Calc. Var. (2006) 27(4): 397–420
DOI 10.1007/s00526-005-0354-5 **Calculus of Variations**

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From 1-homogeneous supremal functionals to difference quotients: relaxation and Γ-convergence

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Abstract In this paper we consider positively 1-homogeneous supremal functionals of the type $F(u) := \sup_\Omega f(x, \nabla u(x))$. We prove that the relaxation $\bar{F}$ is a *difference quotient,* that is

$$\bar{F}(u) = R^{d_F}(u) := \sup_{x,y\in\Omega, x\neq y} \frac{u(x) - u(y)}{d_F(x, y)} \quad \text{for every } u \in W^{1,\infty}(\Omega),$$

where d_F is a geodesic distance associated to F. Moreover we prove that the closure of the class of 1-homogeneous supremal functionals with respect to Γ-convergence is given exactly by the class of difference quotients associated to geodesic distances. This class strictly contains supremal functionals, as the class of geodesic distances strictly contains *intrinsic* distances.

Keywords Variational methods · Supremal functionals · Finsler metric · Relaxation · Γ-convergence

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Introduction

In this paper we are interested in minimization problems for a class of functionals recently called *supremal functionals*, namely functionals $F : W^{1,\infty}(\Omega) \to \mathbb{R}$ of the form

$$F(u) := \sup_{\Omega} f(x, u(x), \nabla u(x)) ,$$

where $\sup_{\Omega}$ means the essential sup on Ω. The model case where $f(x, u, \nabla u) \equiv |\nabla u|$ is related to the classical problem of finding the *best Lipschitz constant* of a function with prescribed boundary data, first considered by McShane in [18].

In order to apply the *direct method* of the calculus of variations the main issue is the lower semicontinuity of F. Semicontinuity properties for supremal functionals have been studied by many authors in the last years; we refer for instance to Barron-Jensen [3], Barron-Liu [5], Barron-Jensen-Wang [4], and to the recent papers by Prinari [19] and Gori-Maggi [17]. In [4] the authors proved a lower semicontinuity result for F under the assumption (called *level convexity)* that the sub levels of $f(x, u, \cdot)$ are convex. As necessary conditions are concerned the only results have been obtained in the one dimensional case in [1] and in [4] under continuity assumptions of f with respect to x and u. In both cases it is stated that if the supremal functional F is lower semicontinuous, then f is level convex in the gradient variable.

In the case of lack of semicontinuity an important step, for the characterization of the minimizing sequences, is to consider the lower semicontinuous envelope of F, i.e., the biggest lower semicontinuous functional smaller than F, the so called *relaxation* of F. In [5] and in [13] it is proved that if f is continuous with respect to x and u, then the relaxation of F is a supremal functional represented by the level convex envelope of f.

In the case of continuity of f, even though the structure of the functional F is supremal, it is still possible to adapt the techniques that are usually used in the case of integral functionals as *blow-up* arguments as well as *zig-zag approximations*. Without the continuity the functional F can be affected by the values of ∇u on arbitrarily small sets and those techniques fail. Our main contribution to the subject is to approach the problem of relaxation and Γ-convergence in any dimension and without any continuity assumption, providing some new geometrical constructions intrinsically related to the *supremal nature* of the functional.

We restrict our analysis to the case *of positively* 1-*homogeneous* supremal functionals of the form

$$F(u) := \sup_{\Omega} f(x, \nabla u(x)) \tag{0.1}$$

and we assume that $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function satisfying

$$\alpha|\xi| \le f(x, \xi) \le \beta|\xi| \quad \text{for a.e } x \in \Omega,\ \text{for every } \xi \in \mathbb{R}^n, \tag{0.2}$$

for some fixed positive constants $\alpha, \beta > 0$, and

$$f(x, t\eta) = |t| f(x, \eta) \quad \text{for a.e. } x \in \Omega, \text{for every } t \in \mathbb{R} \text{ and for every } \eta \in \mathbb{R}^n. \tag{0.3}$$

A natural question is whether in general the relaxation of F is obtained through the convex envelope (which in the case of 1-homogeneous supremal functionals coincides with the level-convex envelope) of the function f which represents F. In Example 3.2 we construct a functional $F(u) := \sup f(x, \nabla u(x))$ such that its relaxation is strictly greater than the functional represented by the convex envelope of f, showing that in general the formula $\bar{F}(u) = \sup f^{**}(x, \nabla u(x))$ is false.

A second unexpected result concerns the study of the asymptotic behavior under Γ-convergence of the class of 1-homogeneous supremal functionals. By means of an example (see Remark 4.5) we show that this class is not closed under Γ-convergence and we prove that its closure is given by the class of *difference quotients* associated to geodesic distances, i.e., functionals of the form

$$R^d(u) := \sup_{x,y\in\Omega, x\neq y} \frac{u(x) - u(y)}{d(x, y)} \quad \text{for every } u \in W^{1,\infty}(\Omega),$$

where d is a geodesic distance equivalent to the Euclidean distance. This class of lower semicontinuous functionals contains any 1-homogeneous supremal functional

$$F(u) = \sup f(x, \nabla u(x))$$

with f convex; more precisely $F = R^{d_F}$, where d_F is defined by

$$d_F(x, y) := \sup\{u(x) - u(y), u \in W^{1,\infty}(\Omega) : F(u) \leq 1\}. \tag{0.4}$$

Actually a difference quotient R^d is a supremal functional represented by a convex function if and only if the distance d is geodesic and satisfies the additional property of being *intrinsic* (the notion of intrinsic distance was introduced by De Cecco-Palmieri [15]; see Definition 1.4). On the other hand there are geodesic (non intrinsic) distances such that the corresponding difference quotient functional can not be written in a supremal form (see Example 2.6).

In view of these results the class of difference quotients seams to be the natural class in which to look for the relaxation of F. Our main result in this direction is the following representation formula for the relaxation $\bar{F}$ of F

$$\bar{F}(u) = R^{d_F}(u) \quad \text{for every } u \in W^{1,\infty}(\Omega),$$

where d_F is given by (0.4). This relaxation formula represents the main tool used in the characterization of the closure under Γ-convergence of 1-homogeneous supremal functionals: the problem reduces to find the closure of the class of distances d_F associated to supremal functionals. In this respect, the main point is that the distance d_F is always geodesic (see Theorem 3.9). In order to show that the supremal functionals are not closed under Γ-convergence, in Remark 4.5 we exhibit a sequence of intrinsic geodesic distances that uniformly converges to a geodesic distance whose difference quotient is not supremal. The same sequence is also a counterexample to Theorem 3.1, i) iff iv), of [7] as it will be explained in [8].

Another consequence of our relaxation formula is that whenever the distance d_F is also intrinsic, then the relaxation $\bar{F}$ is a supremal functional represented by a convex function (see Corollary 3.6). At the present the question whether d_F is in general intrinsic is still open.

Finally let us observe that relations between metric properties and problems in L^∞ were recently used by many authors, in the case of variational problems (see e.g. Buttazzo-De Pascale-Fragalà [7]) and in the study of the viscosity solutions of the so called ∞-Laplacian (see e.g. Aronsson-Crandall- Juutinen [2], Champion-De Pascale [10] and Crandall-Evans-Gariepy [11]). More in general the idea that the metric approach permits to consider situations with lack of regularity is nowadays classical and has been used in many different contexts as, for instance, to treat Hamilton-Jacobi equations with discontinuous Hamiltonian (see Siconolfi [20] and Camilli- Siconolfi [9]).

The paper is organized as follows. In Sect. 1 we recall the main metric notions used in the paper. In Sect. 2 we introduce the class of difference quotient functionals, we prove its main properties and show the relation with the supremal functionals. In Sect. 3 we give a relaxation formula for F,and in Sect. 4 we characterize the closure under Γ-convergence of 1-homogeneous supremal functionals.

1 Preliminaries on geodesic and intrinsic distances

In this section we will recall the main metric notions that will be used in the sequel. We first introduce the notion of geodesic distances. In particular for our setting we will need the definition of intrinsic distance introduced by De Cecco and Palmieri and its main properties (for details see [14–16]).

1.1 Geodesic distances

From now on Ω will be a connected open bounded subset of $\mathbb{R}^n$ with Lipschitz continuous boundary.

We say that a distance $d : \Omega \times \Omega \to [0, +\infty)$ is *geodesic* if

$$d(x, y) = \inf\{\mathcal{L}_d(\gamma) : \gamma \in \Gamma_{x,y}(\Omega)\} \quad \text{for every } x, y \in \Omega.$$

Here $\Gamma_{x,y}(\Omega)$ denotes the set of Lipschitz curves in Ω with end-points x and y, and $\mathcal{L}_d(\gamma)$ denotes the length of the curve γ with respect to the distance d, i.e.,

$$\mathcal{L}_d(\gamma) := \sup\left\{\sum_{i=1}^{k-1} d(\gamma(t_i), \gamma(t_{i+1})) : k \in \mathbb{N}, 0 = t_1 < \ldots < t_i < \ldots < t_k = 1\right\}.$$

Let us denote

$$|x - y|_\Omega = \inf\{\mathcal{L}(\gamma) : \gamma \in \Gamma_{x,y}(\Omega)\},$$

where $\mathcal{L}(\gamma)$ denotes the Euclidean length of γ. Note that by the fact that $\partial\Omega$ is Lipschitz we deduce that there exists a constant $C > 0$ such that $|x - y| \leq |x - y|_\Omega \leq C|x - y|$.

Given two positive constants $0 < \alpha' < \beta'$ we set

$$\begin{aligned} D(\alpha', \beta')(\Omega) := \{d \text{ geodesic distance} : \alpha'|x - y|_\Omega \leq d(x, y) \\ \leq \beta'|x - y|_\Omega \text{ for all } y, x \in \Omega\}. \end{aligned} \tag{1.1}$$

Remark 1.1 (Extension of geodesic distances). Every distance d defined in $\Omega \times \Omega$, satisfying

$$\alpha' |x - y|_\Omega \le d(x, y) \le \beta' |x - y|_\Omega,$$

can be uniquely extended by continuity to $\bar{\Omega} \times \bar{\Omega}$. Moreover if the distance d is geodesic, then its extension (still denoted by d) satisfies

$$d(x, y) = \min\{\mathcal{L}_d(\gamma) : \gamma \in \Gamma_{x,y}(\bar{\Omega})\} \quad \text{for every } x, y \in \bar{\Omega},$$

where $\Gamma_{x,y}(\bar{\Omega})$ denotes now the set of Lipschitz curves in $\bar{\Omega}$ with end-points x and y.

The following proposition states that the class $D(\alpha', \beta') := D(\alpha', \beta')(\bar{\Omega})$ is compact with respect to uniform convergence.

Proposition 1.2 *Let d_n be a sequence of distances in $D(\alpha', \beta')$. Then, up to a subsequence d_n uniformly converge to some distance d in $D(\alpha', \beta')$.*

Definition 1.3 A (convex) *Finsler metric* on Ω is a function $\varphi : \Omega \times \mathbb{R}^N \to [0, +\infty)$, Borel measurable with respect to the first variable and continuous with respect to the second variable, such that the following properties hold:

$\varphi(x, t\eta) = t\varphi(x, \eta)$ for every $x \in \Omega, t > 0, \eta \in \mathbb{R}^n$;
$\alpha'|\eta| \le \varphi(x, \eta) \le \beta'|\eta|$ for every $x \in \Omega, \eta \in \mathbb{R}^n$;
$\varphi(x, \cdot)$ is convex for a.e. $x \in \Omega$;
for every Lipschitz curve $\gamma : [0, 1] \to \Omega$, $\varphi(\gamma(t), \gamma'(t)) = \varphi(\gamma(t), -\gamma'(t))$ for a.e. $t \in [0, 1]$

where α' and β' are two fixed positive constants.

To any geodesic distance in $D(\alpha', \beta')$, we associate a Finsler metric φ_d. Namely for every $x \in \Omega$ and for every direction η we can define the function $\varphi_d(x, \eta)$ as follows

$$\varphi_d(x, \eta) := \limsup_{t \to 0^+} \frac{d(x, x + t\eta)}{t}. \tag{1.2}$$

It turns out that φ_d is a convex Finsler metric. Moreover it can be proved that for every $\gamma \subset \Omega$ we have

$$\mathcal{L}_d(\gamma) = \int_0^1 \varphi_d(\gamma, \gamma')\, dt$$

(see Theorem 2.5 in [16]).

1.2 Intrinsic distances

Definition 1.4 We say that a distance d in $D(\alpha', \beta')$ is *intrinsic* if

$$d(x,y) = \sup_N \inf_{\gamma \in \Gamma^N_{x,y}(\Omega)} \int_0^1 \varphi_d(\gamma, \gamma')\, dt,$$

where the supremum is taken over all subsets N of Ω such that $|N| = 0$ and $\Gamma^N_{x,y}(\Omega)$ denotes the set of all Lipschitz curves in Ω with end-points x and y transversal to N, i.e., such that $\mathcal{H}^1(N \cap \gamma) = 0$, where $\mathcal{H}^1$ denotes the one dimensional Hausdorff measure.

Note that the sup over negligible sets is actually a maximum. We set

$$\tilde{D}(\alpha', \beta') = \{d \in D(\alpha', \beta') : d \text{ is intrinsic}\}. \tag{1.3}$$

To any Finsler metric φ we associate an intrinsic distance δ_φ through the so called *support function* φ^0 of φ, defined by duality as follows

$$\varphi^0(x,\xi) := \sup_{\eta \neq 0} \left\{ \frac{\xi \cdot \eta}{\varphi(x,\eta)} \right\}. \tag{1.4}$$

Clearly, for a.e. $x \in \Omega$, it satisfies the following properties:

$$\begin{aligned} &\varphi^0(x,\cdot) \text{ is convex;} \\ &\varphi^0(x, t\xi) = |t|\varphi^0(x,\xi) \quad \text{for every } t \in \mathbb{R}, \xi \in \mathbb{R}^n\,; \\ &\frac{1}{\beta'}|\xi| \le \varphi^0(x,\xi) \le \frac{1}{\alpha'}|\xi| \quad \text{for every } \xi \in \mathbb{R}^N. \end{aligned}$$

Moreover if φ is convex, then $\varphi^{00} = \varphi$. Now, if φ is equivalent to a Finsler metric (i.e., there exists a Finsler metric $\tilde{\varphi}$ such that φ coincides with $\tilde{\varphi}$ a.e. in Ω), then it is possible to define a distance $\delta_\varphi(x,y)$ in the following way (see [15, 16]):

$$\delta_\varphi(x,y) := \sup\Big\{u(x) - u(y), u \in W^{1,\infty}(\Omega) : \sup_\Omega \varphi^0(x, \nabla u(x)) \le 1\Big\}, \tag{1.5}$$

for every $x, y \in \Omega$. By Theorem 3.7 in [16] $\delta_\varphi(x,y)$ is a geodesic distance and satisfies

$$\delta_\varphi(x,y) = \sup_N \inf_{\gamma \in \Gamma^N_{x,y}(\Omega)} \int_0^1 \varphi(\gamma, \gamma')\, dt. \tag{1.6}$$

The following example shows that in general, if ψ is a Finsler metric, then the derivative φ_{δ_ψ} of δ_ψ can be different from ψ.

Example 1.5 (Example 5.1 in [16]). Let $(a_h)_h$ be dense in $\mathbb{R}$ and let $A \subset \mathbb{R}^2$ be the open set defined by

$$A := \Big\{x = (x_1, x_2) \in \mathbb{R}^2 : \min\Big\{\inf_h |x_1 - a_h|2^h, \inf_h |x_2 - a_h|2^h\Big\} < 1\Big\}.$$

Roughly speaking the set A is given by the union of horizontal and vertical thin strips. Let ψ be the Finsler metric on $\mathbb{R}^2$ defined by

$$\psi(x,v) := \begin{cases} |v| & \text{if } x \in A, \\ 2|v| & \text{otherwise,} \end{cases}$$

and let d be the associated distance (so that $d = \delta_\psi$). Then the derivative φ_d of d is given by

$$\varphi_d(x,v) := \begin{cases} |v| & \text{if } x \in A, \\ |v_1| + |v_2| & \text{otherwise.} \end{cases}$$

In particular, if $x \notin A$ then $\varphi_d(x,v) \neq \psi(x,v)$.
Next proposition states that φ_{δ_ψ} is always lower then ψ.

Proposition 1.6 *Let ψ be a convex Finsler metric. Then*

$$\varphi_{\delta_\psi}(x,\xi) \leq \psi(x,\xi) \quad \textit{for a.e. } x \in \Omega, \quad \textit{for every } \xi \in \mathbb{R}^n.$$

Proof It is enough to prove the inequality for every Lebesgue point $x \in \Omega$ for the function $z \to \psi(z,\xi)$ for every $\xi \in \mathbb{R}^n$ (such points in fact have full measure in Ω). Let us fix a direction $\xi \in \mathbb{R}^n$, and for every $t > 0$ let us denote by γ_t the straight curve joining x with $x + t\xi$. Moreover let N be the negligible set maximizing the right hand side of (1.6). Let us fix $\varepsilon > 0$. By Fubini's Theorem, and using that x is a Lebesgue point, we can easily translate the curve γ_t by a vector r_t such that $|r_t| < t^2$,obtaining a curve $\tilde{\gamma}_t$ satisfying the following-conditions:

(i) $\tilde{\gamma}_t$ is transversal to N;
(ii) $\mathcal{H}^1(\tilde{\gamma}_t \cap \{z \in \Omega : |\psi(z,\xi) - \psi(x,\xi)| > \varepsilon|\xi|\})/t \to 0$ as $t \to 0$.

Using (i) and (ii) we obtain

$$\begin{aligned} \delta_\psi(x, x+t\xi) &\leq \delta_\psi(x, x+r_t) + \int_0^1 \psi(\tilde{\gamma}_t, \tilde{\gamma}_t')\, ds + \delta_\psi(x + r_t + t\xi, x + t\xi) \\ &\leq 2\beta|r_t| + t\psi(x,\xi) + t0(\varepsilon), \end{aligned} \tag{1.7}$$

where $0(\varepsilon) \to 0$ as $\varepsilon \to 0$. Dividing both sides of (1.7) by t, and passing to the limsup as $t \to 0$, in view of the arbitrariness of ε we obtain the thesis. □

Now we are in position to characterize all intrinsic distances.

Proposition 1.7 *Let $d \in D(\alpha', \beta')$. Then the following are equivalent*

(1) $d \in \tilde{D}(\alpha', \beta')$;
(2) $\delta_{\varphi_d} = d$.

Proof (1) $\Rightarrow$ (2) It follows by Theorems 4.5 and 3.7 in [16].
(2) $\Rightarrow$ (1) Set $\psi = \varphi_d$. Since δ_ψ is a geodesic distance we have that for every $x, y \in \Omega$

$$\delta_\psi(x,y) = \inf_{\gamma \in \Gamma_{x,y}(\Omega)} \int_0^1 \varphi_{\delta_\psi}(\gamma, \gamma')\, dt$$

and thus, by Proposition 1.6 and by (1.6), we obtain

$$\begin{aligned}\delta_\psi(x,y) &\leq \sup_N \inf_{\gamma\in\Gamma^N_{x,y}(\Omega)} \int_0^1 \varphi_{\delta_\psi}(\gamma,\gamma')\,dt\\ &\leq \sup_N \inf_{\gamma\in\Gamma^N_{x,y}(\Omega)} \int_0^1 \psi(\gamma,\gamma')\,dt = \delta_\psi(x,y),\end{aligned}$$

i.e.,

$$d(x,y) = \delta_\psi(x,y) = \sup_N \inf_{\gamma\in\Gamma^N_{x,y}(\Omega)} \int_0^1 \varphi_d(\gamma,\gamma')\,dt. \qquad \square$$

We conclude with an example given in [6] which shows that not all geodesic distances are intrinsic.

Example 1.8 Let $\Omega = (-1,1)^2$ and consider the segment $S = (-1,1)\times\{0\}$. Consider the Finsler metric φ defined by

$$\varphi(x,\xi) = \begin{cases} \beta'|\xi| & \text{if } x \in \Omega\backslash S \\ \alpha'|\xi| & \text{if } x \in S, \end{cases}$$

with $0 < \alpha' < \beta'$. Consider now the distance associated to this metric, i.e.,

$$d(x,y) = \inf_{\gamma\in\Gamma_{x,y}(\Omega)} \int_0^1 \varphi(\gamma,\gamma')\,dt$$

This distance is clearly different from β' times the Euclidean distance, in particular for many pairs of points (x,y) near S we have $d(x,y) < \beta'|x-y|$. On the other hand the derivative φ_d of d coincides with $\beta'|\xi|$ in $\Omega\backslash S$ and hence the distance $\delta_{\varphi_d}(x,y)$ coincides with $\beta'|x-y|$. In conclusion the class $\tilde{D}(\alpha',\beta')$ is strictly contained in $D(\alpha',\beta')$.

In the sequel we will use the distance defined in (1.5) also in the case where the functional $\sup_\Omega \varphi^0(x,\nabla u)$ is replaced by some positively 1-homogeneous supremal functional $F(u) = \sup f(x,\nabla u(x))$ with f possibly non convex, satisfying (0.2). In this case we will denote it by d_F, i.e.,

$$d_F(x,y) := \sup\{u(x)-u(y),\, u \in W^{1\infty}(\Omega) : F(u) \leq 1\}. \tag{1.8}$$

By the growth condition on f we clearly have

$$\frac{1}{\beta}|x-y|_\Omega \leq d_F(x,y) \leq \frac{1}{\alpha}|x-y|_\Omega. \tag{1.9}$$

2 Difference quotient functionals

In this section we introduce the class of *difference quotient* functionals, which is the natural setting in the study of relaxation and Γ-convergence of supremal functionals.

For every distance d equivalent to the Euclidean distance let $R^d : W^{1,\infty}(\Omega) \to \mathbb{R}$ be defined by

$$R^d(u) := \sup_{x,y\in\Omega, x\neq y} \frac{u(x)-u(y)}{d(x,y)}. \tag{2.1}$$

The functional R^d is referred to as the *difference quotient* associated to d.

Remark 2.1 Let d_1, d_2 be two distances equivalent to the Euclidean distance. It is easy to see that

$$R^{d_1} = R^{d_2} \quad \text{if and only if} \quad d_1 = d_2. \tag{2.2}$$

In fact for every fixed $z \in \Omega$ the functions $u(\cdot) := d_1(\cdot, z)$ and $v(\cdot) := d_2(\cdot, z)$ belong to $W^{1,\infty}(\Omega)$. To show (2.2) it is sufficient to test the functionals R^{d_1} and R^{d_2} on these functions.

Proposition 2.2 *The difference quotient R^d is lower semicontinuous with respect to the strong convergence in L^∞.*

Proof Let $u \in W^{1,\infty}(\Omega)$ and let $\{u_n\} \subset W^{1,\infty}(\Omega)$ be a sequence converging to u in $L^\infty(\Omega)$. We have that for every $x, y \in \Omega,\ x \neq y$,

$$\frac{u(x)-u(y)}{d(x,y)} = \lim_n \frac{u_n(x)-u_n(y)}{d(x,y)} \le \liminf_n R^d(u_n).$$

Taking the supremum as $x, y \in \Omega$ we get the thesis. □

From now on we will consider supremal functionals of the form

$$F(u) := \sup_\Omega f(x, \nabla u(x)), \tag{2.3}$$

where f satisfies the following growth condition

$$\alpha|\xi| \le f(x,\xi) \le \beta|\xi| \quad \text{for a.e } x \in \Omega, \ \text{for every } \xi \in \mathbb{R}^n \tag{2.4}$$

for some fixed positive constants $0 < \alpha < \beta$, and it is *positively l-homogeneous,* i.e.,

$$f(x, t\eta) = |t| f(x,\eta) \ \text{ for a.e. } x \in \Omega, \ \text{for every } t \in \mathbb{R} \text{ and for every } \eta \in \mathbb{R}^n. \tag{2.5}$$

Remark 2.3 We will see in Remark 3.1 that a supremal functional does not admit a unique representative. Thus we may not expect that if the functional $F(u) := \sup_\Omega f(x, \nabla u(x))$ is positively l-homogeneous (i.e., $F(\lambda u) = |\lambda| F(u)$), then f is positively l-homogeneous. Nevertheless it is easy to see that the function

$$\tilde{f}(x,\xi) := |\xi| f(x, \xi/|\xi|) \quad \forall \xi \in \mathbb{R}^n, \quad \text{for a.e. } x \in \Omega$$

is positively l-homogeneous and always represents F. In the sequel, any positively l-homogeneous functional will be understood to be represented by a positively l-homogeneous function.

The following proposition gives a first important relation between supremal functionals and difference quotient functionals.

Proposition 2.4 *Let F be a 1-homogeneous supremal functional associated to a Carathéodory function f satisfying (2.4), (2.5), and convex with respect to ξ. Then $R^{d_F} = F$, where d_F is the distance defined by (1.8).*

Proof Using that both functionals are positively 1-homogeneous, we have to prove that

$$\sup_{\Omega} f(x, \nabla u(x)) \leq 1 \quad \text{if and only if} \quad \sup_{x,y\in\Omega, x\neq y} \frac{u(x)-u(y)}{d_F(x,y)} \leq 1.$$

By the definition of d_F we have

$$\sup_{\Omega} f(x, \nabla u(x)) \leq 1 \quad \text{implies that} \quad u(x)-u(y) \leq d_F(x,y)$$

for all $x, y \in \Omega$, i.e., $R^{d_F}(u) \leq 1$.

Conversely, let $\varphi := f^0$. Since f is convex, then $\varphi^0 = f$ and thus d_F coincides with the distance δ_φ defined by (1.5). If $R^{d_F}(u) = R^{\delta_\varphi}(u) \leq 1$, then by using Proposition1.6 we obtain that for a.e. $x \in \Omega$ and any $z \in \mathbb{R}^n$,

$$\begin{aligned}\nabla u(x) \cdot z &= \lim_{t\to 0} \frac{u(x+tz)-u(x)}{t} \\ &\leq \limsup_{t\to 0} \frac{\delta_\varphi(x+tz,x)}{t} = \varphi_{\delta_\varphi}(x,z) \leq \varphi(x,z).\end{aligned}$$

Finally, by the definition of φ^0 we deduce $f(x,\nabla u(x)) = \varphi^0(x,\nabla u(x)) \leq 1$ for a.e. $x \in \Omega$ □

The following result shows that the class of supremal functionals represented by a convex function actually coincides with the class of difference quotients associated to an intrinsic distance.

Proposition 2.5 *Let d be a geodesic distance in $D(\alpha', \beta')$ and φ_d be its derivative according to (1.2). The following facts are equivalent*

(1) $d \in \tilde{D}(\alpha', \beta')$;
(2) $R^d(u) = \sup_\Omega \varphi_d^0(x, \nabla u(x))$;
(3) $R^d(u) = \sup_\Omega f(x, \nabla u(x))$, where f is a Carathéodory function, convex with respect to ξ, satisfying (2.5) and (2.4) with $\alpha = \frac{1}{\beta'}$ and $\beta = \frac{1}{\alpha'}$.

Proof (1) ⇒ (2) We have to prove that

$$\sup_{\Omega} \varphi_d^0(x, \nabla u(x)) \leq 1 \quad \text{if and only if} \quad \sup_{x,y\in\Omega, x\neq y} \frac{u(x)-u(y)}{d(x,y)} \leq 1.$$

Since $d \in \tilde{D}(\alpha', \beta')$, by Proposition 1.7 we have that $\delta_{\varphi_d} = d$. Thus by the definition of δ_{φ_d}

$$\sup_{\Omega} \varphi_d^0(x, \nabla u(x)) \leq 1 \quad \text{implies that} \quad u(x)-u(y) \leq \delta_{\varphi_d}(x,y) = d(x,y)$$

for all $x, y \in \Omega$, i.e., $R^d(u) \leq 1$. Conversely, if $R^d(u) \leq 1$ then for a.e. $x \in \Omega$ and any $z \in \mathbb{R}^n$ we have

$$\nabla u(x) \cdot z = \lim_{t \to 0} \frac{u(x+tz) - u(x)}{t} \leq \limsup_{t \to 0} \frac{d(x+tz, x)}{t} = \varphi_d(x, z),$$

and hence, by the definition of φ_d^0, we get $\varphi_d^0(x, \nabla u(x)) \leq 1$ for a.e. $x \in \Omega$.

(2) $\Rightarrow$ (3) The proof of this implication trivially follows by the properties of φ_d^0.

(3) $\Rightarrow$ (1) It is easy to check that

$$\begin{aligned} d(x, y) &= \sup\{u(x) - u(y),\ u \in W^{1,\infty}(\Omega) : R^d(u) \leq 1\} \\ &= \sup\{u(x) - u(y),\ u \in W^{1,\infty}(\Omega) : f(x, \nabla u(x)) \leq 1 \text{ a.e. in } \Omega\} \end{aligned}$$

Since f is convex, $d = \delta_\varphi$ with $\varphi = f^0$. By Proposition 1.6 $\varphi_d \leq \varphi$, and hence $\delta_{\varphi_d} \leq \delta_\varphi = d \leq \delta_{\varphi_d}$. By Proposition 1.7 d is intrinsic, and this concludes the proof. □

A natural question is whether a difference quotient R^d associated to a distance d can be expressed as a supremal functional of the type (2.3), with f possibly non convex in ξ. In the next example we will show that the fact that d is geodesic is not enough to ensure the supremality of R^d.

Example 2.6 Consider the distance $d \in D(\alpha', \beta')$ given in Example 1.8. It is easy to check that

$$\alpha \sup_\Omega |\nabla u| \leq R^d(u) \leq \beta \sup_\Omega |\nabla u|$$

for every $u \in W^{1,\infty}(\Omega)$, with $\alpha = \frac{1}{\beta'}$ and $\beta = \frac{1}{\alpha'}$.

We now prove that R^d can not be written as a supremal functional. Assume by contradiction that

$$R^d(u) = \sup_\Omega g(x, \nabla u(x)),$$

for some Carathéodory function g.

Claim: There exists a set N, with $|N| = 0$, such that $g(x, \xi) \leq \alpha|\xi|$ for every $x \in \Omega \backslash N$ and for every $\xi \in \mathbb{R}^N$. From this claim the conclusion follows immediately taking the function $u(x) = x_1$. In fact we have $R^d(u) = \beta$, which is in contradiction with $g(x, \nabla u(x)) \leq \alpha$ for a.e. $x \in \Omega$.

It remains to prove the claim. By the definition of d it is easy to see that for every $x \in \Omega \backslash S$ there exists a radius $r(x) > 0$ such that $d(x, y) = \beta'|x - y|$ in $B_{r(x)}(x) \times B_{r(x)}(x)$ (where $B_r(x)$ denote the ball of radius r and center x) and hence

$$R^d(u) = \alpha \sup_\Omega |\nabla u| \quad \forall u \in W^{1,\infty}(\Omega), \quad \text{with } \operatorname{supp} u \subseteq B_{r(x)}(x). \tag{2.6}$$

In order to prove the claim it is enough to prove that for any $\xi \in S^1$ there exists a set N_ξ, with $|N_\xi| = 0$, such that

$$g(x, \xi) \leq \alpha \quad \text{in } \Omega \setminus N_\xi. \tag{2.7}$$

Let then assume by contradiction that there exists a vector $\xi \in S^1$ and a set M_ξ, with $|M_\xi| > 0$, such that

$$g(x, \xi) > (\alpha + \varepsilon) \quad \text{in } M_\xi \tag{2.8}$$

for some $\varepsilon > 0$. Now fix a point x_0 in M_ξ such that for every positive r the set $M_\xi \cap B_r(x_0)$ has positive measure (this is always possible because a.e. $x \in M_\xi$ is of density one for M_ξ). We define the function

$$u(x) = \begin{cases} \xi(x - x_0) - r & \text{if } x \in B_r(x_0) \\ \min\left\{ \inf\limits_{y \in B_r(x_0)} (\xi(y - x_0) - r + |x - y|), 0 \right\} & \text{otherwise.} \end{cases}$$

Clearly if r is small enough, then $\operatorname{supp} u \subseteq B_{r(x_0)}(x_0)$ and thus by (2.6) we have $R^d(u) = \alpha \sup_\Omega |\nabla u| = \alpha$, while by (2.8) $\sup g(x, \nabla u(x)) \geq \alpha + \varepsilon$, which gives a contradiction and concludes the proof.

3 Relaxation of supremal functionals

In this section we give a relaxation formula for positively 1-homogeneous supremal functionals with respect to the strong convergence in $L^\infty(\Omega)$, in terms of difference quotient functionals. Given a positively 1-homogeneous supremal functional of the type (2.3), we denote by $\bar{F}$ its relaxation, i.e., its lower semicontinuous envelope with respect to the L^∞-convergence given by

$$\bar{F}(u) := \inf\left\{ \liminf_n F(u_n) : u_n \to u \text{ in } L^\infty(\Omega) \right\}. \tag{3.1}$$

We start observing that the convexity of f is not a necessary condition for the lower semicontinuity of F.

Remark 3.1 A positively 1-homogeneous supremal functional has in general not a unique representation. In fact, under the notation of Example 1.5, using Proposition 2.4 and Proposition 1.7 we deduce that

$$G(u) := \sup_\Omega \psi^0(x, \nabla u) = \sup_\Omega \varphi_d^0(x, \nabla u) \quad \text{for every } u \in W^{1,\infty}(\Omega),$$

and it is easy to check that

$$\psi^0(x, v) := \begin{cases} |v| & \text{if } x \in A, \\ \frac{1}{2}|v| & \text{otherwise,} \end{cases} \quad \text{while}$$

$$\varphi_d^0(x, v) := \begin{cases} |v| & \text{if } x \in A, \\ |v_2| & \text{if } |v_1| < |v_2| \text{ and } x \notin A, \\ |v_1| & \text{otherwise} \end{cases}$$

In particular the functional G is lower semicontinuous and can be represented by any function g, possibly non convex, such that $\psi^0 \leq g \leq \varphi_d^0$.

3.1 A counter example to $\bar{F}(u) = \sup_\Omega f^{**}(x, \nabla u)$

Now we give an example showing that in general the relaxation of F is not obtained through the convexification f^{**} of f with respect to ξ.

Example 3.2 Let us call $\mathcal{G}$ the set of all continuous functions $g : \mathbb{R}^n \to \mathbb{R}$, positively 1-homogeneous and satisfying $\alpha|\xi| \leq g(\xi) \leq \beta|\xi|$ for all $\xi \in \mathbb{R}^n$, and let

$$\mathcal{C} := \{C \subseteq \mathbb{R}^n : C = \{\xi \in \mathbb{R}^n : g(\xi) \leq 1\} \text{ for some } g \in \mathcal{G}\}. \tag{3.2}$$

Note that the sets in $\mathcal{C}$ are closed, star-shaped (with respect to the origin), and that by definition to every $C \in \mathcal{C}$ is associated a function $g \in \mathcal{G}$, which we denote by g_C. Moreover $\mathcal{C}$ is closed for intersection and union.

Let now B be the unit ball in $\mathbb{R}^n$ centered at 0. Then $B \in \mathcal{C}$ with $g_B(\xi) = |\xi|$. Let $H \in \mathcal{C}$ be satisfying the following properties:

1) H is not convex;
2) $H \backslash B \neq \emptyset$ and $B \backslash H \neq \emptyset$;
3) B is contained in the convex hull of H.

Finally let us construct an open and dense set $A \subset \Omega$ with $0 < |A| < |\Omega|$ as follows. Let $\{v_i\}_{i\in\mathbb{N}}$ be a dense subset of ∂B and let $\{p_j\}_{j\in\mathbb{N}}$ be dense in Ω. For a given positive constant $\delta > 0$ we define

$$A := \bigcup_{i,j\in\mathbb{N}} \left\{ x \in \Omega : \operatorname{dist}(x, \{p_i + s v_j, s \in \mathbb{R}\}) < \frac{\delta}{2^{ij}} \right\}.$$

Clearly, if δ is small enough, we have that $0 < |A| < |\Omega|$. Roughly speaking the set A is given by a countable union of thin strips along a dense set of directions.

We consider the functions $f, f_+ : \Omega \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x,\xi) := \begin{cases} g_B(\xi) & \text{if } x \in A; \\ g_H(\xi) & \text{if } x \in \Omega \setminus A. \end{cases} \qquad f_+(x,\xi) := \begin{cases} g_B(\xi) & \text{if } x \in A; \\ g_{H\cap B}(\xi) & \text{if } x \in \Omega \setminus A. \end{cases}$$

The associated supremal functionals are

$$F(u) := \sup_\Omega f(x, \nabla u(x)) \quad \text{and} \quad F_+(x,\xi) := \sup_\Omega f_+(x, \nabla u(x)).$$

Claim: $F = F_+$. Once the claim is proved we can conclude the argument as follows. By the fact that $H \cap B$ is closed, star shaped and strictly contained in B, it is possible to prove that there exists a vector $\xi \in B$ such that ξ is not in the convex hull of $H \cap B$. By property 3) above and by the definition of f_+ and the choice of ξ we have

$$f^{**}(x,\xi) \leq 1 \text{ a.e. on } \Omega \quad \text{while} \quad f_+^{**}(x,\xi) > 1 \text{ a.e. on } \Omega\backslash A. \tag{3.3}$$

Since by the claim $\sup_\Omega f_+^{**}(x, \nabla(\cdot)) \leq F(\cdot)$, and $\sup_\Omega f_+^{**}(x, \nabla(\cdot))$ is lower semicontinuous, we have

$$\sup_\Omega f_+^{**}(x, \nabla(\cdot)) \leq \bar{F}(\cdot).$$

On the other hand, by (3.3) we have

$$\sup_{\Omega} f_{+}^{**}(x, \nabla(\xi \cdot x)) > \sup_{\Omega} f^{**}(x, \nabla(\xi \cdot x)),$$

and therefore $\bar{F}$ is not represented by f^{**}.

It remains the proof of the claim. By construction we have that $F_{+} \geq F$, and so let us assume by contradiction that for some $u \in W^{1,\infty}(\Omega)$ we have

$$F_{+}(u) > 1 \quad \text{while} \quad F(u) < 1. \tag{3.4}$$

This will imply that $\nabla u \in H \setminus \bar{B}$ on a set of positive measure. Therefore there exists a point $x \in \Omega$ of differentiability for u with $|\nabla u(x)| > 1$. To simplify the notation we can assume $x = 0$ and $u(0) = 0$. Let $\{p_n\}$ be a sequence converging to zero, and for every n let us consider the function $u_n : B \to \mathbb{R}$ defined by

$$u_n(x) := \frac{1}{\rho_n} u(\rho_n, x) \quad \text{for every } x \in B.$$

By the definition of A, for every n and for every $\varepsilon > 0$ we can find an open strip L_n^{ε} in B such that $\rho_n L_n^{\varepsilon} \subset A$ and such that L_n^{ε} contains two points a_n^{ε} and b_n^{ε} with

$$\left| a_n^{\varepsilon} - \frac{\nabla u(0)}{|\nabla u(0)|} \right| + \left| b_n^{\varepsilon} - \left(-\frac{\nabla u(0)}{|\nabla u(0)|} \right) \right| \leq \varepsilon. \tag{3.5}$$

By Proposition 2.5, we have that

$$\sup_{L_n^{\varepsilon}} |\nabla u_n(x)| = \sup_{x,y \in L_n^{\varepsilon}} \frac{u_n(x) - u_n(y)}{|x-y|} \geq \frac{u_n(a_n^{\varepsilon}) - u_n(b_n^{\varepsilon})}{|a_n^{\varepsilon} - b_n^{\varepsilon}|}.$$

Using that, by the differentiability of u at 0, $\{u_n\}$ converges to $\nabla u(0) \cdot x$ uniformly, by (3.5) we deduce that, for n big enough,

$$\sup_{L_n^{\varepsilon}} |\nabla u_n(x)| \geq |\nabla u(0)| + o(\varepsilon),$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore, recalling that $|\nabla u(0)| > 1$ we can find ε and n such that $\sup_{L_n^{\varepsilon}} |\nabla u_n(x)| > 1$. We conclude that

$$F(u) \geq \sup_{A} |\nabla u(x)| \geq \sup_{\rho_n L_n^{\varepsilon}} |\nabla u(x)| = \sup_{L_n^{\varepsilon}} |\nabla u_n(x)| > 1,$$

which is in contradiction with (3.4).

Remark 3.3 The main idea of Example 3.2 is that the dense set A does not allow to perform a *zig-zag approximation* on $\Omega \setminus A$ of an affine function $\xi \cdot x$, which uses two gradients $\xi_1, \xi_2 \in H \setminus \bar{B}$, of which $\xi \in B$ is a convex combination.

Note that in Example 3.2 we don't even know whether the relaxation of F is a supremal functional.

3.2 The relaxation formula

In this paragraph we characterize the relaxation of positively 1-homogeneous supremal functionals in terms of associated difference quotient functionals R^{d_F}, where d_F is the distance associated to F as in (1.8). The key step is the the following approximation lemma.

Lemma 3.4 *Let F be a positively 1-homogeneous supremal functional on $W^{1,\infty}(\Omega)$ represented by a Carathéodory function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfying (2.4). Let $v \in W^{1,\infty}(\Omega)$ be such that $R^{d_F}(v) < 1$. Then there exists a sequence $\{v_n\} \subset W^{1,\infty}(\Omega)$ converging to v in $L^\infty(\Omega)$ with $F(v_n) \leq 1$.*

Naively the idea for the construction of the sequence v_n would be to consider a lattice of points on Ω, then in a neighborhood of each point p of the lattice we would like the function v_n to look like a small "cone" given by

$$v(p) + d_F(x, p).$$

Unfortunately, the difference quotient of the above function is less then 1 but it is not clear which is the value of F on it. So, starting from this idea, the right construction will require a finer argument.

Proof Let us fix a positive radius $r > 0$. By the fact that $R^{d_F}(v) < 1$, for every $x, y \in \Omega$ with $|x - y| = r$

$$v(y) - v(x) < d_F(x, y) - \gamma, \tag{3.6}$$

for a positive constant γ depending on r. Let us fix $0 < \varepsilon < \frac{\gamma}{3}$. For every $x \in \Omega$ and for every $y \in \partial B_r(x) \cap \Omega$ (where $B_r(x)$ denotes the ball of radius r centered at x), by the definition of d_F there exists a function $w_r^{x,y} \in W^{1,\infty}(\Omega)$ such that

1. $F(w_r^{x,y}) \leq 1$;
2. $w_r^{x,y}(y) \geq w_r^{x,y}(x) + d_F(x, y) - \varepsilon$;
3. $w_r^{x,y}(x) = v(x)$;

the third property being possible thanks to the translation invariance of the first two. By properties 2), 3) and by (3.6), for every $y \in \partial B_r(x) \cap \Omega$

$$w_r^{x,y}(y) \geq v(x) + d_F(x, y) - \varepsilon > v(y) + \gamma - \varepsilon. \tag{3.7}$$

Note that by property (2.4) we have that $\sup_\Omega |\nabla w_r^{x,y}| < 1/\alpha$, and hence there exists $\delta > 0$ (depending only on ε) such that

$$w_r^{x,y}(z) > v(z) + \gamma - 2\varepsilon > v(z) + \varepsilon \quad \text{for every } z \in \partial B_r(x) \cap \Omega : |z - y| \leq \delta. \tag{3.8}$$

Moreover, since $w_r^{x,y}(x) = v(x)$, there exists $0 < r' < r$ (depending only on ε) such that

$$w_r^{x,y}(z) < v(z) + \varepsilon \quad \text{for every } z \in B_{r'}(x) \cap \Omega. \tag{3.9}$$

For every $x \in \Omega$, let us fix a finite set of points $\{y_1, \ldots, y_N\}$ on $\partial B_r(x) \cap \Omega$ such that

$$\partial B_r(x) \cap \Omega \subset \bigcup_{i=1}^{N} B_\delta(y_i),$$

and let us set the function $w_r^x : B_r(x) \cap \Omega \to \mathbb{R}$ defined by

$$w_r^x(z) := \max_i w_r^{x,y_i}(z) \quad \text{for every } z \in B_r(x) \cap \Omega. \tag{3.10}$$

By construction and by (3.8) and (3.9), we have

1) $\sup_{B_r(x)\cap\Omega} f(z, \nabla w_r^x) \leq 1$;
2) $w_r^x(z) > v(z) + \varepsilon$ for every $z \in \partial B_r(x) \cap \Omega$;
3) $w_r^x(z) < v(z) + \varepsilon$ for every $z \in B_{r'}(x) \cap \Omega$.

Now let Z_r be a finite set of points of Ω such that

$$\Omega \subset \bigcup_{z \in Z_r} B_{r'}(z),$$

and consider the function $w_r : \Omega \to \mathbb{R}$ defined by

$$w_r(x) := \min_{z \in Z_r \cap B_r(x)} w_r^z(x). \tag{3.11}$$

By properties (2) and (3) above it follows that w_r is continuous. Moreover, for almost every x in Ω, $\nabla w_r(x)$ coincides with $\nabla w_r^z(x)$ for some $z \in Z_r$ and this implies that $w_r \in W^{1,\infty}(\Omega)$ and $F(w_r) \leq 1$.

Now let us prove that $\|w_r - v\|_{L^\infty(\Omega)} \to 0$. To this aim, let us fix $x \in \Omega$, and let $z \in B_r(x)$ be such that $w_r(x) = w_r^z(x)$. Recalling that by construction $w_r^z(z) = v(z)$, and using (1.9) we conclude

$$\begin{aligned} |w_r(x) - v(x)| &\leq \left|w_r^z(x) - w_r^z(z)\right| + \left|w_r^z(z) - v(x)\right| \\ &= \left|w_r^z(x) - w_r^z(z)\right| + \left|v(z) - v(x)\right| \\ &\leq 2d_F(x,z) \leq \frac{2C}{\alpha} r. \end{aligned}$$

Therefore, for every $\{r_n\} \to 0$, the sequence $v_n := w_{r_n}$ does the job. □

We are now in a position to give the representation formula for the relaxation.

Theorem 3.5 *Let F be a positively 1-homogeneous supremal functional represented by a function f satisfying property (2.4). Moreover let d_F be the distance associated to F as in (1.8) and let R^{d_F} be the corresponding difference quotient (see (2.1), with d replaced by d_F). Then the relaxation $\bar{F}$ of F with respect to the strong convergence in $L^\infty(\Omega)$ is given by*

$$\bar{F}(u) = R^{d_F}(u) \quad \forall u \in W^{1,\infty}(\Omega).$$

Proof By Proposition 2.2, the functional R^{d_F} is lower semicontinuous, and hence by the definition of the relaxation we get

$$R^{d_F} \leq \bar{F}.$$

In order to prove the inverse inequality, let $v \in W^{1,\infty}(\Omega)$ such that $R^{d_F}(v) < 1$. By Lemma 3.4 there exists a sequence $\{v_n\}$ converging to v in $L^\infty(\Omega)$ with $F(v_n) \leq 1$. In particular,

$$\bar{F}(v) = \inf_{u_n \to v} \liminf_n F(u_n) \leq \liminf_n F(v_n) \leq 1.$$

By the positively 1-homogeneity of $\bar{F}$ and R^{d_F}, and by the arbitrariness of the function satisfying $R^{d_F}(v) < 1$, the proof is concluded. □

The representation formula for the relaxation given by Theorem 3.5 is in accordance with the stability, in terms of Γ-convergence, of the class of difference quotients associated to geodesic distances, considered in the next section. On the other hand a natural question is whether the relaxation of F can be represented as a supremal functional. In view of Proposition 2.5 this is assured by the fact that d_F is an intrinsic distance and this is precisely stated in the following Corollary. We suspect that the distance d_F associated to any supremal functional F is always intrinsic but at the moment this question is still open.

Corollary 3.6 *Let F be a positively 1-homogeneous supremal functional represented by a function f satisfying* (2.4). *Assume that the distance d_F is intrinsic. Then the relaxation $\bar{F}$ of F in the strong topology of L^∞ is given by*

$$\bar{F}(u) = \sup_\Omega \varphi^0_{d_F}(x, \nabla u(x)) \quad \forall u \in W^{1,\infty}(\Omega),$$

where $\varphi^0_{d_F}$ is the support function of the derivative of the distance d_F associated to F as in (1.8)*, according to* (1.4) *and* (1.2)*. Moreover $\varphi^0_{d_F}$ satisfies* (2.4).

Proof The proof is an immediate consequence of Theorem 3.5 and Proposition 2.5. □

3.3 The distance d_F is geodesic

In this paragraph we prove the remarkable fact that the distance d_F is geodesic in Ω. This fact will be crucial in our main results of Sect. 4, on the characterization of the Γ-limit of sequences of supremal functionals.

We need some preliminary constructions. Let d be a distance equivalent to the Euclidean distance. We denote by d_* the distance defined by

$$d_*(x, y) = \inf_{\gamma \in \Gamma_{x,y}(\Omega)} \mathcal{L}_d(\gamma) \quad \text{for every } x, y \in \Omega, \tag{3.12}$$

where $\mathcal{L}_d(\gamma)$ is the length of γ with respect to the distance d, and $\Gamma_{x,y}(\Omega)$ denotes the set of all Lipschitz curves $\gamma : [0, 1] \to \Omega$, with $\gamma(0) = x$ and $\gamma(1) = y$.

Notice that by the fact that Ω is connected it follows that $d_*(x, y) < +\infty$ for any $x, y \in \Omega$. Moreover since $\partial\Omega$ is Lipschitz, there exists a constant $C \geq 1$ such that, chosen two arbitrary points $x, y \in \Omega$ we can find a curve γ which connect them such that the Euclidean length $\mathcal{L}(\gamma)$ of γ satisfies $\mathcal{L}(\gamma) \leq C|x - y|$. From this we can easily deduce that d_* is equivalent to the Euclidean distance and in particular d_* is bounded.

Proposition 3.7 *Let d be a distance equivalent to the Euclidean distance and let d_* be the distance defined by* (3.12). *Then d_* is the smallest geodesic distance greater than or equal to d.*

Proof Let us show that d_* is a geodesic distance. By definition, we have that $d_* > d$ and thus

$$d_*(x, y) = \inf_{\gamma \in \Gamma_{x,y}(\Omega)} \mathcal{L}_d(\gamma) \leq \inf_{\gamma \in \Gamma_{x,y}(\Omega)} \mathcal{L}_{d_*}(\gamma).$$

On the other hand, fix $\varepsilon > 0$ and $\tilde{\gamma} \in \Gamma_{x,y}(\Omega)$ such that $\mathcal{L}_d(\tilde{\gamma}) \leq d_*(x, y) + \varepsilon$. By the definition of $\mathcal{L}_d(\tilde{\gamma})$ there exist $k \in \mathbb{N}$, and $t_1 = 0,\ t_k = 1,\ t_i < t_{i+1}$, such that denoted by $\tilde{\gamma}_i$ the restriction of $\tilde{\gamma}$ to the interval $[t_i, t_{i+1}]$, we have

$$\begin{aligned} \inf_{\gamma \in \Gamma_{x,y}(\Omega)} \mathcal{L}_{d_*}(\gamma) \leq \mathcal{L}_{d_*}(\tilde{\gamma}) &\leq \sum_{i=1}^{k-1} d_*(\tilde{\gamma}(t_i), \tilde{\gamma}(t_{i+1})) + \varepsilon \\ &\leq \sum_{i=1}^{k-1} \mathcal{L}_d(\tilde{\gamma}_i) + \varepsilon = \mathcal{L}_d(\tilde{\gamma}) + \varepsilon \leq d_*(x, y) + 2\varepsilon. \end{aligned}$$

Then the reverse inequality follows by the arbitrariness of ε. It is also very easy to check that d_* is the smallest geodesic distance greater than or equal to d. □

To simplify the notation in what follows and according with Remark 1.1 it is convenient to extend d and d_* from $\Omega \times \Omega$ to $\bar{\Omega} \times \bar{\Omega}$. It is then easy to check that

$$d_*(x, y) = \inf_{\gamma \in \Gamma_{x,y}(\bar{\Omega})} \mathcal{L}_d(\gamma) = \min_{\gamma \in \Gamma_{x,y}(\bar{\Omega})} \mathcal{L}_d(\gamma) \quad \text{for every } x, y \in \bar{\Omega}. \tag{3.13}$$

The advantage of this extension is that the infimum in (3.13) is always achieved.

Now for every positive $\delta \in \mathbb{R}$ we construct an approximation $d^\delta : \Omega \times \Omega \to \mathbb{R}$ of d_*. For every $x \in \Omega$ we define recursively a partition of Ω as follows: we set $C_0^{\delta,x} := \{x\}$,

$$C_1^{\delta,x} := \{y \in \Omega : d(x, y) \leq \delta\}, \tag{3.14}$$

and, assuming to have defined $C_0^{\delta,x}, \ldots, C_{i-1}^{\delta,x}$, we set

$$C_i^{\delta,x} := \Big\{ y \in \Omega \setminus \bigcup_{j-1}^{i-1} C_j^{\delta,x} : d\big(y, C_{i-1}^{\delta,x}\big) \leq \delta \Big\}. \tag{3.15}$$

We define the function

$$d^\delta(x, y) := \delta(i - 1) + d\big(y, C_{i-1}^{\delta,x}\big) \quad \text{if } y \in C_i^{\delta,x}. \tag{3.16}$$

Lemma 3.8 *The sequence of functions $\{d^\delta\}$ uniformly converges to d_*.*

Proof We prove the lemma in two steps.

Step 1. For every $\delta > 0$ *we have* $d^\delta \leq d_*$.

Let $x, y \in \Omega$, let us fix δ, and let $\gamma \in \Gamma_{x,y}(\Omega)$. Moreover let $k \in \mathbb{N}$ be such that $y \in C_k^{\delta,x}$. Let us set

$$t_i := \inf\left\{t \in [0,1] : \gamma(t) \in C_i^{\delta,x}\right\} \quad \text{for every } 1 \leq i \leq k,$$

and $t_{k+1} = 1$. Notice that $t_i < t_{i+1}$, for every $1 \leq i \leq k$. Therefore we have that

$$d^\delta(x,y) \leq \sum_{i-1}^{k} d(\gamma(t_i), \gamma(t_{i+1})) \leq \mathcal{L}_d(\gamma). \tag{3.17}$$

By the arbitrariness of $\gamma \in \Gamma_{x,y}$ and by definition of d_* the step is proved.

Step 2. Up to a subsequence, d^δ *converges uniformly to some* d^0 *with* $d^0 \geq d_*$.

Since the boundary of Ω is Lipschitz regular, the distance d_* is bounded. By the previous step the sequence d^δ is uniformly bounded; therefore by Ascoli-Arzela Theorem d^δ converges uniformly, up to a subsequence, to some function d^0.

It remains to show that $d^0 \geq d_*$. Fix $x, y \in \Omega$. The conclusion follows if we construct a curve $\gamma \in \Gamma_{x,y}(\bar{\Omega})$ with $d^0(x,y) \geq \mathcal{L}_d(\gamma)$. For every $\delta > 0$, let $C_1^{\delta,x}, \dots C_{N_\delta}^{\delta,x}$, be the decomposition of Ω defined in (3.14) and (3.15) with $N_\delta \in \mathbb{N}$ such that $y \in C_{N_\delta}^{\delta,x}$. Let us set $p_0^\delta := x$, and $p_{N_\delta}^\delta := y$. By construction for every $i = 1, \dots, N_\delta - 1$ we can find points $p_i^\delta \in \partial C_i^{\delta,x}$ such that $d(p_i^\delta, p_{i+1}^\delta) = \delta$, if $i = 1, \dots, N_\delta - 2$, and $d(p_{N_\delta - 1}^\delta, y) = d(y, C_{N_\delta - 1}^{\delta,x})$. By the fact that Ω has Lipschitz regular boundary it follows that there exists a positive constant C, independent on δ and i, and a curve γ_i joining p_i^δ with p_{i+1}^δ, such that

$$\mathcal{L}(\gamma_i) \leq C\delta, \tag{3.18}$$

where $\mathcal{L}(\gamma_i)$ denotes the Euclidean length of γ_i. Joining these curves, we obtain a curve $\gamma_\delta : [0,1] \to \bar{\Omega}$ with end-points x and y. In view of Step 1, $N_\delta \delta$ is uniformly bounded; by (3.18) it follows that also γ_δ are uniformly bounded in length. Therefore, if γ_δ are parametrized with constant velocity, they are uniformly Lipschitz continuous, and hence as $\delta \to 0$ they converge uniformly to a Lipschitz continuous function $\gamma : [0,1] \to \bar{\Omega}$. Fixed $0 = t_1 < \dots < t_i < \dots < t_{k+1} = 1 \in [0,1]$, in view of (3.18) we can select points $p_{j_1}^\delta, \dots, p_{j_{k+1}}^\delta$ in $\{p_0^\delta, \dots, p_{N_\delta}^\delta\}$ such that

$$p_{j_i}^\delta \to \gamma(t_i) \quad \text{for every } 1 \leq i \leq k+1.$$

Therefore, we have

$$\sum_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i+1})) = \lim_{\delta \to 0} \sum_{i=1}^{k} d\big(p_{j_i}^\delta, p_{j_{i+1}}^\delta\big) \leq \lim_{\delta \to 0} d^\delta(x,y) = d^0(x,y). \tag{3.19}$$

Taking the supremum in (3.19) over all partitions of [0, 1], we obtain

$$\mathcal{L}_{d_*}(\gamma) \leq d^0(x, y),$$

and this concludes the proof of the step. The conclusion follows immediately combining Step 1 and Step 2. □

We are now in a position to prove the following theorem.

Theorem 3.9 *Let F be a positively 1-homogeneous supremal functional of the form* (2.3) *with f satisfying* (2.4). *Then the distance d_F defined in* (1.8) *is a geodesic distance in $D(\frac{1}{\beta}, \frac{1}{\alpha})$.*

Proof By definition (see (3.12)) we have that $d_F \leq (d_F)_*$. Let us prove that also the reverse inequality holds. Let us fix $x, y \in \Omega$ and a sequence $\delta_n \to 0$. Assume for the moment that the sets $C_i^{\delta_n,x}$ defined in (3.15) are Lipschitz regular. Let us first construct a sequence $\{u_n\} \subset W^{1,\infty}(\Omega)$, with $F(u_n) \leq 1$, such that

$$(u_n(y) - u_n(x)) - d_F^{\delta_n}(x, y) \to 0, \tag{3.20}$$

where $d_F^{\delta_n}$ is defined as in (3.16), with d replaced by d_F and δ replaced by δ_n. To this aim, let $v_n : \Omega \to \mathbb{R}$ be the function $z \to d_F^{\delta_n}(x, z)$. By construction, it is easy to verify that for every i

$$\sup_{p,q \in C_i^{\delta_n,x}} \frac{v_n(p) - v_n(q)}{d_F(p, q)} = 1.$$

Now denote by F_i the restriction of F on $C_i^{\delta_n,x}$. More precisely

$$F_i(u) = \sup_{C_i^{\delta_n,x}} f(x, \nabla u(x)) \quad \forall u \in W^{1,\infty}\big(C_i^{\delta_n,x}\big).$$

By the definition of the intrinsic distance (1.8) we have $d_{F_i}(x, y) > d_F(x, y)$, for every $x, y \in C_i^{\delta_n,x}$, and thus

$$\sup_{p,q \in C_i^{\delta_n,x}} \frac{v_n(p) - v_n(p)}{d_{F_i}(p, q)} \leq 1\,.$$

In view of the Lipschitz regularity of the sets $C_i^{\delta_n,x}$ we may apply Theorem 3.5 to F_i and obtain that $\bar{F}_i(v_n) \leq 1$. Therefore there exists a sequence $\{u_h^{n,i}\}$ in $W^{1,\infty}(C_i^{\delta_n,x})$ such that

$$u_h^{(n,i)} \to v_n \text{ uniformly on } C_i^{\delta_n,x} \quad \text{and} \quad \mathrm{F}_i\big(u_h^{n,i}\big) \leq 1, \tag{3.21}$$

the second property being guaranteed by the 1-homogeneity of F. For every n and i, let $h(n, i)$ be the index such that

$$\big|u_{h(n,i)}^{n,i} - v_n\big| \leq \delta_n/2n \quad \text{on } C_i^{\delta_n,x}. \tag{3.22}$$

Denote $u^{n,i} := u^{n,i}_{h(n,i)}$. We are now in a position to construct the approximating sequence $\{u_n\}$. Since $v_n = \text{const.} = i\delta_n$ on $\partial C_i^{\delta_n,x} \setminus \partial\Omega$ and $u^{n,i}$ are close to v_n, in order to glue the functions $u^{n,i}$ it is enough to slightly translate and then truncate. Namely

$$\begin{aligned} u_n(y) := (u^{n,i}(y) - 2(i-1)\delta_n/n) \vee ((i-1)\delta_n - (2i-3)\delta_n/n) \\ \wedge(i\delta_n - (2i-1)\delta_n/n) \quad \text{if } y \in C_i^{\delta_n,x}. \end{aligned}$$

Using that $|\partial C_i^{\delta_n,x}| = 0$, for every i and n, we have that $F(u_n) = \max_i F_i(u^{n,i})$ and hence, by (3.21), we have $F(u_n) \le 1$. Moreover for every $y \in \Omega$

$$|(u_n(y) - u_n(x)) - d_F^{\delta_n}(x,y)| = |u_n(y) - v_n(y)| \le 2N_{\delta_n}\delta_n/n,$$

which tends to zero as $n \to \infty$ and then (3.20) is proved. Now, by definition (1.8), we have $d_F(x,y) \ge u_n(y) - u_n(x)$ for any n. Therefore, using (3.20) and Lemma 3.8 we obtain

$$d_F(x,y) \ge \lim_{n\to\infty} u_n(y) - u_n(x) = \lim_{n\to\infty} d_F^{\delta_n}(x,y) = (d_F)_*(x,y).$$

The proof that d_F is geodesic is then concluded in the case where the sets $C_i^{\delta_n,x}$ are Lipschitz regular.

In the general case we need to slightly modify the argument. When we introduce the functionals F_i we have to replace the set $C_i^{\delta_n,x}$ with a Lipschitz regular set $\tilde{C}_i^{\delta_n,x} \subseteq C_i^{\delta_n,x}$, with the property that the Hausdorff distance between $\partial\tilde{C}_i^{\delta_n,x} \setminus \partial\Omega$ and $\partial C_i^{\delta_n,x} \setminus \partial\Omega$ is smaller then ε_n, for a suitable choice of ε_n. More precisely we define

$$F_i(u) = \sup_{\tilde{C}_i^{\delta_n,x}} f(x, \nabla u(x)) \quad \forall u \in W^{1,\infty}\big(\tilde{C}_i^{\delta_n,x}\big).$$

Then we can apply Theorem 3.5 and obtain the functions $u^{n,i}$ defined on $\tilde{C}_i^{\delta_n,x}$. If ε_n is small enough, we have

$$\big|u^{n,i}_{h(n,i)} - i\delta_n\big| \le \delta_n/n \quad \text{on } \partial\tilde{C}_i^{\delta_n,x} \big\backslash \partial\Omega$$

This permits to construct as above, by translation and truncation, a function u_n defined on $\cup_i \tilde{C}_i^{\delta_n,x}$ which is constant on $\partial\tilde{C}_i^{\delta_n,x} \setminus \partial\Omega$. This function can be easily extended to a function in $W^{1,\infty}(\Omega)$ which is locally constant on $\Omega \setminus \cup_i \tilde{C}_i^{\delta_n,x}$ and clearly still satisfies (3.20). The conclusion follows as above.

Finally the fact that $d_F \in D(\frac{1}{\beta}, \frac{1}{\alpha})$ follows by extending d_F to $\bar{\Omega} \times \bar{\Omega}$ (see Remark 1.1) and the growth condition (2.4) together with the fact that

$$|x-y|_\Omega = \sup\{u(x)-u(y) : \sup_\Omega |\nabla u| \le 1\}\,. \qquad \square$$

4 Γ-convergence of supremal functionals

The main result of this section is that the closure of 1-homogeneous supremal functionals with respect to Γ-convergence is given by the class of difference quotient functionals associated to a geodesic distance. The following proposition states the Γ-convergence of difference quotients whenever the corresponding distances uniformly converge.

Proposition 4.1 *Let $\{d_n\}$ be a sequence of distances in $D(\alpha', \beta')$. Assume that $\{d_n\}$ converge to some distance $d_\infty \in D(\alpha', \beta')$. Then the functionals R^{d_n} defined in* (2.1) *Γ-converge to R^{d_∞} in $W^{1,\infty}(\Omega)$ with respect to the strong convergence in L^∞.*

Proof Let us prove that, for any sequence $\{u_n\}$ in $W^{1,\infty}(\Omega)$ converging to some u uniformly, we get

$$\liminf_{n\to\infty} R^{d_n}(u_n) \geq R^{d_\infty}(u) \tag{4.1}$$

For every $x, y \in \Omega$ we have that

$$\liminf_{n\to\infty} R^{d_n}(u_n) \geq \liminf_{n\to\infty} \frac{u_n(x) - u_n(y)}{d_n(x, y)} = \frac{u(x) - u(y)}{d_\infty(x, y)}.$$

Taking the supremum in x and y we obtain (4.1).

Now let $u \in W^{1,\infty}(\Omega)$ and assume $R^{d_\infty}(u) < +\infty$. Let us define $u_n \in W^{1,\infty}(\Omega)$ by

$$u_n(x) = \inf_{y\in\Omega}[u(y) + R^{d_\infty}(u) d_n(x, y)]. \tag{4.2}$$

For every ε there exists $z_n \in \Omega$ such that

$$\begin{aligned} 0 &\leq u(x) - u_n(x) = u(x) - \inf_{y\in\Omega}[u(y) + R^{d_\infty}(u) d_n(x, y)] \\ &\leq u(x) - u(z_n) - R^{d_\infty}(u) d_n(x, z_n) + \varepsilon \leq R^{d_\infty}(u)(d_\infty(x, z_n) - d_n(x, z_n)) + \varepsilon. \end{aligned}$$

Thus $\{u_n\}$ converges uniformly to u. On the other hand, by the definition of u_n it is easy to see that $R^{d_n}(u_n) \leq R^{d_\infty}(u)$ for every n, and hence also the Γ-limsup inequality holds. □

Remark 4.2 Note that Proposition 4.1 holds true even if the sequence d_n is a sequence of (not necessarily geodesic) distances on Ω, satisfying $d_n(x, y) \leq M|x - y|$ for every $x, y \in \Omega$ for some positive constant M, and such that d_n converge uniformly to some function d_∞ in $\Omega \times \Omega$.

We immediately deduce the following Γ-convergence result for 1-homogeneous supremal functionals

Theorem 4.3 *Let $F_n : W^{1,\infty}(\Omega) \to \mathbb{R}$ be a sequence of positively 1-homogeneous supremal functionals defined by*

$$F_n(u) := \sup_{\Omega} f_n(x, \nabla u(x)) \quad \textit{for every } u \in W^{1,\infty}(\Omega), \tag{4.3}$$

where f_n *are Carathéodory functions satisfying*

$$\alpha|\xi| \le f_n(x,\xi) \le \beta|\xi| \quad \text{for every } \xi \in \mathbb{R}^n, \quad \textit{for a.e. } x \in \mathbb{R}^n. \tag{4.4}$$

Then there exists a subsequence (still labeled by n) and a difference quotient functional R^d*, with* $d \in D(\alpha', \beta')$*, such that* F_n Γ*-converges to* R^d *in* $W^{1,\infty}(\Omega)$ *with respect to the strong convergence in* $L^\infty(\Omega)$*.*

Proof Clearly we have that F_n Γ-converges to R^d if and only if the relaxation $\bar{F}_n$ of F_n Γ-converges to R^d and hence it is enough to prove the statement for the sequence $\bar{F}_n$.

By Theorem 3.5 we know that the relaxation of F_n is given by $\bar{F}_n = R^{d_n}$, where $d_n = d_{F_n}$ denotes the distance defined by (1.8) corresponding to F_n. By Theorem 3.9 d_n is a geodesic distance in $D(\alpha', \beta')$. By Proposition 1.2, there exists a subsequence (still denoted by d_n) and a distance $d \in D(\alpha', \beta')$ such that $\{d_n\}$ converges uniformly to d. Therefore by Proposition 4.1 the functionals R^{d_n} Γ-converge to R^d in $W^{1,\infty}(\Omega)$ with respect to the strong convergence in L^∞ and this concludes the proof. □

Next Theorem establishes that the class of difference quotients associated to a geodesic distance is the closure of 1-homogeneous supremal functionals with respect to Γ-convergence.

Theorem 4.4 *Let* R^d *be a difference quotient functional associated to a distance* $d \in D(\alpha', \beta')$*. Then there exists a sequence of 1-homogeneous supremal functionals* F_n *of the type* (4.3)*, with* f_n *satisfying* (4.4)*, such that* F_n Γ*-converge to* R^d *in* $W^{1,\infty}(\Omega)$ *with respect to the strong convergence in* $L^\infty(\Omega)$*.*

Proof In view of Corollary 3.6 and Proposition 4.1 the proof reduces to approximate the distance d by a sequence of intrinsic distances $d_n \in \tilde{D}(\alpha', \beta')$ with respect to the uniform convergence. This is a consequence of the fact that every geodesic distance can be approximated by distances associated to smooth Finsler metrics satisfying the same bounds, as proved in [12] [Theorem 4.1]. □

Remark 4.5 (The class of supremal functionals is not closed under Γ-convergence). Note that Theorem 4.4 implies that the class of 1-homogeneous supremal functional is not closed with respect to Γ-convergence. In fact, let $\Omega = (-1,1)^2$ and let d be the non intrinsic distance given in Example 1.8. In Example 2.6 we proved that R^d can not be written in a supremal form. On the other hand, by Theorem 4.4 there exists a sequence of supremal functionals F_n Γ-converging to R^d in $W^{1,\infty}(\Omega)$ with respect to the strong convergence in $L^\infty(\Omega)$.

An explicit sequence of functionals F_n Γ-converging to R^d is the following: let $S = (-1,1) \times \{0\}$ and for every $n \in \mathbb{N}$ let $S_n := (-1,1) \times (-1/n, 1/n)$. Let F_n be the supremal functionals associated to the functions f_n defined by

$$f_n(x,\xi) := \begin{cases} \beta|\xi| & \text{if } x \in S_n, \text{for every } \xi \in \mathbb{R}^n; \\ \alpha|\xi| & \text{if } x \in \Omega \backslash S_n, \text{for every } \xi \in \mathbb{R}^n. \end{cases}$$

It is easy to check that $d_{F_n} \to d$ uniformly, and hence the functionals F_n Γ-converge to R^d in $W^{1,\infty}(\Omega)$ with respect to the strong convergence in $L^\infty(\Omega)$.

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