Calc. Var. (2006) 25(4): 469–489
DOI 10.1007/s00526-005-0350-9

Calculus of Variations

Jean-Christophe Bourgoin

The minimality of the map $\frac{x}{\|x\|}$ for weighted energy

Received: 3 April 2004 / Accepted: 2 March 2005 / Published online: 22 February 2006

Abstract In this paper, we investigate the minimality of the map $\frac{x}{\|x\|}$ from the Euclidean unit ball $\mathbf{B}^n$ to its boundary $\mathbb{S}^{n-1}$ for weighted energy functionals of the type $E_{p,f} = \int_{\mathbf{B}^n} f(r)\|\nabla u\|^p dx$, where f is a non-negative function. We prove that in each of the two following cases:

i) $p = 1$ and f is non-decreasing,
ii) p is integer, $p \leq n-1$ and $f = r^\alpha$ with $\alpha \geq 0$,
the map $\frac{x}{\|x\|}$ minimizes $E_{p,f}$ among the maps in $W^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1})$ which coincide with $\frac{x}{\|x\|}$ on $\partial\mathbf{B}^n$. We also study the case where $f(r) = r^\alpha$ with $-n+2 < \alpha < 0$ and prove that $\frac{x}{\|x\|}$ does not minimize $E_{p,f}$ for α close to $-n+2$ and when $n \geq 6$, for α close to $4-n$.

Mathematics Subject Classification (2000) 58E20; 53C43

Keywords Minimizing map · p-harmonic map · p-energy · Weighted energy

1 Introduction and statement of results

For $n \geq 3$, the map $u_0(x) = \frac{x}{\|x\|} : \mathbf{B}^n \longrightarrow \mathbb{S}^{n-1}$ from the unit ball $\mathbf{B}^n$ of $\mathbb{R}^n$ to its boundary $\mathbb{S}^{n-1}$ plays a crucial role in the study of certain natural energy functionals. In particular, since the works of Hildebrandt, Kaul and Widman ([13]), this map is considered as a natural candidate to realize, for each real number $p \in [1, n)$ the minimum of the p-energy functional,

$$E_p(u) = \int_{\mathbf{B}^n} \|\nabla u\|^p \, dx$$

J.-C. Bourgoin (✉)
Laboratoire de Mathematiques et Physiques Théorique, Université de Tours, Parc Grandmont 37200 TOURS, France
E-mail: Jean-Christophe.Bourgoin@etu.univ.tours.fr

among the maps $u \in W^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1}) = \{u \in W^{1,p}(\mathbf{B}^n, \mathbb{R}^n; \|u\| = 1\ a.e.\}$ satisfying $u(x) = x$ on $\mathbb{S}^{n-1}$.

This question was first treated in the case $p = 2$. Indeed, the minimality of u_0 for E_2 was etablished by Jäger and Kaul ([16]) in dimension $n \geq 7$ and by Brezis, Coron and Lieb in dimension 3 ([2]). In [5], Coron and Gulliver proved the minimality of u_0 for E_p for any integer $p \in \{1, \ldots, n-1\}$ and any dimension $n \geq 3$.

Lin ([17]) has introduced the use of the elegant null Lagrangian method (or calibration method) in this topic. Avellaneda and Lin showed the efficiency of this method in [1] where they give a simpler alternative proof to the Coron-Gulliver result. Note that several results concerning the minimizing properties of p-harmonic diffeomorphisms were also obtained in this way in particular by Coron, Helein and El Soufi, Sandier ([4], [12], [7] and [6]).

The case of non-integer p seemed to be rather difficult. It is only ten years after the Coron-Gulliver article [5], that Hardt, Lin and Wang ([10]) succeeded to prove that, for all $n \geq 3$, the map u_0 minimizes E_p for $p \in [n-1, n)$. Their proof is based on a deep studies of singularities of harmonic and minimizing maps made in the last two decades. In dimension $n \geq 7$, Wang ([20]) and Hong ([14]) have independently proved the minimality of u_0 for any $p \geq 2$ satisfying $p + 2\sqrt{p} \leq n - 2$.

In [15], Hong remarked that the minimality of the p-energy E_p, $p \in (2, n-1]$, is related to the minimization of the following weighted 2-energy:

$$\tilde{E}_p(u) = \int_{\mathbf{B}^n} r^{2-p} \|\nabla u\|^2 dx$$

where $r = \|x\|$. Indeed, using Hlder inequality, it is easy to see that if the map u_0 minimizes $\tilde{E}_p$, then it also minimizes E_p (see [15], p. 465). Unfortunately, as we will see in Corollary 1.1 below, for many values of $p \in (2, n)$, the map u_0 is not a minimizer of $\tilde{E}_p$. Therefore, Theorem 6 of ([15]), asserting that u_0 minimizes $\tilde{E}_p$ seems to be not correct and the question of whether u_0 is a minimizing map of the p-energy E_p for non-integer $p \in (2, n-1)$ is still open[1]

The aim of this paper is to study the minimizing properties of the map u_0 in regard to some weighted energy functionals of the form:

$$E_{p,f}(u) = \int_{\mathbf{B}^n} f(r) \|\nabla u\|^p dx,$$

where $p \in \{1, \ldots, n-1\}$ and $f : [0, 1] \to \mathbb{R}$ is a non-negative non-decreasing continuous function. For $p = 1$, the map u_0 minimizes $E_{1,f}$ for a large class of weights. Indeed, we have the following

Theorem 1.1 *Suppose that f is a non-negative differentiable non-decreasing function. Then the map $u_0 = \frac{x}{\|x\|}$ is a minimizer of the energy $E_{1,f}$, that is, for*

[1] We suspect a problem in Theorem 6 p. 464 of [15]. Indeed the author claims that the quantity $G_{\varphi_1^0,\ldots,\varphi_{n-1}^0}(v, p)$, which represents a weighted energy of the map v on the 3-dimensional cone $\mathcal{C}_0$ in $\mathbf{B}^n$, is uniformly proportional to the weighted energy on the euclidian ball $\mathbf{B}^3$. There is no reason for this fact to be true, the orthogonal projection of $\mathcal{C}_0$ on to $\mathbf{B}^n$ being not homothetic.

any u in $W^{1,1}(\mathbf{B^n}, \mathbb{S}^{n-1})$ *with* $u(x) = x$ *on* $\mathbb{S}^{n-1}$, *we have*

$$\int_{\mathbf{B}^n} f(r)\|\nabla u_0\| dx \leq \int_{\mathbf{B}^n} f(r)\|\nabla u\| dx,$$

Moreover, if f *has no critical points in* $(1, 1)$, *then the map* $u_0 = \frac{x}{\|x\|}$ *is the unique minimizer of the energy* $E_{1,f}$, *that is, the equality in the last inequality holds if and only if* $u = u_0$.

For $p \geq 2$, we restrict ourselves to power functions $f(r) = r^{\alpha}$,

Theorem 1.2 *For any* $\alpha \geq 0$ *and any integer* $p \in \{1, \ldots, n-1\}$, *the map* $u_0 = \frac{x}{\|x\|}$ *is a minimizer of the energy* $E_{p,r^{\alpha}}$ *that is, for any u in* $W^{1,p}(\mathbf{B^n}, \mathbb{S}^{n-1})$ *with* $u(x) = x$ *on* $\mathbb{S}^{n-1}$, *we have,*

$$\int_{\mathbf{B}^n} r^{\alpha}\|\nabla u_0\|^p dx \leq \int_{\mathbf{B}^n} r^{\alpha}\|\nabla u\|^p dx\,.$$

Moreover, if $\alpha > 0$, *then the map* $u_0 = \frac{x}{\|x\|}$ *is the unique minimizer of the energy* $E_{p,r^{\alpha}}$, *that is the equality in the last inequality holds if and only if* $u = u_0$.

The proof of these two theorems is given in Sect. 2. It is based on a construction of an adapted null-Lagrangian. The case of $p = 1$ can be obtained passing through more direct ways and will be treated independently.

The case of weights of the form $f(r) = r^{\alpha}$, with $\alpha < 0$, is treated in Sect. 3. The weighted energy $\int_{\mathbf{B}^n} r^{\alpha}\|\nabla u_0\|^2 dx$ of $u_0 = \frac{x}{\|x\|}$ is finite for $\alpha > -n+2$. Hence we consider the family of maps,

$$u_a(x) = a + \lambda_a(x)(x-a), \quad a \in \mathbf{B}^n,$$

where $\lambda_a(x) \in \mathbb{R}$ is chosen such that $u_a(x) \in \mathbb{S}^{n-1}$ (that is $u_a(x)$ is the intersection point of $\mathbb{S}^{n-1}$ with the half-line of origin a passing by x).

We study the energy $E_{2,r^{\alpha}}(u_a)$ of these maps and deduce the following theorem.

Theorem 1.3 *Suppose that* $n \geq 3$.

(i) For any $a \in \mathbf{B}^n$, $a \neq 0$, *there exists a negative real number* $\alpha_0 \in (-n+2, 0)$, *such that, for any* $\alpha \in (-n+2, \alpha_0]$ *we have*

$$\int_{\mathbf{B}^n} r^{\alpha}\|\nabla u_0\|^2 dx > \int_{\mathbf{B}^n} r^{\alpha}\|\nabla u_a\|^2 dx.$$

(ii) For any integer $n \geq 6$, *there exists* $\alpha_0 \in (4-n, 5-n)$ *such that, for any* $\alpha \in (4-n, \alpha_0)$, *there exists* $a \in \mathbf{B}^n$ *such that,*

$$\int_{\mathbf{B}^n} r^{\alpha}\|\nabla u_0\|^2 dx > \int_{\mathbf{B}^n} r^{\alpha}\|\nabla u_a\|^2 dx.$$

Replacing in Theorem 1.3 α by $2-p$, $p \in (2, n)$, we obtain the following corollary:

Corollary 1.1 *For any* $n \geq 6$, *there exists* $p_0 \in (n-3, n-2)$ *such that, for any* $p \in (p_0, n-2)$ *the map* $u_0 = \frac{x}{\|x\|}$ *does not minimize the functional* $\int_{\mathbf{B}^n} r^{2-p}\|\nabla u\|^2 dx$ *among the maps* $u \in W^{1,2}(\mathbf{B}^n, \mathbb{S}^{n-1})$ *satisfying* $u(x) = x$ *on* $\mathbb{S}^{n-1}$.

2 Proof of theorems 1.1 and 1.2

Consider an integer $p \in \{1, \ldots, n-1\}$ and f a differentiable, non-negative, increasing, and non-identically zero map. We can suppose without loss of generality, that $f(1) = 1$.
For any subset $I = \{i_1, \ldots, i_p\} \subset \{1, \ldots, n-1\}$ with $i_1 < i_2 \ldots < i_p$ and for any map,

$$u = (u_1, \ldots, u_n) : \mathbf{B}^n \longrightarrow \mathbb{S}^{n-1} \quad \text{in } \mathcal{C}^\infty(\mathbf{B}^n, \mathbb{S}^{n-1}) \quad \text{with } u(x) = x \text{ on } \mathbb{S}^{n-1},$$

we consider the n-form:

$$\omega_I(u) = dx_1 \wedge \ldots \wedge d(f(r)u_{i_1}) \wedge \ldots \wedge d(f(r)u_{i_k}) \wedge \ldots \wedge dx_n$$

Lemma 2.1 *We have the identity:*

$$\int_{\mathbf{B}^n} \omega_I(u) = \int_{\mathbf{B}^n} \omega_I(Id) \quad \forall x \in \mathbf{B}^n \quad \textit{where} \quad Id(x) = x.$$

Proof By Stokes theorem, we have:

$$\begin{aligned}
\int_{\mathbf{B}^n} \omega_I(u) &= \int_{\mathbf{B}^n} dx_1 \wedge \ldots \wedge d(f(r)u_{i_1}) \wedge \ldots \wedge d(f(r)u_{i_p}) \wedge \ldots \wedge dx_n \\
&= \int_{\mathbf{B}^n} (-1)^{i_1-1} d(f(r)u_{i_1} dx_1 \wedge \ldots \wedge d(\widehat{f(r)u_{i_1}}) \\
&\qquad \wedge \ldots \wedge d(f(r)u_{i_p}) \wedge \ldots \wedge dx_n) \\
&= \int_{\mathbb{S}^{n-1}} (-1)^{i_1-1} x_{i_1} dx_1 \wedge \ldots \wedge d(\widehat{f(r)u_{i_1}}) \\
&\qquad \wedge \ldots \wedge d(f(r)u_{i_p}) \wedge \ldots \wedge dx_n.
\end{aligned}$$

Indeed, on $\mathbb{S}^{n-1}$, we have $f(r)u_{i_1} = x_{i_1}$ ($r = 1$, $f(1) = 1$ and $u(x) = x$). Iterating, we get the designed identities. Consider the n-form:

$$S(u) = \sum_{|I|=p} w_I(u)$$

By Lemma 2.1, we have:

$$\int_{\mathbf{B}^n} S(u) = \sum_{|I|=p} \int_{\mathbf{B}^n} w_I(u) = \sum_{|I|=p} \int_{\mathbf{B}^n} dx = C_n^p \frac{|\mathbb{S}^{n-1}|}{n},$$

where $|\mathbb{S}^{n-1}|$ is the Lebesgue measure of the sphere. □

Lemma 2.2 *The n-form $S(u)$ is $O(n)-equivariant$, that is, for any rotation R in $O(n)$, we have:*

$$S({}^tRuR)({}^tRx) = S(u)(x) \quad \forall x \in \mathbf{B}^n.$$

Proof Consider $S(u)(x)(e_1, \ldots, e_n)$ where $(e_1, \ldots, e_n)$ is the stantard basis of $\mathbb{R}^n$ and notice that it is equal to $(-1)^n$ times the $(p+1)^{th}$ coefficient of the polynomial $P(\lambda) = \det(Jac(fu)(x) - \lambda Id)$ which does not change when we replace fu by ${}^tRfuR$. □

For any $x \in \mathbf{B}^n$, let $R \in O(n)$ be such that ${}^tRu(x) = e_n = (0, \ldots, 0, 1)$. Consider $y = {}^tRx$, $v = {}^tRuR$, so that:

$$v(y) = e_n, \quad d({}^tRuR)(y)(\mathbb{R}^n) \subset e_n^\perp \quad \text{that is} \quad \frac{\partial v_n}{\partial x_j}(y) = 0 \quad \forall j \in \{1, \ldots, n\}.$$

Lemma 2.3 *Let $a_1, \ldots, a_n$ be n non-negative numbers, and $p \in \{1, \ldots, n-1\}$. Then:*

$$\sum_{i_1 < \ldots < i_p} a_{i_1} \ldots a_{i_p} \le \frac{1}{(n-1)^p} C_{n-1}^p \left(\sum_{j=1}^{n-1} a_j \right)^p .$$

Proof See for instance Hardy coll. [4], theorem 52. □

Let $I = \{i_1, \ldots, i_p\} \subset \{1, \ldots, n\}$. We have:
if $i_p \neq n$,

$$\begin{aligned}\omega_I(v)(y) &= \big(dx_1 \wedge \ldots \wedge d(f(r)v_{i_1}) \wedge \ldots \wedge d(f(r)v_{i_k}) \wedge \ldots \wedge dx_n\big)(y) \\ &= |f(r)|^p \big(dx_1 \wedge \ldots \wedge dv_{i_1} \wedge \ldots \wedge dv_{i_k} \wedge \ldots \wedge dx_n\big)(y).\end{aligned}$$

Indeed, $\forall j \le n-1$, $d(f(r)v_j(y)) = d(f(r))v_j(y) + f(r)dv_j(y) = f(r)dv_j(y)$ since $v(y) = e_n$.
If $i_p = n$,

$$\omega_I(v)(y) = |f(r)|^{p-1}\big(dx_1 \wedge \ldots \wedge dv_{i_1} \wedge \ldots \wedge df\big)(y).$$

Indeed, $d(f(r)v_n)(y) = df(y)v_n(y) + f(r)dv_n(y) = df(y)$ (as $dv(y) \subset e_n^\perp$). The Hadamard inequality gives:

$$\begin{aligned}|S(v)(y)| = \Big|\sum_{|I|=p} \omega_I(v)(y)\Big| &\le |f(r)|^p \sum_{1 \le i_1 < i_2 < \ldots < i_p \le n-1} \|dx_1\| \ldots \|dv_{i_1}\| \\ &\qquad \ldots \|dv_{i_p}\| \ldots \|dx_n\|(y) \\ &\quad + |f(r)|^{p-1} \sum_{1 \le i_1 < i_2 < \ldots < i_{p-1} \le n-1} \|dx_1\| \ldots \|dv_{i_1}\| \\ &\qquad \ldots \|dv_{i_p}\| \ldots \|df\|(y) \\ &\le |f(r)|^p \Bigg(\sum_{1 \le i_1 < i_2 < \ldots < i_p \le n-1} \|dx_1\|^2 \ldots \|dv_{i_1}\|^2 \\ &\qquad \ldots \|dv_{i_p}\|^2 \ldots \|dx_n\|^2(y) \Bigg)^{\frac{1}{2}} \big(C_n^p\big)^{\frac{1}{2}} \\ &\quad + f'(r)f(r)^{p-1} \sum_{1 \le i_1 < i_2 < \ldots < i_{p-1} \le n-1} \|dx_1\| \ldots \|dv_{i_1}\| \\ &\qquad \ldots \|dv_{i_p}\|(y).\end{aligned}$$

The Hardy inequality gives, after integration and using the fact that $\|\nabla u\| = \|\nabla v\|$,

$$\frac{C_n^p}{n}|\mathbb{S}^{n-1}| \leq \frac{C_{n-1}^p}{(n-1)^{p/2}} \int_{\mathbf{B}^n} f^p(r)\|\nabla u\|^p dx + \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \int_{\mathbf{B}^n} f'(r) f^{p-1}(r)\|\nabla u\|^{p-1} dx. \quad (1)$$

Remark : If f' is positive and if equality holds in (1), then, $\forall i \leq n-1$, $y_i = 0$ and $y_n = \pm\frac{x}{\|x\|}$, which implies that $u(x) = \pm\frac{x}{\|x\|}$.

Proof of the Theorem 1.1 Inequality (1) give

$$|\mathbb{S}^{n-1}| \leq \sqrt{n-1} \int_{\mathbf{B}^n} f(r)\|\nabla u\| dx + \int_{\mathbf{B}^n} f'(r)\, dx.$$

Hence:

$$\int_{\mathbf{B}^n} f\|\nabla u\| dx \geq \frac{|\mathbb{S}^{n-1}|}{\sqrt{n-1}} \left(1 - \int_0^1 f'(r) r^{n-1} dr\right)$$

$$\int_{\mathbf{B}^n} f\|\nabla u\| dx \geq \sqrt{n-1}|\mathbb{S}^{n-1}| \int_0^1 f(r) r^{n-2} dr = \int_{\mathbf{B}^n} f(r)\|\nabla u_0\|\, dx.$$

To see the uniqueness il suffices to refer to the remark above. It gives that for any $x \in \mathbf{B}^n$, $u(x) = \frac{x}{\|x\|}$ or $u(x) = -\frac{x}{\|x\|}$. As $u(x) = x$ on the unit sphere, we have, for any $x \in \mathbf{B}^n \backslash \{0\}$, $u(x) = \frac{x}{\|x\|}$. □

Proof of the Theorem 1.2 Let α be a positive real number. From inequality (1) we have:

$$\frac{C_n^p}{n}|\mathbb{S}^{n-1}| \leq \frac{C_{n-1}^p}{(n-1)^{p/2}} \int_{\mathbf{B}^n} r^{\alpha p}\|\nabla u\|^p dx + \alpha \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \int_{\mathbf{B}^n} r^{\alpha p-1}\|\nabla u\|^{p-1} dx.$$

By Hölder inequality, we have, setting $q = \frac{p}{p-1}$:

$$\begin{aligned}
\frac{C_n^p}{n}|\mathbb{S}^{n-1}| &\leq \frac{C_{n-1}^p}{(n-1)^{p/2}} \int_{\mathbf{B}^n} r^{\alpha p}\|\nabla u\|^p dx \\
&\quad + \alpha \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \left(\int_{\mathbf{B}^n} r^{p(\alpha-1)} dx\right)^{1/p} \left(\int_{\mathbf{B}^n} r^{\alpha p}\|\nabla u\|^p dx\right)^{1/q} \\
&\leq \frac{C_{n-1}^p}{(n-1)^{p/2}} \int_{\mathbf{B}^n} r^{\alpha p}\|\nabla u\|^p dx \\
&\quad + \alpha \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \frac{|\mathbb{S}^{n-1}|^{1/p}}{(n+p(\alpha-1))^{1/p}} \left(\int_{\mathbf{B}^n} r^{\alpha p}\|\nabla u\|^p dx\right)^{1/q}
\end{aligned}$$

Consider the polynomial function:

$$P(t) = \frac{C_{n-1}^{p}}{(n-1)^{p/2}} t^q + \alpha \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \frac{|\mathbb{S}^{n-1}|^{1/p}}{(n+p(\alpha-1))^{1/p}} t - \frac{C_n^p}{n} |\mathbb{S}^{n-1}|.$$

Setting $A = (\int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u\|^p)^{1/q}$ and $B = (\int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u_0\|^p)^{1/q}$, we get $P(A) \geq 0$ while

$$\begin{aligned} P(B) &= \frac{C_{n-1}^{p-1}}{n+p(\alpha-1)} |\mathbb{S}^{n-1}| + \alpha \frac{C_{n-1}^{p-1}}{n+p(\alpha-1)} |\mathbb{S}^{n-1}| - \frac{C_n^p}{n} |\mathbb{S}^{n-1}| \\ &= \frac{C_{n-1}^{p-1}}{n+p(\alpha-1)} |\mathbb{S}^{n-1}| \left(\frac{n-p}{n} + \alpha - \frac{C_n^p}{nC_{n-1}^{p-1}} (n+p(\alpha-1)) \right) = 0. \end{aligned}$$

On the other hand, $\forall t \geq 0$, $P'(t) > 0$. Hence, P is increasing in $[0, +\infty)$ and is equal to zero only for B. Necessarily, we have $A \geq B$.
Moreover, if $\alpha > 0$, $A = B$ implies that equality in the inequality (1) holds. Referring to the remark above, and as $u_0(x) = x$ on the sphere, we have $u = u_0 = \frac{x}{\|x\|}$. Replacing α by α/p we finish the prove of the theorem. □

3 The energy of a natural family of maps

Let $a = (\theta, \ldots, 0)$ be a point of $\mathbf{B}^n$ with $0 < \theta < 1$ and consider the map,

$$u_a(x) = a + \lambda_a(x)(x-a),$$

where $\lambda_a(x) > 0$ is chosen so that $u_a(x) \in \mathbb{S}^{n-1}$ for any $x \in \mathbf{B}^n \setminus \{0\}$,

$$\lambda_a(x) = \frac{\sqrt{\Delta_a(x)} - (a|x-a)}{\|x-a\|^2}$$

and

$$\Delta_a(x) = (1 - \|a\|^2)\|x-a\|^2 + (a|x-a)^2.$$

Notice that $u_a(x) = x$ as soon as x is on the sphere. If we denote by $\{e_i\}_{i \in \{1,\ldots,n\}}$ the standard basis of $\mathbb{R}^n$, then, $\forall i \leq n$, we have,

$$\begin{aligned} \|du_a(x).e_i\|^2 = &\left(\frac{\sqrt{\Delta_a} - (a|x-a)}{\|x-a\|^2} \right)^2 \\ &+ \Bigg[-2 \frac{(x-a|e_i)}{\|x-a\|^4} \big(\sqrt{\Delta_a} - (a|x-a) \big) \\ &+ \frac{(1-\|a\|^2)(x-a|e_i) + (x-a|a)(a|e_i)}{\sqrt{\Delta_a}\|x-a\|^2} \\ &- \frac{(a|e_i)}{\|x-a\|^2} \Bigg]^2 \|x-a\|^2 \end{aligned}$$

$$+2\Big(\frac{\sqrt{\Delta_a}-(a|x-a)}{\|x-a\|^2}\Big)\Big(-2\frac{(x-a|e_i)}{\|x-a\|^4}\Big(\sqrt{\Delta_a}-(a|x-a)\Big)$$
$$+\quad\frac{(1-\|a\|^2)(x-a|e_i)+(x-a|a)(a|e_i)}{\sqrt{\Delta_a}\|x-a\|^2}$$
$$-\frac{(a|e_i)}{\|x-a\|^2}\Big)(x-a|e_i).$$

Let us prove that, for each $\alpha\in(-n,0)$, $\int_{\mathbf{B}^n} r^\alpha\|\nabla u_a\|dx$ is finite. Consider the map:

$$\begin{aligned} F:\mathbb{R}^+\times\mathbb{S}^{n-1} &\longrightarrow \mathbb{R}^n\\ (r,s) &\longmapsto a+rs=x. \end{aligned}$$

Then, we have,

$$F^*\big(\|\nabla u_a\|^2dx\big)=\frac{1}{r^2}\sum_{i=1}^n H_{i,a}(s)\,r^{n-1}dr\wedge ds,$$

where $H_{i,a}(s)$ is given on the sphere by,

$$\begin{aligned} H_{i,a}(s) &= ((1-\|a\|^2+(a|s)^2)^{1/2}-(a|s))^2\\ &+\Big[-2(s|e_i)((1-\|a\|^2+(a|s)^2)^{1/2}-(s|a))\\ &+\frac{(1-\|a\|^2)(s|e_i)+(a|e_i)(s|a)}{(1-\|a\|^2+(a|s)^2)^{1/2}}-(a|e_i)\Big]^2\\ &+2((1-\|a\|^2+(a|s)^2)^{1/2}-(a|s))\\ &\times\Big(-2(s|e_i)((1-\|a\|^2+(a|s)^2)^{1/2}-(s|a))\\ &+\frac{(1-\|a\|^2)(s|e_i)+(a|e_i)(s|a)}{(1-\|a\|^2+(a|s)^2)^{1/2}}-(a|e_i)\Big)(s|e_i). \end{aligned}$$

It is clear that $H_{i,a}(s)$ is continuous on $\mathbb{S}^{n-1}$. Therefore, near the point a, as $n\geq 3$, the map $\|x\|^\alpha\|\nabla u_a\|$ is integrable. Furthermore, near the point 0, as $\alpha>-n$, this map is also integrable. In conclusion, for any $\alpha\in(-n,0)$, the energy $E_{r^\alpha,2}(u_a)$ is finite.

Proof of Theorem 1.3(i) Since we have

$$E_{2,r^\alpha}(u_0)=\int_{\mathbf{B}^n}\|x\|^\alpha\|\nabla u_0\|^2dx=\frac{|\mathbb{S}^{n-1}|(n-1)}{n+\alpha-2},$$

the energy $E_{2,r^\alpha}(u_0)$ goes to infinity as $\alpha\to -n+2$. On the other hand, as the energy $E_{2,r^\alpha}(u_a)$ is continuous in α, there exists a real number $\alpha_0\in(-n+2,0)$ such that, $\forall\alpha,\ 2-n<\alpha\leq\alpha_0$,

$$\int_{\mathbf{B}^n}\|x\|^\alpha\|\nabla u_0\|^2\,dx>\int_{\mathbf{B}^n}\|x\|^\alpha\|\nabla u_a\|^2\,dx.$$

□

Proof of Theorem 1.3(ii) Since $a = (\theta, 0, \dots, 0)$, we will study the function,

$$G(\theta) = E_{2,r^\alpha}(u_a) = \int_{\mathbf{B}^n} r^\alpha \|\nabla u_a\|^2 \, dx.$$

Precisely, we will show that for any $\alpha \in (5-n, 4-n)$, G is two times differentiable at $\theta = 0$ with $\frac{dG}{d\theta}(0) = 0$ and, when α is sufficiently close to $4 - n$, $\frac{d^2G}{d\theta^2}(0) < 0$. Assertion (ii) of Theorem 1.3 then follows immediately. We have,

$$\begin{aligned}
H_{i,a}(s) = H_{i,\theta}(s) &= \left(\sqrt{1-\theta^2+\theta^2 s_1^2} - \theta s_1\right)^2 \\
&+ \left(-2s_i\left(\sqrt{1-\theta^2+\theta^2 s_1^2} - \theta s_1\right) + \frac{(1-\theta^2)s_i + \delta_{i1}\theta^2 s_1}{\sqrt{1-\theta^2+\theta^2 s_1^2}} - \delta_{i1}\theta\right)^2 \\
&+ 2\left(\sqrt{1-\theta^2+\theta^2 s_1^2} - \theta s_1\right)\left(-2s_i\left(\sqrt{1-\theta^2+\theta^2 s_1^2} - \theta s_1\right)\right. \\
&\left. + \frac{(1-\theta^2)s_i + \delta_{i1}\theta^2 s_1}{\sqrt{1-\theta^2+\theta^2 s_1^2}} - \delta_{i1}\theta\right) s_i,
\end{aligned}$$

where $\delta_{ij} = 0$ if $i \neq j$ and 0 else.
We notice that $H_{i,\theta}(s)$ is bounded on $[0, 1] \times \mathbb{S}^{n-1}$. Indeed, for all $x, y, z \in [0, 1]$, excepting $(x, y) = (0, 1)$, we have,

$$\left|\frac{x}{\sqrt{1 - y^2 + y^2x^2)}}\right| \leq 1 \quad \text{and} \quad \left|\frac{(1 - y^2)z}{\sqrt{1 - y^2 + y^2x^2)}}\right| \leq 1.$$

Then, for almost all $(s, \theta) \in \mathbb{S}^{n-1} \times [0, 1]$, we have,

$$\left|\frac{(1-\theta^2)s_i + \delta_{i1}\theta^2 s_1}{\sqrt{1-\theta^2+\theta^2 s_1^2}}\right| \leq 1,$$

and the others terms are continuous in $[0, 1] \times \mathbb{S}^{n-1}$.
We have,

$$\begin{aligned}
E_{2,r^\alpha}(u_a) &= \int_{\mathbf{B}^n} \|x\|^\alpha \|\nabla u_a\|^2 dx = \int_{\mathbf{B}^n} \|a + rs\|^\alpha r^{n-3} H(\theta, s), dr ds \\
&= \int_{\mathbf{S}^{n-1}} H(\theta, s)\left(\int_0^{\gamma_\theta(s)} \left((r + \theta s_1)^2 + \theta^2\left(1 - s_1^2\right)\right)^{\alpha/2} r^{n-3} dr\right) ds,
\end{aligned}$$

where $\gamma_\theta(s) = \sqrt{1-\theta^2+\theta^2 s_1^2} - \theta s_1$ and $H(\theta, s) = \Sigma_{i=1}^n H_{i,\theta}(s)$. We notice that

$H(\theta, s)$ is indefinitely differentiable in $(-1/2, 1/2) \times \mathbb{S}^{n-1}$. Let C_n be a positive real number so that, $\forall(\theta, s) \in (-1/2, 1/2) \times \mathbb{S}^{n-1}$

$$|H(\theta, s)| \leq C_n, \left|\frac{\partial H(\theta, s)}{\partial\theta}\right| \leq C_n, \left|\frac{\partial^2 H(\theta, s)}{\partial\theta^2}\right| \leq C_n.$$

Furthermore, we have,

$$H(\theta, s) = (n–1)-2(n–1)s_1\theta+\big((2n–3)s_1^2-n+2\big)\theta^2+o(\theta^2). \tag{A}$$

Let us set $\rho = r + \theta s_1$, $\beta(\theta, s) = \sqrt{1-\theta^2+\theta^2 s_1^2}$ and

$$F(\theta, s) = \int_{\theta s_1}^{\beta(\theta,s)} (\rho - \theta s_1)^{n-3}\big(\rho^2 + \theta^2\big(1 - s_1^2\big)\big)^{\alpha/2}\, d\rho.$$

Notice that $\rho \in [-1, 3]$. Then, $G(\theta) = \int_{\mathbb{S}^{n-1}} H(\theta, s)F(\theta, s)ds$. Let us set $g(\rho, \theta, s) = (\rho - \theta s_1)^{n-3}(\rho^2 + \theta^2(1 - s_1^2))^{\alpha/2}$. □

Lemma 3.1 *The map $\theta \mapsto G(\theta)$ is continuous on $(-1/2, 1/2)$ and continuously differentiable on $(-1/2, 1/2)\backslash\{0\}$ for any $\alpha > 3-n$.*

Proof We have, $\forall s \in \mathbb{S}^{n-1}\backslash\{(\pm1, 0, \ldots, 0)\}$,

$$\frac{(\rho - \theta s_1)^2}{\big(\rho^2 + \theta^2\big(1 - s_1^2\big)\big)} \leq \frac{2}{1 - s_1^2} \tag{3.1}$$

Indeed, $(1-s_1^2)(\rho-\theta s_1)^2 \leq 2(1-s_1^2)(\rho^2+\theta^2) \leq 2(\rho^2+\theta^2(1-s_1^2))$. And then,

$$g(\rho, \theta, s) \leq \frac{2^{\frac{n-3}{2}}}{\big(1 - s_1^2\big)^{\frac{n-3}{2}}}\big(\rho^2 + \theta^2\big(1 - s_1^2\big)\big)^{\frac{\alpha+n-3}{2}}. \tag{3.2}$$

Since $\alpha > 3 - n$ we deduce that the map $(\rho, \theta) \to g(\rho, \theta, s)$ is continuous on $(-1/2, 1/2) \times [-1, 3]$. Hence, the map $z \mapsto \int_0^z g(\rho, \theta, s)\, d\rho$ is differentiable on $[-1, 3]$ and,

$$\frac{\partial}{\partial z}\int_0^z g(\rho, \theta, s)d\rho = g(z, \theta, s).$$

Furthermore, for any $\rho \in [-1, 3]$, the map $\theta \mapsto g(\rho, \theta, s)$ is differentiable and

$$\begin{aligned}\frac{\partial g}{\partial\theta}(\rho, \theta, s) = &-(n - 3)s_1(\rho - \theta s_1)^{n-4}\big(\rho^2 + \theta^2\big(1 - s_1^2\big)\big)^{\frac{\alpha}{2}}\\ &+\frac{\alpha}{2}(\rho - \theta s_1)^{n-3}2\theta\big(1 - s_1^2\big)\big(\rho^2 + \theta^2\big(1 - s_1^2\big)\big)^{\frac{\alpha}{2}-1}.\end{aligned}$$

Let a, b be two real in $(0, 1/2)$ with $a < b$. We have for any $|\theta| \in (a, b)$, for any $s \in \mathbb{S}^{n-1}\backslash\{(\pm1, 0, \ldots, 0)\}$,

$$\begin{aligned}\left|\frac{\partial g}{\partial\theta}(\rho, \theta, s)\right| \leq\ &(n - 3)4^{n-4}\big(a^2\big(1 - s_1^2\big)\big)^{\frac{\alpha}{2}}\\ &+|\alpha|4^{n-3}\big(1 - s_1^2\big)\big(a^2(1 - s_1^2)\big)^{\frac{\alpha}{2}-1}.\end{aligned} \tag{3.3}$$

This shows that $\theta \mapsto \int_0^z g(\rho,\theta,s)d\rho$ is differentiable on $(-1/2,1/2)\setminus\{0\}$ and

$$\frac{\partial}{\partial\theta}\int_0^z g(\rho,\theta,s)\,d\rho = \int_0^z \frac{\partial g}{\partial\theta}(\rho,\theta,s)\,d\rho.$$

Moreover the map $(z,\theta) \mapsto \int_0^z \frac{\partial g}{\partial\theta}(\rho,\theta,s)d\rho$ is continuous in $[-1,3]\times(-1/2,1/2)\setminus\{0\}$. Indeed, $\theta \mapsto \frac{\partial g}{\partial\theta}(\rho,\theta,s)$ is clearly continuous on $(-1/2,1/2)\setminus\{0\}$ and from (3.3) and by Lebesgue Theorem, $\theta \mapsto \int_0^z \frac{\partial g}{\partial\theta}(\rho,\theta,s)d\rho$ is continuous on $(-1/2,1/2)\setminus\{0\}$. Then, for any $\epsilon > 0$, we will have for any sufficiently small h, k,

$$\left|\int_0^{z+h}\frac{\partial g}{\partial\theta}(\rho,\theta+k,s)\,d\rho - \int_0^z \frac{\partial g}{\partial\theta}(\rho,\theta,s)\,d\rho\right| \le \left|\int_0^z \frac{\partial g}{\partial\theta}(\rho,\theta+k,s)d\rho - \int_0^z \frac{\partial g}{\partial\theta}(\rho,\theta,s)d\rho\right| + \left|\int_z^{z+h}\frac{\partial g}{\partial\theta}(\rho,\theta+k,s)d\rho\right| \le \epsilon.$$

The map $(z,\theta)\mapsto \int_0^z g(\rho,\theta,s)d\rho$ is differentiable on $[-1,3]\times(-1/2,1/2)\setminus\{0\}$ and the map $\theta\mapsto F(\theta,s)$ is differentiable in $(-1/2,1/2)\setminus\{0\}$ and for any $\theta\in(-1/2,1/2)\setminus\{0\}$,

$$\begin{aligned}\frac{\partial F}{\partial\theta}(\theta,s) &= \frac{\partial\beta}{\partial\theta}(\theta,s)g(\beta(\theta,s),\theta,s) - s_1 g(\theta s_1,\theta,s) + \int_{\theta s_1}^{\beta(\theta,s)}\frac{\partial g}{\partial\theta}(\rho,\theta,s)d\rho\\ &= \frac{\theta\left(s_1^2-1\right)}{\left(1-\theta^2+\theta^2 s_1^2\right)^{1/2}}\left(\left(1-\theta^2+\theta^2 s_1^2\right)^{1/2}-\theta s_1\right)^{n-3}\\ &\quad + \int_{\theta s_1}^{\beta(\theta,s)} g_1(\rho,\theta,s)d\rho + \int_{\theta s_1}^{\beta(\theta,s)} g_2(\rho,\theta,s)d\rho,\end{aligned}$$

where,

$$g_1(\rho,\theta,s) = -(n-3)s_1(\rho-\theta s_1)^{n-4}\left(\rho^2+\theta^2\left(1-s_1^2\right)\right)^{\frac{\alpha}{2}}$$

and

$$g_2(\rho,\theta,s) = \frac{\alpha}{2}(\rho-\theta s_1)^{n-3}2\theta\left(1-s_1^2\right)\left(\rho^2+\theta^2\left(1-s_1^2\right)\right)^{\frac{\alpha}{2}-1}.$$

Now, the map $\theta\mapsto F(\theta,s)$ is continuous on $(-1/2,1/2)$. Indeed, since the map $\theta\mapsto g(\rho,\theta,s)d\rho$ is continuous on $(-1/2,1/2)$ and from (3.2) $\theta\mapsto \int_0^z g(\rho,\theta,s)d\rho$ is continuous on $(-1/2,1/2)$. Then, for any $\epsilon>0$, we have $\forall h,k$ sufficiently small,

$$\left|\int_0^{z+h} g(\rho,\theta+k,s)d\rho - \int_0^z g(\rho,\theta,s)d\rho\right| \le \left|\int_0^z g(\rho,\theta+k,s)d\rho - \int_0^z g(\rho,\theta,s)d\rho\right| + \left|\int_z^{z+h} g(\rho,\theta+k,s)d\rho\right| \le \epsilon.$$

Then, the map $(z,\theta)\mapsto\int_0^z g(\rho,\theta,s)d\rho$ is continuous on $[-1,3]\times(-1/2,1/2)$ and consequently $\theta\mapsto F(\theta,s)$ is continuous on $(-1/2,1/2)$.

Now, we know that $\theta \mapsto H(\theta, s)F(\theta, s)$ is continuous on $(-1/2, 1/2)$ and differentiable on $(-1/2, 1/2) \setminus \{0\}$. Furthermore from (3.2), we have, for any $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$,

$$|H(\theta, s)F(\theta, s)| \leq 3.2^{\frac{n-3}{2}} 10^{\frac{\alpha+n-3}{2}} C_n . \frac{1}{\left(1 - s_1^2\right)^{\frac{n-3}{2}}}. \tag{3.4}$$

$$\left| \frac{\partial H}{\partial \theta}(\theta, s)F(\theta, s) \right| \leq 3.2^{\frac{n-3}{2}} 10^{\frac{\alpha+n-3}{2}} C_n . \frac{1}{\left(1 - s_1^2\right)^{\frac{n-3}{2}}}. \tag{3.5}$$

Consider the map $\eta : (\theta, s) \mapsto \eta(\theta, s) = \frac{\theta(s_1^2-1)}{\sqrt{1-\theta^2+\theta^2 s_1^2}}((1-\theta^2+\theta^2 s_1^2)^{1/2} - \theta s_1)^{n-3}$. This map is indefinitely differentiable on $(-1/2, 1/2) \times \mathbb{S}^{n-1}$. Let B_n be a positive real number so that, $\forall (\theta, s) \in (-1/2, 1/2) \times \mathbb{S}^{n-1}$,

$$|\eta(\theta, s)| \leq B_n \qquad \left| \frac{\partial \eta}{\partial \theta}(\theta, s) \right| \leq B_n .$$

Considering $a, b \in (0, 1/2)$ with $a < b$ we have, for any $\theta \in (a, b)$, for any $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$,

$$\begin{aligned} \left| H(\theta, s) \frac{\partial F}{\partial \theta}(\theta, s) \right| \leq \big(B_n &+ 3(n-3).4^{n-4}.a^{\alpha}\left(1 - s_1^2\right)^{\frac{\alpha}{2}} \\ &+ |3\alpha|.4^{n-3} a^{\alpha-1}\left(1 - s_1^2\right)^{\frac{\alpha}{2}}\big) C_n . \end{aligned} \tag{3.6}$$

Since the maps $s \mapsto \frac{1}{(1-s_1^2)^{\frac{n-3}{2}}}$ and $s \mapsto (1 - s_1^2)^{\frac{\alpha}{2}}$ are integrable on $\mathbb{S}^{n-1}$, we deduce that $\theta \mapsto G(\theta)$ is continuous on $(-1/2, 1/2)$ and continuously differentiable on $(-1/2, 1/2) \setminus \{0\}$. □

Lemma 3.2 *The map $\theta \mapsto G(\theta)$ is differentiable at* 0 *and* $\frac{dG}{d\theta}(0) = 0$.

Proof Since for any $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$, $\theta \mapsto F(\theta, s)$ is continuous on $(-1/2, 1/2)$ from (A) we have,

$$\begin{aligned} \frac{\partial H}{\partial \theta}(\theta, s)F(\theta, s) \underset{\theta \to 0}{\longrightarrow} \frac{\partial H}{\partial \theta}(0, s)F(0, s) &= -2(n-1)s_1 \int_0^1 \rho^{n-3+\alpha} d\rho \\ &= \frac{-2(n-1)s_1}{n-2+\alpha}. \end{aligned}$$

From (1.5) and Lebesgue Theorem we have,

$$\int_{\mathbb{S}^{n-1}} \frac{\partial H}{\partial \theta}(\theta, s)F(\theta, s) ds \underset{\theta \to 0}{\longrightarrow} \int_{\mathbb{S}^{n-1}} \frac{-2(n-1)s_1}{n-2+\alpha} ds = 0.$$

Moreover, it is clear that,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s)\eta(\theta, s) ds \underset{\theta \to 0}{\longrightarrow} 0.$$

Let $J(m,n)$ be the integral,

$$J(m,n) = \int_{\frac{s_1}{\sqrt{1-s_1^2}}}^{\sqrt{\frac{1}{\theta^2\left(1-s_1^2\right)}-1}} \left(\sqrt{1-s_1^2}\,t - s_1\right)^m (t^2+1)^n \, dt.$$

Notice that $J(m,n)$ converges as θ goes to 0 if and only if $m+2n<-1$.
Consider the change of variables $\rho = t\theta\sqrt{1-s_1^2}$ if $\theta>0$. If $\theta<0$, then we set $\rho = -t\theta\sqrt{1-s_1^2}$ and conclusion will be the same. Hence, we assume that $\theta>0$. Then,

$$\int_{\theta s_1}^{\beta(\theta,s)} g_1(\rho,\theta,s)d\rho = -(n-3)s_1\left(1-s_1^2\right)^{\frac{1+\alpha}{2}}\theta^{n-3+\alpha} J\left(n-4, \frac{\alpha}{2}\right).$$

$$\int_{\theta s_1}^{\beta(\theta,s)} g_2(\rho,\theta,s)d\rho = \alpha\theta^{n-3+\alpha}\left(1-s_1^2\right)^{\frac{1+\alpha}{2}} J\left(n-3, \frac{\alpha}{2}-1\right).$$

First case: $\alpha \geq 4-n$.
$J(n-4,\frac{\alpha}{2})$ and $J(n-3,\frac{\alpha}{2}-1)$ go to $+\infty$ as $\theta\to 0$. Furthermore, we have,

$$J\left(n-4,\frac{\alpha}{2}\right) \underset{0}{\sim} \left(1-s_1^2\right)^{\frac{n-4}{2}} \int_{\frac{s_1}{\sqrt{1-s_1^2}}}^{\sqrt{\frac{1}{\theta^2\left(1-s_1^2\right)}-1}} t^{n-4+\alpha}\, dt$$

$$J\left(n-4,\frac{\alpha}{2}\right) \underset{0}{\sim} \frac{1}{n-3+\alpha}\frac{1}{\theta^{n-3+\alpha}}\left(1-s_1^2\right)^{\frac{-1-\alpha}{2}}.$$

Since $t^{n+\alpha-5}$ may be equal to zero at zero, we write,

$$J\left(n-3,\frac{\alpha}{2}-1\right) = \int_{\frac{s_1}{\sqrt{1-s_1^2}}}^{1} \left(\sqrt{1-s_1^2}\,t-s_1\right)^{n-3}(t^2+1)^{\frac{\alpha}{2}-1}\,dt$$
$$+ \int_1^{\sqrt{\frac{1}{\theta^2\left(1-s_1^2\right)}-1}} \left(\sqrt{1-s_1^2}\,t-s_1\right)^{n-3}(t^2+1)^{\frac{\alpha}{2}-1}\,dt.$$

We have

$$J\left(n-3,\frac{\alpha}{2}-1\right) \underset{0}{\sim} \left(1-s_1^2\right)^{\frac{n-3}{2}} \int_1^{\sqrt{\frac{1}{\theta^2\left(1-s_1^2\right)}-1}} t^{n-5+\alpha}\,dt.$$

Then, if $\alpha \neq 4-n$,

$$J\left(n-3,\frac{\alpha}{2}-1\right) \underset{0}{\sim} \frac{1}{n-4+\alpha}\frac{1}{\theta^{n-4+\alpha}}\left(1-s_1^2\right)^{\frac{1-\alpha}{2}},$$

and note that if $\alpha = 4-n$, $J(n-3, \frac{\alpha}{2}-1) \sim_0 -(1-s_1^2)^{\frac{n-3}{2}} \ln(\theta^2(1-s_1^2))$. Hence, by (A) we have,

$$H(\theta, s) \int_{\theta s_1}^{\beta(\theta,s)} g_1(\rho, \theta, s) d\rho = -H(\theta, s)(n-3) s_1 \big(1-s_1^2\big)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} I_1 \times \underset{\theta\to 0}{\longrightarrow} -\frac{(n-3)(n-1)}{n-3+\alpha} s_1,$$

and

$$H(\theta, s) \int_{\theta s_1}^{\beta(\theta,s)} g_2(\rho, \theta, s) d\rho = H(\theta, s) \alpha \big(1-s_1^2\big)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} I_2 \underset{\theta\to 0}{\longrightarrow} 0.$$

Observe that $\frac{|s_1|}{\sqrt{1-s_1^2}} \leq \sqrt{\frac{1}{\theta^2(1-s_1^2)} - 1}$. Indeed, $s_1^2\theta^2 \leq 1 - \theta^2 + \theta^2 s_1^2$. It follows from (1.1) that

$$(\rho - \theta s_1)^{n-4} \leq \frac{2^{\frac{n-4}{2}}}{\big(1-s_1^2\big)^{\frac{n-4}{2}}} \big(\rho^2 + \theta^2\big(1-s_1^2\big)\big)^{\frac{n-4}{2}}.$$

Recall that $\rho = t\theta\sqrt{1-s_1^2}$. Since $\alpha \geq 4-n$, we have, for any $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \ldots, 0)\}$,

$$\begin{aligned}
\left| H(\theta, s) \int_{\theta s_1}^{\beta(\theta,s)} g_1(\rho, \theta, s) d\rho \right| &\leq 2C_n(n-3)\, 2^{\frac{n-4}{2}} \big(1-s_1^2\big)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\
&\quad \times \int_0^{\sqrt{\frac{1}{\theta^2\left(1-s_1^2\right)}-1}} (t^2+1)^{\frac{n-4+\alpha}{2}}\, dt \\
&\leq C_n(n-3) 2^{\frac{n-2}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\
&\quad \times \sqrt{\frac{1}{\theta^2(1-s_1^2)} - 1} \left(\frac{1}{\theta^2\big(1-s_1^2\big)}\right)^{\frac{n-4+\alpha}{2}} \\
&\leq C_n(n-3) 2^{\frac{n-2}{2}} \big(1-s_1^2\big)^{\frac{-n+4}{2}} \sqrt{1-\theta^2\big(1-s_1^2\big)} \\
&\leq C_n(n-3) 2^{\frac{n-1}{2}} \big(1-s_1^2\big)^{\frac{-n+4}{2}}.
\end{aligned}$$

Since $s \mapsto (1-s_1^2)^{\frac{-n+4}{2}}$ is integrable on $\mathbb{S}^{n-1}$, by Lebesgue Theorem we have,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \int_{\theta s_1}^{\beta(\theta,s)} g_1(\rho, \theta, s)\, d\rho ds \underset{\theta\to 0}{\longrightarrow} - \int_{\mathbb{S}^{n-1}} \frac{(n-3)(n-1)}{n-3+\alpha} s_1 ds = 0.$$

Moreover, we have, for any $s \in \mathbb{S}^{n-1}\setminus\{(\pm 1, 0, \dots, 0)\}$, since $\alpha + n - 5 \geq 0$,

$$\begin{aligned}\left| H(\theta, s) \int_{\theta s_1}^{\beta(\theta,s)} g_2(\rho, \theta, s) d\rho \right| &\leq 2C_n|\alpha| 2^{\frac{n-3}{2}} \left(1 - s_1^2\right)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\ &\quad \times \int_0^{\sqrt{\frac{1}{\theta^2\left(1-s_1^2\right)}-1}} (t^2+1)^{\frac{n-5+\alpha}{2}} dt \\ &\leq C_n|\alpha| 2^{\frac{n-2}{2}} \left(1 - s_1^2\right)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\ &\quad \times \int_0^{\sqrt{\frac{1}{\theta^2\left(1-s_1^2\right)}-1}} \frac{1}{(t^2+1)} dt \left(\frac{1}{\theta^2\left(1-s_1^2\right)}\right)^{\frac{n-3+\alpha}{2}} \\ &\leq C_n|\alpha| 2^{\frac{n-2}{2}} \frac{\pi}{2} \left(1 - s_1^2\right)^{\frac{-n+4}{2}}.\end{aligned}$$

Then, by Lebesgue Theorem,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \int_{\theta s_1}^{\beta(\theta,s)} g_2(\rho, \theta, s) d\rho ds \underset{\theta \to 0}{\longrightarrow} 0.$$

Second case: $3 - n < \alpha < 4 - n$.
For the same reasons that when $\alpha \geq 4 - n$, we have,

$$H(\theta, s) \int_{\theta s_1}^{\beta(\theta,s)} g_1(\rho, \theta, s) d\rho \underset{\theta \to 0}{\longrightarrow} -\frac{(n-3)(n-1)}{n-3+\alpha} s_1.$$

Furthermore, as $4 - n > \alpha > 3 - n$, $\forall s \in \mathbb{S}^{n-1}\setminus\{(-1, 0, \dots, 0), (1, 0, \dots, 0)\}$,

$$\begin{aligned}\left| H(\theta, s) \int_{\theta s_1}^{\beta(\theta,s)} g_1(\rho, \theta, s) d\rho \right| &\leq 2C_n(n-3) 2^{\frac{n-4}{2}} \left(1 - s_1^2\right)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\ &\quad \times \int_0^{\sqrt{\frac{1}{\theta^2\left(1-s_1^2\right)}-1}} (t^2+1)^{\frac{n-4+\alpha}{2}} dt \\ &\leq C_n(n-3) 2^{\frac{n-2}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\ &\quad \times \int_0^{\sqrt{\frac{1}{\theta^2\left(1-s_1^2\right)}-1}} (t^2)^{\frac{n-4+\alpha}{2}} dt \\ &\leq \frac{C_n(n-3) 2^{\frac{n-2}{2}} \left(1 - s_1^2\right)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha}}{n-3+\alpha} \\ &\quad \times \left(\frac{1}{\theta^2\left(1-s_1^2\right)} - 1\right)^{\frac{n-3+\alpha}{2}}. \\ &\leq \frac{C_n(n-3) 2^{\frac{2n-7+\alpha}{2}} \left(1 - s_1^2\right)^{\frac{4-n}{2}}}{n-3+\alpha}.\end{aligned}$$

Then, by Lebesgue Theorem,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \int_{\theta s_1}^{\beta(\theta,s)} g_1(\rho, \theta, s) d\rho ds \underset{\theta\to 0}{\longrightarrow} -\int_{\mathbb{S}^{n-1}} \frac{(n-3)(n-1)}{n-3+\alpha} s_1 ds = 0.$$

Moreover, $J(n-3, \frac{\alpha}{2}-1)$ is finite when $\theta \to 0$ then, as $\alpha > 3-n$, Furthermore,

$$H(\theta, s) \int_{\theta s_1}^{\beta(\theta,s)} g_2(\rho, \theta, s) d\rho ds \underset{\theta\to 0}{\longrightarrow} 0.$$

$$\begin{aligned}\left| H(\theta, s) \int_{\theta s_1}^{\beta(\theta,s)} g_2(\rho, \theta, s) d\rho \right| &\le 2C_n|\alpha| 2^{\frac{n-3}{2}} \left(1-s_1^2\right)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\ &\quad \times \int_0^{\sqrt{\frac{1}{\theta^2\left(1-s_1^2\right)}-1}} (t^2+1)^{\frac{n+\alpha-5}{2}} dt \\ &\le C_n|\alpha| 2^{\frac{n-1}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\ &\quad \times \int_0^{\sqrt{\frac{1}{\theta^2\left(1-s_1^2\right)}-1}} \frac{1}{(t^2+1)} d\left(\frac{1}{\theta^2\left(1-s_1^2\right)}\right)^{\frac{n-3+\alpha}{2}} \\ &\le C_n|\alpha| 2^{\frac{n-1}{2}} \left(1-s_1^2\right)^{\frac{-n+4}{2}} \int_0^{+\infty} \frac{1}{(t^2+1)} dt.\end{aligned}$$

Then, by Lebesgue Theorem,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \int_{\theta s_1}^{\beta(\theta,s)} g_2(\rho, \theta, s) d\rho ds \underset{\theta\to 0}{\longrightarrow} 0.$$

Finally, we have

$$\frac{dG}{d\theta}(\theta) \underset{\theta\to 0}{\longrightarrow} 0.$$

By Lemma 3.1 we deduce that G is differentiable at 0 and $\frac{dG}{d\theta}(0) = 0$. □

Lemma 3.3 *The map $\theta \to G(\theta)$ is two times differentiable on* $(-1/2, 1/2)\setminus\{0\}$.

Proof We know that the map $\theta \to \frac{\partial H}{\partial \theta}(\theta, s) F(\theta, s)$ is differentiable on $(-1/2, 1/2)\setminus\{0\}$. The maps $\theta \to \eta(\theta, s)$, $\theta \to g_1(\rho, \theta, s)$, $\theta \to g_2(\rho, \theta, s)$ are differentiable on $(-1/2, 1/2)\setminus\{0\}$. We have,

$$\begin{aligned}\frac{\partial \eta}{\partial \theta}(\theta, s) = &\frac{\left(s_1^2-1\right)\sqrt{1-\theta^2+\theta^2 s_1^2} - \theta\left(s_1^2-1\right)\frac{\theta\left(s_1^2-1\right)}{\sqrt{1-\theta^2+\theta^2 s_1^2}}}{1-\theta^2+\theta^2 s_1^2} \\ &\qquad \times\left(\sqrt{1-\theta^2+\theta^2 s_1^2} - \theta s_1\right)^{n-3} \\ &+ \frac{(n-3)\theta\left(s_1^2-1\right)}{\sqrt{1-\theta^2+\theta^2 s_1^2}} \left(\frac{\theta\left(s_1^2-1\right)}{\sqrt{1-\theta^2+\theta^2 s_1^2}} - s_1\right) \\ &\qquad \times\left(\sqrt{1-\theta^2+\theta^2 s_1^2} - \theta s_1\right)^{n-4}.\end{aligned}$$

$$\frac{\partial g_1}{\partial\theta}(\rho,\theta,s) = (n-3)(n-4)s_1^2(\rho-\theta s_1)^{n-5}\big(\rho^2+\theta^2\big(1-s_1^2\big)\big)^{\frac{\alpha}{2}} - \alpha(n-3)s_1\big(1-s_1^2\big)\theta(\rho-\theta s_1)^{n-4}\big(\rho^2+\theta^2\big(1-s_1^2\big)\big)^{\frac{\alpha}{2}-1}.$$

$$\frac{\partial g_2}{\partial\theta}(\rho,\theta,s) = -\alpha(n-3)s_1\big(1-s_1^2\big)\theta(\rho-\theta s_1)^{n-4}(\rho^2+\theta^2\big(1-s_1^2\big))^{\frac{\alpha}{2}-1} + \alpha(\alpha-2)\big(1-s_1^2\big)^2\theta^2(\rho-\theta s_1)^{n-3}(\rho^2+\theta^2\big(1-s_1^2\big))^{\frac{\alpha}{2}-2} + \alpha\big(1-s_1^2\big)(\rho-\theta s_1)^{n-3}\big(\rho^2+\theta^2\big(1-s_1^2\big)\big)^{\frac{\alpha}{2}-1}.$$

We set,

$$\begin{aligned}
g_{11}(\rho,\theta,s) &= (n-3)(n-4)s_1^2(\rho-\theta s_1)^{n-5}\big(\rho^2+\theta^2\big(1-s_1^2\big)\big)^{\frac{\alpha}{2}},\\
g_{12}(\rho,\theta,s) &= -2\alpha(n-3)s_1\big(1-s_1^2\big)\theta(\rho-\theta s_1)^{n-4}\big(\rho^2+\theta^2\big(1-s_1^2\big)\big)^{\frac{\alpha}{2}-1}.\\
g_{21}(\rho,\theta,s) &= \alpha(\alpha-2)\big(1-s_1^2\big)^2\theta^2(\rho-\theta s_1)^{n-3}\big(\rho^2+\theta^2\big(1-s_1^2\big)\big)^{\frac{\alpha}{2}-2},\\
g_{22}(\rho,\theta,s) &= \alpha\big(1-s_1^2\big)(\rho-\theta s_1)^{n-3}\big(\rho^2+\theta^2\big(1-s_1^2\big)\big)^{\frac{\alpha}{2}-1}.
\end{aligned}$$

Let $a, b \in (0, 1/2)$ with $a < b$. We have, $\forall s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$,

$$\left|\frac{\partial g_1}{\partial\theta}(\rho,\theta,s)\right| \le (n-3)(n-4)4^{n-5}a^{\alpha}\big(1-s_1^2\big)^{\frac{\alpha}{2}} + |\alpha|(n-3)4^{n-4}a^{\alpha-1}\big(1-s_1^2\big)^{\frac{\alpha}{2}}. \tag{3.7}$$

$$\left|\frac{\partial g_2}{\partial\theta}(\rho,\theta,s)\right| \le |\alpha(\alpha-2)|4^{n-3}a^{\alpha-2}\big(1-s_1^2\big)^{\frac{\alpha}{2}} + |\alpha|4^{n-3}a^{\alpha-1}\big(1-s_1^2\big)^{\frac{\alpha}{2}} + |\alpha|(n-3)4^{n-4}a^{\alpha-1}\big(1-s_1^2\big)^{\frac{\alpha}{2}}. \tag{3.8}$$

Then, for any $i \in \{1, 2\}$, the maps $\theta \mapsto \int_0^z g_i(\rho,\theta,s)d\rho$ is differentiable on $(0, 1/2)$, and

$$\frac{\partial}{\partial\theta}\int_0^z g_i(\rho,\theta,s)d\rho = \int_0^z \frac{\partial g_i}{\partial\theta}(\rho,\theta,s)\,d\rho.$$

Furthermore, for any $i \in \{1, 2\}$, $\theta \mapsto \frac{\partial g_i}{\partial\theta}(\rho,\theta,s)$ is continuous on $(-1/2, 1/2)\backslash\{0\}$, then, $\theta \mapsto \int_0^z \frac{\partial g_i}{\partial\theta}(\rho,\theta,s)d\rho$, is continuous on $(-1/2, 1/2)\backslash\{0\}$. Hence, for any $i \in \{1, 2\}$ and for any $\epsilon > 0$, we have $\forall h, k$ two sufficiently small,

$$\left|\int_0^{z+h}\frac{\partial g_i}{\partial\theta}(\rho,\theta+k,s)d\rho - \int_0^z \frac{\partial g_i}{\partial\theta}(\rho,\theta,s)d\rho\right| \le \left|\int_0^z \frac{\partial g_i}{\partial\theta}(\rho,\theta+k,s)d\rho - \int_0^z \frac{\partial g_i}{\partial\theta}(\rho,\theta,s)\,d\rho\right| + \left|\int_z^{z+h}\frac{\partial g_i}{\partial\theta}(\rho,\theta+k,s)\,d\rho\right| \le \epsilon.$$

This proves that for any $i \in \{1,2\}$, $(z,\theta) \mapsto \int_0^z \frac{\partial g_i}{\partial \theta}(\rho,\theta,s)\,d\rho$ is continuous on $[-1,3]\times(-1/2,1/2)\backslash\{0\}$. Moreover, for any $i \in \{1,2\}$ the map $\rho \mapsto g_i(\rho,\theta,s)$ is continuous on $[-1,3]$ for any $\theta \in (-1/2,1/2)\backslash\{0\}$. Then, $z \mapsto \int_0^z g_i(\rho,\theta,s)\,d\rho$ is differentiable on $[-1,3]$ for any $\theta \in (-1/2,1/2)\backslash\{0\}$ and $\frac{\partial}{\partial z}\int_0^z g_i(\rho,\theta,s)\,d\rho = g_i(z,\theta,s)$.

Since $(z,\theta) \mapsto g_i(z,\theta)$ is continuous on $[-1,3]\times(-1/2,1/2)\backslash\{0\}$ we finally deduce that for any $i \in \{1,2\}$, $\theta \mapsto \int_{\theta s_1}^{\beta(\theta,s)} g_i(\rho,\theta,s)\,d\rho$ is differentiable on $(-1/2,1/2)\backslash\{0\}$ and,

$$\begin{aligned}\sum_{i=1}^{2}\frac{\partial}{\partial\theta}\int_{\theta s_1}^{\beta(\theta,s)} g_i(\rho,\theta,s)\,d\rho &= \frac{\theta\big(s_1^2-1\big)}{\sqrt{1-\theta^2+\theta^2 s_1^2}}\\ &\quad\times\Big(-(n-3)s_1\big(\sqrt{1-\theta^2+\theta^2 s_1^2}-\theta s_1\big)^{n-4}\\ &\qquad+\alpha\theta\big(1-s_1^2\big)\Big(\sqrt{1-\theta^2+\theta^2 s_1^2}-\theta s_1\Big)^{n-3}\Big)\\ &\quad+\sum_{i=1}^{2}\int_{\theta s_1}^{\beta(\theta,s)}\frac{\partial^2 g_i}{\partial^2\theta}(\rho,\theta,s)\,d\rho.\end{aligned}$$

We deduce that $\theta \mapsto \frac{\partial F}{\partial \theta}$ is differentiable in $(-1/2,1/2)\backslash\{0\}$. Moreover, we see that the map,

$$\begin{aligned}\theta \mapsto \lambda(\theta,s) = \frac{\theta\big(s_1^2-1\big)}{\sqrt{1-\theta^2+\theta^2 s_1^2}}&\Big(-(n-3)s_1\Big(\sqrt{1-\theta^2+\theta^2 s_1^2}-\theta s_1\Big)^{n-4}\\ &+\alpha\theta\big(1-s_1^2\big)\Big(\sqrt{1-\theta^2+\theta^2 s_1^2}-\theta s_1\Big)^{n-3}\Big)\end{aligned}$$

is indefinitely differentiable on $(-1/2,1/2)\times\mathbb{S}^{n-1}$. Then, by (1.1), (1.2), (1.8), (1.7), (1.3) and (A), for any $a,b \in (0,1/2)$, $a < b$ there exists constants $K_{1,n,ab,\alpha}, K_{2,n,ab,\alpha}, K_{3,n,ab,\alpha}$ so that, for any $|\theta| \in (a,b)$, for any $s \in \mathbb{S}^{n-1}\setminus\{(\pm 1,0,\dots,0)\}$,

$$\left|\frac{\partial^2 HF}{\partial\theta^2}(\theta,s)\right| \le K_{1,n,ab,\alpha}\big(1-s_1^2\big)^{\frac{\alpha}{2}} + K_{2,n,ab,\alpha}\big(1-s_1^2\big)^{\frac{3-n}{2}} + K_{3,n,ab,\alpha}.$$

We deduce by Lebesgue Theorem that the map $\theta \mapsto E(\theta)$ is two times differentiable on $(-1/2,1/2)\backslash\{0\}$ and,

$$\frac{d^2 G}{d\theta^2}(\theta) = \int_{\mathbb{S}^{n-1}}\frac{\partial^2 HF}{\partial\theta^2}(\theta,s))ds. \qquad \square$$

Lemma 3.4 *If* $5-n > \alpha > 4-n$*, the map* $\theta \mapsto G(\theta)$ *is two times differentiable at* 0.

Proof Suppose that $\alpha \in (4-n, 5-n)$. As in Lemma 3.5, we can see that,

$$\int_{\mathbb{S}^{n-1}} \frac{\partial^2 H}{\partial \theta^2}(\theta, s) F(\theta, s)\, ds \underset{\theta \to 0}{\longrightarrow} \int_{\mathbb{S}^{n-1}} \frac{1}{2} \frac{(2n-3)s_1^2 - (n-2)}{n-2+\alpha}\, ds$$
$$= \frac{-n^2+4n-3}{2n(n-2+\alpha)} |\mathbb{S}^{n-1}|,$$

$$\int_{\mathbb{S}^{n-1}} \frac{\partial H}{\partial \theta}(\theta, s) \eta(\theta, s)\, ds \underset{\theta \to 0}{\longrightarrow} 0,$$

$$\int_{\mathbb{S}^{n-1}} \frac{\partial H}{\partial \theta}(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s)\, d\rho \underset{\theta \to 0}{\longrightarrow} \int_{\mathbb{S}^{n-1}} \frac{2(n-3)(n-1)}{n-3+\alpha} s_1^2$$
$$= \frac{2(n-3)(n-1)}{n(n-3+\alpha)} |\mathbb{S}^{n-1}|,$$

$$\int_{\mathbb{S}^{n-1}} \frac{\partial H}{\partial \theta}(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s)\, d\rho \underset{\theta \to 0}{\longrightarrow} 0,$$

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \frac{\partial \eta}{\partial \theta}(\theta, s)\, ds \underset{\theta \to 0}{\longrightarrow} \int_{\mathbb{S}^{n-1}} (n-1)\big(s_1^2 - 1\big)\, ds = \frac{-(n-1)^2}{n} |\mathbb{S}^{n-1}|,$$

and

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \lambda(\theta, s)\, ds \underset{\theta \to 0}{\longrightarrow} 0.$$

As in Lemma 3.5, we set $\rho = \sqrt{1 - s_1^2}\theta t$ if $\theta > 0$. Hence,

$$\int_{\theta s_1}^{\beta(\theta, s)} g_{11}(\rho, \theta, s) d\rho = (n-3)(n-4) s_1^2 \big(1 - s_1^2\big)^{\frac{1+\alpha}{2}} \theta^{n-4+\alpha} J\Big(n-5, \frac{\alpha}{2}\Big).$$

$$\int_{\theta s_1}^{\beta(\theta s)} g_{12}(\rho, \theta, s) d\rho = -2\alpha(n-3) s_1 \big(1 - s_1^2\big)^{\frac{1+\alpha}{2}} \theta^{n-4+\alpha} J\Big(n-4, \frac{\alpha}{2} - 1\Big)$$

$$\int_{\theta s_1}^{\beta(\theta s)} g_{21}(\rho, \theta, s) d\rho = \alpha(\alpha - 2) \big(1 - s_1^2\big)^{\frac{1+\alpha}{2}} \theta^{n-4+\alpha} J\Big(n-3, \frac{\alpha}{2} - 2\Big)$$

$$\int_{\theta s_1}^{\beta(\theta s)} g_{22}(\rho, \theta, s) d\rho = \alpha \big(1 - s_1^2\big)^{\frac{1+\alpha}{2}} \theta^{n-4+\alpha} J\Big(n-3, \frac{\alpha}{2} - 1\Big)$$

Since $\alpha \in (4-n, 5-n)$, the integrals $J(n-5, \frac{\alpha}{2})$ and $J(n-3, \frac{\alpha}{2} - 1)$ are infinite and we have,

$$J\Big(n-5, \frac{\alpha}{2}\Big) \underset{0}{\sim} \frac{\big(1 - s_1^2\big)^{\frac{-1-\alpha}{2}} \theta^{-n-\alpha+4}}{n-4+\alpha}, \quad J\Big(n-3, \frac{\alpha}{2} - 1\Big) \underset{0}{\sim} \frac{\big(1 - s_1^2\big)^{\frac{1-\alpha}{2}} \theta^{-n-\alpha+4}}{n-4+\alpha}$$

And the integrals $J(n-4, \frac{\alpha}{2}-1)$ and $J(n-3, \frac{\alpha}{2}-2)$ are finite. Then,

$$\int_{\theta s_1}^{\beta(\theta,s)} g_{11}(\rho,\theta,s)d\rho \underset{\theta\mapsto 0}{\longrightarrow} \frac{(n-3)(n-4)s_1^2}{n-4+\alpha},$$

$$\int_{\theta s_1}^{\beta(\theta,s)} g_{22}(\rho,\theta,s)d\rho \underset{\theta\mapsto 0}{\longrightarrow} \frac{\alpha\left(1-s_1^2\right)}{n-4+\alpha}.$$

$$\int_{\theta s_1}^{\beta(\theta,s)} g_{12}(\rho,\theta,s)d\rho \underset{\theta\mapsto 0}{\longrightarrow} 0, \int_{\theta s_1}^{\beta(\theta,s)} g_{21}(\rho,\theta,s)d\rho \underset{\theta\mapsto 0}{\longrightarrow} 0$$

Moreover, we can see that, for any $i, j \in \{1,2\}$, for any $(\theta, s) \in (-1/2, 1/2) \times \mathbb{S}^{n-1} \backslash \{(\pm 1, 0, \dots, 0)\}$,

$$H(\theta,s)\int_{\theta s_1}^{\beta(\theta s)} g_{ij}(\rho,\theta,s)d\rho \leq C_{n,\alpha}\left(1-s_1^2\right)^{\frac{5-n}{2}} + D_{n,\alpha}\left(1-s_1^2\right)^{\frac{\alpha+1}{2}}.$$

where $C_{n,\alpha}$ and $D_{n,\alpha}$ are two constants independent of θ. By Lebesgue Theorem we deduce that,

$$\int_{\mathbb{S}^{n-1}} H(\theta,s)\frac{\partial^2 F}{\partial\theta^2}(\theta,s)\,ds \underset{\theta\to 0}{\longrightarrow} \frac{-(n-1)^2}{n}|\mathbb{S}^{n-1}| + (n-1)\frac{(n-3)(n-4)+\alpha(n-1)}{n(n-4+\alpha)}|\mathbb{S}^{n-1}|.$$

By Lemmas 1.1–1.3, $\theta \mapsto G(\theta) \in \mathcal{C}^1((-1/2,1/2),\mathbb{R})$ and is two times differentiable on $(-1/2,1/2)\backslash\{0\}$. Furthermore, when $\alpha \in (4-n, 5-n)$, as the limit of $\frac{d^2G}{d\theta^2}(\theta)$ exists as $\theta \to 0$, we have $\theta \mapsto G(\theta)$ is two times differentiable on $(-1/2, 1/2)$. □

Proof of ii) Assume that $\alpha \in (4-n, 5-n)$, by Lemma 1.1–1.4, we have,

$$G(\theta) = G(0) + \frac{1}{2}\frac{d^2G}{d\theta^2}(0) + o(\theta^2).$$

Furthermore we have,

$$\frac{d^2G}{d\theta^2}(0) = \frac{-n^2+4n-3}{2n(n-2+\alpha)}|\mathbb{S}^{n-1}| + \frac{2(n-3)(n-1)}{n(n-3+\alpha)}|\mathbb{S}^{n-1}| + \frac{-(n-1)^2}{n}|\mathbb{S}^{n-1}| + (n-1)\frac{(n-3)(n-4)+\alpha(n-1)}{n(n-4+\alpha)}|\mathbb{S}^{n-1}|.$$

We have, for any $n \geq 6$.

$$(n-3)(n-4)+\alpha(n-1) \underset{\alpha\mapsto 4-n}{\longrightarrow} -2(n-4) < 0.$$

Then,

$$\frac{(n-3)(n-4)+\alpha(n-1)}{n(n-4+\alpha)} \underset{\alpha\mapsto_{>}4-n}{\longrightarrow} -\infty, \text{ and } \frac{d^2G}{d^2\theta}(0) \underset{\alpha\mapsto_{>}4-n}{\longrightarrow} -\infty.$$

Hence, there is α_0 such that, for any $\alpha \in (4-n, \alpha_0)$, $G(\theta) < G(0)$ for θ sufficiently small, that is,

$$G(\theta) = E_{2,r^\alpha}(u_a) = \int_{\mathbf{B}^n} r^\alpha \|\nabla u_a\|^2 dx < G(0) = \int_{\mathbf{B}^n} r^\alpha \|\nabla u_0\|^2 dx. \qquad \square$$

Acknowledgements The author would express his grateful to Professor Ahmad El Soufi and Professor Etienne Sandier for their helpful advice.

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