater than N (that is, $\lim_{u\to\infty} f(\lambda u)/f(u) = \lambda^q$, for every $\lambda > 0$). Using regular variation theory, we treat both cases: $b > 0$ on $\partial\Omega$ and $b \equiv 0$ on $\partial\Omega$.

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1 Introduction and main results

Let Ω be a smooth, strictly convex, bounded domain in $\mathbb{R}^N$ with $N \geq 2$.

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The Dirichlet problem for the Monge–Ampère equation (that is, find $u \in C^\infty(\overline{\Omega})$ such that

$$\det D^2 u = G(x, u, Du) > 0 \quad \text{in } \Omega, \qquad u = \varphi \quad \text{on } \partial\Omega, \tag{1.1}$$

where G and φ are smooth) has been studied extensively, see [5,8,18,24,25,31,37–42]. Using barrier techniques and maximum principles, it is shown in [5] that the solvability reduces to the existence of smooth convex subsolutions. In the case of a non-smooth data, P.-L. Lions [24] proved, via a penalization method, that the solvability of the Dirichlet problem can be reduced to the existence of a generalized subsolution in the sense of A. D. Aleksandrov. A condition under which such a subsolution exists was also provided. Necessary and sufficient conditions for the existence of solutions are studied in [40].

Boundary value problems for Hessian equations (involving the k-Hessian operator $S_k(D^2 u)$ where $k \in \{1, \ldots, N\}$ and S_k is the kth elementary symmetric function of the eigenvalues of the Hessian matrix D^2u of u) received increasing attention in recent years. We refer the reader to [8,17,18,37–42]. The Laplace operator and the Monge–Ampère operator are well-known examples of Hessian operators corresponding to $k = 1$ and $k = N$, respectively.

Our purpose here is to investigate the Monge–Ampère equation

$$\det D^2 u = b(x) f(u) > 0 \quad \text{in } \Omega, \tag{1.2}$$

subject to an infinite Dirichlet boundary condition

$$u(x) \to \infty \ \text{ as } d(x) := \operatorname{dist}(x, \partial\Omega) \to 0, \tag{1.3}$$

where we assume, throughout, that $f \in C[0, \infty) \cap C^\infty(0, \infty)$ is positive increasing such that $f(0) = 0$ and $b \in C^\infty(\overline{\Omega})$ is positive in Ω.

Problems of this type have first been considered by Cheng and Yau [6,7] (with $f(u) = e^{Ku}$ in bounded convex domains and with $b(x)f(u) = e^{2u}$ in unbounded domains). The Monge–Ampère equation with boundary blow-up has been treated in [23,29] and [19], while the more general case of Hessian equations has been studied in [34] and [14]. We refer to [36] for recent new results on boundary blow-up problems for k-curvature equations.

The study of boundary blow-up solutions has been initiated by Bieberbach [3] and Rademacher [32] for the equation $\Delta u = e^u$ in a smooth bounded domain in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Since then many papers have been dedicated to resolving existence, uniqueness and asymptotic behaviour issues for blow-up solutions of semilinear/quasilinear elliptic equations; see e.g., [1,2,10–13,16,22,26–28,30] and their references.

If Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ and f_1 satisfies

$$\begin{cases} f_1 \text{ is locally Lipschitz continuous on } [0, \infty), \\ \text{positive and non-decreasing on } (0, \infty) \text{ with } f_1(0) = 0, \end{cases} \tag{H_0}$$

then, $\Delta u = f_1(u)$ in Ω, subject to (1.3), possesses positive $C^2(\Omega)$-solutions if and only if f_1 satisfies the Keller–Osserman condition (see [22,30]):

$$\int_1^\infty \frac{dt}{\sqrt{F_1(t)}} < \infty, \quad \text{where } F_1(t) = \int_0^t f_1(s)\, ds. \tag{H_1}$$

We first establish the existence of smooth blow-up solutions of (1.2). By a *blow-up solution* of (1.2) we mean any positive $C^2(\Omega)$-solution of (1.2)+(1.3).

Theorem 1.1 (Existence) *Let Ω be a smooth, strictly convex, bounded domain in $\mathbb{R}^N$ with $N \geq 2$. Suppose that there exists a function f_0 on $[0, \infty)$ such that:*

(A_1) $f_0(u) \leq f(u)$, *for every* $u > 0$;
(A_2) $f_0^{1/N}$ *satisfies* (H_0) *and* (H_1), *that is they hold with* $f_0^{1/N}$ *instead of* f_1;
(A_3) $f_0^{-1/N}$ *is convex on* $(0, \infty)$.

Then, (1.2) *admits strictly convex blow-up solutions in* $C^\infty(\Omega)$.

In particular, the assumptions (A_2) and (A_3) are met by the following functions:

(i) $f_0(u) = (e^u - 1)^p$, for every $p > 0$;
(ii) $f_0(u) = u^q$ with $q > N$;
(iii) $f_0(u) = u^N[\ln(u + 1)]^\beta$ with $\beta > 2N$.

When $f(u) = u^q$ with $q > N$, Theorem 1.1 was established in [23]. More general existence results are obtained in Theorems 1.1 and 1.2 of [19], whose assumption (1.3) is of the type (A_1) with f_0 in the special case (ii). However, this kind of growth restriction on f is not optimal as proved by taking in our Theorem 1.1 the particular case $f = f_0$ and f_0 as in (iii) above.

Our proof rests upon the solvability of the Dirichlet problem for the Monge–Ampère equation and the convexity of the minimal positive solution $u_{\min}$ of $\Delta u = f_0^{1/N}(u)$ in Ω, subject to (1.3). The existence of $u_{\min}$ is ensured by (A_2), while its convexity follows from Theorem 3.1 in [14], which requires (A_3). A comparison principle (Proposition 2.4) and the arithmetic-geometric inequality (3.3) for convex functions will be used to conclude the existence of a blow-up solution of (1.2).

If b is a *positive* smooth function on $\overline{\Omega}$ and $f(u) = u^q$ with $q > N$, then it is shown in [23] that (1.2) admits a unique blow-up solution $u \in C^\infty(\Omega)$. Moreover, there exist positive constants c_1 and c_2 such that for $x \in \Omega$

$$c_1[d(x)]^{-\alpha} \leq u(x) \leq c_2[d(x)]^{-\alpha}, \quad \text{where } \alpha = (N + 1)/(q - N). \tag{1.4}$$

Our next aim is to establish the asymptotic behaviour near $\partial\Omega$ of the blow-up solutions of (1.2) in a general setting when b is allowed to *vanish on the whole boundary* $\partial\Omega$. This corresponds to a critical case, $0 \cdot \infty$ on $\partial\Omega$, which arises in the right-hand side of (1.2).

To remove the positivity restriction of b on $\partial\Omega$, which appears in previous papers on the topic (such as [14,23,29,34]), we will proceed in a substantially different manner. As a special feature, we will present our main results (Theorems 1.2 and 1.6) in connection with regular variation theory arising in probability theory (see Sect. 4). Let us recall here the definition of a regularly varying function, while more details on the regular variation theory can be found in Sect. 4.

A positive measurable function R defined on $[A, \infty)$, for some $A > 0$, is called *regularly varying* (*at infinity*) *with index* $q \in \mathbb{R}$, written $R \in RV_q$, provided that

$$\lim_{u\to\infty} \frac{R(\lambda u)}{R(u)} = \lambda^q, \quad \text{for all } \lambda > 0. \tag{1.5}$$

When the index q is zero, we say that the function is *slowly varying*.

Note that $R \in RV_q$ if and only if $L(u) := R(u)/u^q$ is slowly varying.

Notation. If H is a non-decreasing function on $\mathbb{R}$, then we denote by $H^{\leftarrow}$ the (left continuous) inverse of H (see [33]), that is

$$H^{\leftarrow}(y) = \inf\{s : H(s) \geq y\}.$$

By $f_1(u) \sim f_2(u)$ as $u \to u_0 \in \overline{\mathbb{R}}$ we mean that $f_1(u)/f_2(u) \to 1$ as $u \to u_0$.

If $\alpha > 0$ is sufficiently large, we define

$$\mathcal{P}(u) = \sup\left\{ \frac{f(y)}{y^N} : \alpha \le y \le u \right\}, \quad \text{for } u \ge \alpha. \tag{1.6}$$

For an open bounded subset Ω of $\mathbb{R}^N$ with boundary of class C^2 and every $x \in \partial\Omega$, we denote by $\rho_1(x), \ldots, \rho_{N-1}(x)$ the principal curvatures of $\partial\Omega$ at x. For $m \in \{1, \ldots, N-1\}$, we define the mth curvature $\sigma_m(x)$ of $\partial\Omega$ at x by

$$\sigma_m(x) = S_m(\rho_1(x), \ldots, \rho_{N-1}(x)) = \sum_{1 \le i_1 < \cdots < i_m \le N-1} \rho_{i_1}(x) \cdots \rho_{i_m}(x).$$

Recall that Ω (as above) is said to be m-convex if $\sigma_j(x) \ge 0$, for every $x \in \partial\Omega$ and every $j \in \{1, \ldots, m\}$, and it is called strictly m-convex if it is m-convex and $\sigma_m(x) > 0$ for every $x \in \partial\Omega$. In particular, the (strict) $(N-1)$-convexity for domains is equivalent to the usual (strict) convexity.

Let $\mathcal{K}_\ell$ denote the set of all positive non-decreasing C^1-functions k defined on $(0, \nu)$, for some $\nu > 0$, for which there exists

$$\lim_{t \to 0} \left(\frac{K(t)}{k(t)} \right)' = \ell, \quad \text{where } K(t) = \int_0^t k(s)\, ds. \tag{1.7}$$

Note that $\ell \in [0, 1]$ and $\lim_{t \to 0} K(t)/k(t) = 0$, for every $k \in \mathcal{K}_\ell$. A complete characterization of $\mathcal{K}_\ell$ (according to $\ell \ne 0$ or $\ell = 0$) is provided by [13] (see [9] for a more general result and a different approach).

It is easy to check that the following functions belong to $\mathcal{K}_\ell$ with the specified ℓ:

(a) $k(t) = (-1/\ln t)^p$ with $\ell = 1$,
(b) $k(t) = t^p$ with $\ell = 1/(p+1)$,
(c) $k(t) = e^{-1/t^p}$ with $\ell = 0$, where $p > 0$ is arbitrary in (a)–(c).

The class $\mathcal{K}_\ell$ will be used to model the behaviour of b near $\partial\Omega$ (see (1.8)).

The next result establishes lower and upper estimates for the growth of the blow-up solutions of (1.2) near $\partial\Omega$ when $f \in RV_q$ with $q > N$ and (1.8) holds.

Theorem 1.2 (Asymptotic behaviour) *Let $N \ge 2$ and Ω be a smooth, strictly convex, bounded domain in $\mathbb{R}^N$. Assume that $f \in RV_q$ with $q > N$ and there exists $k \in \mathcal{K}_\ell$ such that*

$$0 < \beta^- = \liminf_{d(x) \to 0} \frac{b(x)}{k^{N+1}(d(x))} \quad \text{and} \quad \limsup_{d(x) \to 0} \frac{b(x)}{k^{N+1}(d(x))} = \beta^+ < \infty. \tag{1.8}$$

Then, every strictly convex blow-up solution u_∞ of (1.2) satisfies

$$\xi^- \le \liminf_{d(x) \to 0} \frac{u_\infty(x)}{\phi(d(x))} \quad \text{and} \quad \limsup_{d(x) \to 0} \frac{u_\infty(x)}{\phi(d(x))} \le \xi^+, \tag{1.9}$$

where ϕ is defined by

$$\phi(t) = \mathcal{P}^{\leftarrow}([K(t)]^{-N-1}), \quad \text{for } t > 0 \text{ small}, \tag{1.10}$$

and $\xi^\pm$ are positive constants given by

$$\frac{(\xi^+)^{N-q}}{\beta^-} \max_{\partial\Omega} \sigma_{N-1} = \frac{(\xi^-)^{N-q}}{\beta^+} \min_{\partial\Omega} \sigma_{N-1} = \frac{[(q-N)/(N+1)]^{N+1}}{1 + \ell(q-N)/(N+1)}. \tag{1.11}$$

Remark 1.3 In the setting of Theorem 1.2, the limit $\lim_{d(x)\to 0} u_\infty(x)/\phi(d(x))$ exists provided that Ω is a ball and (1.8) holds with $\beta^- = \beta^+ \in (0,\infty)$. The latter condition is equivalent to saying that

$$b(x) \sim k^{N+1}(d(x)) \text{ as } d(x) \to 0, \quad \text{for some } k \in \mathcal{K}_\ell. \tag{1.12}$$

More exactly, when Ω is a ball of radius $R > 0$, Theorem 1.2 reads as follows.

Corollary 1.4 *Let $\Omega \subset \mathbb{R}^N$ be a ball of radius $R > 0$ and $f \in RV_q$ with $q > N$. If (1.12) holds, then every strictly convex blow-up solution u_∞ of (1.2) satisfies*

$$u_\infty(x) \sim \xi\phi(d(x)) \text{ as } d(x) \to 0, \tag{1.13}$$

where ϕ is defined by (1.10) and ξ is given by

$$\xi = \left\{ \frac{[(q-N)/(N+1)]^{N+1} R^{N-1}}{1 + \ell(q-N)/(N+1)} \right\}^{1/(N-q)}. \tag{1.14}$$

Remark 1.5 If $f \in RV_q$ $(q > N)$ and $k \in \mathcal{K}_\ell$, then the explosion rate of $\phi(t)$ at $t = 0$ (ϕ defined by (1.10)) is significantly faster when $\ell = 0$. A precise description of the variation of $\phi(1/u)$ at $u = \infty$ is given by Proposition 5.7 according to $\ell = 0$ (when $\phi(1/u) \notin RV_m$, for every $m \in \mathbb{R}$) or $\ell \neq 0$ (when $\phi(1/u) \in RV_{(N+1)/[\ell(q-N)]}$).

We now assert that, under slightly more restrictive conditions than those in Theorem 1.2, there is at most one strictly convex blow-up solution of (1.2).

Theorem 1.6 (Uniqueness) *Let Ω be a smooth, strictly convex, bounded domain in $\mathbb{R}^N$ ($N \geq 2$). Suppose $f \in RV_q$ with $q > N$ and $f(u)/u^N$ is increasing on $(0,\infty)$.*

Then, (1.2) has at most one strictly convex blow-up solution provided that either

(i) *b is positive on $\overline{\Omega}$ or*
(ii) *b is zero on $\partial\Omega$, Ω is a ball of radius $R > 0$ and (1.12) holds.*

Remark 1.7 In view of Corollary 1.4, the case (ii) of Theorem 1.6 yields a precise asymptotic behaviour of any strictly convex blow-up solution of (1.2) at $\partial\Omega$. This fact is essentially used to prove the claim of Theorem 1.6. In contrast to this appears case (i), when we conclude that any two strictly convex blow-up solutions of (1.2) must coincide without using any a priori blow-up estimates near the boundary. Our argument modifies an idea in [23], which treats the case (i) for $f(u) = u^q$ for every $u > 0$ (with $q > N$).

We see that $f \in RV_q$ if and only if it can be written as

$$f(u) = T(u)u^q \exp\left(\int_D^u \frac{\varepsilon(t)}{t}\right), \quad u \geq D, \tag{1.15}$$

for some $D > 0$, where $\varepsilon \in C[D,\infty)$ satisfies $\lim_{u\to\infty} \varepsilon(u) = 0$ and $T(u)$ is measurable on $[D,\infty)$ such that $\lim_{u\to\infty} T(u) = \widehat{T} \in (0,\infty)$ (use Proposition 4.7 with $L(u) = f(u)/u^q$). If f is of the form (1.15) with $T(u) = \text{Const.} > 0$, then we say that f is *normalised regularly varying of index q* (and write $f \in NRV_q$).

Remark 1.8 If $f \in NRV_q$ with $q > N$, then $f(u)/u^N$ is increasing for $u > 0$ large. Hence, $\mathcal{P}(u)$ defined by (1.6) coincides with $f(u)/u^N$.

In Sect. 5 we promote the use of regular variation theory (initiated by Karamata [20,21]) and its extensions (due de Haan [15]) to gain insight into the blow-up rate of the solutions of (1.2) at $\partial\Omega$. Our results treat the case $f \in RV_q$ $(q > N)$ for various decay rates of b at $\partial\Omega$, as illustrated below:

(i) $b(x) \sim [-1/\ln d(x)]^{p(N+1)}$ as $d(x) \to 0$;
(ii) $b(x) \sim [d(x)]^{p(N+1)}$ as $d(x) \to 0$;
(iii) $b(x) \sim e^{-(N+1)/[d(x)]^p}$ as $d(x) \to 0$;

where p is a positive constant (see Example 5.9 in Sect. 5).

The use of regular variation theory in the study of the blow-up solutions to semilinear elliptic equations originates with [11,12] (see also [10] or [13]). These papers are concerned with the uniqueness and asymptotics of the blow-up solutions to (1.2) with the Laplacian instead of the Monge–Ampère operator. We point out a significant difference which arises between these two cases: the first term in the asymptotic expansion near $\partial\Omega$ of the blow-up solution in the former case (Laplacian) is independent of the geometry of the boundary, whereas in the latter case (Monge–Ampère operator) we prove here the involvement of the boundary through its Gauss curvature.

The plan of this paper is as follows. In Sect. 2 we prove a general formula for $\det D^2h(g(x))$ (where $g \in C^2(\Omega)$ and $h \in C^2(\mathbb{R})$), which is pertaining to our local argument near $\partial\Omega$ in the proof of Theorem 1.2. In Sect. 3 we deduce the existence assertion of Theorem 1.1. In Sect. 4 we provide the necessary definitions and properties from regular variation theory. In Sect. 5 we discuss the asymptotic properties of the function ϕ involved in the asymptotic formulas of Theorem 1.2 and Corollary 1.4. Sections 6 and 7 are dedicated respectively to the proofs of Theorems 1.2 and 1.6.

2 Basics

Proposition 2.1 *Let Ω be an open subset of $\mathbb{R}^N$ with $N \geq 2$. If $g \in C^2(\Omega)$ and $h \in C^2(\mathbb{R})$, then the following holds*

$$\begin{aligned}\det D^2h(g(x)) =& [h'(g(x))]^{N-1}h''(g(x)) < \mathrm{Co}\big(D^2g(x)\big)Dg(x), Dg(x) > \\ &+ [h'(g(x))]^N \det D^2g(x), \quad \forall x \in \Omega,\end{aligned} \tag{2.1}$$

where $Dg(x) = \mathrm{col}\left(\frac{\partial g(x)}{\partial x_1}, \ldots, \frac{\partial g(x)}{\partial x_N}\right)$ and $\mathrm{Co}\big(D^2g(x)\big)$ denotes the cofactor matrix of $D^2g(x)$.

Proof Let $x \in \Omega$ be fixed. For every integers i, j between 1 and N, we have

$$\frac{\partial^2 h(g(x))}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(h'(g(x))\frac{\partial g(x)}{\partial x_j}\right) = h''(g(x))\frac{\partial g(x)}{\partial x_i}\frac{\partial g(x)}{\partial x_j} + h'(g(x))\frac{\partial^2 g(x)}{\partial x_i \partial x_j}.$$

This shows that

$$D^2h(g(x)) = h''(g(x))Dg(x) \otimes Dg(x) + h'(g(x))D^2g(x).$$

Since the determinant is linear in each of its columns, we can write the determinant of $D^2h(g(x))$ as the sum of 2^N determinants, where each summand has the jth column either

$$h''(g(x))\frac{\partial g(x)}{\partial x_j}\mathrm{col}\left(\frac{\partial g(x)}{\partial x_1}, \frac{\partial g(x)}{\partial x_2}, \ldots, \frac{\partial g(x)}{\partial x_N}\right) \tag{2.2}$$

or

$$h'(g(x))\mathrm{col}\left(\frac{\partial^2 g(x)}{\partial x_1 \partial x_j}, \frac{\partial^2 g(x)}{\partial x_2 \partial x_j} \dots, \frac{\partial^2 g(x)}{\partial x_N \partial x_j}\right). \tag{2.3}$$

We denote by M_j the matrix whose jth column is of the form (2.2) and the rest of its columns are of the type (2.3). By expanding the determinant of M_j along the jth column, we find

$$\det M_j = [h'(g(x))]^{N-1} h''(g(x)) \frac{\partial g(x)}{\partial x_j} \sum_{i=1}^{N} \frac{\partial g(x)}{\partial x_i} C_{ij}(x),$$

where $C_{ij}(x)$ stands for the cofactor of the (i, j)th entry of the symmetric matrix $D^2 g(x)$. Thus, we have

$$\sum_{j=1}^{N} \det M_j = [h'(g(x))]^{N-1} h''(g(x)) \left\langle \mathrm{Co}(D^2 g(x)) Dg(x), Dg(x) \right\rangle. \tag{2.4}$$

If M_0 denotes the matrix with all its columns of the type (2.3), then

$$\det M_0 = [h'(g(x))]^N \det(D^2 g(x)). \tag{2.5}$$

Since the determinant of any matrix with two different columns of the type (2.2) is zero, we infer that

$$\det D^2 h(g(x)) = \det M_0 + \sum_{j=1}^{N} \det M_j.$$

From (2.4) and (2.5), we conclude the proof of (2.1). □

For $\mu > 0$, we set $\Gamma_\mu = \{x \in \overline{\Omega} : \ d(x) < \mu\}$.

Remark 2.2 If Ω is bounded and $\partial\Omega \in C^k$ for $k \geq 2$, then there exists a positive constant μ depending on Ω such that $d \in C^k(\Gamma_\mu)$ (cf. Lemma 14.16 in [17]).

Corollary 2.3 *Let Ω be bounded with $\partial\Omega \in C^k$ for $k \geq 2$. Assume that $\mu > 0$ is small such that $d \in C^2(\Gamma_\mu)$ and h is a C^2-function on $(0, \mu)$. Let $x_0 \in \Gamma_\mu \setminus \partial\Omega$ and $y_0 \in \partial\Omega$ be such that $|x_0 - y_0| = d(x_0)$. Then, we have*

$$\det D^2 h(d(x_0)) = [-h'(d(x_0))]^{N-1} h''(d(x_0)) \Pi_{i=1}^{N-1} \frac{\rho_i(y_0)}{1 - \rho_i(y_0)\, d(x_0)}, \tag{2.6}$$

where $\rho_1(y_0), \dots, \rho_{N-1}(y_0)$ are the principal curvatures of $\partial\Omega$ at y_0.

Proof Lemma 14.17 in [17] gives the expression of the Hessian matrix of d at x_0 in terms of a principal coordinate system at y_0, namely

$$[D^2 d(x_0)] = \mathrm{diag}\left[\frac{-\rho_1(y_0)}{1 - \rho_1(y_0)\, d(x_0)}, \dots, \frac{-\rho_{N-1}(y_0)}{1 - \rho_{N-1}(y_0)\, d(x_0)}, 0\right]. \tag{2.7}$$

Since

$$Dd(x_0) = \mathrm{col}(0, \dots, 0, 1),$$

we obtain

$$\left\langle \mathrm{Co}(D^2 d(x_0)) Dd(x_0), Dd(x_0) \right\rangle = (-1)^{N-1} \Pi_{i=1}^{N-1} \frac{\rho_i(y_0)}{1 - \rho_i(y_0)\, d(x_0)}. \tag{2.8}$$

Applying Proposition 2.1 with $g(x) = d(x)$ and using (2.7), (2.8) we derive (2.6). □

Proposition 2.4 (Comparison principle) *Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$ and let $\underline{u}, \overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$. Suppose $g(x, u)$ is defined for $x \in \Omega$ and u in some interval containing the ranges of $\underline{u}$ and $\overline{u}$.*

If the following holds:

(i) $g(x, u)$ *is increasing in u for all $x \in \Omega$,*
(ii) *the matrix* $\left[\frac{\partial^2 \underline{u}}{\partial x_i \partial x_j}\right]$ *is positive definite in Ω,*
(iii) $\det D^2 \underline{u}(x) \geq g(x, \underline{u}(x))$ *and* $\det D^2 \overline{u}(x) \leq g(x, \overline{u}(x))$, *for every $x \in \Omega$,*
(iv) $\underline{u}(x) \leq \overline{u}(x)$, *for every $x \in \partial\Omega$,*

then we have

$$\underline{u}(x) \leq \overline{u}(x), \quad \forall x \in \overline{\Omega}.$$

For the proof of Proposition 2.4 we refer to Lemma 2.1 in [23].

3 Proof of Theorem 1.1

By Theorem 7.1 in [5] we see that the following problem

$$\begin{cases} \det D^2 u = (b(x) + 1/n) f(u) & \text{in } \Omega, \\ u = n \geq 1 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

possesses a unique strictly convex solution $u_n \in C^\infty(\overline{\Omega})$.

Since $u_n \leq u_{n+1}$ on $\partial\Omega$ and

$$\det D^2 u_{n+1} = [b(x) + 1/(n+1)] f(u_{n+1}) \leq (b(x) + 1/n) f(u_{n+1}) \quad \text{in } \Omega,$$

by Proposition 2.4 it follows that $u_n \leq u_{n+1}$ in Ω.

We next show that the sequence $(u_n)_{n\geq 1}$ is uniformly bounded from above on every compact set D included in Ω. We distinguish two cases:

Case 1 $b > 0$ on $\partial\Omega$. Then $b_0 := \min_{\overline{\Omega}} b$ is positive. From (A_2) it follows that the boundary blow-up problem

$$\begin{cases} \Delta u = N b_0^{1/N} f_0^{1/N}(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

admits a minimal positive $C^2(\Omega)$-solution, say $u_\star$. Since (A_3) holds, by Theorem 3.1 in [14] we infer that $u_\star$ is convex.

Recall now the arithmetic–geometric inequality for C^2-convex functions v in Ω:

$$\det D^2 v \leq \left(\frac{\Delta v}{N}\right)^N \quad \text{in } \Omega. \tag{3.3}$$

Applying (3.3) for $u_\star$ and using (A_1), we deduce

$$\det D^2 u_\star \leq \left(\frac{\Delta u_\star}{N}\right)^N = b_0 f_0(u_\star) \leq (b(x) + 1/n) f(u_\star) \quad \text{in } \Omega. \tag{3.4}$$

By (3.1) and (3.2), we have $n = u_n < u_\star = \infty$ on $\partial\Omega$. Thus, using (3.4) and Proposition 2.4, we deduce that $u_n \leq u_\star$ in Ω, for every $n \geq 1$.

Case 2 $b \geq 0$ on $\partial\Omega$. Let D be an arbitrary compact set included in Ω and let $\tau > 0$ be small such that $D \subset \Omega^\tau$, where $\Omega^\tau = \{x \in \Omega : \ d(x) > \tau\}$. Set $b_\tau := \min\{b(x) : \ x \in \overline{\Omega^\tau}\}$. Let $v_\star$ denote the minimal positive solution of (3.2) where Ω and b_0 are replaced by Ω^τ and b_τ, respectively. From Case I above we obtain that $v_\star$ is convex in Ω^τ and $u_n \leq v_\star$ in Ω^τ, for every $n \geq 1$.

Consequently, in both cases we have proved that the pointwise limit $U(x) := \lim_{n\to\infty} u_n(x)$ exists, for every $x \in \Omega$. Using now an argument as in [23] (proof of Theorem 2.1) or [29] (proof of Theorem 2.4), we deduce that $U \in C^\infty(\Omega)$ and $\det D^2 U = b(x) f(U)$ in Ω. This finishes the proof of Theorem 1.1. □

4 Regular variation theory

We give a brief account of the definitions and properties of regularly varying functions involved in this article (see [4,33] or [35]).

Definition 4.1 A positive measurable function R defined on $[A, \infty)$, for some $A > 0$, is called *regularly varying* (*at infinity*) *with index* $q \in \mathbb{R}$, written $R \in RV_q$, provided that

$$\lim_{u\to\infty} \frac{R(\lambda u)}{R(u)} = \lambda^q, \quad \text{for all } \lambda > 0. \tag{4.1}$$

When the index q is zero, we say that the function is *slowly varying*.

We make the convention not to mention “at infinity” from now on.

Note that if $R \in RV_q$, then $L(u) := R(u)/u^q$ is a slowly varying function.

Example 4.1 The following functions are slowly varying:

(1) Any measurable function on $[A, \infty)$ which has a positive limit at infinity.
(2) The logarithm $\log u$, its iterates $\log_m u$ and powers of $\log_m u$.
(3) $\exp\{(\log u)^\alpha\}$ with $\alpha \in (0, 1)$.

Proposition 4.2 (Uniform Convergence Theorem) *If L is slowly varying then $L(\lambda u)/L(u) \to 1$ as $u \to \infty$ holds uniformly on each compact λ-set in* $(0, \infty)$.

Proposition 4.3 (Elementary properties of slowly varying functions) *Assume that L is slowly varying. The following hold*:

(i) $\log L(u)/\log u \to 0$ *as* $u \to \infty$*;*
(ii) *For any* $\alpha > 0$, $u^\alpha L(u) \to \infty$, $u^{-\alpha} L(u) \to 0$ *as* $u \to \infty$*;*
(iii) $(L(u))^\alpha$ *varies slowly for every* $\alpha \in \mathbb{R}$*;*
(iv) *If* L_1 *varies slowly, so do* $L(u)L_1(u)$ *and* $L(u) + L_1(u)$.

Remark 4.4 Assume that $R \in RV_q$. If $q > 0$ (resp., $q < 0$), then $\lim_{u\to\infty} R(u) = \infty$ (resp., 0). However, if $q = 0$ then the behavior of R at infinity cannot be completely described. For instance, $L(u) = \exp\left\{(\log u)^{1/3} \cos((\log u)^{1/3})\right\}$ is slowly varying with

$$\liminf_{u\to\infty} L(u) = 0, \ \limsup_{u\to\infty} L(u) = \infty.$$

Proposition 4.5 (Karamata's Theorem; direct half) *Let* $R \in RV_q$ *be locally bounded in* $[A, \infty)$. *Then*

(i) *for any* $j \geq -(q+1)$,

$$\lim_{u\to\infty} \frac{u^{j+1}R(u)}{\int_A^u x^j R(x)\,dx} = j+q+1. \tag{4.2}$$

(ii) *for any* $j < -(q+1)$ *(and for* $j = -(q+1)$ *if* $\int^\infty x^{-(q+1)}R(x)\,dx < \infty$*)*

$$\lim_{u\to\infty} \frac{u^{j+1}R(u)}{\int_u^\infty x^j R(x)\,dx} = -(j+q+1). \tag{4.3}$$

Proposition 4.6 (Karamata's Theorem; converse half) *Let* R *be positive and locally integrable in* $[A,\infty)$.

(i) *If* (4.2) *holds for some* $j > -(q+1)$, *then* $R \in RV_q$.
(ii) *If* (4.3) *is satisfied for some* $j < -(q+1)$, *then* $R \in RV_q$.

Proposition 4.7 (Representation Theorem) *A function* $L(u)$ *is slowly varying if and only if it can be written in the form*

$$L(u) = M(u)\exp\left\{\int_B^u \frac{\varepsilon(t)}{t}\,dt\right\} \quad (u \geq B) \tag{4.4}$$

for some $B > 0$, *where* $\varepsilon \in C[B,\infty)$ *satisfies* $\lim_{u\to\infty}\varepsilon(u) = 0$ *and* $M(u)$ *is measurable on* $[B,\infty)$ *such that* $\lim_{u\to\infty} M(u) := \widehat{M} \in (0,\infty)$.

By (4.4), we see that $L(u) \sim \widehat{L}(u)$ as $u \to \infty$, where

$$\widehat{L}(u) = \widehat{M}\exp\left\{\int_B^u \frac{\varepsilon(t)}{t}\,dt\right\} \quad (u \geq B). \tag{4.5}$$

Of course, $\widehat{L}(u)$ is a slowly varying function, whose benefit is a C^1-regularity such that $\varepsilon(u) = u\widehat{L}'(u)/\widehat{L}(u)$, for each $u \geq B$.

A function $\widehat{L}(u)$ of the form (4.5) will be called a *normalised* slowly varying function. Moreover, any function $\widehat{L} \in C^1[B,\infty)$ which is positive and satisfies

$$\lim_{u\to\infty} u\widehat{L}'(u)/\widehat{L}(u) = 0 \tag{4.6}$$

is a normalised slowly varying function.

In general, if $\widehat{R}(u)/u^q$ ($q \in \mathbb{R}$) is a normalised slowly varying function, then we call $\widehat{R}(u)$ a *normalised regularly varying function of index* q and denote $\widehat{R} \in NRV_q$.

Notice that $NRV_q \subset RV_q$, since the function $f(u) = u^q + \sin(u^{q+1})$ (defined for large u) is an example that belongs to RV_q but not to NRV_q.

A function $\widehat{R} \in RV_q$ belongs to NRV_q if and only if

$$\widehat{R} \in C^1[B,\infty),\ \text{for some } B > 0, \quad \text{and} \quad \lim_{u\to\infty} u\widehat{R}'(u)/\widehat{R}(u) = q.$$

Remark 4.8 For any $R \in RV_q$, there exists $\widehat{R} \in NRV_q$ such that $\widehat{R}(u)/R(u) \to 1$ as $u \to \infty$. Indeed, let $L(u) := R(u)/u^q$ and use Proposition 4.7 to find $\widehat{L}(u)$ as above. Set $\widehat{R}(u) = u^q\widehat{L}(u)$. Then, we have

$$\widehat{R} \in C^1, \quad \lim_{u\to\infty}\frac{\widehat{R}(u)}{R(u)} = 1, \quad \lim_{u\to\infty}\frac{u\widehat{R}'(u)}{\widehat{R}(u)} = q + \lim_{u\to\infty}\frac{u\widehat{L}'(u)}{\widehat{L}(u)} = q.$$

Proposition 4.9 (Proposition 0.8 in [33]) *We have*

(i) *If* $R \in RV_q$, *then* $\lim_{u\to\infty} \log R(u)/\log u = q$.
(ii) *If* $R_1 \in RV_{q_1}$ *and* $R_2 \in RV_{q_2}$ *with* $\lim_{u\to\infty} R_2(u) = \infty$, *then*

$$R_1 \circ R_2 \in RV_{q_1 q_2}.$$

(iii) *Suppose* R *is non-decreasing and* $R \in RV_q$, $0 < q < \infty$. *Then*

$$R^{\leftarrow} \in RV_{q^{-1}}.$$

(iv) *Suppose* R_1, R_2 *are non-decreasing and* q*-varying with* $q \in (0, \infty)$. *Then, for* $c \in (0, \infty)$, *we have*

$$\lim_{u\to\infty} \frac{R_1(u)}{R_2(u)} = c \quad \text{if and only if} \quad \lim_{u\to\infty} \frac{R_1^{\leftarrow}(u)}{R_2^{\leftarrow}(u)} = c^{-1/q}.$$

The next result shows that any function R varying regularly with non-zero index is asymptotic to a monotone function.

Proposition 4.10 (see Theorem 1.5.3 in [4]) *Let* $R \in RV_q$ *and choose* $B \geq 0$ *so that* R *is locally bounded on* $[B, \infty)$. *If* $q > 0$, *then*

(a) $\overline{R}(u) := \sup\{R(y) : B \leq y \leq u\} \sim R(u)$ *as* $u \to \infty$,
(b) $\underline{R}(u) := \inf\{R(y) : y \geq u\} \sim R(u)$ *as* $u \to \infty$.

If $q < 0$, *then*

(c) $\sup\{R(y) : y \geq u\} \sim R(u)$ *as* $u \to \infty$,
(d) $\inf\{R(y) : B \leq y \leq u\} \sim R(u)$ *as* $u \to \infty$.

5 Asymptotic properties of ϕ

The aim of this section is to give an insight into the asymptotic properties of $\phi(t)$ (in (1.10)) at the origin. An important role in this pursuit is played by Karamata's theory of regular variation and its extensions.

Lemma 5.1 *Let* $k \in \mathcal{K}_\ell$ *and* $f \in RV_q$ *with* $q > N$. *If* ϕ *is defined by* (1.10), *then there exists a function* $\psi \in C^2(0, \tau)$ *with* $\tau > 0$ *which satisfies* $\lim_{t\to 0} \psi(t)/\phi(t) = 1$ *and the following*:

(i) $\lim_{t\to 0} \dfrac{\psi(t)\psi''(t)}{[\psi'(t)]^2} = 1 + \dfrac{(q-N)\ell}{N+1}$,

(ii) $\lim_{t\to 0} \dfrac{[-\psi'(t)]^{N-1}\psi''(t)}{k^{N+1}(t) f(\psi(t))} = \left(\dfrac{N+1}{q-N}\right)^{N+1} \left[1 + \dfrac{(q-N)\ell}{N+1}\right]$,

where ℓ *appears in* (1.7).

Proof (i) Denote $g(u) = f(u)/u^N$. Since $g \in RV_{q-N}$ and $q > N$, by Proposition 4.10 we have $\lim_{u\to\infty} g(u)/\mathcal{P}(u) = 1$. By Remark 4.8 we infer that there exists a function $\widehat{g} \in C^2(0, \tau)$ such that $\lim_{u\to\infty} \widehat{g}(u)/g(u) = 1$ and

$$\lim_{u\to\infty} \frac{u\widehat{g}'(u)}{\widehat{g}(u)} = q - N, \quad \lim_{u\to\infty} \frac{u\widehat{g}''(u)}{\widehat{g}'(u)} = q - N - 1. \tag{5.1}$$

We define ψ as follows

$$\widehat{g}(\psi(t)) = [K(t)]^{-N-1} \quad \text{for } t > 0 \text{ small}. \tag{5.2}$$

From (1.10) and Proposition 4.9, we see that $\lim_{t\to 0} \psi(t)/\phi(t) = 1$.

By differentiating (5.2) we obtain

$$\widehat{g}'(\psi(t))\psi'(t) = -(N+1)[K(t)]^{-N-2}k(t), \quad \text{for } t > 0 \text{ small}. \tag{5.3}$$

This, jointly with (5.1) and (5.2), shows that

$$\frac{\psi'(t)}{\psi(t)} \sim \frac{-(N+1)}{q-N}\frac{k(t)}{K(t)} \quad \text{as } t \to 0. \tag{5.4}$$

We differentiate (5.3), then use (1.7) and (5.1) to deduce that as $t \to 0$

$$\begin{aligned} &\widehat{g}'(\psi(t))\frac{[\psi'(t)]^2}{\psi(t)}\left(q - N - 1 + \frac{\psi(t)\psi''(t)}{[\psi'(t)]^2}\right) \\ &\sim (N+1)(N+1+\ell)k^2(t)[K(t)]^{-N-3}. \end{aligned} \tag{5.5}$$

The assertion of (i) follows now from (5.3)–(5.5).

(ii) From (5.2) and (5.4), we find

$$\lim_{t\to 0}\left[-\frac{\psi'(t)}{\psi(t)}\right]^{N+1}\frac{1}{k^{N+1}(t)\widehat{g}(\psi(t))} = \left(\frac{N+1}{q-N}\right)^{N+1}.$$

This, combined with (i), proves the claim of (ii). □

The next result has been proved in [13] (see [9] for a different proof).

Proposition 5.2 *The following hold*:

(i) $k \in \mathcal{K}_\ell$ *with* $\ell \neq 0$ *if and only if* k *is non-decreasing on some interval* $(0, \nu)$ *with* $\nu > 0$ *and* $u \longmapsto k(1/u)$ *belongs to* $NRV_{1-1/\ell}$.

(ii) $k \in \mathcal{K}_\ell$ *with* $\ell = 0$ *if and only if* K *is of the form*

$$K(t) = d_0 \exp\left(-\int_t^{d_1} \frac{ds}{\zeta_0(s)}\right), \quad 0 < t < d_1, \tag{5.6}$$

for some positive constants d_0, d_1 *and a positive function* ζ_0 *in* $C^1(0, d_1)$ *such that* $\lim_{t\to 0^+} \zeta_0'(t) = 0$.

To describe the variation of ϕ at zero, we need some concepts that are naturally extending regular variation theory. For the reader's convenience, we recall below some definitions and results to be found elsewhere.

Definition 5.1 A positive measurable function R defined on a neighborhood of ∞ is called rapidly varying at infinity of index ∞ (notation $R \in RV_\infty$) if

$$\lim_{u\to\infty} R(\lambda u)/R(u) = \begin{cases} 0 & \text{if } \lambda \in (0,1), \\ 1 & \text{if } \lambda = 1, \\ \infty & \text{if } \lambda = \infty, \end{cases} \tag{5.7}$$

and is called rapidly varying at infinity of index $-\infty$ (notation $R \in RV_{-\infty}$) if

$$\lim_{u\to\infty} R(\lambda u)/R(u) = \begin{cases} \infty & \text{if } \lambda \in (0,1), \\ 1 & \text{if } \lambda = 1, \\ 0 & \text{if } \lambda = \infty. \end{cases} \tag{5.8}$$

Example 5.3 The function $g(u) = e^u$ is rapidly varying at ∞ of index ∞, while $g(u) = e^{-u}$ is rapidly varying at ∞ of index $-\infty$.

An important subclass of functions rapidly varying at infinity is represented by that of Γ-varying functions introduced by de Haan ([15]) (see also [33,4]).

Definition 5.2 ([33]) A non-decreasing function U defined on an interval (A, ∞) is Γ-varying at ∞ if $\lim_{x\to\infty} U(x) = \infty$ and there exists a positive function χ defined on (A, ∞) such that

$$\lim_{x\to\infty} \frac{U(x+\lambda\chi(x))}{U(x)} = e^{\lambda}, \quad \forall \lambda \in \mathbb{R}. \tag{5.9}$$

The function χ is called an *auxiliary function* and is unique up to asymptotic equivalence. If (5.9) is satisfied for χ_1 and χ_2 then $\chi_1(x) \sim \chi_2(x)$ as $x \to \infty$. Conversely, if (5.9) is fulfilled for χ and $\chi_1(x) \sim \chi(x)$ as $x \to \infty$, then (5.9) also holds with χ_1.

Example 5.4 ([15]) The following functions U satisfy (5.9) with the specified auxiliary functions χ:

(1) $U(x) = \exp(x^p)$ for $p > 0$ with $\chi(x) = \begin{cases} 1 & \text{for } x \le 0, \\ p^{-1}x^{1-p} & \text{for } x > 0. \end{cases}$

(2) $U(x) = \exp(x \log_+ x)$ with $\chi(x) = \begin{cases} 1 & \text{for } x \le 1, \\ (\log x)^{-1} & \text{for } x > 1. \end{cases}$

(3) $U(x) = \exp(e^x)$ with $\chi(x) = e^{-x}$.

More examples of Γ-varying functions can be constructed using the next result.

Proposition 5.5 (Theorem 1.5.6 in [15]) *If U_1 is monotone and regularly varying of index $\rho > 0$ and $U_2 \in \Gamma$ with auxiliary function χ, then U defined by*

$$U(x) = U_1(U_2(x)) \quad \textit{for large } x > 0$$

belongs to Γ with auxiliary function $(1/\rho)\chi$.

Remark 5.6 If U belongs to Γ, then U is rapidly varying at infinity of index ∞ (see Proposition 3.10.3 in [4]).

We are now ready to analyze the variation of $\phi(1/u)$ at $u = \infty$, where ϕ is defined by (1.10). Assuming that $f \in RV_q$ with $q > N$, we will see that $\phi(1/u)$ is Γ-varying at $u = \infty$ if $k \in \mathcal{K}_0$, in contrast to the case $k \in \mathcal{K}_\ell$ with $\ell \neq 0$ when $\phi(1/u)$ is regularly varying at $u = \infty$ of index $(N+1)/[\ell(q-N)]$.

Proposition 5.7 (Variation speed of ϕ) *Assume that $f \in RV_q$ with $q > N$ and $k \in \mathcal{K}_\ell$. The following hold*:

(i) *If $\ell \neq 0$, then $u \longmapsto \phi(1/u) \in RV_{(N+1)/[\ell(q-N)]}$*;

(ii) *If* $\ell = 0$, *then* $u \longmapsto \phi(1/u) \notin RV_m$, *for every* $m \in \mathbb{R}$. *In this case,* $\phi(1/u)$ *is* Γ*-varying at* $u = \infty$ *with the auxiliary function*

$$\frac{(q-N)u^2 K(1/u)}{(N+1)k(1/u)}.$$

(iii) *Let* $\widehat{f}(u) \sim f(u)$ *as* $u \to \infty$ *be such that* $J(u) := \widehat{f}(u)/u^N$ *is non-decreasing for large* $u > 0$. *If* $r(t) \sim CK^{-N-1}(t)$ *as* $t \to 0$, *for some constant* $C > 0$, *then* $\widehat{\phi}(t) \sim C^{1/(q-N)}\phi(t)$ *as* $t \to 0$, *where* $\widehat{\phi}(t)$ *is defined by*

$$\widehat{\phi}(t) = J^{\leftarrow}(r(t)) \quad \text{for small } t > 0. \tag{5.10}$$

Proof By Propositions 4.9 and 4.10, we have

$$\mathcal{P}(u) \sim f(u)/u^N \text{ as } u \to \infty \quad \text{and} \quad \mathcal{P}^{\leftarrow} \in RV_{1/(q-N)}. \tag{5.11}$$

(i) If $\ell \neq 0$, then $u \longmapsto k(1/u) \in NRV_{(\ell-1)/\ell}$ (cf. Proposition 5.2 (i)). Hence, we have $u \longmapsto K(1/u) \in RV_{-1/\ell}$. Since $\phi(1/u) = \mathcal{P}^{\leftarrow}([K(1/u)]^{-N-1})$, the assertion of (i) follows now from (5.11) and Proposition 4.9 (ii).

(ii) If $\ell = 0$, then by Proposition 5.2 and [33, p. 106] we obtain $[K(1/u)]^{-N-1}$ is Γ-varying at $u = \infty$ with the auxiliary function $\zeta(u)$ given by

$$\zeta(u) = \frac{u^2 K(1/u)}{(N+1)k(1/u)}.$$

In particular, $u \longmapsto k(1/u)$ is rapidly varying at ∞ with index $-\infty$. It follows that $\phi(1/u) \notin RV_m$, for every $m \in \mathbb{R}$. By (5.11) and Proposition 5.5 we conclude that $\phi(1/u)$ is Γ-varying at $u = \infty$ with the auxiliary function $(q-N)\zeta(u)$.

(iii) From (5.11) and Proposition 4.9, we have $J^{\leftarrow} \in RV_{1/(q-N)}$ and $J^{\leftarrow}(u) \sim \mathcal{P}^{\leftarrow}(u)$ as $u \to \infty$. Using $r(t) \sim CK^{-N-1}(t)$ as $t \to 0$ and Proposition 4.2, we conclude the proof of Proposition 5.7. □

Remark 5.8 The function $r(t)$ defined, for small $t > 0$, as follows

$$r(t) = \begin{cases} [\ell t k(t)]^{-N-1} & \text{if } k \in \mathcal{K}_\ell \text{ with } \ell \neq 0, \\ [k^2(t)/k'(t)]^{-N-1} & \text{if } k \in \mathcal{K}_0, \end{cases}$$

possesses the property that $r(t) \sim [K(t)]^{-N-1}$ as $t \to 0$.

Keeping in mind that the asymptotic behaviour of ϕ is of interest, we can simplify the calculation by using Remark 5.8 and Proposition 5.7 (iii).

Example 5.9 Let $f(u) \sim u^q$ as $u \to \infty$, for some $q > N$. If $p > 0$, then

(1) $k(t) = (-1/\ln t)^p \in \mathcal{K}_1$ and $\phi(1/u) \sim (u \ln u)^{(N+1)/(q-N)}$ as $u \to \infty$.
(2) $k(t) = t^p \in \mathcal{K}_{1/(p+1)}$ and $\phi(1/u) \sim [(p+1)u^{p+1}]^{(N+1)/(q-N)}$ as $u \to \infty$.
(3) $k(t) = e^{-1/t^p} \in \mathcal{K}_0$ and $\phi(1/u) \sim [pu^{p+1}e^{u^p}]^{(N+1)/(q-N)}$ as $u \to \infty$.

Clearly, (1) and (2) in the above example illustrate Proposition 5.7 (i), whereas (3) agrees with the findings of Proposition 5.7 (ii). Indeed, by Example 5.4 (1) and Proposition 5.5, we remark that

$$\left[pu^{p+1}e^{u^p}\right]^{(N+1)/(q-N)}$$

is Γ-varying at $u = \infty$ with the auxiliary function

$$\frac{(q-N)u^{1-p}}{p(N+1)}.$$

On the other hand,

$$\frac{(q-N)u^2K(1/u)}{(N+1)k(1/u)} \sim \frac{(q-N)u^{1-p}}{p(N+1)} \quad \text{as } u \to \infty,$$

since, by Remark 5.8, we have

$$\frac{K(t)}{k(t)} \sim \frac{k(t)}{k'(t)} = \frac{t^{p+1}}{p} \quad \text{as } t \to 0.$$

6 Proof of Theorem 1.2

Fix $\epsilon \in (0, 1/2)$. We choose $\delta > 0$ small enough such that

(a) k is non-decreasing on $(0, 2\delta)$.
(b) $\beta^-(1-\epsilon)k^{N+1}(d(x)) \le b(x) \le \beta^+(1+\epsilon)k^{N+1}(d(x))$, for every $x \in \Omega_{2\delta}$, where for $\lambda > 0$ we set

$$\Omega_\lambda = \{x \in \Omega : \ d(x) < \lambda\}.$$

(c) $d(x)$ is a C^2-function on $\Gamma_{2\delta} = \{x \in \overline{\Omega} : \ d(x) < 2\delta\}$.
(d) $\psi' < 0$ on $(0, 2\delta)$ and $\psi, \psi'' > 0$ on $(0, 2\delta)$, where ψ is as in Lemma 5.1.
(e) $\Pi_{i=1}^{N-1}(1 - \rho_i(y)d(x)) > 1 - \epsilon$, for every $x \in \Omega_{2\delta}$. Recall that $\rho_i(y)$ (with $i \in \{1, \ldots, N-1\}$) denote the principal curvatures of $\partial\Omega$ at y, where $y \in \partial\Omega$ is such that $|x - y| = d(x)$.

Fix $\tau \in (0, \delta)$. With $\xi^\pm$ given by (1.11), we set

$$\eta^\pm = [(1 \mp \epsilon)(1 \mp 2\epsilon)]^{1/(N-q)}\,\xi^\pm. \tag{6.1}$$

Let us now define

$$\begin{cases} v_\tau^+(x) = \eta^+\psi(d(x) - \tau), & \forall x \in \Omega_{2\delta} \setminus \overline{\Omega}_\tau, \\ v_\tau^-(x) = \eta^-\psi(d(x) + \tau), & \forall x \in \Omega_{2\delta - \tau}. \end{cases}$$

Step 1 We prove that, near the boundary, v_τ^+ (resp., v_τ^-) is an upper (resp., lower) solution of (1.2), that is

$$\begin{cases} \det D^2 v_\tau^+(x) \le b(x) f(v_\tau^+(x)), & \forall x \in \Omega_{2\delta} \setminus \overline{\Omega}_\tau, \\ \det D^2 v_\tau^-(x) \ge b(x) f(v_\tau^-(x)), & \forall x \in \Omega_{2\delta - \tau}. \end{cases} \tag{6.2}$$

By (a) and (b), it suffices to show that

$$\begin{cases} \det D^2 v_\tau^+(x) \le \beta^-(1-\epsilon)k^{N+1}(d(x) - \tau) f(v_\tau^+(x)), & \forall x \in \Omega_{2\delta} \setminus \overline{\Omega}_\tau, \\ \det D^2 v_\tau^-(x) \ge \beta^+(1+\epsilon)k^{N+1}(d(x) + \tau) f(v_\tau^-(x)), & \forall x \in \Omega_{2\delta - \tau}. \end{cases} \tag{6.3}$$

We denote by

$$m^+ = \max_{y \in \partial\Omega} \sigma_{N-1}(y) \quad \text{and} \quad m^- = \min_{y \in \partial\Omega} \sigma_{N-1}(y).$$

Using Corollary 2.3 and (e), we obtain

$$\det D^2 v_\tau^+(x) = (\eta^+)^N [-\psi'(d(x)-\tau)]^{N-1} \psi''(d(x)-\tau) \frac{\sigma_{N-1}(y)}{\Pi_{i=1}^{N-1}[1-\rho_i(y)(d(x)-\sigma)]}$$
$$\leq \frac{(\eta^+)^N}{1-\epsilon} m^+ [-\psi'(d(x)-\tau)]^{N-1}\psi''(d(x)-\tau), \quad \forall x \in \Omega_{2\delta} \setminus \overline{\Omega}_\tau.$$

Similarly, we have

$$\det D^2 v_\tau^-(x) = (\eta^-)^N [-\psi'(d(x)+\tau)]^{N-1} \psi''(d(x)+\tau) \frac{\sigma_{N-1}(y)}{\Pi_{i=1}^{N-1}[1-\rho_i(y)(d(x)+\sigma)]}$$
$$\geq \frac{(\eta^-)^N}{1+\epsilon} m^- [-\psi'(d(x)+\tau)]^{N-1}\psi''(d(x)+\tau), \quad \forall x \in \Omega_{2\delta-\tau}.$$

Therefore, to deduce (6.3) it is enough to establish

$$\lim_{t\to 0} (\eta^\pm)^N \frac{m^\pm}{\beta^\mp} \frac{[-\psi'(t)]^{N-1}\psi''(t)}{k^{N+1}(t) f(\eta^\pm \psi(t))} = (1 \mp \epsilon)(1 \mp 2\epsilon). \tag{6.4}$$

Since $f \in RV_q$, (6.4) is valid thanks to Lemma 5.1 and our choice of $\eta^\pm$ in (6.1).

Step 2 *Every strictly convex blow-up solution u_∞ of* (1.2) *satisfies* (1.9).

Let $C = \max_{d(x)=\delta} u_\infty(x)$. Notice that

$$\begin{cases} v_\tau^+(x) + C = \infty > u_\infty(x), & \forall x \in \Omega \text{ with } d(x) = \tau, \\ v_\tau^+(x) + C \geq u_\infty(x), & \forall x \in \Omega \text{ with } d(x) = \delta. \end{cases}$$

Using (6.2) we deduce that, for every $x \in \Omega_\delta \setminus \overline{\Omega}_\tau$,

$$\det D^2(v_\tau^+(x) + C) = \det D^2(v_\tau^+(x)) \leq b(x) f(v_\tau^+(x)) \leq b(x) f(v_\tau^+(x) + C).$$

Since u_∞ is a solution to (1.2), by Proposition 2.4 we find

$$v_\tau^+(x) + C \geq u_\infty(x), \quad \forall x \in \Omega_\delta \setminus \overline{\Omega}_\tau. \tag{6.5}$$

We set $C' = \xi^- \psi(\delta)$. Hence, we have $C' \geq v_\tau^-(x)$ for every $x \in \Omega$ with $d(x) = \delta - \tau$. It follows that

$$u_\infty(x) + C' \geq v_\tau^-(x), \quad \forall x \in \partial\Omega_{\delta-\tau}.$$

We see that, for every $x \in \Omega_{\delta-\tau}$,

$$\det D^2(u_\infty(x) + C') = \det D^2(u_\infty(x)) = b(x) f(u_\infty(x)) \leq b(x) f(u_\infty(x) + C'),$$

while by (6.2) we have

$$\det D^2 v_\tau^-(x) \geq b(x) f(v_\tau^-(x)), \quad \forall x \in \Omega_{\delta-\tau}.$$

Using again Proposition 2.4, we infer that

$$u_\infty(x) + C' \geq v_\tau^-(x), \quad \forall x \in \Omega_{\delta-\tau}. \tag{6.6}$$

By (6.5) and (6.6), letting $\tau \to 0$ we obtain

$$\begin{cases} [(1+\epsilon)(1+2\epsilon)]^{1/(N-q)} \xi^- \psi(d(x)) - C' \leq u_\infty(x), & \forall x \in \Omega_\delta, \\ u_\infty(x) \leq [(1-\epsilon)(1-2\epsilon)]^{1/(N-q)} \xi^+ \psi(d(x)) + C, & \forall x \in \Omega_\delta. \end{cases} \tag{6.7}$$

Dividing by $\psi(d(x))$ and letting $d(x) \to 0$, we obtain

$$\begin{cases} \liminf\limits_{d(x)\to 0} \dfrac{u_\infty(x)}{\psi(d(x))} \geq [(1+\epsilon)(1+2\epsilon)]^{1/(N-q)}\xi^-, \\ \limsup\limits_{d(x)\to 0} \dfrac{u_\infty(x)}{\psi(d(x))} \leq [(1-\epsilon)(1-2\epsilon)]^{1/(N-q)}\xi^+. \end{cases} \tag{6.8}$$

Since $\epsilon > 0$ is arbitrary, we let $\epsilon \to 0$ and conclude (1.9). This completes the proof of Theorem 1.2. □

7 Proof of Theorem 1.6

We divide the proof into two steps:

Step 1 *For every strictly convex blow-up solutions u_1, u_2 of* (1.2)*, it holds*

$$\lim_{d(x)\to 0} u_1(x)/u_2(x) = 1.$$

Our argument is different depending on whether (i) or (ii) is satisfied:

Case (i) $b > 0$ on $\overline{\Omega}$.

Since u_1 and u_2 are arbitrary, it suffices to show that

$$\liminf_{d(x)\to 0} u_1(x)/u_2(x) \geq 1. \tag{7.1}$$

Without loss of generality, we can assume that 0 belongs to Ω.

Let $\epsilon \in (0, 1)$ be fixed and let $\lambda > 1$ be close to 1.

For a subset ω of $\mathbb{R}^N$, we denote by

$$(1/\lambda)\omega = \{(1/\lambda)x : x \in \omega\}.$$

We set

$$C_\lambda = \left[(1+\epsilon)\lambda^{2N} \max_{x\in(1/\lambda)\overline{\Omega}} \left(\frac{b(\lambda x)}{b(x)}\right)\right]^{1/(q-N)}. \tag{7.2}$$

Notice that $C_\lambda \to (1+\epsilon)^{1/(q-N)}$ as $\lambda \to 1$. Hence, by Proposition 4.2 and $\lim_{d(x)\to 0} u_1(x) = \infty$, we deduce that there exists $\delta = \delta(\epsilon) > 0$, which is independent of λ, such that

$$C_\lambda^q \frac{f(u_1(x))}{f(C_\lambda u_1(x))} \leq 1+\epsilon, \quad \forall x \in \Omega_\delta \quad \text{and} \quad \lambda > 1 \text{ close to } 1. \tag{7.3}$$

We now define U_λ as follows

$$U_\lambda(x) = C_\lambda u_1(\lambda x), \quad \forall x \in (1/\lambda)\Omega_\delta. \tag{7.4}$$

We assert that U_λ satisfies

$$\det D^2 U_\lambda(x) \leq b(x) f(U_\lambda(x)), \quad \forall x \in (1/\lambda)\Omega_\delta. \tag{7.5}$$

Indeed, by (7.2)–(7.4) we infer that, for every $x \in (1/\lambda)\Omega_\delta$,

$$\begin{aligned} \det D^2 U_\lambda(x) &= \lambda^{2N} C_\lambda^N b(\lambda x) f(u_1(\lambda x)) \\ &\leq \lambda^{2N} C_\lambda^{N-q}(1+\epsilon) b(\lambda x) f(C_\lambda u_1(\lambda x)) \\ &\leq b(x) f(C_\lambda u_1(\lambda x)) = b(x) f(U_\lambda(x)). \end{aligned}$$

Since f is increasing on $(0, \infty)$, it follows that (7.5) holds when $U_\lambda(x)$ is replaced by $U_\lambda(x) + M$, for every constant $M > 0$. Notice also that $U_\lambda(x) = \infty > u_2(x)$, for every $x \in (1/\lambda)\partial\Omega$. Moreover, $x \in (1/\lambda)\partial\Omega$ implies that $d(x) < \delta$ (as $\lambda > 1$ is close to 1). Thus, if we choose $M > 0$ large enough (e.g., $M = \max_{d(x)=\delta} u_2(x)$), then by Proposition 2.4 we obtain

$$U_\lambda(x) + M \geq u_2(x), \quad \forall x \in \Omega_\delta \cap (1/\lambda)\Omega_\delta. \tag{7.6}$$

Letting $\lambda \to 1$ in (7.6), we find

$$(1+\epsilon)^{1/(q-N)} u_1(x) + M \geq u_2(x), \quad \forall x \in \Omega_\delta.$$

This implies that

$$\liminf_{d(x)\to 0} \frac{u_1(x)}{u_2(x)} \geq (1+\epsilon)^{1/(N-q)}.$$

Since $\epsilon > 0$ is arbitrary, letting $\epsilon \to 0$ we conclude (7.1).

Case (ii) $b \equiv 0$ on $\partial\Omega$, Ω is a ball of radius $R > 0$ and (1.12) holds.

By Corollary 1.4, every strictly convex blow-up solution u_∞ of (1.2) satisfies

$$\lim_{d(x)\to 0} \frac{u_\infty(x)}{\phi(d(x))} = \left\{ \frac{[(q-N)/(N+1)]^{N+1} R^{N-1}}{[1 + \ell(q-N)/(N+1)]} \right\}^{1/(N-q)}, \tag{7.7}$$

where ϕ is defined by (1.10) and ℓ appears in (1.7).

Hence, the assertion of Step 1 is proved in both situations (i) and (ii).

Step 2 *There is at most one strictly convex blow-up solution of* (1.2).

If u_1, u_2 are arbitrary strictly convex blow-up solutions of (1.2), it suffices to show that $u_1 \leq u_2$ in Ω.

Fix $\epsilon > 0$. By Step 1 we infer that

$$\lim_{d(x)\to 0} [u_1(x) - (1+\epsilon)u_2(x)] = -\infty. \tag{7.8}$$

Since $f(u)/u^N$ is increasing on $(0, \infty)$, we deduce that

$$\begin{aligned} \det D^2((1+\epsilon)u_2(x)) &= (1+\epsilon)^N \det D^2 u_2(x) \\ &= (1+\epsilon)^N b(x) f(u_2(x)) \\ &\leq b(x) f((1+\epsilon)u_2(x)), \quad \forall x \in \Omega. \end{aligned} \tag{7.9}$$

By (7.8), (7.9) and Proposition 2.4, we find $u_1 \leq (1+\epsilon)u_2$ in Ω. Letting $\epsilon \to 0$ we obtain $u_1 \leq u_2$ in Ω. This completes the proof of Theorem 1.6. □

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