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Calculus of Variations

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Boundary Harnack inequality and a priori estimates of singular solutions of quasilinear elliptic equations

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1 Introduction

Let Ω be a domain is $\mathbb{R}^N$, $N \geq 2$, d a locally bounded and measurable function defined in Ω and p a real number larger than 1. This article deals with the study of positive solutions of

$$-\mathrm{div}(|Du|^{p-2}Du) - d(x)u^{p-1} = 0 \quad \text{in } \Omega \tag{1.1}$$

which admit an isolated singularity on the boundary of Ω. It is known since the starting pioneering work of Serrin [12] that one of the main goals for studying the regularity of solutions of quasilinear equations consists in obtaining Harnack inequalities. The simplest form of this inequality is the following: Assume $B_{2r} \subset\subset \Omega$ and $d \in L^\infty(B_{2r})$, then there exists a constant $C = C(N, p, r(\|d\|_{L^\infty(B_{2r})})^{1/p}) \geq 1$ such that any nonnegative solution u of (1.1) in B_{2r} satisfies

$$u(x) \leq Cu(y) \quad \forall (x, y) \in B_r \times B_r. \tag{1.2}$$

Actually this inequality is valid for a much wider class of operators in divergence form with a power-type growth. Among the important consequences of this inequality are the Hölder continuity of the weak solutions of (1.1) and the two-side estimate of solutions admitting an isolated singularity. Among more sophisticated consequences are the obtention of local upper estimates of solutions of the same equation near a singular point. This program has been carried out by Gidas and Spruck for equation

$$-\Delta u = u^q \tag{1.3}$$

in the case $N \geq 2$ and $1 < q < (N+2)/(N-2)$ [6], and recently by Serrin and Zhou [13] for equation

$$-\mathrm{div}(|Du|^{p-2}Du) = u^q \tag{1.4}$$

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in the case $N \geq p$ and $p-1 < q < Np/(N-p)-1$. A third type of applications of Harnack inequality linked to the notion of isotropy leads to the description of positive isolated singularities of solutions. This was carried out by Véron [18] for

$$-\Delta u + u^q = 0 \tag{1.5}$$

in the case $1 < q < N/(N-2)$, and by Friedman and Véron [5] for

$$-\mathrm{div}(|Du|^{p-2}Du) + u^q = 0 \tag{1.6}$$

when $p-1 < q < N(p-1)/(N-p)$. When the singularity of u is not an internal point but a boundary point, the situation is more complicated and the mere inequality (1.2) with only one function has no meaning. Boundary Harnack inequalities which deals with two nonnegative solutions of (1.1) vanishing on a part of the boundary asserts that the two solutions must vanish at the same rate. For linear second order elliptic equations they are used for studying the properties of the harmonic measure [3] (see also [1]). For p-harmonic function in a ball, a sketch of construction is given by Manfredi and Weitsman [10] in order to obtain Fatou type results. In this article we consider singular solutions of (1.1) with a singular potential type reaction term. The first result we prove is the following: *Assume $\partial\Omega$ is C^2 and d is measurable, locally bounded in $\overline{\Omega}\setminus\{a\}$ for some $a \in \partial\Omega$ and satisfies*

$$|d(x)| \leq C_0 |x-a|^{-p} \quad a.\,e.\ in\ \ B_R(a) \cap \Omega \tag{1.7}$$

for some $C_0, R > 0$ and $p > 1$. Then there exists a positive constant C depending also on N, p and C_0 such that if $u \in C^1(\overline{\Omega} \setminus \{a\})$ is a nonnegative solution of (1.1) vanishing on $\partial\Omega \setminus \{a\}$, there holds

$$\frac{u(y)}{C\rho(y)} \leq \frac{u(x)}{\rho(x)} \leq \frac{Cu(y)}{\rho(y)} \qquad \forall (x,y) \in \Omega \times \Omega \quad s.t.\ |x| = |y|. \tag{1.8}$$

where $\rho(.)$ is the distance function to $\partial\Omega$. Another form of this estimate, usually called boundary Harnack inequality, asserts that if u_1 and u_2 are two nonnegative solutions of (1.1) vanishing on $\partial\Omega \setminus \{a\}$, there holds

$$\frac{u_1(x)}{Cu_1(y)} \leq \frac{u_2(x)}{u_2(y)} \leq \frac{Cu_1(x)}{u_1(y)} \quad \forall (x,y) \in \Omega \times \Omega \quad s.t.\ |x| = |y|. \tag{1.9}$$

for some structural constant $C > 0$. Another consequence of the construction leading to (1.8) is the existence of a power-like a priori estimate: *Assume Ω is a bounded C^2 domain with $a \in \partial\Omega$, $A \in \Omega$ is an arbitrary point and d a measurable function such that*

$$|d(x)| \leq C_0 |x-a|^{-p} \quad a.e.\ in\ \ \Omega. \tag{1.10}$$

Then there exist two positive constants $\alpha > 0$ depending on N, p, Ω and C_0, and C depending on the same parameters and also on A such that, any nonnegative solution $u \in C(\overline{\Omega} \setminus \{a\}) \cap W^{1,p}_{loc}(\overline{\Omega} \setminus \{a\}))$ which vanishes on $\partial\Omega \setminus \{a\}$ verifies

$$u(x) \leq C \frac{\rho(x)}{|x-a|^{\alpha+1}} u(A) \qquad \forall x \in \overline{\Omega} \setminus \{a\}. \tag{1.11}$$

The precise value of α is unknown and surely difficult to know explicitely, even in the simplest case when u is a p-harmonic function. In several cases the value of α is associated to the construction of separable p-harmonic functions called the spherical p-harmonics. Another striking applications of the boundary Harnack principle deals with the structure of the set of positive singular solutions. We prove the following: *Let Ω be C^2 and bounded, $a \in \partial\Omega$ and d satisfies (1.10). Assume also that the operator $v \mapsto -\mathrm{div}(|Dv|^{p-2}Dv) - d(x)v^{p-1}$ satisfies the comparison principle in $\Omega \setminus B_\epsilon(a)$ for any $\epsilon > 0$, among nonnegative solutions which vanishes on $\partial\Omega \setminus B_\epsilon(a)$. If u and v are two positive solutions of (1.1) in Ω which vanish on $\partial\Omega \setminus \{a\}$, there exists $k > 0$ such that*

$$k^{-1}u(x) \leq v(x) \leq ku(x) \quad \forall x \in \Omega. \tag{1.12}$$

Furthermore, if we assume also either $p = 2$, either $p > 2$ and u has no critical point in Ω, or $1 < p < 2$ and $d \geq 0$, there exists $k > 0$ such that

$$v(x) = ku(x) \quad \forall x \in \Omega. \tag{1.13}$$

In the last section we give some partial results concerning the existence of singular solutions of equations of type (1.1) and their link with separable solutions which are solutions under the form $u(x) = |x|^{-\gamma}\phi(x/|x|)$. If $d \equiv 0$ such specific solutions, studied by Kroll and Maz'ya [8], Tolksdorff [15], Kichenassamy and Véron [7], are called spherical p-harmonics.

Our paper is organized as follows: 1- Introduction. 2 The boundary Harnack principle. 3 The set of singular solutions. 4 Existence of singular solutions. 5 References.

2 The boundary Harnack principle

In this section we consider nonnegative solutions of

$$-\mathrm{div}(|Du|^{p-2}Du) - d(x)u^{p-1} = 0 \tag{2.1}$$

in a domain Ω which may be Lipschitz continuous or C^2. The function d, is supposed to be measurable and singular in the sense that it satisfies

$$|d(x)| \leq C_0 |x-a|^{-p} \quad \text{a.e.} \ \in \Omega \tag{2.2}$$

for some point $a \in \partial\Omega$ and some $C_0 \geq 0$. By a solution of (2.1) vanishing on $\partial\Omega \setminus \{a\}$, we mean a $u \in C(\bar{\Omega} \setminus \{a\})$ such that $Du \in L^p(K)$ for every K compact, $K \subset \bar{\Omega} \setminus \{a\}$ which verifies

$$\int_\Omega (|Du|^{p-2}Du.D\zeta + d(x)u^{p-1}\zeta)\,dx = 0 \tag{2.3}$$

for every $\zeta \in C^1(\bar{\Omega})$, with compact support in $\bar{\Omega} \setminus \{a\}$.

2.1 Estimates near the boundary in Lipschitz domains

Let Ω be a bounded domain in $\mathbb{R}^N$ with a Lipschitz continuous boundary. Then there exist $m > 1$ and $r_0 > 0$ such that for any $Q \in \partial\Omega$ there exists an isometry $\mathcal{I}_Q$ in $\mathbb{R}^N$ and a Lipschitz continuous real valued function ϕ defined in $\mathbb{R}^{N-1}$ such that

$$|\phi(x) - \phi(y)| \leq m\,|x-y| \quad \forall (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}, \tag{2.4}$$

and

$$B_{2r_0}(Q) \cap \{x = (x', x_N) = (x_1, \ldots, x_N) : x_N \geq \phi(x')\} = \mathcal{I}_Q(\Omega \cap B_{2r_0}(Q)).$$

For any $A \in B_{r_0(Q)} \cap \partial\Omega$, $A = (a', \phi(a'), r > 0$ and $\gamma > 0$, we denote by $C_{A,r,\gamma}$ the truncated cone

$$C_{A,r,\gamma} = \{x = (x', x_N) : x_N > \phi(a')\,,\ \gamma|x' - a'| \leq x_N - \phi(a')\} \cap B_r(A)$$

The opening angle of this cone is $\theta_\gamma = \tan^{-1}(1/\gamma)$ Clearly, for every $\gamma \geq m$ and $0 < r \leq r_0$, $\mathcal{I}_Q^{-1}(C_{A,r,\gamma})$ is included into Ω. Up to an orthogonal change of variable, we shall assume that $\mathcal{I}_Q = Id$. We denote also by $\rho(x)$ the distance from x to $\partial\Omega$. The next result is a standard geometric construction which can be found in [1]

Lemma 2.1 *Let $Q \in \partial\Omega$ and $0 < r < r_0/5$ and $h > 1$ an integer. There exists $N_0 \in \mathbb{N}$ depending only on m such that for any points x and y in $\Omega \cap B_{3r/2}(Q)$ verifying $\min\{\rho(x), \rho(y)\} \geq r/2^h$, there exists a connected chain of balls $B_1, \ldots B_j$ with $j \leq N_0 h$ such that*

$$x \in B_1,\ \ y \in B_j,\ \ B_i \cap B_{i+1} \neq \emptyset \text{ and } 2B_i \subset B_{2r}(Q) \cap \Omega \text{ for } 1 \leq i \leq j-1. \tag{2.5}$$

Lemma 2.2 *Assume u is a nonnegative solution of (2.1) in $B_{2r}(P)$ where $|a - P| \geq 4r$. Then there exists a positive constant c_1 depending on p, N and C_0 such that*

$$u(x) \leq c_1 u(y) \quad \forall (x, y) \in B_r(P) \times B_r(P). \tag{2.6}$$

Proof By a result of Trudinger [16, Theorem 1.1], if u is a nonnegative solution of (2.1) in $B_{2r}(P)$ there exists a constant C' depending on N, p, r and $\||d|^{1/p}\|_{L^\infty(B_{2r}(P))}$ such that (2.6) holds. Furthermore $C' \leq C_2 \exp(C_3 r\||d|^{1/p}\|_{L^\infty(B_{2r}(P))})$ where C_2 and C_3 depend on N and p. This implies (2.6) since, by (2.2), $r\||d|^{1/p}\|_{L^\infty(B_{2r}(P))}$ remains bounded by a constant depending on p and C_0. □

Up to a translation, we shall assume that the singular boundary point a is the origin of coordinates.

Lemma 2.3 *Assume Ω is as in Lemma 2.1 with $Q \neq 0 \in \partial\Omega$, $0 < r \leq \min\{r_0, |Q|/4\}$ and u is a nonnegative solution of (2.1) in $B_{2r}(Q)$. Then there exists a positive constant $c_2 > 1$ depending on p, N, m and C_0 such that*

$$u(x) \leq c_2^h u(y), \tag{2.7}$$

for every x and y in $B_{3r/2}(Q) \cap \Omega$ such that $\min\{\rho(x), \rho(y)\} \geq r/2^h$ for some $h \in \mathbb{N}$.

Proof By Lemma 2.1 there exists $N_0 \in \mathbb{N}^*$ and a connected chain of $j \le N_0 h$ balls B_i with respective radii r_i and centers x_i, satisfying (2.5). Thus

$$\max_{B_i} u \le c_1 \min_{B_i} u \quad \forall i = 1, \ldots, j, \tag{2.8}$$

by the previous lemma. Therefore (2.7) holds with $c_2 = c_1^{N_0}$. □

Lemma 2.4 *Let $0 < r \le |Q|/4$ and u be a nonnegative solution of (2.1) in $B_{2r}(Q) \cap \Omega$ which vanishes on $\partial\Omega \cap B_{2r}(Q)$. If $P \in \partial\Omega \cap B_r(Q)$ and $0 < s < r/(1+m)$ so that $B_s(P) \subset B_{2r}(Q)$, there exist two positive constants δ and c_3 depending on N, p, m and C_0 such that*

$$u(x) \le c_3 \frac{|x-P|^\delta}{s^\delta} M_{s,P}(u) \tag{2.9}$$

for all $x \in B_s(P) \cap \Omega$, where $M_{s,P}(u) = \max\{u(z) : z \in B_s(P) \cap \Omega\}$.

Proof Since $\partial\Omega$ is Lipschitz, it is regular in the sense that there exists $\theta > 0$, $s_1 > 0$ such that

$$meas\,(\Omega^c \cap B_s(y)) \ge \theta\, meas\,(B_s(y)), \quad \forall y \in \partial\Omega,\ \forall 0 < s < s_1.$$

By [16, Theorem 4.2] there exists $\delta \in (0,1)$ depending on p, N, C_0, θ and s_1, such that for any $y \in \partial\Omega$, there holds

$$|u(z) - u(z')| \le C \left(\frac{s}{s_1}\right)^{\delta(1-\gamma)} \quad \forall (z, z') \in B_s(y) \cap \Omega \times B_s(y) \cap \Omega, \tag{2.10}$$

where C depends on p, N, C_0 and $\sup_{B_{s_1}(y)\cap\Omega} u = M_{s_1,y}(u)$. Because the equation is homogeneous with respect to u, this local estimate is invariant if we replace u by $\tilde{u} = u/M_{s_1,y}(u)$. Thus the dependence is homogeneous of degree 1 with respect to $M_{s_1,y}(u)$, which implies

$$|u(z) - u(z')| \le C' \left(\frac{s}{s_1}\right)^{\delta(1-\gamma)} M_{s_1,y}(u) \quad \forall (z, z') \in B_s(y) \cap \Omega \times B_s(y) \cap \Omega. \tag{2.11}$$

Taking $z' = P = y$, $s = |x - P|$, we derive (2.9). □

If $X \in B_{2r_0}(0) \cap \partial\Omega$ and $r > 0$, we denote by $A_r(X)$ the point with coordinates $(x', \phi(x') + r)$. The next result is the key point in the construction. Although it follows [1], we give the proof for the sake of completeness.

Lemma 2.5 *Let $0 < r \le \min\{2r_0, |Q|/8, s_1, 2^5\}$ and u be a nonnegative solution of (2.1) in $B_{2r}(Q) \cap \Omega$ which vanishes on $\partial\Omega \cap B_{2r}(Q)$. Then there exists a positive constant c_4 depending only on N, p, m and C_0 such that*

$$u(x) \le c_4 u(A_{r/2}(Q)) \quad \forall x \in B_r(Q) \cap \Omega. \tag{2.12}$$

Proof The proof is by contradiction. We first notice from (2.9) that if $P \in B_{2r}(Q) \cap \partial\Omega$ verifies $B_s(P) \cap \Omega \subset B_{2r}(Q) \cap \Omega$ and if $c_5 = (2c_3)^{1/\delta}$, there holds

$$M_{s/c_5,P}(u) \leq \frac{1}{2} M_{s,P}(u). \tag{2.13}$$

By Lemma 2.3, if $y \in B_{3r/2}(Q)$ satisfies $u(y) > c_2^h u_{A_{r/2}(Q)}$, it means that $\rho(y) < r/2^h$. Let $M > 0$ such that $2^M > c_2$ (defined in Lemma 2.3), $N = \max\{1 + \mathbb{E}(6 + M \ln c_5 / \ln 2), M + 5\}$, so that $2^N > 2^6 c_5^M$, and $c_4 = c_2^N$. Let u be a positive solution of (2.1) vanishing on $B_{2r}(Q) \cap \partial\Omega$ which satisfies

$$u(Y_0) > c_2^N u(A_{r/2}(Q)), \tag{2.14}$$

for some $Y_0 \in B_r(Q) \cap \Omega$. Then $\rho(Y_0) < r/2^N$. Let $Q_0 \in \partial\Omega$ such that $\rho(Y_0) = |Y_0 - Q_0|$. Then

$$|Q - Q_0| \leq |Y_0 - Q_0| + |Y_0 - Q| \leq r/2^N + r \leq r(1 + 2^{-5})$$

Therefore $Q_0 \in B_{3r/2}(Q) \cap \partial\Omega$. Set $s_2 = c_5^M r/2^N$, then $B_{s_2}(Q_0) \subset B_{r(1+2^{-5}+2^{-6})}(Q) \subset B_{3r/2}(Q)$ because $s_2 \leq 2^{-6}$ by the choice of N. Applying (2.13) with $s = s_2$ yields to

$$\begin{aligned} M_{s_2,Q_0}(u) &\geq 2^M M_{s_2/c_5^M,Q_0}(u) \geq 2^M u(Y_0) \geq 2^M c_2^N u(A_{r/2}(Q)) \\ &\geq c_2^{N+1} u(A_{r/2}(Q)), \end{aligned}$$

since $|Y_0 - Q_0| \leq r/2^N = s_2/c_5^M$ and $2^M > c_2$. Hence we can choose $Y_1 \in \overline{B_{s_2}(Q_0) \cap \Omega}$ which realizes $M_{s_2,Q_0}(u)$ and this implies that $\rho(Y_1) < r/2^{N+1}$. A point $Q_1 \in \partial\Omega$ such that $\rho(Y_1) = |Y_1 - Q_1|$ satisfies also

$$|Q - Q_1| \leq |Q - Q_0| + |Q_0 - Q_1| \leq r(1 + 2^{-5} + 2^{-6}).$$

Now

$$\begin{aligned} M_{s_2/2,Q_1}(u) &\geq 2^M M_{r/2^{N+1},Q_1}(u) \geq 2^M u(Y_1) \\ &\geq 2^{2M} c_2^N u(A_{r/2}(Q)) \geq 2^{N+2} A_{r/2}(Q). \end{aligned}$$

Iterating this procedure, we construct two sequences $\{Y_k\}$ of points such that $\rho(Y_k) < r/2^{k+N}$ and $\{Q_k\}$ such that $Q_k \in \partial\Omega$ and $|Q - Q_k| \leq r(1 + 2(2^{-5} + 2^{-6} + \cdots + 2^{-5-k})) < 3r/2$ and

$$u(Y_k) \geq 2^{N+k} A_{r/2}(Q) \quad \forall k \in \mathbb{N}^*.$$

Since $Y_k \in B_{3r/2}$ and $\rho(Y_k) \to 0$ as $k \to \infty$ we get a contradiction with the fact that $A_{r/2}(Q) > 0$. □

Remark The proof of the previous lemma shows that estimate (2.12) is valid for a much more general class of equations under the form

$$-\text{div}A(x, u, Du) + B(x, u, Du) = 0 \tag{2.15}$$

where A and B are respectively vector and real valued Caratheodory functions defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ and verifying, for some constants $\gamma > 0$ and $a_0,\ a_1,\ C_0 \geq 0$,

$$A(x, r, q).q \geq \gamma\, |q|^p\,,$$

$$|A(x, r, q)| \leq a_0\, |q|^{p-1} + a_1\, |r|^{p-1}\,,$$

and

$$|B(x, r, q)| \leq C_0\, |r|^{p-1}\, |x|^{-p}\,,$$

for $(x, r, q) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$.

2.2 Estimates near the boundary in C^2 domains

From now we assume that Ω is a bounded domain with a C^2 boundary. For any $x \in \partial\Omega$, we denote by ν_x the normal unit outward vector to $\partial\Omega$ at x. Let $R_0 > 0$ be such that for any $x \in \partial\Omega$, the two balls $B_{R_0}(x - R_0\nu_x)$ and $B_{R_0}(x + R_0\nu_x)$ are subsets of Ω and $\bar{\Omega}^c$ respectively. If $P \in \partial\Omega$, we denote by $N_r(P)$ and $\mathcal{N}_r(P)$ the points $P - r\nu_P$ and $= P + r\nu_P$. Notice that $r \leq R_0$ implies $\rho(N_r(P)) = \rho(\mathcal{N}_r(P)) = r$.

Lemma 2.6 *Let $Q \in \partial\Omega \setminus \{0\}$, $0 < r \leq \min\{R_0/2, |Q|/2\}$ and u be a nonnegative solution of (2.1) in $B_{2r}(Q) \cap \Omega$ which vanishes on $B_{2r}(Q) \cap \partial\Omega$. Then there exist $b \in (0, 2/3)$ and $c_6 > 0$ depending respectively on N, p and C_0 and N, p, R_0 and C_0 such that*

$$\frac{t}{c_6 r} \leq \frac{u(N_t(P))}{u(N_{r/2}(Q))} \leq \frac{c_6 t}{r} \tag{2.16}$$

for any $P \in B_r(Q) \cap \partial\Omega$ and $0 \leq t \leq rb/2$.

Proof Up to a dilation, we can assume that $|Q| = 1$, since if we replace x by $x/|Q|$, Eq. (2.1) and the estimates (2.16) are structuraly invariant (which means that C_0 and the C_i are unchanged), while the curvature constant R_0 is replaced by $R_0/|Q|$ which is no harm since Ω is bounded.

Step 1 The lower bound. For $a > 0$ and $\alpha > 0$ to be made precise later on, let us define

$$v(x) = V(|x - N_{r/2}(P)|) = \frac{e^{-a(|x-N_{r/2}(P)|/r)^\alpha} - e^{-a/2^\alpha}}{e^{-a/4^\alpha} - e^{-a/2^\alpha}}$$

for $x \in B_{r/2}(N_{r/2}(P)) \cap B_{r/4}(P)$. We set $s = |x - N_{r/2}(P)|$. Since $|Q| = 1$, the function d satisfies $-\tilde{C}_0 \leq d(x) \leq \tilde{C}_0$. Next

$$\begin{aligned}
&-\text{div}(|Dv|^{p-2}\, Dv) + \tilde{C}_0 v^{p-1} \\
&\quad = -\left|V'\right|^{p-2}\big((p-1)V'' + (N-1)V'/s\big) + \tilde{C}_0 V^{p-1}.
\end{aligned}$$

Therefore this last expression will be nonpositive if and only if

$$(p-1)\left(\frac{a\alpha s^{\alpha}}{r^{\alpha}}+1-\alpha\right)+1-N$$
$$\geq \tilde{C}_0\left(\frac{a\alpha}{r^{\alpha}}\right)^{1-p} e^{(p-1)a(s/r)^{\alpha}} s^{\theta}\left(e^{-a(s/r)^{\alpha}}-e^{-a/2^{\alpha}}\right)^{p-1} \tag{2.17}$$

where $\theta = p + (1-p)\alpha$. But $\theta = 0$ by choosing $\alpha = p/(p-1)$, thus (2.17) is equivalent to

$$(p-1)\left(\frac{a\alpha s^{\alpha}}{r^{\alpha}}+1-\alpha\right)+1-N \geq \tilde{C}_0\left(\frac{a\alpha}{r^{\alpha}}\right)^{1-p}\left(1-e^{a(1/4^{\alpha}-1/2^{\alpha})}\right)^{p-1}.$$

If $r/4 \leq s \leq r/2 \leq 1/4$,

$$(p-1)\left(\frac{a\alpha s^{\alpha}}{r^{\alpha}}+1-\alpha\right)+1-N \geq (p-1)\left(\frac{a\alpha}{4^{\alpha}}+1-\alpha\right)+1-N = \frac{pa}{4^{\alpha}}-N,$$

while

$$\left(\frac{a\alpha}{r^{\alpha}}\right)^{1-p}\left(1-e^{a(1/4^{\alpha}-1/2^{\alpha})}\right)^{p-1} \leq \left(\frac{a\alpha}{r^{\alpha}}\right)^{1-p} \leq a^{1-p}.$$

Therefore, if we choose a such that

$$a^{p-1}\left(\frac{ap}{4^{\alpha}}-N\right) \geq \tilde{C}_0, \tag{2.18}$$

we derive

$$-\mathrm{div}(|Dv|^{p-2}Dv)+\tilde{C}_0 v^{p-1} \leq 0 \tag{2.19}$$

in $B_{r/2}(N_{r/2}(P)) \cap B_{r/4}(P)$. Furthermore $\rho(x) \geq r/16$ for any $x \in \partial B_{r/4}(P) \cap B_{r/2}(N_{r/2}(P))$, therefore

$$u(x) \geq c_2^{-4} u(N_{r/2}(P)) v(x), \tag{2.20}$$

by Lemma 2.3 and since $v \leq 1$. Because u is a supersolution for (2.19), we obtain that (2.20) holds for any $x \in B_{r/4}(P) \cap B_{r/2}(N_{r/2}(P))$. Finally

$$v(x) \geq \frac{e^{-a/2^{\alpha}}(2^{-\alpha}-(s/r)^{\alpha})}{e^{-a/4^{\alpha}}-e^{-a/2^{\alpha}}} \geq C(a,\alpha)(1-(1-2t/r)^{\alpha}) \geq \frac{C'(a,\alpha)t}{r}$$

if $x = N_t(P)$ with $0 \leq t \leq r/2$. This gives the left-hand side of (2.16).

Step 2 The upper bound. Let $b \in (0, 2/3]$ be a parameter to be made precise later on. By the exterior sphere condition, $B_{3br}(\mathcal{N}_{3rb}(P)) \subset \bar{\Omega}^c$. Let ϕ_1 be the first eigenfunction of the p-Laplace operator in $B_3 \setminus \bar{B}_1$ with Dirichlet boundary conditions and λ_1 the corresponding eigenvalue. It is well known that ϕ_1 is radial. We normalize ϕ_1 by $\phi_1(y) = 1$ on $\{y : |y| = 2\}$ (notice that ϕ_1 is radial) and set

$$\phi_{rb}(x) = \phi_1\left(\frac{|x-\mathcal{N}_{rb}(P)|}{rb}\right),$$

thus

$$-div(|D\phi_{rb}|^{p-2}D\phi_{rb}) = \frac{\lambda_1}{(rb)^p}\phi_{rb}^{p-1}$$

in $B_{3rb}(\mathcal{N}_{rb}(P)) \setminus \bar{B}_{rb}(\mathcal{N}_{rb}(P))$ and vanishes on the boundary of this domain. For b small enough $\lambda_1/(rb)^p \geq 1 + \tilde{C}_0$ for any $r \in (0, 1/2]$, thus

$$-\mathrm{div}(|D\phi_{rb}|^{p-2} D\phi_{rb}) - \tilde{C}_0 \phi_{rb}^{p-1} \geq \phi_{rb}^{p-1} \tag{2.21}$$

in $\Omega \cap B_{3rb}(\mathcal{N}_{rb}(P)) \setminus \bar{B}_{rb}(\mathcal{N}_{rb}(P)) \supseteq \Omega \cap B_{2rb}(\mathcal{N}_{rb}(P))$ while u verifies

$$-\mathrm{div}(|Du|^{p-2} Du) - \tilde{C}_0 u^{p-1} \leq 0 \tag{2.22}$$

in the same domain. We can also take $b > 0$ such that $B_{2br}(\mathcal{N}_{br}(P)) \subset B_r(Q)$, thus

$$u(x) \leq c_4 u(N_{r/2}(Q))$$

for $x \in \partial B_{2rb}(\mathcal{N}_{rb}(P)) \cap \Omega$ by Lemma 2.5. Now the function $\tilde{\phi}_{rb} = c_4 u(N_{r/2}(Q))\phi_{rb}$ satisfies (2.21) in $\Omega \cap B_{2rb}(\mathcal{N}_{rb}(P))$ and dominates u on $\partial(\Omega \cap B_{2rb}(\mathcal{N}_{rb}(P))) = (\partial B_{2rb}(\mathcal{N}_{rb}(P)) \cap \Omega) \cup (B_{2rb}(\mathcal{N}_{rb}(P)) \cap \partial\Omega)$. By the Diaz-Saa inequality [4]

$$\int_{\Omega \cap B_{2rb}(\mathcal{N}_{rb}(P))} \left(\frac{\mathrm{div}(|Du^{p-2}|Du)}{u^{p-1}} - \frac{\mathrm{div}(|D\tilde{\phi}_{rb}|^{p-2} D\tilde{\phi}_{rb})}{\tilde{\phi}_{rb}^{p-1}} \right) \left(u^p - \tilde{\phi}_{rb}^p\right)_+ dx \leq 0,$$

valid because $(u^p - \phi_{rb}^p)_+ \in W_0^{1,p}(\Omega \cap B_{2rb}(\mathcal{N}_{rb}(P)))$. Therefore

$$\int_{\Omega \cap B_{2rb}(\mathcal{N}_{rb}(P))} \left(u^p - \tilde{\phi}_{rb}^p\right)_+ dx \leq 0,$$

from which follows the inequality $u \leq \tilde{\phi}_{rb}$ in $\Omega \cap B_{2rb}(\mathcal{N}_{rb}(P))$. In particular

$$u(N_t(P)) \leq c_4 \phi_1 \left(\frac{|N_t(P) - \mathcal{N}_{rb}(P)|}{rb} \right) u(N_{r/2}(Q)).$$

Since $\phi_1(s) \leq C(s-1)$ for $s \in [1, 2]$, we obtain the right-hand side of (2.16). □

The main result of this section is the following

Theorem 2.7 *There exists two constants $\alpha > 0$ and $c_7 > 0$ depending on N, p, C_0 and N, p, C_0 and R_0 respectively such that if u is any nonnegative solution of (2.1) vanishing on $\partial\Omega \setminus \{0\}$ there holds*

$$\frac{1}{c_7} \rho(x) |x|^{\alpha - 1} u(A) \leq u(x) \leq c_7 \rho(x) |x|^{-\alpha - 1} u(A) \tag{2.23}$$

for any $x \in \Omega$, where A is a fixed point in Ω such that $\rho(A) \geq R_0$.

Proof Step 1: Tangential estimate. Let $x \in \Omega$ such that $|x| = 2r \leq R_0$ and $\rho(x) = t < br/2$. Let $Q \in \partial\Omega \setminus \{0\}$ such that $|Q| = |x|$ and $x \in B_r(Q)$, the previous lemma implies

$$\frac{2}{c_6 |x|} \rho(x) u(N_{r/2}(Q)) \leq u(x) \leq \frac{2c_6}{|x|} \rho(x) u(N_{r/2}(Q)). \tag{2.24}$$

There exists a fixed integer $k > 2$ such that we can connect two points lying on $\partial B_{2r}(0) \cap \partial\Omega$ by k connected balls B_j ($j = 1, \dots, k$) with radius $r/4$ and center on $\partial B_{2r}(0)$. In particular we can connect $N_{r/2}(Q)$ with $N_{2r}(0) = -2r\nu_0$ and all the balls can be taken such that the distance of their center to $\partial\Omega$ be larger that $r/2$. Since by Lemma 2.2 there holds

$$\sup_{B_j} u \le c_1 \inf_{B_j} u \quad \forall j = 1, \dots k,$$

we derive

$$\frac{2}{c_1^k c_6\,|x|}\rho(x)u(N_{2r}(0)) \le u(x) \le \frac{2c_1^k c_6}{|x|}\rho(x)u(N_{2r}(0)). \tag{2.25}$$

Let $A_0 = -R_0\nu_0$, $b_1 = -2r\nu_0 = N_{2r}(0)$, for $\ell \ge 2$, $b_\ell = -2(1 + 3(2^{\ell-1} - 1)/2)r\nu_0$ and $r_\ell = 2^{\ell-1}r$. Applying again Lemma 2.2 in $B_{2r_\ell}(b_\ell) \subset \Omega$, we have

$$\sup_{B_{r_\ell}(b_\ell)} u \le c_1 \inf_{B_{r_\ell}(b_\ell)} u \quad \forall \ell = 1, 2, \dots \tag{2.26}$$

Let τ be the solution of

$$2(1 + 3(2^{\tau-1} - 1)/2)r = R_0 \Leftrightarrow \tau = \frac{\ln(R_0 + r) - \ln 3r}{\ln 2} + 1$$

and $\ell_0 = \mathbb{E}(\tau) + 1$, then $A_0 \in B_{r_{\ell_0}}(b_{\ell_0})$, and the combination of (2.25) and (2.26) (applied ℓ_0 times) yields to

$$\frac{1}{c_1^{k+\ell_0} c_6\,|x|}\rho(x)u(A_0) \le u(x) \le \frac{c_1^{k+\ell_0} c_6}{|x|}\rho(x)u(A_0). \tag{2.27}$$

Since $r \le R_0/2$, the computation of τ yields to

$$2^\tau = \frac{2(R_0 + r)}{3r} \le \frac{R_0}{r} : c_1^\tau \le \left(\frac{R_0}{r}\right)^{\ln c_1/\ln 2}.$$

This implies (2.23) with $\alpha = \ln c_1/\ln 2$.

Step 2: Internal estimate. If $x \in \Omega$ satisfies $|x| \le R_0$ and $\rho(x) \ge b/4\ absx$, we can directly procede without using Lemma 2.6. Using internal Harnack inequality (2.6) and connecting x to $N_r(0)$ and then to A_0 we obtain

$$\frac{1}{c_7}\,|x|^\alpha\, u(A) \le u(x) \le c_7\,|x|^{-\alpha}\, u(A), \tag{2.28}$$

from which (2.24) is derived since $\rho(x) \ge b\,|x|/4$. Finally, if $|x| \ge R_0$ and $\rho(x) \le R_0$, we can replace the singular point 0 by a regular point $B \in \partial\Omega$ such that $|x - B| \le R_0$. The previous procedure leads to the same estimate. At end, if $\rho(x) > R_0$ we apply again the internal Harnack inequality (2.6). Since Ω is bounded, x and A_0 can be joined by at most $d = 2\,\mathrm{diam}\,(\Omega)/R_0$ balls B_i with radius $R_0/2$ and center b_i satisfying $\rho(b_i) \ge R_0$. Then using d times (2.6) yields to (2.24). □

Remark If $p = 2$ and d is regular, it is proved that Lemma 2.6 holds even if $\partial\Omega$ is Lipschitz continuous. The previous proof is adapted by replacing the doubling property of the radius on the connecting balls $B_{r_\ell}(b_\ell)$ by radii such that $r_{\ell+1} = \beta r_\ell$, where $\beta > 0$ depends on the opening of the standard cone C associated to the inside cone property of Ω. This observation shows that in the general case $p \neq 2$ and d singular, the validity of Lemma 2.6 implies Theorem 2.7 when Ω is a bounded domain satisfying the inside cone property.

Remark In the case $p = 2$ and $\lim_{x\to 0} |x|^2 d(x) = 0$ the value of α is known and equal to $N - 1$. When $p \neq 2$ the value of α is unknown, even in the case where $d = 0$.

The next result is a consequence of the method used in the proof of Theorem 2.7.

Theorem 2.8 *Let $u \in C^1(\bar{\Omega} \setminus \{0\})$ be a positive solutions of (2.1) vanishing on $B_{2R_0} \cap (\partial\Omega \setminus \{0\})$. Then there exists a constant $c_9 > 0$ depending on p, N, C_0 and R_0 such that*

$$\frac{1}{c_9}\frac{u(y)}{\rho(y)} \leq \frac{u(x)}{\rho(x)} \leq c_9\frac{u(y)}{\rho(y)}, \tag{2.29}$$

for every x and y in $B_{R_0}(0) \cap \Omega$ satisfying $|y|/2 \leq |x| \leq 2|y|$.

Proof By (2.25) we have

$$\frac{1}{c'|x|}u(N_{|x|}(0)) \leq \frac{u(x)}{\rho(x)} \leq \frac{c'}{|x|}u(N_{|x|}(0)).$$

for any $x \in \Omega$ such that $|x| \leq R_0/2$ and $\rho(x) \leq b|x|/4$. If we assume that $x \in \Omega \cap B_{R_0/2}(0)$ verifies $|x| \leq R_0/2$ and $\rho(x) > b|x|/4$ we can connect x to $N_{|x|}(0)$ by a fixed number n of balls of radius $b|x|/8$ with their center at a distance to $\partial\Omega$ larger than $b|x|$. The classical Harnack inequality yields to

$$\frac{1}{c_1^n}u(N_{|x|}(0)) \leq u(x) \leq c_1^n u(N_{|x|}(0)).$$

Since $\rho(x) \leq |x| \leq \rho(x)/b$, we obtain, for any $x \in B_{R_0/2}(0) \cap \Omega$,

$$\frac{1}{c_8|x|}\rho(x)u(N_{|x|}(0)) \leq u(x) \leq \frac{c_8}{|x|}\rho(x)u(N_{|x|}(0)). \tag{2.30}$$

where c_8 depends on p, N, C_0 and R_0. By Harnack inequality, we can replace $u(N_{|x|}(0))$ by $u(N_s(0))$ for any $|x|/2 \leq s \leq 2|x|$ and get

$$\frac{1}{c_1c_8|x|}\rho(x)u(N_s(0)) \leq u(x) \leq \frac{c_1c_8}{|x|}\rho(x)u(N_s(0)). \tag{2.31}$$

If $y \in B_{R_0/2}(0) \cap \Omega$ satisfies $|x|/2 \leq |y| \leq |x|$, we apply twice (2.31) and we get (2.30) with $c_9 = c_1^2c_8^2$. □

Another consequence of this method and of Lemma 2.2 and Lemma 2.5 is the

Theorem 2.9 *There exists a constant c_9' depending on N, p, C_0 and R_0 such that any $u \in C^1(\bar{\Omega} \setminus \{0\})$ be a positive solutions of (2.1) vanishing on $B_{2R_0} \cap (\partial\Omega \setminus \{0\})$ verifies*

$$u(x) \leq c_9' u(N_r(0)) \tag{2.32}$$

for every $0 < r \leq R_0/2$ and any $x \in \Omega \cap B_{2r}(0) \setminus B_{r/2}(0)$.

Remark Since Lemma 2.2 and Lemma 2.5 are valid in Lipschitz continuous domains and the construction of connected chain of balls too by Lemma 2.1, the above inequality remains valid if Ω is Lipschitz continuous.

The next result is known as the boundary Harnack inequality.

Theorem 2.10 *Let $Q \in \partial\Omega$, $0 < r \leq \min\{R_0/2, |Q|/2\}$, and u_1 and u_2 be two nonnegative solutions of (2.1) in $B_{2r}(Q) \cap \Omega$ which vanish on $B_{2r}(Q) \cap \partial\Omega$. Then there exists $c_{10} > 0$ depending respectively on N, p and C_0 such that*

$$\frac{1}{c_{10}} \frac{u_1(x)}{u_1(y)} \leq \frac{u_2(x)}{u_2(y)} \leq c_{10} \frac{u_1(x)}{u_1(y)} \tag{2.33}$$

for any $x, y \in B_r(Q) \cap \Omega$.

Proof If $x \in B_r(Q) \cap \Omega$ satisfies $\rho(x) \leq br/2$, we denote by $P_x = P$ the unique projection of x on $\partial\Omega$ and put $t = \rho(x)$. By (2.16),

$$\frac{t}{c_6 r} \leq \frac{u_i(x)}{u_i(N_{r/2}(Q))} \leq \frac{c_6 t}{r} \tag{2.34}$$

$$\frac{1}{c_6^2} \frac{u_1(x)}{u_1(N_{r/2}(Q))} \leq \frac{u_2(x)}{u_2(N_{r/2}(Q))} \leq c_6^2 \frac{u_1(x)}{u_1(N_{r/2}(Q))} \tag{2.35}$$

from which (2.33) is derived with a first constant $c_{10} = c_6^4$. Next, if $x \in B_r(Q) \cap \Omega$ satisfies $\rho(x) > br/2$, we denote $\beta = 2 + \mathbb{E}(-\ln b / \ln 2)$, thus $\rho(x) > r/2^\beta$. By 2.3

$$\frac{1}{c_2^\beta} \leq \frac{u_i(x)}{u_i(N_{r/2}(Q))} \leq c_2^\beta, \tag{2.36}$$

for $i = 1, 2$. Therefore (2.35) holds with c_6^2 replaced by c_2^β. Finally (2.33) is verified with $c_{10} = \max\{c_6^4, c_2^{2\beta}\}$. □

The next result is another form of the boundary Harnack inequality

Theorem 2.11 *Let $u_i \in C^1(\bar{\Omega} \setminus \{0\})$ ($i = 1, 2$) be two nonnegative solutions of (2.1) vanishing on $B_{2R_0} \cap (\partial\Omega \setminus \{0\})$. Then there exists $c_{11} > 0$ depending respectively on N, p and C_0 such that for any $r \leq R_0$*

$$\begin{aligned} &\sup\left(\frac{u_1(x)}{u_2(x)} : x \in \Omega \cap (B_r(0) \setminus B_{r/2}(0)\right) \\ &\quad \leq c_{11} \inf\left(\frac{u_1(x)}{u_2(x)} : x \in \Omega \cap (B_r(0) \setminus B_{r/2}(0)\right). \end{aligned} \tag{2.37}$$

Proof Applying twice 2.8, we get

$$\frac{1}{c_9^2}\frac{u_1(x)}{u_1(y)} \leq \frac{u_2(x)}{u_2(y)} \leq c_9^2\frac{u_1(x)}{u_1(y)}, \tag{2.38}$$

for any x and y such that $|x|/2 \leq |y| \leq 2|x|$. Equivalently

$$\frac{1}{c_9^2}\frac{u_1(x)}{u_2(x)} \leq \frac{u_1(y)}{u_2(y)} \leq c_9^2\frac{u_1(x)}{u_2(x)}, \tag{2.39}$$

which the claim with $c_{11} = c_9^2$. □

3 The set of singular solutions

We still assume that Ω is a bounded domain with a C^2 boundary containing the singular point 0. We introduce the following assumption on the function d.

Definition 3.1 A measurable function d satisfying (2.2) with $a = 0 \in \partial\Omega$ is said to satisfy the local comparison principle in Ω if, for any $\epsilon >_{\_} 0$ and any $u_i \in C^1(\bar{\Omega}_\epsilon)$ $(i = 1, 2)$ nonnegative solutions of (2.1) in $\Omega_\epsilon = \Omega \setminus \bar{B}_e(0)$ which vanish on $\partial^*\Omega_\epsilon = \partial\Omega \setminus B_\epsilon(0)$, $u_1(x) \geq u_2(x)$ on $\Omega \cap \partial B_\epsilon(0)$ implies $u_1 \geq u_2$ in $\bar{\Omega}_\epsilon$.

Clearly, if d is nonpositive it satisfies the local comparison principle. However there are many other cases, depending either on the value of C_0 or the rate of blow-up of d near 0 which insure this principle.

Theorem 3.2 *Assume d satisfies the local comparison principle and there exists a nonnegative nonzero solution u to (2.1) in Ω which vanishes on $\partial\Omega \setminus \{0\}$. If v is any other nonnegative solution of (2.1) in Ω vanishing on $\partial\Omega \setminus \{0\}$ there exists $k \geq 0$ such that $v \leq ku$.*

Proof Since any nontrivial nonnegative solution is positive by Harnack inequalities we can assume that both u and v are positive in Ω. We denote by $\mathcal{H}$ the set of $h > 0$ such that $v < hu$ in Ω and we assume that $\mathcal{H}$ is empty otherwhile the results is proved. Then for any $n \in \mathbb{N}_*$ there exists $x_n \in \Omega$ such that $v(x_n) \geq nu(x_n)$. We can assume that $x_n \to \xi$ for some $\xi \in \bar{\Omega}$. Clearly $\xi \in \Omega$ is impossible. Let us assume first that $\xi \in \partial\Omega \setminus \{0\}$ and denote by ξ_n the projection of x_n onto $\partial\Omega$. Thus

$$\frac{v(x_n) - v(\xi_n)}{\rho(x_n)} \geq n\frac{u(x_n) - u(\xi_n)}{\rho(x_n)}.$$

Because u and v are C^1 in $\bar{\Omega} \setminus \{0\}$,

$$\lim_{n\to\infty}\frac{v(x_n) - v(\xi_n)}{\rho(x_n)} = \frac{\partial v}{\partial \nu_\xi}(\xi) \quad\text{and}\quad \lim_{n\to\infty}\frac{u(x_n) - u(\xi_n)}{\rho(x_n)} = \frac{\partial u}{\partial \nu_\xi}(\xi).$$

Since Hopf boundary lemma is valid (see [15], and [11] for a thoroughful treatment of strong maximum principle), the two above normal derivative at ξ are negative, which leads to a contradiction. Thus we are left with the case $x_n \to 0$. Set $r_n = |x_n|$. By Theorem 2.11

$$\inf\left\{\frac{v(x)}{u(x)} : |x| = r_n\right\} \geq c_{11}^{-1}\frac{v(x_n)}{u(x_n)} \geq c_{11}^{-1}n$$

By the local comparison principle assumption, $v \geq nc_{11}^{-1}u$ in Ω_{r_n}. This again leads to a contradiction. □

The next statement is useful to characterize unbounded solutions

Proposition 3.3 *Assume u is a nonnegative solution of (2.1) vanishing on* $\partial\Omega\setminus\{0\}$*, unbounded and without extremal points near* 0*. Then*

$$\lim_{x\to 0}\frac{|x|\,u(x)}{\rho(x)} = \infty. \tag{3.1}$$

Proof Assume that (3.1) is not true. Then there exist a sequence $\{s_n\}$ converging to 0 and a constant $M > 0$ such that

$$\sup\left\{\frac{|x|\,u(x)}{\rho(x)} : |x| = s_n\right\} \leq M.$$

Therefore $\sup\{u(x) : |x| = s_n\} \leq M$. Because u has no extremal points near 0, say in $B_{s_0}(0)$ for some $s_0 > 0$, the maximum of u in $\Omega \cap (B_{s_0}(0) \setminus B_{s_n}(0))$ is achieved either on $|x| = s_0$ or on $|x| = s_n$. Therefore

$$\begin{aligned}&\max\{u(x) : x \in \Omega \cap (B_{s_0}(0) \setminus B_{s_n}(0))\}\\&\quad\leq \max\{M, \max\{u(x) : x \in \Omega \cap \partial B_{s_0}(0)\}t\} = M.\end{aligned}$$

Since this is valid for any n, it implies that u is bounded in Ω, contradiction. □

Such a solution is called a *singular solution*. The next result, which extends a previous result in [2], made more precise the statement of Theorem 3.2.

Theorem 3.4 *Assume d satisfies the local comparison principle and there exists a positive singular solution u to (2.1) in* Ω *vanishing on* $\partial\Omega \setminus \{0\}$*. Assume also either* $1 < p \leq 2$ *and* $d \geq 0$*, or* $p > 2$*, u admits no critical point in* Ω *and*

$$\liminf_{x\to 0}\frac{|x|\,|Du(x)|}{u(x)} > 0. \tag{3.2}$$

If v is any other positive solution of (2.1) in Ω *vanishing on* $\partial\Omega \setminus \{0\}$ *there exists* $k \geq 0$ *such that* $v = ku$*.*

Proof Let us assume that v is not zero. By Theorem 3.2 there exists a minimal $k > 0$ such that $v \leq ku$. As in the proof of Theorem 3.2 the following holds:

(i) either the graphs of v and ku are tangent at some $\xi \in \Omega$. If we set $w = ku - v$, then $w(\xi) = 0$, and

$$-\mathcal{L}w - Dw = 0 \tag{3.3}$$

where $\mathcal{L}$ is a linear elliptic operator and $D = d(x)(k^{p-1}u^{p-1} - v^{p-1})/w$. Since $ku(\xi) = v(\xi) > 0$, D is locally bounded near ξ. If $p > 2$ and u admits no critical point in Ω, $\mathcal{L}$ is uniformly elliptic ([5, 12]) for details in a similar situation). Thus the strong maximum principles holds and w is locally zero. Since Ω is connected $w \equiv 0$ in Ω. If $1 < p \leq 2$, the strong maximum principle holds to and we have the same conclusion.

(ii) either the graphs of v and ku are not tangent inside Ω, but tangent on $\partial\Omega \setminus \{0\}$. Since the normal derivatives of ku and v at ξ coincide, $\mathcal{L}$ is uniformly elliptic. If $p \geq 2$ the coefficient D is locally bounded. If $1 < p \leq 2$ this is not the case but D remains nonnegative. In both case Hopf maximum principle applies and yields to $\partial w/\partial \nu_\xi(\xi) < 0$. This is again a contradiction.

(iii) or $v < ku$ in Ω, $\partial v/\partial \nu > k\partial u/\partial \nu$ on $\partial\Omega \setminus \{0\}$ and there exists a sequence $\{x_n\} \subset \Omega$ converging to 0 such that

$$\lim_{n\to\infty} \frac{v(x_n)}{u(x_n)} = k.$$

Furthermore we can assume that

$$\frac{v(x_n)}{u(x_n)} = \sup\left\{\frac{v(x)}{u(x)} : |x| = |x_n| =: r_n\right\}$$

Put $a_n = \max\{u(x) : |x| = r_n\}$. By Theorem 2.9 there exists $c'_9 > 0$ depending on N, p, C_0 and R_0 such that

$$u(N_{r_n}(0)) \leq a_n \leq c'_9 u(N_{r_n}(0)), \tag{3.4}$$

which implies

$$\max\{u(x) : r_n/2 \leq |x| \leq 2r_n\} \leq c'_9 a_n. \tag{3.5}$$

We set $u_n(x) = u(r_n x)/a_n$, $v_n(x) = v(r_n x)/a_n$ and $d_n(x) = r_n^p d(r_n x)$. Then both u_n and v_n are solutions of

$$-\mathrm{div}(|Df|^{p-2} Df) - d_n f^{p-1} = 0$$

in $\Omega_n = \Omega/r_n$ and vanish on $\partial\Omega_n \setminus \{0\}$. By (3.5), u_n and v_n are uniformly bounded in $\tilde{\Omega}_n = \Omega_n \cap (B_2(0) \setminus \bar{B}_{1/2}(0))$. Since $\partial\Omega_n \cap (B_2(0) \setminus \bar{B}_{1/2}(0))$ is uniformly C^2 we deduce by the degenerate elliptic equations theory [9] that, up to subsequences, u_n and v_n converge in the $C^1_{loc}(\bar{\Omega}_n \cap (B_2(0) \setminus \bar{B}_{1/2}(0)))$-topology to functions U and V which satisfy

$$-\mathrm{div}(|Df|^{p-2} Df) - d_\infty f^{p-1} = 0$$

in $H \cap (B_2(0) \setminus \bar{B}_{1/2}(0)))$, where H is the half space $\{\eta \in \mathbb{R}^N : \eta.\nu_0 < 0\}$ and d_∞ is some weak limit of d_n in the weak-star topology of L^∞. Moreover, if $p > 2$, (3.2), jointly with (3.4) and (3.5), implies that

$$|Du_n(x)| = \frac{r_n Du(r_n x)}{a_n} \geq \gamma \tag{3.6}$$

for $2/5 \leq |x| \leq 8/5$, where $\gamma > 0$. We put

$$\ell_n = \inf\left\{\frac{v(x)}{u(x)} : |x| = |x_n| =: r_n\right\} \leq k$$

and $\xi_n = x_n/r_n$. Up to another choice of subsequence, we can also assume that $\ell_n \to \ell$ and $\xi_n \to \xi$ with $\xi = 1$. Furthermore $V \leq kU$, $V(\xi) = kU(\xi)$ and, if $\xi \in \partial H \cap (B_2(0) \setminus \bar{B}_{1/2}(0)))$,

$$\frac{\partial V}{\partial \nu_0}(\xi) = k\frac{\partial U}{\partial \nu_0}(\xi) < 0.$$

In this case, and more generally if the coincidence set Ξ of V and kU has a nonempty intersection with $\partial H \cap (B_2(0) \setminus \bar{B}_{1/2}(0)))$, Hopf boundary lemma applies and implies that $V = kV$ in the whole domain. If this is not the case we use (3.6) to conclude again by the strong maximum principle that $V = kU$ in $H \cap (B_2(0) \setminus \bar{B}_{1/2}(0)))$. Therefore $\ell = k$ and for any $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $n \geq n_\epsilon$ implies

$$(k-\epsilon)u(x) \leq v(x) \leq ku(x) \quad \forall x \in \Omega \cap \partial B_{r_n}(0).$$

By the local comparison principle the same estimate holds in Ω_{r_n}. Since this is valid for any n and any ϵ, we conclude that $v = ku$. □

4 Existence of singular solutions

4.1 Separable solutions

The existence of N-dimensional regular separable p-harmonic functions associated to cones is due to Tolksdorff [15]. Extension to singular function is proved in [19]. These solutions are obtained as follows: *Let $(r,\sigma) \in \mathbb{R}_+ \times S^{N-1}$ be the spherical coordinates in $\mathbb{R}^N$ and $S \subset S^{N-1}$ a smooth spherical domain. Then there exists two couples (γ_S, ψ_S) and (β_S, ϕ_S), where γ_S and β_S are positive real numbers and ψ_S and ϕ_S belong to $C^2(\bar{S})$ and vanish on ∂S, such that*

$$U_S = r^{\gamma_S}\psi_S \quad \text{and} \quad V_S = r^{-\beta_S}\phi_S, \tag{4.1}$$

are p-harmonic functions in the cone $C_S = \{(r,\sigma) : r > 0,\ \sigma \in S\}$. These couples are unique up to homothety over ψ_S and ϕ_S. Furthermore the following equation holds

$$\begin{cases} -\operatorname{div}\left((a^2\eta^2 + |\nabla\eta|^2)^{(p-2)/2}\nabla\eta\right) = \lambda(a)(a^2\eta^2 + |\nabla\eta|^2)^{(p-2)/2}\eta & \text{in } S \\ \eta = 0 \quad \text{on } \partial S. \end{cases} \tag{4.2}$$

where $\lambda(a) = a(a(p-1) + p - N)$ if $(a,\eta) = (\beta_S, \phi_S)$, and $\lambda(a) = a(a(p-1) + N - p)$ if $(a,\eta) = (\gamma_S, \psi_S)$.

If $p \neq 2$ and $N \neq 2$, γ_S and β_S are unknown except if $S = S_+^{N-1} = S^{N-1} \cap \{x = (x', x_N) : x_N > 0 \in \mathbb{R}^N\}$, in which case $\gamma_S = 1$ and $\psi_S = x_N$. If $N = 2$, equation (4.2) is completely integrable and the values of the γ_S and β_S are known ([8, 7]). When $p \neq 2$, the existence of solutions to (4.2) is not easy since this is not a variational problem on S. Tolksdorff's method ([15]) is based upon a N-dimensional shooting argument: he constructs the solution v of

$$\begin{cases} -\mathrm{div}\left(|Dv|^{p-2}Dv\right) = 0 & \text{in } C_S^1 = C_S \cap \{x : |x| \geq 1\} \\ v = (2-|x|)_+ & \text{on } \partial C_S^1. \end{cases} \tag{4.3}$$

Then he proves, thanks to an equivalence principle, that the function v stabilizes at infinity under the asymptotic form $v(x) \approx |x|^{-\beta}\phi(x/|x|)$, with $\beta > 0$, which gives (4.2) and the function V_S. The domain S characterizes the exponent β. The same argument applies if (4.3) is replaced by

$$\begin{cases} -\mathrm{div}\left(|Dv|^{p-2}Dv\right) + \dfrac{cu^{p-1}}{|x|^p} = 0 & \text{in } C_S^1 = C_S \cap \{x : |x| \geq 1\} \\ v = (2-|x|)_+ & \text{on } \partial C_S^1. \end{cases} \tag{4.4}$$

with $c > 0$. This gives rise to a solution of (4.4) in C_S under the form $V_{S,c} = r^{-\beta_{c,S}}\eta$ where $\beta_{c,S} > 0$ and

$$\begin{cases} -\mathrm{div}\left((\beta_{c,S}^2\eta^2 + |\nabla\eta|^2)^{(p-2)/2}\nabla\eta\right) & \\ +c\eta^{p-1} = \lambda(\beta_{c,S})(\beta_{c,S}^2\eta^2 + |\nabla\eta|^2)^{(p-2)/2}\eta & \text{in } S \\ \eta = 0 & \text{on } \partial S. \end{cases} \tag{4.5}$$

With these considerations we can construct singular solutions of (2.1) under a restrictive geometry assumption on Ω, by taking $S = S_+^{N-1}$, the upper half unit sphere.

Theorem 4.1 *Assume $d(x) = -c\,|x|^{-p}$ with $c \geq 0$ and Ω is a bounded domain with a C^2 boundary containing 0. Assume also $\partial\Omega$ is flat in a neighborhood of 0. Then there exists a positive solution of (2.1) which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies*

$$\lim_{\substack{x\to 0 \\ x/|x|\to\sigma}} |x|^{\beta_{c,S_+^{N-1}}} u(x) = \eta(\sigma) \tag{4.6}$$

uniformly for $\sigma \in S_+^{N-1}$, where η is a positive solution of (4.5).

Proof We denote by $\Lambda := \partial H$ the hyperplane $\{x : x.\nu_0 = 0\}$. Since Ω is flat in a neighborhood of 0, there exists $\gamma > 0$ such that $D_\gamma(0) = B_\gamma(0) \cap \Lambda \subset \partial\Omega$. Let $K = \max\{V_{c,S_+^{N-1}}(x) : x \in \partial\Omega \setminus D_\gamma(0)\}$ and $k = \min\{V_{c,S_+^{N-1}}(x) : x \in \partial\Omega \setminus D_\gamma(0)\}$. Since $V_{c,S_+^{N-1}} : x \mapsto |x|^{-\beta_{c,S_+^{N-1}}}\eta(x/|x|)$ is a singular solution of

$$-\mathrm{div}(|Dv|^{p-2}Dv) + \frac{c}{|x|^p}|v|^{p-2}v = 0 \tag{4.7}$$

which vanishes on $\Lambda \cap \partial\Omega \setminus \{0\}$ and is positive in H, $K > 0$ and $k \leq 0$. Furthermore $V_{c,S_+^{N-1}} - K$ (resp. $V_{c,S_+^{N-1}} - k$) is a subsolution (resp. a supersolution) of (2.1) which is nonpositive (resp. nonnegative) on $\partial\Omega \setminus \{0\}$. For any $\epsilon > 0$ let u_ϵ be the solution of

$$\begin{cases} -\mathrm{div}(|Du_\epsilon|^{p-2} Du_\epsilon) + \dfrac{c}{|x|^p} |u_\epsilon|^{p-2} u_\epsilon = 0 & \text{in } \Omega \setminus B_\epsilon(0) \\ u_\epsilon = 0 & \text{on } \partial\Omega \cap B^c_\epsilon(0) \\ u_\epsilon = V_{c,S_+^{N-1}} & \text{on } \Omega \cap \partial B_\epsilon(0). \end{cases} \tag{4.8}$$

Then $(V_{c,S_+^{N-1}} - K)_+ \leq u_\epsilon \leq V_{c,S_+^{N-1}} - k$. Since u_ϵ is locally uniformly bounded in $\bar{\Omega} \setminus \{0\}$, it follows, by the classical regularity theory for degenerate equations that it is relatively compact in the $C^1_{loc}(\bar{\Omega} \setminus \{0\})$-topology. Thus, up to some subsequence, u_ϵ converges to some u and u is a solution of (2.1) which vanishes on $\partial\Omega \setminus \{0\}$ and satisfies (4.6). □

Remark If $N = p$ the set of p-harmonic functions is invariant under the Möebius group, and in particular under the transformation $x \mapsto \mathcal{I}(x) = x/|x|^2$ which preserves C_S. In such a case $\beta_{S_+^{N-1}} = 1$. By using the transformation $\mathcal{I}$ it is possible to prove (see [2]) that there exist positive N-harmonic functions in any bounded domain Ω having a singularity at a point a of the boundary and vanishing on $\partial\Omega \setminus \{a\}$.

Remark When $p = 2$ it is possible to prove the existence of a singular solution to

$$\begin{cases} -\Delta u + d(x)u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \setminus \{a\} \end{cases} \tag{4.9}$$

where $a \in \partial\Omega$, for any C^2 domain Ω and any d locally bounded in $\bar{\Omega} \setminus \{a\}$ such that

$$-\infty < \liminf_{x\to a} |x - a|^2 d(x) \leq \limsup_{x\to a} |x - a|^2 d(x) < N^2/4.$$

We conjecture that such a result holds for (2.1) and $p \neq 2$ although the precise upper limit as $x \to a$ of $|x - a|^p d(x)$. We believe that at least if $N \geq p$ and $\limsup_{x\to a} |x - a|^2 d(x) \leq ((N - p)/p)^p$, (the Hardy constant for $W^{1,p}$ in $\mathbb{R}^N$), such a singular solution do exist.

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