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Calculus of Variations

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Regularity of the inverse of spatial mappings with finite distortion

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1 Introduction

In a recent paper [4], Hencl and Koskela came up with a very elegant question. Namely, they studied the regularity of f^{-1} for a homeomorphism f of the Sobolev class $\mathscr{W}^{1,p}_{\text{loc}}(\Omega, \Omega')$ where $p \geq 1$ and Ω and Ω' are domains in the plane. In general, even the inverse of a Lipschitz homeomorphism can fail to belong to $\mathscr{W}^{1,1}_{\text{loc}}$, see [4]. Analyzing the situation more carefully, one may notice a crucial aspect: the differential of the mapping does not vanish in the zero set of the Jacobian. For the non-negative Jacobian determinant almost everywhere[1] there doesn't exist a measurable function $K_\circ : \Omega \to [0, \infty)$, which is finite almost everywhere so that

$$|Df(x)|^n \leq K_\circ(x)\, J(x, f). \tag{1}$$

Avoiding this phenomenon, Hencl and Koskela were able to prove that the inverse lies in $\mathscr{W}^{1,1}_{\text{loc}}$ under the minimal possible regularity assumption of a mapping.

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[1] The assumtion $J(x, f) \geq 0$ a.e is not restrictive. In this paper mappings are differentiable almost everywhere and homeomorphisms; thus, either the Jacobian of the mapping is non-negative or non-positive almost everywhere, therefore, for simplicity, we can assume that $J(x, f) \geq 0$ almost everywhere.

Theorem 1.1 (HenclandKoskela) *Let Ω and Ω' be domains in $\mathbb{R}^2$. Suppose that $f \in \mathscr{W}^{1,1}_{loc}(\Omega, \Omega')$ is a homeomorphism of finite distortion such that f satisfies the distortion inequality (1). Then $f^{-1} \in \mathscr{W}^{1,1}_{loc}(\Omega', \Omega)$ and f^{-1} is a mapping of finite distortion.*

The purpose of this paper is to carry out Theorem 1.1 to higher dimensions. Before arriving at the results, it is necessary to recall a few basic definitions. First, $f : \Omega \to \mathbb{R}^n$ is *a mapping of finite distortion* if the following three conditions are satisfied:

(i) $f \in \mathscr{W}^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$,
(ii) The Jacobian determinant $J(x, f)$ of f is locally integrable,
(iii) There is a measurable function $K_\circ = K_\circ(x)$, finite almost everywhere, such that f satisfies the distortion inequality

$$|Df(x)|^n \leq K_\circ(x)\, J(x, f) \qquad \text{a.e. } x \in \Omega. \tag{2}$$

Above we used the operator norm of the differential matrix, defined by $|Df(x)| = \sup\{|Df(x)h|\ |h| = 1\}$. Geometrically, inequality (2) means that at almost every point $x \in \Omega$ the differential $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$ deforms the unit sphere onto an ellipsoid whose eccentricity is controlled by $K_o(x)$. Thus, in particular, the case $K_o = 1$ results in conformal deformations. Also assumption (ii) becomes natural because the Jacobian of a weakly differentiable homeomorphism is always locally integrable. The smallest function $K_\circ(x)$ for which the distortion inequality (2) holds is called *the outer distortion function* of f, defined by

$$K_\circ(x, f) = \begin{cases} \dfrac{|Df(x)|^n}{J(x, f)}, & \text{if } J(x, f) > 0 \\ 1, & \text{otherwise.} \end{cases} \tag{3}$$

For more about the theory of mappings with finite distortion, please refer to the monograph by Iwaniec and Martin [5].

Theorem 1.2 *Suppose a homeomorphism $f : \Omega \to \Omega'$ belongs to the Sobolev class $\mathscr{W}^{1,p}_{loc}(\Omega, \Omega')$ for some $p > n - 1$ and f has finite distortion. Then f^{-1} is a mapping of finite distortion. Furthermore, f^{-1} is differentiable almost everywhere.*

This result has been known under the natural Sobolev regularity $\mathscr{W}^{1,n}_{\text{loc}}$-assumption of the mapping [12] and [9]. Roughly speaking, this is the minimal regularity assumption that guarantees integration by parts against the Jacobian determinant i.e. one can apply Stokes' theorem. In this sense, it seems very surprising that we can relax the Sobolev regularity assumption here. The proof by Hencl and Koskela is based on the approximation argument. They construct Lipschitz approximations to f^{-1} and prove that the approximations converge to the inverse. In higher dimensions, auxiliary estimates for the inverse are more complicated, and proving that the approximation to f^{-1} converges is no longer as easy as in the case $n = 2$.

To overcome these difficulties we implement a new method in approaching the problem. We believe that these techniques will be very useful in the future studies of mappings with finite distortion.

In developing a theory of mappings with finite distortion, we are usually faced on the following question: How regular does the distortion function have to be to guarantee the desired properties for the mapping or the inverse? Here the natural Sobolev regularity $\mathscr{W}^{1,n}_{\rm loc}$ also plays a special role, making it useful to know under which assumption of the distortion function the differential of the inverse mapping has the Sobolev n-regularity. The estimate (28) in the proof of Theorem 1.2 guarantees that this is the case provided $K_\circ(\cdot, f) \in L^{n-1}(\Omega)$; however, knowing a priori that $f^{-1} \in \mathscr{W}^{1,n}_{\rm loc}(\Omega', \Omega)$, it is possible to prove the identity

$$\int_{\Omega'} |Df^{-1}(y)|^n \, dy = \int_\Omega K_I(x, f)\, dx. \tag{4}$$

This identity has been known under the natural Sobolev regularity $f \in \mathscr{W}^{1,n}_{\rm loc}(\Omega, \Omega')$ [7] and [1]. In [7], it was shown that the n-regularity assumption can be slightly relaxed. It is enough to assume that $|Df|^n \log^{-1}(1 + |Df|)$ is locally integrable, primarily because we can still integrate by parts against the non-negative Jacobian determinant. This is not true when we only assume that $f \in \mathscr{W}^{1,p}_{\rm loc}(\Omega, \Omega')$ for some $p < n$. This identity plays a central role in examining extremal mappings of finite distortion in [1]. In the non-linear elasticity we have information on the differential, and on its cofactor matrix $D^\sharp f$. For example, Šverák, Müller, Qi, and Yan in [9, 11] studied topological properties of the class

$$\mathcal{A}^+_{p,\frac{n}{n-1}}(\Omega) = \left\{ f \in W^{1,p}_{loc}(\Omega, \mathbb{R}^n) : \quad |D^\sharp f| \in L^{\frac{n}{n-1}}_{loc}(\Omega) \text{ and } J(x, f) > 0 \text{ a.e.} \right\}.$$

Thus, we are led to an examination of the *inner distortion function* $K_I(\cdot, f)$, defined by

$$K_I(x, f) = \begin{cases} \dfrac{|D^\sharp f(x)|^n}{J(x, f)^{n-1}}, & \text{if } J(x, f) > 0 \\ 1, & \text{if } |D^\sharp f(x)| = 0 \\ \infty, & \text{if } J(x, f) = 0 \text{ and } |D^\sharp f(x)| \neq 0. \end{cases} \tag{5}$$

Geometrically, the cofactor matrix $D^\sharp f$ made up of cofactors of Df controls the change of surface elements, Df controls the change of line elements, and $J(\cdot, f)$ controls the change of volume. Inequality (2) translates into a similar one for $D^\sharp f$,

$$|D^\sharp f(x)|^n \le K_I(x, f)\, J(x, f)^{n-1} = K_I(x, f) \det D^\sharp f(x) \qquad \text{a.e.} \tag{6}$$

where $1 \le K_I(x, f) \le K_\circ^{n-1}(x, f) < \infty$, see [5, p. 109]. We observe, however, that in dimension $n \ge 3$ this does not imply the outer distortion inequality (2). For example, consider $f(x_1, \ldots, x_n) = (x_1, 0, \ldots, 0)$. Also, notice that in dimension 2 both distortion functions coincide, i.e. $K_I(x, f) = K_\circ(x, f)$ almost everywhere. Thus, the following question is raised; is it true that $|Df^{-1}| \in L^n(\Omega)$ if we only assume that the inner distortion function is integrable? This question is highly motivated by studing extremal mappings of finite distortion in higher dimensions.

For more details, please refer to [1]. Our second theorem gives an affirmative answer to the question.

Theorem 1.3 *Suppose $f \in \mathscr{W}^{1,p}_{loc}(\Omega, \Omega')$, $p > n-1$, is a homeomorphism of finite distortion with*

$$\int_\Omega K_I(x, f)\, dx < \infty. \tag{7}$$

Then the inverse map $f^{-1} : \Omega' \to \Omega$ belongs to $\mathscr{W}^{1,n}_{loc}(\Omega', \Omega)$, has finite distortion and we have

$$\int_{\Omega'} |Df^{-1}(y)|^n\, dy = \int_\Omega K_I(x, f)\, dx. \tag{8}$$

We believe that Theorems 1.2 and 1.3 hold under the regularity assumption $p = n-1$ in the higher dimension cases ($n \geq 3$) as well, but we have not been able to verify these.

2 Regularity of the inverse mapping

Our starting point is to prove the following auxiliary inequality which is a counterpart of the corresponding planar estimate from [4], see also [8]. If $n = 2$, then we can choose $p = 1$ in the inequality. This is the reason why in Theorem 1.1 it is enough to assume the Sobolev regularity $n - 1 = 1$ in this case.

Lemma 2.1 *Suppose a homeomorphism f belongs to the Sobolev class $\mathscr{W}^{1,p}_{loc}$ (Ω, Ω') for some $p > n-1$. Then*

$$\operatorname{diam}(f^{-1}(\mathbb{B}_r)) \leq C_p(n) r^{1-n} |f^{-1}(\mathbb{B}_{2r})|^{\frac{p-n+1}{p}} \left(\int_{f^{-1}(\mathbb{B}_{2r})} |Df|^p \right)^{\frac{n-1}{p}} \tag{9}$$

for all balls $\mathbb{B}_r = \mathbb{B}(y, r)$ such that $\mathbb{B}_{3r} = \mathbb{B}(y, 3r) \subset \Omega'$.

Before giving the proof of this lemma, we recall a well-known fact that a function in the Sobolev class $\mathscr{W}^{1,p}_{\mathrm{loc}}(\Omega)$ where $\Omega \subset \mathbb{R}^n$, is Hölder continuous with exponent $1 - n/p$ provided $p > n$. More precisely, we have the following oscillation lemma, see for example [2, p. 143].

Lemma 2.2 *Let u be a function belonging to the Sobolev class $\mathscr{W}^{1,p}_{loc}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ and $p > n$. Then*

$$|u(x) - u(y)| \leq C_p(n) s^{1-\frac{n}{p}} \left(\int_{\mathbb{B}_s} |\nabla u|^p \right)^{\frac{1}{p}} \tag{10}$$

for almost every $x, y \in \mathbb{B}_s = \mathbb{B}(z, s) \subset \Omega$.

Proof of Lemma 2.1 We may assume that $\operatorname{diam} f^{-1}(\mathbb{B}_r) = d$ and that the set $\overline{f^{-1}(\mathbb{B}_r)}$ contains the origin and the point $(0, 0, \ldots, 0, d)$. For $0 < t < d$, we write

$$L_t = \left\{x \in f^{-1}(\mathbb{B}_{2r});\quad x_n = t\right\} \tag{11}$$

and fix a point $z(t) \in L_t \cap f^{-1}(\mathbb{B}_r)$. Also, we choose a radius $s(t) > 0$ such that a $(n-1)$-dimensional ball $B^{n-1}(z(t), s(t)) \subset L_t$ and $\partial B^{n-1}(z(t), s(t)) \cap \partial f^{-1}(\mathbb{B}_{2r})$ is non-empty. For shorter notation we write $B^t = B^{n-1}(z(t), s(t))$. Fix $p > n-1$ and applying Lemma 2.2, for almost every $t \in (0, d)$ we have

$$r \le C_p(n)\, s(t)^{1-\frac{n-1}{p}} \left(\int_{B^t} |Df|^p\right)^{\frac{1}{p}}. \tag{12}$$

This estimate can be equivalently written as

$$s(t)^{n-p-1} \le [C_p(n)]^p\, r^{-p} \int_{B^t} |Df|^p. \tag{13}$$

Integrating this inequality from 0 to d over t's we find that

$$\int_0^d s(t)^{n-p-1}\, dt \le [C_p(n)]^p\, r^{-p} \int_{f^{-1}(\mathbb{B}_{2r})} |Df|^p. \tag{14}$$

Here we used the fact that $B_t \subset L_t$. For estimating the integral on the left hand side, we set a positive number $\alpha = (n-1)(p-n+1)/p$ and compute

$$\begin{aligned} d^{\frac{p}{n-1}} &= \left(\int_0^d s(t)^{-\alpha} s(t)^{\alpha}\, dt\right)^{\frac{p}{n-1}} \le \left(\int_0^d s^{-\alpha\frac{p}{n-1}}\right)\left(\int_0^d s^{\alpha\frac{p}{p-n+1}}\right)^{\frac{p-n+1}{n-1}} \\ &= \left(\int_0^d s^{n-p-1}\right)\left(\int_0^d s^{n-1}\right)^{\frac{p-n+1}{n-1}} \\ &\le \left(\int_0^d s(t)^{n-p-1}\, dt\right) |f^{-1}(\mathbb{B}_{2r})|^{\frac{p-n+1}{n-1}}. \end{aligned} \tag{15}$$

Here we again used the fact that $B_t \subset L_t$. Combining the previous estimate with (14), the claim follows.

3 Proof of Theorem 1.2

Before jumping into the proof, we state a crucial version of the area formula for us. Let $\mathbb{G} \subset \Omega$ be a measurable set. Suppose that g is a homeomorphism such that g is differentiable at every point of $\mathbb{G}$. Let u be a non-negative Borel-measurable function in $\mathbb{R}^n$. Then

$$\int_{\mathbb{G}} u(g(x))|J(x, g)|\, dx \le \int_{\mathbb{R}^n} u(y)\, dy. \tag{16}$$

This follows from [3, Theorem 3.1.8] together with the area formula for Lipschitz mappings.

Under the assumptions of Theorem 1.2, the estimate (9) gets the following slightly weaker form:

$$\operatorname{diam}(f^{-1}(\mathbb{B}_r)) \le C_p(n) r^{1-n} |f^{-1}(\mathbb{B}_{2r})|^{\frac{p-n+1}{p}} \left(\int_{\mathbb{B}_{2r}} \Psi(y)\, dy \right)^{\frac{n-1}{p}} \tag{17}$$

where

$$\Psi(y) = \frac{|(Df)(f^{-1}(y))|^p}{J(f^{-1}(y), f)} \chi_{f(\mathbb{A})}(y) \tag{18}$$

and

$$\mathbb{A} = \{x \in \Omega; \quad f \text{ is differentiable at } x \text{ and } J(x, f) > 0\}.$$

Indeed, since f has finite distortion i.e. the differential of f vanishes a.e. in the zero set of the Jacobian determinant and a homeomorphism in the Sobolev class $\mathscr{W}^{1,p}_{\mathrm{loc}}(\Omega, \Omega')$, $p > n-1$, is differentiable almost everywhere we find that

$$Df(x) = 0 \quad \text{almost everywhere in } \Omega \setminus \mathbb{A}. \tag{19}$$

This elegant differentiable result goes back to the work of Väisälä [13]. Now, the inequality (17) follows from the change of variable formula (16) for f. The estimate (17) is only slightly weaker in terms of that the function Ψ is locally integrable in Ω': For every $y \in f(\mathbb{A})$, f is differentiable at $f^{-1}(y)$ and $J(f^{-1}(y), f) > 0$. Therefore, f^{-1} is differentiable at y and

$$J(y, f^{-1}) = \frac{1}{J(f^{-1}(y), f)} \qquad \text{for all } y \in f(\mathbb{A}). \tag{20}$$

Therefore, for every open subset of Ω', say $\tilde{\Omega}'$, we have

$$\begin{aligned} \int_{\tilde{\Omega}'} \Psi(y)\, dy &= \int_{\tilde{\Omega}'} |(Df)(f^{-1}(y))|^p J(y, f^{-1}) \chi_{f(\mathbb{A})}(y)\, dy \\ &\le \int_{f^{-1}(\tilde{\Omega}')} |Df|^p, \end{aligned} \tag{21}$$

as desired. Here, we also applied the change of variable formula (16) for the mapping f^{-1}.

Step 1 Our first step is to show that f^{-1} is differentiable almost everywhere and f^{-1} has finite distortion. For that, we remind the reader that the volume derivate of f^{-1} at y, denote by $\mu'_{f^{-1}}(y)$, is finite almost everywhere [14, Theorem 24.2], that is,

$$\mu'_{f^{-1}}(y) = \lim_{r \to 0} \frac{|f^{-1}(\mathbb{B}(y, r))|}{|\mathbb{B}(y, r)|} < \infty \qquad \text{a.e. } y \in \Omega'. \tag{22}$$

By Rademacher-Stepanov Theorem, see [14, Theorem 29.1], f^{-1} is differentiable almost everywhere provided that

$$\limsup_{r \to 0} \frac{\operatorname{diam} f^{-1}(\mathbb{B}(y, r))}{r} < \infty \qquad \text{a.e. } y \in \Omega'. \tag{23}$$

Fix a point $y \in \Omega'$ such that

$$\mu'_{f^{-1}}(y) < \infty \tag{24}$$

and

$$\limsup_{r\to 0} \fint_{\mathbb{B}(y,r)} \Psi(z)\, dz = \Psi(y) < \infty. \tag{25}$$

Notice, that almost every point in Ω' satisfies these conditions. For shorter the notation we write $\mathbb{B}_s = \mathbb{B}(y, s)$, for $s > 0$. The estimate (17) implies

$$\begin{aligned}\limsup_{r\to 0} \frac{\operatorname{diam} f^{-1}(\mathbb{B}_r)}{r} &\le C_p(n) \limsup_{r\to 0} \left(\fint_{\mathbb{B}_{2r}} \Psi \right)^{\frac{n-1}{p}} \left(\frac{|f^{-1}(\mathbb{B}_{2r})|}{|\mathbb{B}_{2r}|} \right)^{\frac{p-n+1}{p}} \\ &\le C_p(n) [\Psi(y)]^{\frac{n-1}{p}} [\mu'_{f^{-1}}(y)]^{\frac{p-n+1}{p}}.\end{aligned} \tag{26}$$

Hence f^{-1} is diferentiable almost everywhere in Ω'. Let $y \in \Omega'$ be a point where f^{-1} is differentiable, then the above estimate together with the definition of Ψ gives

$$\begin{aligned}|Df^{-1}(y)| &\le C_p(n)\, [\Psi(y)]^{\frac{n-1}{p}}\, J(y, f^{-1})^{\frac{p-n+1}{p}} \\ &= C_p(n) \frac{|(Df)(f^{-1}(y))|^{n-1}}{J(f^{-1}(y), f)^{\frac{n-1}{p}}} J(y, f^{-1})^{\frac{p-n+1}{p}} \chi_{f(\mathbb{A})}(y) \\ &= C_p(n) |(Df)(f^{-1}(y))|^{n-1} J(y, f^{-1}) \chi_{f(\mathbb{A})}(y).\end{aligned} \tag{27}$$

Here we also used the identity (20). Combining (27) with the assumption that f has finite distortion we find that

$$\begin{aligned}|Df^{-1}(y)| &\le C_p(n) K_\circ(f^{-1}(y), f)^{\frac{n-1}{n}} J(f^{-1}(y), f)^{\frac{n-1}{n}} J(y, f^{-1}) \chi_{f(\mathbb{A})}(y) \\ &= C_p(n) K_\circ(f^{-1}(y), f)^{\frac{n-1}{n}} \chi_{f(\mathbb{A})}(y)\, J(y, f^{-1})^{\frac{1}{n}}\end{aligned} \tag{28}$$

Therefore, f^{-1} has finite distortion.

Step 2 Next, we will prove that $f^{-1} \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\Omega', \Omega)$. For that we need only to show that f^{-1} is ACL i.e. absolutely continuous on almost all lines parallel to the coordinate axes, since the estimate (27) guarantees the local integrability of the partial derivatives. Indeed, applying the area formula for f^{-1} we see that one immediately. For the ACL-property, fix an open cube $\mathbb{Q}$ such that $\overline{\mathbb{Q}} \subset \Omega'$ it suffices to show that f^{-1} is ACL in this cube. By symmetry it is enough to consider line segments parallel to the x_n-axis. Assume that $\mathbb{Q} = \mathbb{Q}_0 \times \mathbb{I}$, where $\mathbb{Q}_0$ is an open cube in $\mathbb{R}^{n-1}$ and $\mathbb{I} = (a, b) \subset \mathbb{R}$. Next, for each Borel set $\mathbb{E} \subset \mathbb{Q}_0$ we define

$$\eta(\mathbb{E}) = |f^{-1}(\mathbb{E} \times \mathbb{I})|. \tag{29}$$

By Lebesgue theorem [14, Theorem 23.5], the set function η has a finite derivative $\eta'(y)$ for almost every $y \in \mathbb{Q}_0$, that is,

$$\eta'(y) = \limsup_{r\to 0} \frac{\eta(\mathbb{B}^{n-1}(y, r))}{r^{n-1}} < \infty. \tag{30}$$

Denote by $\mathbb{V}_0$ the measure zero set in $\mathbb{Q}_0$ where η' does not exist or is infinity. Also, we know that Ψ, defined in (18), is locally integrable function in Ω'. Therefore, there exists a measurable zero set in $\mathbb{Q}_0$, denoted by $\mathbb{V}_1$, such that

$$\int_{\mathbb{I}} \Psi(y,t)\,dt < \infty \qquad \text{for all } y \in \mathbb{V}_1 \tag{31}$$

Let $\mathcal{A}(\mathbb{I}) = \mathcal{A}$ be a finite union of closed intervals in $\mathbb{I}$ whose end-points are rational numbers. Clearly, this set is countable. For all $I \in \mathcal{A}(\mathbb{I})$ we define a function $\psi_I : \mathbb{Q}_0 \to \mathbb{R}$ such that

$$\psi_I(x) = \int_I \Psi(y,t)\,dt \qquad \text{when } y \in \mathbb{Q}_0 \setminus \mathbb{V}_1.$$

Now by Fubini $\psi_I \in L^1(\mathbb{Q}_0)$. Thus almost every $y \in \mathbb{Q}_0$ is a Lebesgue point for ψ_I. Denote by $\mathbb{V}_I$ the measure zero set of non-Lebesgue points. Now

$$\mathbb{V} = \mathbb{V}_0 \cup \mathbb{V}_1 \cup \left(\bigcup_{I \in \mathcal{A}} \mathbb{V}_I \right)$$

has a measure zero, because it is a countable union of sets of measure zero.

Fix $y \in \mathbb{Q}_0 \setminus \mathbb{V}$. We will prove that f^{-1} is absolutely continuous on the segment $y \times \mathbb{I}$, which proves the claim. Let $\{\triangle_i\}_{i=1}^{\ell}$, $\triangle_i = [a_i, b_i]$, be an union of closed intervals on $\mathbb{I}$ whose interiors are mutually disjoint and whose endpoints are rational numbers.

Fix a natural number β such that

$$\frac{1}{\beta} < \frac{\min\{\operatorname{dist}\big(\cup \triangle_i, \partial\mathbb{Q}\big), b_1 - a_1, \dots, b_\ell - a_\ell\}}{17}.$$

Now a standard covering argument [14, Lemma 31.1] gives us a number δ depends on β such that for $0 < r < \delta$ we have a covering $B_1^i, \dots, B_{\kappa_i}^i$ of $\triangle_i$ which has properties

- $\operatorname{diam}(B_j^i) = 2r$
- overlapping of B_j^i is at most 4
- $B_j^i \subset \frac{1}{\beta}$-neighborhood of $\triangle_i$.

Define balls $\mathbb{B}_j^i = \mathbb{B}(x_j^i, r)$ such that $y \times B_j^i$ is the diameter of the ball. For shorter the notation we write $\mathbb{A}_j^i = f^{-1}\mathbb{B}(x_j^i, 2r)$. Now, applying the estimate (17) and

Hölder's inequality we have

$$
\begin{aligned}
\sum_{i=1}^{\ell} |f^{-1}(y, a_i) - |f^{-1}(y, b_i)| \;&\leq\; \sum_{i=1}^{\ell}\sum_{j=1}^{\kappa_i} \operatorname{diam}\big(f^{-1}\mathbb{B}_j^i\big) \\
&\leq C_p(n) \sum_{i=1}^{\ell}\sum_{j=1}^{\kappa_i} r^{1-n} \big|\mathbb{A}_j^i\big|^{\frac{p-n+1}{p}} \left(\int_{2\mathbb{B}_j^i} \Psi(y)\,dy\right)^{\frac{n-1}{p}} \\
&= C_p(n) \sum_{i=1}^{\ell}\sum_{j=1}^{\kappa_i} \left(\frac{|\mathbb{A}_j^i|}{r^{n-1}}\right)^{\frac{p-n+1}{p}} \left(\frac{1}{r^{n-1}}\int_{2\mathbb{B}_j^i} \Psi(y)\,dy\right)^{\frac{n-1}{p}} \\
&\leq C_p(n) \left(\frac{1}{r^{n-1}} \sum_{i=1}^{\ell}\sum_{j=1}^{\kappa_i} |\mathbb{A}_j^i|\right)^{\frac{p-n+1}{p}} \left(\frac{1}{r^{n-1}} \sum_{i=1}^{\ell}\sum_{j=1}^{\kappa_i}\int_{2\mathbb{B}_j^i} \Psi(y)\,dy\right)^{\frac{n-1}{p}}.
\end{aligned}
$$

From the geometry of our covering, it follows that the overlapping of $2\mathbb{B}_j^i$ is at most 14. Therefore, we have

$$
\sum_{i=1}^{\ell}\sum_{j=1}^{\kappa_i} \big|\mathbb{A}_j^i\big| \leq 14\eta(\mathbb{B}^{n-1}(y, r)) \tag{32}
$$

and

$$
\sum_{i=1}^{\ell}\sum_{j=1}^{\kappa_i} \int_{2\mathbb{B}_j^i} \Psi(y)\,dy \leq 14 \int_{\mathbb{B}_{2r}\times I^\beta} \Psi(y)\,dy \tag{33}
$$

where $\mathbb{B}_{2r} = \mathbb{B}^{n-1}(y, 2r)$ and $I^\beta = \cup_{i=1}^{\ell}[a_i - 1/\beta, b_i + 1/\beta]$. By taking upper limit when $r \to 0$ and then letting $\beta \to \infty$ we conclude that

$$
\sum_{i=1}^{\ell} |f^{-1}(y, a_i) - f^{-1}(y, b_i)| \leq C_p(n)[\eta'(y)]^{\frac{p-n+1}{p}} \left(\int_{\cup_{i=1}^{\ell}[a_i, b_i]} \Psi(y, t)dt\right)^{\frac{n-1}{p}}.
$$

This estimate holds for rational numbers $a_j, b_j \in \mathbb{I}$. By continuity of f^{-1} it holds for all possible choices of $a_j, b_j \in \mathbb{I}$. Therefore, f^{-1} is absolutely continuous on the line segment $y \times \mathbb{I}$, as desired.

3.1 Proof of Theorem 1.3

Theorem 1.2 gives that f^{-1} is differentiable almost everywhere in Ω' and has finite distortion. Therefore, the measure of the set

$$
\mathbb{O}' = \{y \in \Omega';\; J(y, f^{-1}) > 0 \text{ and } f^{-1} \text{ is differentiable at } y\} \tag{34}
$$

equals $|\{y \in \Omega' ;\;\; Df^{-1}(y) \neq 0\}|$ and so

$$
\int_{\Omega'} |Df^{-1}(y)|^n\,dy = \int_{\mathbb{O}'} |Df^{-1}(y)|^n\,dy. \tag{35}
$$

Arguing the same way as in (20), we find that

$$\int_{\mathbb{O}'} |Df^{-1}(y)|^n \, dy = \int_{\mathbb{O}'} |Df^{-1}(y)|^n J(f^{-1}(y), f) J(y, f^{-1}) \, dy. \tag{36}$$

Applying (16) for f^{-1}, we have

$$\int_{\mathbb{O}'} |Df^{-1}(y)|^n J(f^{-1}(y), f) J(y, f^{-1}) \, dy \leq \int_{\Omega} |Df^{-1}\big(f(x)\big)|^n J(x, f) \, dx. \tag{37}$$

Above estimates guarantee that

$$\int_{\Omega'} |Df^{-1}(y)|^n \, dy \leq \int_{\Omega} |Df^{-1}(f(x))|^n J(x, f) \, dx. \tag{38}$$

On the other hand, the change of variable (16) for f gives

$$\int_{\Omega'} |Df^{-1}(y)|^n \, dy \geq \int_{\Omega} |Df^{-1}(f(x))|^n J(x, f) \, dx. \tag{39}$$

Combining this with (38) we find that

$$\int_{\Omega'} |Df^{-1}(y)|^n \, dy = \int_{\Omega} |Df^{-1}\big(f(x)\big)|^n J(x, f) \, dx. \tag{40}$$

By [6, Theorem 1.2.], the Jacobian determinant of f is strictly positive almost everywhere. Here we also used the point-wise inequlity $K_\circ(x, f) \leq K_I^{n-1}(x, f)$ for f with finite distortion, see [5, p. 109]. The familiar Cramer's rule gives

$$\begin{aligned} \int_{\Omega} |Df^{-1}(f(x))|^n J(x, f) \, dx &= \int_{\Omega} \frac{|D^\sharp f(x)|^n}{J(x, f)^{n-1}} \, dx \\ &= \int_{\Omega} K_I(x, f) \, dx. \end{aligned} \tag{41}$$

The desired identity follows.

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