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Calculus of Variations

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Boundary vortices in thin magnetic films

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Abstract We study the asymptotic behavior of a family of functional describing the formation of topologically induced boundary vortices in thin magnetic films. We obtain convergence results for sequences of minimizers and some classes of stationary points, and relate the limiting behavior to a finite dimensional problem, the renormalized energy associated to the vortices.

Keywords Micromagnetism · Ginzburg-Landau type vortices

Mathematics Subject Classification (2000) 35B25; 82D40

1 Introduction

In this article we analyze the behavior as $\varepsilon \to 0$ of the functionals

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon}\int_{\partial\Omega} \sin^2(u-g)\,d\mathscr{H}^1, \tag{1.1}$$

where Ω is a simply connected domain in $\mathbb{R}^2$, and $g : \partial\Omega \to \mathbb{R}$ a function such that $e^{ig} : \partial\Omega \to S^1$ is a map of degree $D \neq 0$. We show convergence results for sequences of minimizers and stationary points of not too high energy. The limit functions are harmonic functions with boundary singularities. For certain cases, in particular for minimizers, we give an asymptotic expansion for the energy, showing that the singular part of the energy depends only on the number of such singularities, while their interaction energy is described by a renormalized energy occurring as the first nonsingular term in the expansion, similar to results obtained by Bethuel-Brezis-Hélein for the Ginzburg-Landau energy.

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The motivation to study the functionals (1.1) comes from micromagnetism. Kohn and Slastikov [6] were able to show that it arises as a thin-film limit of the micromagnetic energy functional given by

$$E(m) = w^2 \int_{\Omega_h} |\nabla m|^2 + \int_{\mathbb{R}^3} |\nabla U|^2, \tag{1.2}$$

where $\Omega_h = \Omega \times (0, h)$ is a Lipschitz domain in $\mathbb{R}^3$, $m : \Omega_h \to S^2$, and U is related to m via the static Maxwell equation $\Delta U = \operatorname{div}(m\chi_{\Omega_h})$. The number w is a material parameter, called the *exchange length*. We have neglected crystal anisotropy here, which amounts to considering so-called *soft* magnetic films, and have not included the interaction with an external magnetic field.

Depending on the relation between the length scales w, h and $\ell = \operatorname{diam} \Omega$, many scaling limits of (1.2) can be considered, see [4] for an overview. We will set $\ell = 1$ by choice of units (so h really denotes the aspect ratio), and concentrate on thin films, i.e. $h \to 0$. One of the first results in this direction is due to Gioia and James [5], who studied the case where $h \to 0$ while w stays constant. The resulting limiting theory predicts the limit magnetization to be constant. However, w is usually also small, and so it is useful to have theories that treat w as another small parameter.

Kohn and Slastikov [6] studied the regimes $\frac{w^2}{h|\log h|} \to \infty$ and $\frac{w^2}{h|\log h|} \to \alpha \in (0, \infty)$ and could show Γ-convergence of appropriate rescalings of the micromagnetic energy to limiting reduced energy functionals. In the first case, the limit energy is finite only on constant in-plane magnetizations $m \equiv \overline{m} \in S^1$, and given by

$$\frac{1}{2\pi} \int_{\partial\Omega} (\overline{m} \cdot \nu)^2 d\mathscr{H}^1,$$

where ν denotes the outer normal to $\partial\Omega$. In the second case, the magnetization is still forced to be in-plane and unit length, but need not be constant. The energy is given by

$$\mathscr{E}^\alpha(m) = \alpha \int_\Omega |\nabla m|^2 + \frac{1}{2\pi} \int_{\partial\Omega} (m \cdot \nu)^2. \tag{1.3}$$

In the borderline case where $\frac{w^2}{h}$ is constant, Moser [10, 11] was able to show a convergence result for minimizers of (1.2) and could show the formation of boundary vortices.

We investigate the behavior as $\alpha \to 0$ (i.e. $\frac{w^2}{h|\log h|} \to 0$) of $\frac{1}{\alpha}\mathscr{E}^\alpha$ which can be seen as connecting the results of Kohn and Slastikov to that of Moser. The functionals (1.1) correspond to those of (1.3) after the substitutions $m = e^{iu}$ and $\nu = ie^{ig}$.

Let us explain a bit how these functionals are similar to the Ginzburg-Landau functional of [1]. With $m_0 = \tau$ being a continuous unit tangent field to $\partial\Omega$, we are (after recaling and renaming variables) considering the variational problem for $m : \Omega \to \mathbb{R}^2$: Minimize

$$\frac{1}{2} \int_\Omega |\nabla m|^2 + \frac{1}{2\varepsilon} \int_{\partial\Omega} (1 - (m \cdot m_0)^2) d\mathscr{H}^1$$

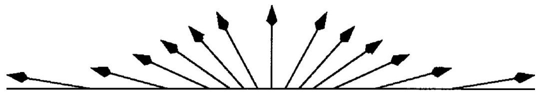

Fig. 1 A boundary vortex

subject to $|m| = 1$ in Ω as $\varepsilon \to 0$. This problem has an interior constraint and a boundary penalty.

Bethuel, Brezis and Hélein [1] studied the behavior as $\varepsilon \to 0$ of

$$\frac{1}{2}\int_{\Omega} |\nabla m|^2 + \frac{1}{4\varepsilon^2}\int_{\Omega} \left(1 - |m|^2\right)^2$$

subject to $m = m_0$ on $\partial\Omega$, so this problem has a boundary constraint and an interior penalty.

Common to both problems is that, as long as m_0 has nonzero topological degree, there is no map in $H^1(\Omega, \mathbb{R}^2)$ that satisfies the constraint and makes the penalty term zero. This is due to the fact that a continuous map $v : \partial\Omega \to S^1$ can be extended to a continuous map $\overline{v} : \overline{\Omega} \to S^1$ if and only if $\deg(v) = 0$. Although H^1 maps need not be continuous, the argument still carries through to show that there is not even an extension of finite H^1 energy. Both problems are thus forced to develop singularities as $\varepsilon \to 0$, and the minimum energy will become unbounded.

We call the singularities of both problems *vortices*, since minimizers converge as $\varepsilon \to 0$ to maps that have the form $\frac{z-a_i}{|z-a_i|}$ near the singularities a_i. In the Ginzburg-Landau case, these vortices are interior and each carries a topological degree of 1; in our case, the singularities lie on the boundary, and we only see one half of the vortex. Each "boundary vortex" corresponds to a transition from m_0 to $-m_0$ or vice versa, and can be viewed as carrying $\frac{1}{2}$ topological charge. More detailed analysis (see Proposition 6.2) actually shows that the minimizers for small ε look like a standard vortex $\frac{z}{|z|}$ placed at distance ε outside the domain, see Fig. 1, where the domain is above the line and m_0 is its tangent. Since both our functional and the simplified Ginzburg-Landau functional exhibit formation of singularities due to the same topological reason, it is perhaps not too surprising that there are similarities in both the methods used and the results obtained.

Our main results in this paper are Theorem 4.2, where we prove subconvergence of minimizers and isolation of vortices, Theorem 5.4 where we obtain subconvergence for stationary points satisfying a natural logarithmic energy bound, and finally Theorem 8.6 where we give an asymptotic expansion of the energy along a converging sequence with isolated vortices. The energy is given by a singular part depending only on the number of the vortices, and an $O(1)$ part that depends on the position of the vortices and can be calculated via the solution of a linear boundary value problem.

Our approach to convergence theorems for minimizers follows the ideas of Bethuel-Brezis-Hélein [1] and Struwe [13]. There are also similarities to the approach of Moser [11] who combined interior and boundary vortices, but without calculating a renormalized energy. A different view of (1.1) was pursued in [7], where the functional was reduced to a nonlocal one on the boundary, and a Γ-convergence theorem for the natural scaling was proved. In [7], we could treat an arbitrary continuous periodic potential Φ with $\Phi^{-1}(0) = \pi\mathbb{Z}$. In the present paper, we rely on $\Phi(t) = \sin^2 t$ because we use the uniqueness result of Toland [15]. However, after submission of this article, we learnt that X. Cabré and J. Solà-Morales [3] have recently studied the half-space solutions corresponding to a large class of potentials and proved a quite general uniqueness theorem. Independently from our work, X. Cabré and N. Cónsul have also derived a renormalized energy for a class of related problems, see [2].

2 Conventions and basic results

We will use the expression "a sequence $\varepsilon \to 0$" meaning any sequence $\varepsilon_j \to 0$ that will then be regarded as fixed, and subsequences will be taken from this fixed sequence.

We will use $B_R^+(z_0)$ with $z_0 = (x_0, y_0)$ to denote the half-ball $\{z \in \mathbb{R}^2 : |z - z_0| < R, y > y_0\}$, and abbreviate $B_R^+ = B_R^+(0)$. The symbol Γ_R will usually denote the flat part of ∂B_R^+.

We usually omit to explicitly mention the measure when writing integrals, unless there is possibility of confusion. Integrals over 2-dimensional sets like B_R^+, Ω etc. are thus implicitly meant to be w.r.t. 2-dimensional Lebesgue measure, while integrals over 1-dimensional sets such as Γ_R, $\partial\Omega$ or $\partial B_R \cap \Omega$ are w.r.t. 1-dimensional Hausdorff measure $\mathscr{H}^1$.

For the convenience of the reader, we collect some results on existence and regularity results for minimizers and stationary point of (1.1) whose proofs are relatively straightforward.

Proposition 2.1 *For all $\varepsilon > 0$, the functional E_ε attains its minimum.*

Proposition 2.2 *Stationary points of E_ε satisfy the equation*

$$\int_\Omega \nabla u \cdot \nabla \varphi + \frac{1}{2\varepsilon} \int_{\partial\Omega} \sin(2(u-g))\varphi = 0 \tag{2.1}$$

for all $\varphi \in H^1(\Omega)$. Any solution u of (2.1) is of class $H^2(\Omega)$, and is a strong solution of the equations

$$\Delta u = 0 \qquad in\ \Omega \tag{2.2}$$

$$\frac{\partial u}{\partial \nu} = -\frac{1}{2\varepsilon} \sin 2(u-g) \quad on\ \partial\Omega. \tag{2.3}$$

If in addition $\partial\Omega \in C^{k+1}$ and $g \in C^k$ (i.e. $e^{ig} \in C^k$), then $u \in H^{k+1}(\Omega)$. In particular, if $\partial\Omega$ and g are C^∞, then $u \in C^\infty(\overline{\Omega})$. If $\partial\Omega$ and g are real analytic, then also u is real analytic up to the boundary.

Proof The H^k regularity can be proved by a difference quotient argument. The claim about the analyticity follows from [9]. □

3 Localization of vortices

In this section we show that for sequences (u_ε) of stationary points of E_ε that satisfy an energy bound

$$E_\varepsilon(u_\varepsilon) \le M \log \frac{1}{\varepsilon}, \tag{3.1}$$

the *approximate vortex set* $S_\varepsilon := \{z \in \partial\Omega : \sin^2\big(u(x) - g(x)\big) \ge \frac{1}{4}\}$ can be covered by a bounded number of ε-balls.

In order to see that the assumption (3.1) is reasonable, we show that it holds true for minimizers:

Proposition 3.1 *There is a constant $C_1 = C_1(\Omega, g)$ such that any sequence of minimizers (u_ε) of E_ε satisfies*

$$E_\varepsilon(u_\varepsilon) \le \pi D \log \frac{1}{\varepsilon} + C_1. \tag{3.2}$$

Proof It suffices to construct one sequence of functions (v_ε) satisfying this bound. To this end, choose $2D$ distinct points $a_1, \ldots, a_{2D} \in \partial\Omega$ and let $0 < R < \frac{1}{2}\min_{i\neq j}|a_i - a_j|$. We construct the comparison function v_ε separately inside $B_R(a_i) \cap \Omega$ and in the rest of the domain. Setting $a_i = 0$ without loss of generality, we can assume R to be so small that $\Omega \cap B_R = \{re^{i\vartheta} : \vartheta_1(r) < \vartheta < \vartheta_2(r), 0 < r < R\}$ with $|\vartheta_j'| \le c$ and so $|\vartheta_2(r) - \vartheta_1(r) - \pi| \le cr$. With $h_1(r) = g(e^{i\vartheta_1(r)}) + k\pi$ and $h_2(r) = g(e^{i\vartheta_2(r)}) + (k-1)\pi$, $k \in \mathbb{Z}$, we define v_ε in $\Omega \cap (B_R \setminus B_\varepsilon)$ as

$$v_\varepsilon(re^{i\vartheta}) = \frac{h_2(r) - h_1(r)}{\vartheta_2(r) - \vartheta_1(r)}(\vartheta - \vartheta_1(r)) + h_1(r).$$

Note that this function satisfies $\sin^2(v_\varepsilon - g) = 0$ on $B_R \cap \partial\Omega$. Expressing the Dirichlet integral in polar coordinates, it is then easy to see that the part corresponding to the radial derivative is bounded independently of ε. The tangential derivative yields the term

$$\frac{1}{2}\int_\varepsilon^R \int_{\vartheta_1}^{\vartheta_2} \frac{1}{r^2}\left(\frac{h_2 - h_1}{\vartheta_2 - \vartheta_1}\right)^2 r\,dr\,d\vartheta = \frac{1}{2}\int_\varepsilon^R \frac{(h_2 - h_1)^2}{r(\vartheta_2 - \vartheta_1)}dr \le \frac{1}{2}\int_\varepsilon^R \frac{(\pi + cr)^2}{\pi - cr}dr,$$

and this can be estimated by $\frac{\pi}{2}\log\frac{R}{\varepsilon} + C$. Inside $B_\varepsilon \cap \Omega$, we will have to violate the condition $\sin^2(v_\varepsilon - g) = 0$ in order to obtain a function with bounded Dirichlet energy. By scaling, it is easy to see that a continuation of v_ε with uniformly bounded Dirichlet integral exists, and since $\mathscr{H}^1(\partial\Omega \cap B_\varepsilon) \le c\varepsilon$, this shows $E_\varepsilon(v_\varepsilon; B_R \cap \Omega) \le \frac{\pi}{2}\log\frac{R}{\varepsilon} + c$. Choosing the constants $k = k_i$ near each a_i appropriately and using a harmonic continuation of $v_\varepsilon|_{\partial B_R(a_i)}$ and $g + k_i\pi$ to $\Omega \setminus \cup B_R(a_i)$, we finally can combine everything to a comparison function satisfying (3.2). □

As in the proofs for corresponding results in Ginzburg-Landau vortices [1], [13], a central point in obtaining estimates is a Rellich-Pohožaev identity. We state it in the following form:

Lemma 3.2 *Assume that* $\Omega \subset \mathbb{R}^2$ *is a Lipschitz domain,* $u \in H^2(\Omega)$ *is harmonic, and* $w \in C^1(\overline{\Omega}, \mathbb{C})$ *is holomorphic inside* Ω*. Then*

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu}(w \cdot \nabla u) = \frac{1}{2}\int_{\partial\Omega}(w \cdot \nu)|\nabla u|^2, \tag{3.3}$$

where ν *denotes the outer normal to* $\partial\Omega$*.*

Proof For any $u \in H^2(\Omega)$, it is easy to prove by direct calculation and using the Cauchy-Riemann equations for w that

$$\nabla u \cdot \nabla(w \cdot \nabla u) = \frac{1}{2}\operatorname{div}(w|\nabla u|^2).$$

Integrating by parts $\int_\Omega \Delta u(w \cdot \nabla u) = 0$ and using the last identity, (3.3) now follows easily from the Gauß-Green theorem. □

We note the following important consequence of (3.3):

Lemma 3.3 *Let* Ω *be a* strongly star-shaped *Lipschitz domain, i.e. assume there exists a* $p \in \Omega$ *and* $k > 0$ *such that* $(z - p) \cdot \nu \geq k|z - p|$ *for all* $z \in \partial\Omega$*. Assume* $u \in H^2(\Omega)$ *is harmonic. Then there exist constants* $0 < c < C$ *depending only on* k *such that*

$$c\int_{\partial\Omega}\left|\frac{\partial u}{\partial \tau}\right|^2 \leq \int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 \leq C\int_{\partial\Omega}\left|\frac{\partial u}{\partial \tau}\right|^2, \tag{3.4}$$

where $\frac{\partial u}{\partial \tau}$ *denotes the tangential derivative.*

Proof With p being a star point as above that we assume to be 0 without loss of generality, we use the Rellich-Pohožaev identity (3.3) with $w(z) = z$. This shows

$$\int_{\partial\Omega}\frac{\partial u}{\partial \nu} z \cdot \nabla u = \frac{1}{2}\int_{\partial\Omega}(z \cdot \nu)|\nabla u|^2.$$

From the decomposition $\nabla u = \frac{\partial u}{\partial \nu}\nu + \frac{\partial u}{\partial \tau}\tau$ we obtain

$$\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 z \cdot \nu + \frac{\partial u}{\partial \nu}\frac{\partial u}{\partial \tau} z \cdot \tau = \frac{1}{2}\int_{\partial\Omega}\left(\left|\frac{\partial u}{\partial \tau}\right|^2 + \left|\frac{\partial u}{\partial \nu}\right|^2\right) z \cdot \nu$$

from which it follows that

$$\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 z \cdot \nu = \int_{\partial\Omega}\left|\frac{\partial u}{\partial \tau}\right|^2 z \cdot \nu - 2\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\frac{\partial u}{\partial \tau}\right| z \cdot \tau,$$

and now we can use the lower bound $z \cdot \nu \geq k|z|$, $|\tau| = 1$ and the inequality $2AB \leq \alpha A^2 + \alpha^{-1}B^2$ to finish the proof. □

In the following we will derive estimates relating the penalty term and the following radial derivative of the energy:

Definition 3.4 For $z_0 \in \partial\Omega$, $\varepsilon > 0$ and $u \in H^2(\Omega)$ define for all $\rho > 0$

$$A(\rho) = A_{u,\varepsilon,z_0}(\rho) = \rho \int_{\partial B_\rho(z_0)\cap\Omega} |\nabla u|^2 d\mathscr{H}^1 + \frac{\rho}{\varepsilon}\int_{\partial B_\rho(z_0)\cap\partial\Omega} \sin^2(u-g)d\mathscr{H}^0. \tag{3.5}$$

Proposition 3.5 *There exist* $\varepsilon_0 > 0$ *and* $C_2 > 0$ *depending only on* Ω *and* g *such that for all* $\varepsilon < \varepsilon_0$, $\rho < \varepsilon^{3/4}$, *any stationary point* u *of* E_ε, *and any* $z_0 \in \partial\Omega$, *the following inequality holds:*

$$\frac{1}{2\varepsilon}\int_{\Gamma_\rho(z_0)} \sin^2(u-g) \le A_{u,\varepsilon,z_0}(\rho) + C_2\sqrt{\varepsilon}, \tag{3.6}$$

where $\Gamma_\rho(z_0) = \partial\Omega \cap B_\rho(z_0)$.

Proof We choose ε_0 so small that for all $\rho < \varepsilon_0^{3/4}$ and all $z_0 \in \partial\Omega$, $\omega_\rho(z_0) = \Omega\cap B_\rho(z_0)$ is strongly star-shaped in the sense of Lemma 3.3 with respect to some $p_\rho \in \omega_\rho(z_0)$, with a $k > 0$ that can be chosen uniformly in ρ and z_0. In addition, we assume by using $\partial\Omega \in C^2$ and choosing ε_0 sufficiently small that there exists a vector field $Z \in C^1(\overline{\Omega}, \mathbb{R}^2)$ with the property that for $|z-z_0| < \varepsilon_0^{3/4}$, there hold $Z\cdot\nu = 0$ on $\partial\Omega$ and the inequalities $|Z-z| \le C|z-z_0|^2$ and $|\nabla Z - \mathrm{id}\,| \le C|z-z_0|$. Setting $z_0 = 0$ for convenience, we multiply $\Delta u = 0$ with $z\cdot\nabla u$ and obtain by integration by parts over ω_ρ the relation

$$\int_{\omega_\rho} \nabla u\cdot\nabla(z\cdot\nabla u) = \int_{\partial\omega_\rho} \frac{\partial u}{\partial\nu} z\cdot\nabla u.$$

We use (3.3) and split $z = Z + (z-Z)$ on Γ_ρ. This yields

$$\frac{1}{2}\int_{\partial\omega_\rho} (z\cdot\nu)|\nabla u|^2 = \rho\int_{\partial B_\rho\cap\Omega} \left|\frac{\partial u}{\partial\nu}\right|^2 + \int_{\Gamma_\rho} \frac{\partial u}{\partial\nu} Z\cdot\nabla u + \int_{\Gamma_\rho}\frac{\partial u}{\partial\nu}(z-Z)\cdot\nabla u.$$

Noting that $Z\cdot\nabla u = (Z\cdot\tau)\frac{\partial u}{\partial\tau}$, where τ is a tangent field to $\partial\Omega$, we can integrate the term involving Z by parts and obtain using (2.3)

$$\begin{aligned}\int_{\Gamma_\rho}\frac{\partial u}{\partial\nu} Z\cdot\nabla u &= -\frac{1}{2\varepsilon}\int_{\Gamma_\rho} \sin 2(u-g)\frac{\partial u}{\partial\tau}(Z\cdot\tau)\\ &= -\frac{1}{2\varepsilon}\int_{\partial\Omega\cap\partial B_\rho} \sin^2(u-g)(Z\cdot\tau)d\mathscr{H}^0\\ &\quad +\frac{1}{2\varepsilon}\int_{\Gamma_\rho}\sin^2(u-g)\frac{\partial}{\partial\tau}(Z\cdot\tau) - \frac{1}{2\varepsilon}\int_{\Gamma_\rho}\sin 2(u-g)\frac{\partial g}{\partial\tau}(Z\cdot\tau).\end{aligned}$$

Combining this with the results above shows

$$\begin{aligned}\frac{1}{2\varepsilon}\int_{\Gamma_\rho}\sin^2(u-g)\frac{\partial}{\partial\tau}(Z\cdot\tau) &= \frac{1}{2\varepsilon}\int_{\partial\Omega\cap\partial B_\rho}\sin^2(u-g)Z\cdot\tau\, d\mathscr{H}^0\\ &\quad+\frac{1}{2\varepsilon}\int_{\Gamma_\rho}\sin 2(u-g)Z\cdot\tau g' + \frac{1}{2}\int_{\partial\omega_\rho} z\cdot\nu|\nabla u|^2\\ &\quad+\int_{\partial\omega_\rho}\frac{\partial u}{\partial\nu}(Z-z)\cdot\nabla u - \rho\int_{\partial B_\rho\cap\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2.\end{aligned}$$

Dropping the final term due to its sign, using the assumptions on Z, a C^1 bound on g, and $|z \cdot \nu| \leq C\rho^2$ on Γ_ρ, we obtain that

$$(1 - C\rho)\frac{1}{2\varepsilon}\int_{\Gamma_\rho} \sin^2(u-g) \leq \frac{1+C\rho}{2}\frac{\rho}{\varepsilon}\int_{\partial\Omega\cap\partial B_\rho} \sin^2(u-g)d\mathscr{H}^0$$
$$+ \frac{1}{\varepsilon}c(g)\rho^2 + \left(C\rho^2 + \frac{\rho}{2}\right)\int_{\partial B_\rho\cap\Omega} |\nabla u|^2 + C\rho^2\int_{\Gamma_\rho} |\nabla u|^2.$$

By the star-shapedness of ω_ρ and Lemma 3.3 we have the estimate

$$\int_{\Gamma_\rho} |\nabla u|^2 \leq C\left(\int_{\Gamma_\rho}\left|\frac{\partial u}{\partial \nu}\right|^2 + \int_{\partial B_\rho\cap\Omega} |\nabla u|^2\right)$$

and by (2.3), we can estimate

$$\int_{\Gamma_\rho}\left|\frac{\partial u}{\partial \nu}\right|^2 = \frac{1}{4\varepsilon^2}\int_{\Gamma_\rho} 4\sin^2(u-g)\cos^2(u-g) \leq \frac{2}{\varepsilon}\left(\frac{1}{2\varepsilon}\int_{\Gamma_\rho}\sin^2(u-g)\right).$$

Combining terms, we obtain

$$\left(1 - C\rho - \frac{C\rho^2}{\varepsilon}\right)\frac{1}{2\varepsilon}\int_{\Gamma_\rho} \sin^2(u-g) \leq \frac{1+C\rho}{2}\frac{\rho}{\varepsilon}\int_{\partial\Omega\cap\partial B_\rho} \sin^2(u-g)d\mathscr{H}^0$$
$$+ \left(C\rho^2 + \frac{\rho}{2}\right)\int_{\partial B_\rho\cap\Omega} |\nabla u|^2 + \frac{C\rho^2}{\varepsilon},$$

and from this we can deduce the claim for $\varepsilon < \rho < \varepsilon^{3/4}$ and $\varepsilon < \varepsilon_0$ sufficiently small. □

This leads to the following criterion for vortex-free parts of the boundary:

Proposition 3.6 *There are constants $\gamma > 0$ and $C_3 > 0$ depending on Ω and g such that for every $z_0 \in \partial\Omega$, $\varepsilon < \varepsilon_0$ (with ε_0 from Proposition 3.5), $\rho < \varepsilon^{3/4}$, and every stationary point u of E_ε satisfying $A_{u,\varepsilon,z_0}(\rho) < \gamma$, there holds*

$$\sup_{\Gamma_{\rho/2}(z_0)} \sin^2(u-g) < \frac{1}{4} \tag{3.7}$$

and

$$\frac{1}{2\varepsilon}\int_{\Gamma_\rho(z_0)} \sin^2(u-g) \leq C_3. \tag{3.8}$$

Proof By Lemma 3.3, we can estimate

$$\int_{\Gamma_\rho}\left|\frac{\partial u}{\partial \tau}\right|^2 \leq C\int_{\partial\omega_\rho}\left|\frac{\partial u}{\partial \nu}\right|^2 \leq C\int_{\partial B_\rho\cap\Omega}|\nabla u|^2 + C\int_{\Gamma_\rho}\left|\frac{\partial u}{\partial \nu}\right|^2. \tag{3.9}$$

From (3.6), the definition of A in (3.5) and $A(\rho) < \gamma$, we thus can estimate, using (2.3) as in the last proof and Sobolev embedding in one dimension

$$[u]^2_{C^{0,1/2}(\Gamma_\rho)} \le C \int_{\Gamma_\rho} \left|\frac{\partial u}{\partial \tau}\right|^2 \le C \left(\frac{1}{\rho} A(\rho) + \frac{1}{\varepsilon^2} \int_{\Gamma_\rho} \sin^2(u-g)\right)$$
$$\le \frac{C}{\varepsilon}\big(2\gamma + C_2\sqrt{\varepsilon_0}\big).$$

Assume now that $\sin^2(u(z) - g(z)) \ge \frac{1}{4}$ for some $z \in \Gamma_{\rho/2}$. Then by the last equation and the differentiability of g, there holds $\sin^2(u(z') - g(z')) \ge \frac{1}{8}$ at least for $|z - z'| \le \frac{\varepsilon}{C(\gamma + \sqrt{\varepsilon_0})}$, where the latter term is $\ge \frac{\varepsilon}{2}$ if we choose ε_0 and γ sufficiently small. We estimate $\int_{\Gamma_\rho} \sin^2(u-g)$ from below:

$$\frac{1}{2\varepsilon}\int_{\Gamma_\rho} \sin^2(u-g) \ge \frac{1}{2\varepsilon}\frac{1}{8}\frac{\varepsilon}{2} \ge \frac{1}{32}.$$

On the other hand, we have by Proposition 3.5 the upper bound $\gamma + C_2\sqrt{\varepsilon_0}$, and now choosing γ and ε_0 sufficiently small leads to a contradiction. □

Lemma 3.7 *Let (u_ε) be a sequence of stationary points of E_ε satisfying the logarithmic energy bound $E_\varepsilon(u_\varepsilon) \le M \log \frac{1}{\varepsilon}$. Then for any $z_0 \in \partial\Omega$, the function $A(\rho) = A_{u_\varepsilon,\varepsilon,z_0}(\rho)$ defined as in (3.5) satisfies*

$$\inf_{\varepsilon^{6/7} \le \rho \le \varepsilon^{5/6}} A(\rho) \le \frac{84}{\log \frac{1}{\varepsilon}} E_\varepsilon(u_\varepsilon; \Omega \cap B_{\varepsilon^{5/6}}(z_0)) \le 84M \tag{3.10}$$

and

$$\inf_{5\varepsilon^{5/6} \le \rho \le 5\varepsilon^{4/5}} A(\rho) \le 60M. \tag{3.11}$$

Proof The first claim follows from the calculation

$$M \log\frac{1}{\varepsilon} \ge E_\varepsilon(u_\varepsilon; \Omega \cap B_{\varepsilon^{5/6}}) \ge \frac{1}{2}\int_{\varepsilon^{6/7}}^{\varepsilon^{5/6}} \frac{A(\rho)}{\rho} d\rho \ge \frac{1}{2}(\inf A) \log\frac{\varepsilon^{5/6}}{\varepsilon^{6/7}} = \frac{\inf A}{84} \log\frac{1}{\varepsilon}.$$

The inequality (3.11) follows in a similar manner. □

As in Ginzburg-Landau theory (see e.g. the lecture notes of Rivière [12]) sets that carry a small amount of energy do not contain vortices:

Lemma 3.8 **(η-compactness)** *There exist constants $\eta_0, \varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and $\rho < \varepsilon^{3/4}$, and every stationary point u of E_ε satisfying for some $z_0 \in \partial\Omega$ the inequality*

$$E_\varepsilon(u; B_\rho(z_0) \cap \Omega) \le \eta_0 \log\frac{\rho}{\varepsilon},$$

there holds $\sin^2(u - g) < \frac{1}{4}$ on $B_{\rho/2}(z_0) \cap \partial\Omega$.

Proof By virtually the same argument as above, we obtain around any $z \in B_{\rho/2}(z_0) \cap \partial\Omega$ that

$$\eta_0 \log \frac{\rho}{\varepsilon} \geq \frac{1}{2} \int_{\varepsilon/2}^{\varepsilon^{3/4}/2} \frac{A(r)}{r} dr \geq \frac{1}{8} (\inf A) \log \frac{1}{\varepsilon},$$

hence

$$\inf_{\varepsilon/2 < \sigma < \varepsilon^{3/4}/2} A(\sigma) \leq 8 \frac{\log \frac{\rho}{\varepsilon}}{\log \frac{1}{\varepsilon}} < 8\eta_0.$$

We can now choose η_0 sufficiently small so that Proposition 3.6 implies the claim. □

Proposition 3.9 *There is a constant $N = N(g, \Omega, M)$ such that for any sequence of stationary points u_ε satisfying the energy bound $E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$, the approximate vortex set S_ε can be covered by at most N balls of radius ε, such that the $\varepsilon/5$ balls around the same centers are disjoint.*

Proof For $z \in S_\varepsilon$, we choose by virtue of Proposition 3.6 and Lemma 3.7 a radius $\rho \in [\varepsilon^{6/7}, \varepsilon^{5/6}]$ such that

$$\frac{84}{\log \frac{1}{\varepsilon}} E_\varepsilon(u_\varepsilon; \Omega \cap B_{\varepsilon^{4/5}}(z)) \geq A_{u_\varepsilon,\varepsilon,z}(\rho) \geq \gamma. \tag{3.12}$$

Choose by Vitali's $5r$ covering lemma $z_j = z_j^\varepsilon \in S_\varepsilon$, $j \in J_\varepsilon$, such that $S_\varepsilon \subset \cup_{j \in J_\varepsilon} B_{5\varepsilon^{4/5}}(z_j)$, and such that the $B_{\varepsilon^{4/5}}(z_j)$ are disjoint. Then (3.12) shows that

$$|J_\varepsilon| \leq \frac{84M}{\gamma}. \tag{3.13}$$

We now choose radii $\rho_j \in [5\varepsilon^{5/6}, 5\varepsilon^{4/5}]$ such that $A_{u_\varepsilon,\varepsilon,z_j}(\rho_j) \leq 60M$. Using Proposition 3.6 we obtain

$$\frac{1}{2\varepsilon} \int_{\partial\Omega \cap B_{\rho_j}(z_j)} \sin^2(u_\varepsilon - g) \leq C,$$

and now by the same argument as in the proof of Proposition 3.6, we see

$$[u_\varepsilon]_{C^{0,1/2}(\partial\Omega \cap B_{\rho_j}(z_j))} \leq \frac{C}{\sqrt{\varepsilon}}, \tag{3.14}$$

and this again implies

$$\frac{1}{2\varepsilon} \int_{\partial\Omega \cap B_{\varepsilon/5}(z_j)} \sin^2(u_\varepsilon - g) \geq c > 0. \tag{3.15}$$

Using once more the $5r$ lemma, we can choose $z_k = z_k^\varepsilon$, $k \in K_\varepsilon$ such that $B_{\varepsilon/5}(z_k)$ are disjoint and $B_\varepsilon(z_k)$ cover S_ε. By (3.13) and (3.15) we now have

$$c|K_\varepsilon| \leq \sum_{k \in K_\varepsilon} \frac{1}{\varepsilon} \int_{\partial\Omega \cap B_{\varepsilon/5}(z_j)} \sin^2(u_\varepsilon - g) \leq \sum_{j \in J_\varepsilon} \frac{1}{\varepsilon} \int_{\partial\Omega \cap B_{\rho_j}(z_j)} \leq \frac{84CM}{\gamma},$$

which implies the claim. □

For comparison arguments we shall need the following lower bound for the energy on half-annuli:

Proposition 3.10 *Let* $0 < \rho < R \leq R_0$, R_0 *sufficiently small,* $z_0 \in \partial\Omega$, *w.l.o.g.* $z_0 = 0$. *Assume* $D_{R,\rho} = (B_R \setminus \overline{B_\rho}) \cap \Omega = \{re^{i\vartheta} : \vartheta_1(r) < \vartheta < \vartheta_2(r), \rho < r < R\}$ *with* $|\vartheta_2(r) - \vartheta_1(r) - \pi| \leq Cr$. *Assume also that for* $j = 1, 2$ *there holds* $(u - g)(re^{i\vartheta_j(r)}) \in (k_j\pi - \delta, k_j\pi + \delta)$ *for some* $k_j \in \mathbb{Z}$ *and* $\delta \in (0, \frac{\pi}{2})$. *Then there is a constant depending on* R_0 *(which in turn depends on* Ω*) and* g *so that for any such function* u, *its energy is bounded below as*

$$E_\varepsilon(u; D_{R,\rho}) \geq \frac{\pi}{2}(k_2 - k_1)^2 \log\frac{R}{\rho} - C(k_2 - k_1)^2\left(R + \frac{\varepsilon}{\rho}\right). \tag{3.16}$$

Proof We will use the abbreviations $u_j(r) = u(re^{i\vartheta_j(r)})$ and $g_j = g(re^{i\vartheta_j(r)})$ for the functions on the two boundary components. We also assume w.l.o.g. $k_1 = k$ and $k_2 = 0$. Using polar coordinates, disregarding the radial derivative and by use of Hölder's inequality, we calculate

$$\begin{aligned}\int_{D_{R,\rho}} |\nabla u|^2 &\geq \int_\rho^R \frac{1}{r}\int_{\vartheta_1}^{\vartheta_2}\left|\frac{\partial u}{\partial\vartheta}\right|^2 d\vartheta\, dr\\ &\geq \int_\rho^R \frac{1}{\vartheta_2 - \vartheta_1}\left(\int_{\vartheta_1}^{\vartheta_2}\left|\frac{\partial u}{\partial\vartheta}\right|\right)^2 \geq \int_\rho^R \frac{(u_1 - u_2)^2}{r(\pi + cr)}dr.\end{aligned}$$

We rewrite $u_1 - u_2 = k\pi - (u_1 - g_1 - k\pi) - (u_2 - g_2) - (g_1 - g_2)$. Using the lower bound $\sin^2(t - k_i\pi) \geq \sigma t^2$ valid for $|t| < \delta$ with some $\sigma = \sigma(\delta)$, we can thus estimate

$$\begin{aligned}E_\varepsilon(u; D_{R,\rho}) \geq &\frac{1}{2}\int_\rho^R \frac{1}{r(\pi + cr)}(k\pi - (g_1 - g_2) - ((u_1 - g_1 - k\pi) - (u_2 - g_2)))^2\\ &+ \frac{\sigma}{\varepsilon}\big((u_1 - g_1 - k\pi)^2 + (u_2 - g_2)^2\big)^2 dr.\end{aligned}$$

On the last term, we use the inequality $a^2 + b^2 \geq \frac{1}{2}(a+b)^2$ with $a = u_1 - g_1 - k\pi$ and $b = u_2 - g_2$. Then we use the inequality $\alpha(A - B)^2 + \beta B^2 \geq \frac{1}{\frac{1}{\alpha}+\frac{1}{\beta}}A^2$ that can be obtained by optimizing over B on $A = (k\pi - (g_1 - g_2)$ and $B = (u_1 - g_1 - k\pi) + (u_2 - g_2)$. This yields using also a C^1 bound on g

$$E_\varepsilon(u; D_{R,\rho}) \geq \frac{1}{2}\int_\rho^R \frac{(k\pi - cr)^2}{r(\pi + cr) + \frac{4\varepsilon}{\sigma}}dr.$$

After subtraction of $\frac{k^2\pi}{2r}$, the integral of the difference can then be estimated by

$$-C|k|(R - \rho) - Ck^2\varepsilon\left(\frac{1}{\rho} - \frac{1}{R}\right)$$

which implies the claim. □

4 Convergence results by comparison arguments

In this section, we assume u_ε to be stationary points of E_ε satsfying an upper bound

$$E_\varepsilon(u_\varepsilon) \le \pi D \log \frac{1}{\varepsilon} + C_0 \tag{4.1}$$

for some constant C_0, where D is the degree of e^{ig}. This bound holds true for minimizers by Proposition 3.1. We will use the following notation. By Proposition 3.9, there exist $a_j^\varepsilon \in \partial\Omega$, $1 \le j \le N_\varepsilon \le N$ such that the approximate vortex set S_ε satisfies $S_\varepsilon \subset \cup_{1\le j\le N_\varepsilon} B_\varepsilon(a_j^\varepsilon)$. Passing to a subsequence of $\varepsilon \to 0$, we can assume that $N_\varepsilon = N_0$ is constant and $a_j^\varepsilon \to a_j^0$ as $\varepsilon \to 0$. Note that the a_j^0 need not be distinct. We define for $0 < \sigma < \frac{1}{2}\min_{a_j^0 \ne a_{j'}^0} \operatorname{dist}(a_j^0, a_{j'}^0)$ the sets $\Omega_\sigma^\varepsilon = \Omega \setminus \cup_j B_\sigma(a_j^\varepsilon)$ and $\Omega_\sigma^0 = \Omega \setminus \cup_j B_\sigma(a_j^0)$. With this setup (and this subsequence) we have the following bounds:

Proposition 4.1 *There is a constant $C = C(g, \Omega, C_0)$ such that $E_\varepsilon(u_\varepsilon; \Omega_\sigma^\varepsilon) \le \pi D \log \frac{1}{\sigma} + C$.*

Proof We follow closely the proof of Proposition 3.3 in [13]. We write x_j for a_j^ε and set $\mathcal{N} = \{1, \dots, N_0\}$. Let $\mathcal{R}_\varepsilon^\sigma$ denote the set of radii in $[\varepsilon, \sigma]$ such that $\partial B_R(x_j) \cap B_\varepsilon(x_\ell) = \emptyset$ for $j \ne \ell$ and such that there exists for $R \in \mathcal{R}_\varepsilon^\sigma$ a $\mathcal{N}_R \subset \mathcal{N}$ with the properties that $(B_R(x_j))_{j\in\mathcal{N}_R}$ is disjoint, $\mathcal{N}_R \subset \mathcal{N}_{R'}$ for $R' \le R$ and $\cup_{j\in\mathcal{N}} B_\varepsilon(x_j) \subset \cup_{j\in\mathcal{N}_R} B_R(x_j)$. It is possible to show that $\mathcal{R}_\varepsilon^\sigma = \cup_{m=1}^M [\alpha_m, \beta_m]$, where for $R = \alpha_m$, there exists $\ell \notin \mathcal{N}_R$ with $\overline{B_\varepsilon(x_\ell)} \setminus \cup_{j\in\mathcal{N}_R} B_R(x_j) \ne \emptyset$, and for $R = \beta_m$, there exist $j \ne \ell \in \mathcal{N}_R$ with $\partial B_R(x_j) \cap \overline{B_R(x_\ell)} \ne \emptyset$. Then $\mathcal{N}_R = \mathcal{N}^m$ is constant for $R \in [\alpha_m, \beta_m]$ and $\mathcal{N}^{m+1} \subsetneq \mathcal{N}^m$ so $M \le N_0$. In addition, there exists a constant $K = K(N_0)$ such that $\alpha_1 \le K\varepsilon$, $\beta_M \ge \frac{\sigma}{K}$ and $\alpha_{m+1} \le K\beta_m$, since never more than N balls can touch.

On the half-annuli $D_{\beta_m,\alpha_m}(x_j)$ for $j \in \mathcal{N}^m$, we apply Proposition 3.10 with a jump height $\varkappa_{m,j}$ that satisfies $\sum_m \sum_{j\in\mathcal{N}^m} \varkappa_{m,j}^2 \ge \left|\sum_m \sum_{j\in\mathcal{N}^m} \varkappa_{m,j}\right| = 2D$. This leads to the estimate

$$\begin{aligned} E_\varepsilon\big(u_\varepsilon; \Omega_\sigma^\varepsilon\big) &\le E_\varepsilon(u; \Omega) - \sum_{m=1}^M \sum_{j\in\mathcal{N}^m} E_\varepsilon(u; D_{\beta_m,\alpha_m}(x_j)) \\ &\le \pi D \log \frac{1}{\varepsilon} + C_0 - \sum_m \sum_j \frac{\pi}{2}\varkappa_{m,j}^2 \left(\log \frac{\beta_m}{\alpha_m} - C\right) \\ &\le \pi D \log \frac{1}{\varepsilon} + C - \pi D \sum_m (\log \beta_m - \log \alpha_m) \\ &\le \pi D \log \frac{1}{\sigma} + C. \end{aligned}$$

□

Theorem 4.2 *Let (u_ε) be a sequence of critical points satisfying the energy bound $E_\varepsilon(u_\varepsilon) \le \pi D \log \frac{1}{\varepsilon} + C_0$. Then there is a subsequence and $N = 2D$ points*

$a_1, \dots, a_N \in \partial\Omega$ such that

$$\int_{\Omega'} |\nabla u_\varepsilon|^2 \le M(\Omega') < \infty \tag{4.2}$$

for all open Ω' with $\overline{\Omega'} \subset \overline{\Omega} \setminus \{a_1, \dots, a_N\}$. Additionally, there hold the bounds

$$\int_{\Omega} |\nabla u_\varepsilon|^p \le C(p) \tag{4.3}$$

uniformly in ε for all $1 \le p < 2$. In particular, after adding a suitable $z_\varepsilon \in 2\pi\mathbb{Z}$, a subsequence of (u_ε) converges weakly in H^1_{loc} and $W^{1,p}$, $p < 2$, to a harmonic function u_*. The limit has the properties that $(u_* - g)$ is piecewise constant on $\partial\Omega \setminus \{a_1, \dots, a_N\}$, with values in $\pi\mathbb{Z}$, and jumps by $-\pi$ at the points a_j.

Proof We use the setup described at the beginning of this section. In particular, we use the points a_j^0 as defined there. Note that for $\varepsilon < \varepsilon_0(\sigma)$, there holds $\Omega_\sigma^0 \subset \Omega^\varepsilon_{\sigma/2}$ and so by Proposition 4.1,

$$\int_{\Omega_\sigma^0} |\nabla u_\varepsilon|^2 \le 2E_\varepsilon\big(u_\varepsilon; \Omega^\varepsilon_{\sigma/2}\big) \le 2\pi D \log\frac{2}{\sigma} + C, \tag{4.4}$$

which proves (4.2). To obtain the L^p bounds (4.3), fix a $\sigma > 0$ and $1 \le p < 2$. Then by Hölder's inequality and Proposition 4.1

$$\begin{aligned}
\int_\Omega |\nabla u_\varepsilon|^p &\le \int_{\Omega^\varepsilon_\sigma} |\nabla u_\varepsilon|^p + \sum_{\ell=1}^\infty \int_{\Omega^\varepsilon_{2^{-\ell}\sigma} \setminus \Omega^\varepsilon_{2^{-\ell+1}\sigma}} |\nabla u_\varepsilon|^p \\
&\le C + \sum_{\ell=1}^\infty \big|\Omega^\varepsilon_{2^{-\ell}\sigma} \setminus \Omega^\varepsilon_{2^{-\ell+1}\sigma}\big|^{1-p/2} \left(\int_{\Omega^\varepsilon_{2^{-\ell}\sigma}} |\nabla u_\varepsilon|^2\right)^{p/2} \\
&\le C + c\sum_{\ell=1}^\infty 2^{-(1-p/2)\ell}\left(2\pi D \log\frac{1}{2^\ell \sigma} + C\right)^{p/2} \le C,
\end{aligned}$$

since the sum converges by the root test. From this L^p gradient bound, we obtain the weak compactness up to translation by Poincaré's inequality. The weak limit u_* is harmonic since $\int_\Omega \nabla u_* \cdot \nabla\varphi = \lim_{\varepsilon\to 0} \int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$. That the boundary values satisfy $u_* - g \in \pi\mathbb{Z}$ with possible jumps at the a_i follows from $\int_{\partial\Omega} \sin^2(u_\varepsilon - g) \to 0$ and $(u_\varepsilon - g)$ being close to $\pi\mathbb{Z}$ outside the approximate vortex set S_ε.

We still have to prove that $N = 2D$ and that there is a jump by $-\pi$ at each of the points a_j. To see this, we note that we can localize parts of the proof of Proposition 4.1 around a_j^ε to obtain

$$E_\varepsilon\big(u_\varepsilon; \Omega \cap B_\eta\big(a_j^\varepsilon\big)\big) \ge \frac{\pi}{2} \sum_m \varkappa_{m,j}^2 \log\frac{1}{\varepsilon} - C(\eta)$$

The jump of u_* at a_j^0 is $-\pi d_j$, where $d_j = \sum_m \varkappa_{m,j}$ so $\sum_m \varkappa_{m,j}^2 \geq |d_j|$. The upper bound on the energy now implies

$$\sum_j |d_j| \leq 2D + \frac{C(\eta)}{\log \frac{1}{\varepsilon}}.$$

Letting $\varepsilon \to 0$ we obtain $\Sigma_j |d_j| \leq 2D = \Sigma_j d_j$, which proves $d_j \geq 0$. Since by the lower bound argument, the energy around those a_j^0 with $d_j > 0$ already suffices to make up for the singular part of the energy, we can use the η-compactness lemma 3.8 to see that $d_j = 0$ is impossible.

To finish the proof, we need to show $d_j = 1$. To this end, we compare the energy of u_ε to that of u_*. Letting $\varepsilon \to 0$ in (4.4) and using the weak lower semicontinuity of the Dirichlet integral, we have

$$\int_{\Omega_\sigma^0} |\nabla u_*|^2 \leq 2\pi D \log \frac{1}{\sigma} + C.$$

On the other hand, Proposition 7.1 shows that for σ sufficiently small,

$$\int_{\Omega_\sigma^0} |\nabla u_*|^2 \geq \pi \sum_j d_j^2 \log \frac{1}{\sigma} - C.$$

Combining these estimates shows $\Sigma_j (d_j^2 - d_j) \leq 0$. Since $d_j \neq 0$, it follows that $d_j = 1$ for all j. □

5 Convergence results by PDE arguments

The $W^{1,p}$ convergence results of the previous section also hold for general stationary points where upper and lower energy bounds do not match as those for minimizers do. Away from the vortices, there also holds convergence in higher norms.

Proposition 5.1 *There is a constant $C > 0$ such that for every sequence of stationary points u_ε satisfying the energy bound $E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$, there holds*

$$\limsup_{\varepsilon \to 0} \operatorname*{osc}_{\overline{\Omega}} u_\varepsilon \leq C$$

In particular, by adding a suitable sequence $z_\varepsilon \in 2\pi\mathbb{Z}$, the u_ε themselves can be assumed to be uniformly bounded in L^∞.

Proof From Proposition 3.9 we know that there exist a bounded number of points $a_i^\varepsilon \in \partial\Omega$ such that $|\sin(u_\varepsilon - g)| < \frac{1}{2}$ outside $\cup_i B_\varepsilon(a_i^\varepsilon)$, so the oscillation there is bounded. Inside $B_\varepsilon(a_i^\varepsilon) \cap \partial\Omega$, the oscillation is bounded since there we have $[u_\varepsilon]_{C^{0,1/2}} \leq \frac{C}{\sqrt{\varepsilon}}$ as follows from the proof of Proposition 3.9. By the maximum principle, the bounds extend to $\overline{\Omega}$. □

Proposition 5.2 *Let $u = u_\varepsilon$ be a stationary point of E_ε and let $z_0 \in \partial\Omega$, w.l.o.g. $z_0 = 0$. Let $R > 0$ be such that $B_R \cap S_\varepsilon = \emptyset$, where S_ε is the approximate vortex set. Let $G \in H^1(B_R)$ be a function with $G|_{\partial\Omega\cap B_R(z)} = g$, and let $k \in \mathbb{Z}$ such that $|u - g - k\pi| \le \arcsin\frac{1}{2}$. Then for any $\vartheta < 1$ there holds*

$$\int_{B_{\vartheta R}\cap\Omega} |\nabla u|^2 + \frac{1}{\varepsilon}\int_{\partial\Omega\cap B_{\vartheta R}} (u - g - k\pi)^2 \le C \tag{5.1}$$

Proof We test the equation (2.1) with $\eta^2(u - G - k\pi)$. This yields

$$0 = \int_\Omega \eta^2|\nabla u|^2 + \frac{1}{2\varepsilon}\int_{\partial\Omega} \eta^2 \sin 2(u - g - k\pi)(u - g - k\pi) + 2\int_\Omega \eta(u - G - k\pi)\nabla u \cdot \nabla\eta + \int_\Omega \eta^2 \nabla u \cdot \nabla G.$$

By the monotonicity $\sin 2(u - g - k\pi)(u - g - k\pi) \ge c(u - g - k\pi)^2$ that holds true by choice of k since $S_\varepsilon \cap B_R = \emptyset$ and by aid of Young's inequality, we obtain

$$\int_\Omega \eta^2|\nabla u|^2 + \frac{c}{\varepsilon}\int_{\partial\Omega} |u - g - k\pi|^2 \le C\int_\Omega |\nabla\eta|^2(u - G - k\pi)^2 + \eta^2|\nabla G|^2.$$

Choosing a standard cut-off function η satisfying $\eta = 1$ on $B_{\vartheta R}$ and $\eta = 0$ outside B_R with $|\nabla\eta| \le \frac{C}{R(1-\vartheta)}$, we obtain the result. □

Proposition 5.3 *Let u_ε be stationary points of E_ε satisfying $E_\varepsilon(u_\varepsilon) \le M \log\frac{1}{\varepsilon}$. Assume (by aid of Proposition 3.9) that the approximate vortex set S_ε is covered by $\bigcup_{j=1}^N B_\varepsilon(a_j^\varepsilon)$. Then for any $\sigma > 0$, the energy of u_ε on $\Omega_\sigma^\varepsilon = \Omega \setminus \bigcup_j B_\sigma(a_j^\varepsilon)$ can be estimated as*

$$E_\varepsilon\big(u_\varepsilon; \Omega_\sigma^\varepsilon\big) \le C \log\frac{1}{\sigma}. \tag{5.2}$$

Proof This follows from Proposition 5.2 since the part of $\Omega_\sigma^\varepsilon$ near the boundary can always be covered by a logarithmical number of balls, see Fig. 2. In the remaining sector, classical interior gradient bounds for harmonic functions also show logarithmic bounds. If many vortices are close together, we can combine the bounds obtained near each vortex similar to the argument in the proof of Proposition 4.1. □

Theorem 5.4 *There is for $1 \le p < 2$ a constant $C = C(g, p, M, \Omega)$ such that for every sequence u_ε of stationary points of E_ε satisfying $E_\varepsilon(u_\varepsilon) \le M \log\frac{1}{\varepsilon}$, there holds*

$$\int_\Omega |\nabla u_\varepsilon|^p \le C. \tag{5.3}$$

Proof This follows exactly as in the proof of Theorem 4.2 from the estimate (5.2). □

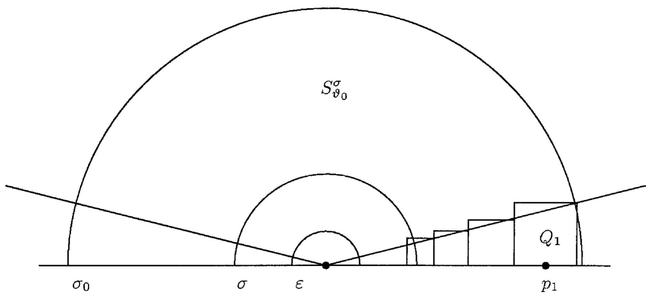

Fig. 2 Construction for the proof of Proposition 5.3: The set $B^+_{\sigma_0} \setminus B^+_{\sigma}$ is covered by the circular sector $S^{\sigma}_{\vartheta_0}$ and some squares Q_i in geometrical progression that can thus be covered by $C(\vartheta_0) \log \frac{\sigma_0}{\sigma}$ half-balls not touching B^+_{ε}.

Remark 5.5 Theorem 5.4 can fail without the assumption $E_\varepsilon(u_\varepsilon) \leq M \log \frac{1}{\varepsilon}$ that provides an a priori bound on the number of vortices. Counterexamples (with $\Omega = B_1(0)$ and $g = 0$) can be constructed by conformally mapping the periodic half-space solutions given by Toland [15] to the unit disk, see [8, Sect. 5.5] for some more explicit calculations. This is in contrast to Ginzburg-Landau theory, where the logarithmic a priori bound always holds in the case of a starshaped domain [1, Theorem X.1].

To obtain bounds in higher norms, we will flatten the boundary and use a harmonic extension of the forcing function g. By changing variables we obtain

Proposition 5.6 *Let $z_0 \in \partial\Omega$, w.l.o.g $z_0 = 0$ and $\partial\Omega$ has horizontal tangent at 0. Then for $\rho > 0$ sufficiently small, the part of Ω near z_0 can be written as a graph of a C^1 function γ over its tangent plane, so $\Psi : \overline{B^+_\rho} \to \overline{\Omega}$, $\Psi(x, y) = (x, y + \gamma(x))$ is a diffeomorphism of $\overline{B^+_\rho}$ onto a (closed) relative neighborhood of z_0 in $\overline{\Omega}$.*

Let u_ε be a stationary point of E_ε and G a harmonic extension of g to $\Psi(B^+_\rho)$ with bounded Dirichlet integral. Then the function $w_\varepsilon = (u_\varepsilon - G) \circ \Psi$ solves the PDE

$$\int_{B^+_\rho} a_{ij} \partial_i w_\varepsilon \partial_j \varphi + \int_{\Gamma_\rho} \left(\frac{1}{2\varepsilon} \sin 2w_\varepsilon + h \right) b\varphi = 0 \tag{5.4}$$

for all $\varphi \in H^1(B^+_\rho)$ that vanish near ∂B_ρ, where $(a_{ij}) = \begin{pmatrix} 1 & -\gamma' \\ -\gamma' & 1+\gamma'^2 \end{pmatrix}$, $b = \sqrt{1 + \gamma'^2}$ and $h = \frac{\partial G}{\partial \nu} \circ \Psi^{-1}$.

Proposition 5.7 *Let $w = w_\varepsilon$ be a solution of (5.4) and $R > 0$ such that $\sin^2 w < \frac{1}{4}$ on Γ_R. Then for $\vartheta < 1$*

$$\int_{B^+_{\vartheta R}} |\nabla^2 w|^2 + \frac{1}{\varepsilon} \int_{\Gamma_{\vartheta R}} \left| \frac{\partial w}{\partial \tau} \right|^2 \leq C(\vartheta, R). \tag{5.5}$$

Proof For ease of presentation, assume $\partial\Omega$ is already flat, and $g = 0$. Then $a_{ij} = \delta_{ij}$, $b = 1$, $h = 0$. We differentiate (5.4) and test with $\eta^2 \partial_1 w$, where η denotes the usual cut-off function that is 0 outside B_R^+, 1 inside $B_{\vartheta R}^+$, and satisfies $|\nabla\eta| \leq \frac{C}{R(1-\vartheta)}$. This shows

$$\int_{B_R^+} \nabla\partial_1 w \nabla(\eta^2 \partial_1 w) + \frac{1}{2\varepsilon}\int_{\Gamma_R} \partial_1(\sin 2w)\eta^2\partial_1 w = 0 \tag{5.6}$$

We now use $\partial_1(\sin 2w) = 2\cos(2w)\partial_1 w$ and $2\cos(2w) = 2(1-2\sin^2 w) \geq 1$ to obtain by Young's inequality for any α

$$\int_{B_R^+} \eta^2|\nabla\partial_1 w|^2 + \frac{1}{\varepsilon}\int_{\Gamma_R} \eta^2|\partial_1 w|^2 \leq \alpha \int_{B_R^+} \eta^2|\nabla\partial_1 w|^2 + \frac{1}{\alpha}\int_{B_R^+} |\nabla\eta|^2|\nabla w|^2 \tag{5.7}$$

Using the result of Proposition 5.2 and choosing $\alpha < 1$, we obtain the claimed bound for $\partial_1\nabla w$. We can extend this to the full second gradient since $\partial_{22} w = -\partial_{11} w$ and hence $|\partial_1\nabla w| = |\partial_2\nabla w|$. □

Proposition 5.8 *Let u_ε be stationary points of E_ε satisfying $E_\varepsilon(u_\varepsilon) \leq M \log\frac{1}{\varepsilon}$. Assume the approximate vortex set S_ε is covered by $B_\varepsilon(a_j^\varepsilon)$, with $a_j^\varepsilon \to a_j^0$ as $\varepsilon \to 0$. Then on $\Omega_\sigma = \Omega \setminus \bigcup B_\sigma(a_j^0)$ there holds the estimate*

$$\limsup_{\varepsilon\to 0} \int_{\Omega_\sigma} |\nabla^2 u_\varepsilon|^2 \leq C(\sigma) \tag{5.8}$$

as well as

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{\partial\Omega\cap\partial\Omega_\sigma} \sin^2(u_\varepsilon - g) = 0. \tag{5.9}$$

Proof The first claim follows from (5.5) and a covering argument. For the second, we observe that the H^2 bound implies weak H^2 convergence $u_\varepsilon \to u_*$ and thus also $\frac{\partial u_\varepsilon}{\partial\nu} \to \frac{\partial u_*}{\partial\nu}$ in L^2. Now we have (with $\Gamma = \partial\Omega \cap \partial\Omega_\sigma$)

$$\frac{1}{\varepsilon}\int_\Gamma \sin^2(u_\varepsilon - g) \leq \frac{C}{\varepsilon}\int_\Gamma \sin^2(u_\varepsilon - g)\cos^2(u_\varepsilon - g) = C\varepsilon\int_\Gamma \left|\frac{\partial u_\varepsilon}{\partial\nu}\right|^2,$$

which tends to 0 by the convergence of $\frac{\partial u_\varepsilon}{\partial\nu}$. □

6 The half-space solutions

Blow-up of the solutions of (2.2)–(2.3) at scale ε will lead to a half-space problem. The resulting equation is the Peierls-Nabarro equation known from the theory of crystal dislocations, and its solutions have been classified by Toland [15]. We will use the following essential uniqueness result:

Theorem 6.1 (Toland [15]) *Let u be a bounded solution of*

$$\Delta u = 0 \qquad \text{in } \mathbb{R}^2_+ \tag{6.1}$$

$$\frac{\partial u}{\partial\nu} = -\frac{1}{2}\sin 2u \quad \text{on } \mathbb{R}. \tag{6.2}$$

Then either u is periodic or constant, or there exist $n \in \mathbb{Z}$, $a \in \mathbb{R}$, and a sign such that

$$u(x, y) = \pm \arctan \frac{x + a}{y + 1} + \pi n + \frac{\pi}{2} \tag{6.3}$$

Proposition 6.2 *Assume u_ε are stationary points of E_ε satisfying $E_\varepsilon(u_\varepsilon) \le M \log \frac{1}{\varepsilon}$, let $z_0 \in \partial\Omega$ and define $w_\varepsilon := (u_\varepsilon - G) \circ \Psi$ as above. Then the functions $V_\varepsilon(z) = w_\varepsilon(\varepsilon z)$ converge weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ to a nonperiodic solution of (6.1)–(6.2).*

Proof As in the proof of Proposition 5.3, the energy of w_ε satisfies a logarithmic energy bound $\le C \log \frac{\rho}{\varepsilon}$, giving a local bound for the Dirichlet energy of V_ε in B_R^+.

It follows that $V_\varepsilon \rightharpoonup V$ in $H^1(B_R)$ for every R. Since V_ε satisfies the PDE

$$\int_{B^+_{\rho/\varepsilon}} a^\varepsilon_{ij} \partial_i V_\varepsilon \partial_j \varphi + \int_{\Gamma_{\rho/\varepsilon}} \left(\frac{1}{2} \sin 2V_\varepsilon + h_\varepsilon \right) b^\varepsilon \varphi = 0 \tag{6.4}$$

for all $\varphi \in H^1(B^+_{\rho/\varepsilon})$ vanishing near $\partial B_{\rho/\varepsilon}$, where $a^\varepsilon_{ij}(z) = a_{ij}(\varepsilon z)$, $b^\varepsilon(z) = b^\varepsilon(\varepsilon z)$ and $h_\varepsilon(z) = \varepsilon(\frac{\partial G}{\partial \nu} \circ \Psi)(\varepsilon z)$. Letting $\varepsilon \to 0$, we have $a^\varepsilon_{ij} \to \delta_{ij}$, $b^\varepsilon \to 1$, and $h_\varepsilon \to 0$ uniformly in every B_R^+. Passing to the limit in (6.4) we thus obtain that V satisfies the weak form of (6.1)– (6.2). The limit cannot be periodic since this and the strong convergence $V_\varepsilon \to V$ in $L^2_{\mathrm{loc}}(\mathbb{R})$ would otherwise contradict Proposition 3.9 for ε sufficiently small. □

Corollary 6.3 *If (u_ε) are stationary points of E_ε with $E_\varepsilon(u_\varepsilon) \le M \log \frac{1}{\varepsilon}$, then the approximate vortex set S_ε can be covered (for a subsequence) by disjoint balls $B_{\sigma\varepsilon}(a^\varepsilon_j)$ for some $a^\varepsilon_j \in \partial\Omega$ and some $\sigma > 0$. If (u_ε) have been minimizers or local minimizers (i.e. with respect to variations of small support) then a^ε_j converge to distinct points a^0_j as $\varepsilon \to 0$.*

Proof It follows via (3.15) from (3.14) and the fact that the only possible $L^2_{\mathrm{loc}}(\mathbb{R})$ limits of the blowup of Proposition 6.2 around a point are constant or the Toland solution that for ε small enough, around every interval in $S_\varepsilon \subset \partial\Omega$ we need to have a jump of $\pm\pi$. Two such intervals need to be an asymptotic distance bigger than $K\varepsilon$ apart since otherwise the ε-scale blowup would converge to a solution that shifts twice by π, at a distance K. Such a solution does not exist by Theorem 6.1. Hence, $\sigma\varepsilon$-balls will be eventually disjoint. They have to cover for some σ by Proposition 3.9.

The second part follows as in the proof of Theorem 4.2. □

7 The renormalized energy

In this section we calculate the energy of the possible limit functions, and obtain a singular term plus a renormalized energy W that depends on the position of the singularities. Furthermore, we calculate the gradient of this energy.

Proposition 7.1 (Energy expansion for limit functions) *Let $\vec{a} = (a_i)$ with $i = 1, \dots, N$ be a collection of distinct points in $\partial\Omega$, $d_i \in \mathbb{Z}$ with $\Sigma_i d_i = 2D$, and let u_* be a harmonic function such that $u_* - g \in \pi\mathbb{Z}$ on $\partial\Omega$ and u_* jumps by $-d_i\pi$ at the points a_i. Then the Dirichlet energy of u_* in the domain $\Omega_\rho = \Omega \setminus \cup_{i=1}^N B_\rho(a_i)$ has the following asymptotic expansion as $\rho \to 0$:*

$$\frac{1}{2}\int_{\Omega_\rho} |\nabla u_*|^2 = \frac{\pi}{2}\sum_{i=1}^N d_i^2 \log\frac{1}{\rho} + W + O(\rho\log\rho), \tag{7.1}$$

where W is the renormalized energy *corresponding to (a_i, d_i) that can be calculated by the expression*

$$W = -\pi \sum_{1\le i<j\le N} d_i d_j \log|a_i - a_j| + \frac{1}{2}\int_{\partial\Omega} V g' - \frac{\pi}{2}\sum_{j=1}^N d_j R(a_j). \tag{7.2}$$

Here V denotes a solution of the inhomogeneous Neumann problem

$$\Delta V = 0 \qquad \text{in } \Omega \tag{7.3}$$

$$\frac{\partial V}{\partial \nu} = g' - \pi\sum_{j=1}^N d_j\delta_{a_j} \quad \text{on } \partial\Omega, \tag{7.4}$$

and R is a harmonic function, continuous on $\overline{\Omega}$ and given by

$$R(z) = V(z) - \sum_{j=1}^N d_j \log|z - a_j|. \tag{7.5}$$

Proof Similar to the corresponding proof for interior vortices (where the lifting of e^{iu} to u can be done only locally) in [1, Chapter I], this follows by observing that V and u_* are harmonic conjugates, hence have the same energy. The energy of V is then calculated using (7.5). □

Proposition 7.2 *The gradient of W at a point $\vec{a} = (a_i)$ (with fixed $\vec{d} = (d_i)$) is given by*

$$\nabla_{\vec{a}} W(\vec{a}) = (f_i(a_i)), \tag{7.6}$$

where $f_i(z) = \frac{\partial}{\partial\nu}(u_*(z) - d_i \arg(z - a_i))$.

Proof We follow the calculations in [1, pp. 87–89]. Fix all but one of the points (a_i), say a_j. Now let $\Phi_b(z)$ denote for $b \in \partial\Omega$ the solution (normalized to have mean 0 on $\partial\Omega$) of

$$\Delta\Phi_b = 0 \qquad \text{in } \Omega \tag{7.7}$$

$$\frac{\partial\Phi_b}{\partial\nu} = g' - \pi\sum_{k\ne j} d_k\delta_{a_k} - \pi d_j\delta_b \quad \text{on } \partial\Omega. \tag{7.8}$$

We also set $L(z) = \sum_{k \neq j} d_k \log |z - a_k|$ and $\Psi_b = \Phi_b - L$. Both Ψ_b and L are harmonic in Ω, and the normal derivatives are given by

$$\frac{\partial}{\partial \nu} L = -\pi \sum_{k \neq j} d_k \delta_{a_k} + h$$

for some bounded function h, and

$$\frac{\partial}{\partial \nu} \Psi_b = g' - h - \pi d_j \delta_b.$$

We also set $R_b(z) = \Psi_b(z) - d_j \log |z - b|$. R_b is C^1 on $\overline{\Omega}$. Finally, for $b \neq b'$ we define another harmonic function ζ by

$$\zeta(z) = R_{b'}(z) - R_b(z) + d_j \log \frac{|z - b'|}{|z - b|} = \Psi_{b'}(z) - \Psi_b(z) = \Phi_{b'}(z) - \Phi_b(z).$$

The normal derivative of ζ is then

$$\frac{\partial \zeta}{\partial \nu} = -\pi d_j (\delta_{b'} - \delta_b).$$

Now we use Green's identity on Ψ_b and $\Psi_{b'}$ for $b \neq b'$, from which we obtain $\int_{\partial\Omega} (\Psi_{b'} \frac{\partial}{\partial \nu} \Psi_b - \Psi_b \frac{\partial}{\partial \nu} \Psi_{b'}) = 0$. Hence

$$\int_{\partial\Omega} (\Psi_b' - \Psi_b)(g' - h) = \pi d_j (\Psi_{b'}(b) - \Psi_b(b')) = \pi d_j (R_{b'}(b) - R_b(b')). \quad (7.9)$$

From using Green's identity on the harmonic functions L and ζ we obtain

$$-\pi d_j (L(b') - L(b)) = \int_{\partial\Omega} (\Psi_{b'} - \Psi_b) h - \pi \sum_{k \neq j} d_k (R_{b'}(a_k) - R_b(a_k)) - \pi d_j \sum_{k \neq j} d_k \log \frac{|a_k - b'|}{|a_k - b|}.$$

Expanding out L, the logarithmic terms cancel, and so

$$\int_{\partial\Omega} (\Psi_{b'} - \Psi_b) h = \pi \sum_{k \neq j} d_k (R_{b'}(a_k) - R_b(a_k)).$$

By (7.9) and the definition of Ψ, this shows

$$\pi \sum_{k \neq j} d_k (R_{b'}(a_k) - R_b(a_k)) + \pi d_j (R_{b'}(b) - R_b(b')) = \int_{\partial\Omega} (\Phi_{b'} - \Phi_b) g'. \quad (7.10)$$

Differentiating (7.10) with respect to $b \in \partial\Omega$ and setting $b' = b$, we see (with $\partial_b R_b$, $\partial_z R_b$ denoting tangential differentiation with respect to the subscript and the argument, respectively)

$$-\pi \sum_{k \neq j} d_k \partial_b R_b(a_k) + \pi d_j (\partial_z R_b(b) - \partial_b R_b(b)) + \int_{\partial\Omega} \partial_b \Phi_b g' = 0. \quad (7.11)$$

Now W is given by

$$W = -\pi \sum_{k \neq j} d_k d_j \log|b - a_k| - \frac{\pi}{2} \sum_{k \neq j} \sum_{\ell \neq j,k} d_k d_\ell \log|a_k - a_\ell| + \frac{1}{2} \int_{\partial\Omega} \Phi_b g' - \frac{\pi}{2} \sum_{k \neq j} d_k R_b(a_k) - \frac{\pi}{2} d_j R_b(b).$$

Differentiating tangentially with respect to b, we obtain

$$\partial_b W = -\pi \sum_{k \neq j} d_k d_j \frac{(b - a_k) \cdot \tau}{|b - a_k|^2} + \frac{1}{2} \int_{\partial\Omega} \partial_b \Phi_b g' - \frac{\pi}{2} \sum_{k \neq j} d_k \partial_b R_b(a_k) - \frac{\pi}{2} d_j (\partial_z R_b(b) + \partial_b R_b(b)).$$

Using (7.11), we can simplify this to

$$\partial_b W = -\pi \sum_{k \neq j} d_k d_j \frac{(b - a_k) \cdot \tau}{|b - a_k|^2} - \pi d_j \partial_z R_b(b).$$

On the other hand, the tangential derivative of the function

$$S_j(z) = \Phi_b(z) - d_j \log|z - b| = R_b(z) + \sum_{k \neq j} d_k \log|z - a_k|$$

at b is given by

$$\frac{\partial}{\partial\tau} S_j(b) = \sum_{k \neq j} d_k \frac{(b - a_k) \cdot \tau}{|b - a_k|^2} + \partial_z R_b(b),$$

so $\partial_b W = -\pi \frac{\partial}{\partial\tau} S_j(b)$. Since the u_* corresponding to $\vec{a}$ and Φ_b are conjugate harmonic (and so are arg and log),

$$\frac{\partial}{\partial\nu}(u_* - d_j \arg(z - b)) = -\frac{\partial}{\partial\tau} S_j,$$

so summing up, the derivative of W with respect to changing $b = a_j$ is given by

$$\frac{\partial}{\partial a_j} W(\mathrm{a}) = \pi \frac{\partial}{\partial\nu}(u_*(z) - d_j \arg(z - a_j)). \tag{7.12}$$

□

8 Energy expansion for isolated vortices

In this section we consider stationary points of E_ε such that S_ε can be covered by balls $B_{\sigma\varepsilon}(a_j^\varepsilon)$ with $a_j^\varepsilon \to a_j^0$ that are distinct, such that the jump d_j near these points is ± 1 (this is true for minimizers by Corollary 6.3), and obtain that there is an asymptotic expansion of the energy in terms of the renormalized energy of the last section.

To this end, let us assume w_ε to be a solution of (5.4) around some point z_0 that satisfies $|\sin 2w_\varepsilon| < \frac{1}{4}$ on $\Gamma_{R_0} \setminus \Gamma_{\sigma\varepsilon}$ and $\sup_{B_R^+} |w_\varepsilon| \leq C$. Assume in addition that $|w(\pm x, 0) - k_\pm \pi| < \frac{1}{4}$ for $x \in (\sigma\varepsilon, R_0)$, with $k_\pm \in \mathbb{Z}$ and $|k_+ - k_-| = 1$. These assumptions are valid for minimizers by Corollary 6.3.

Let $\overline{w}_\varepsilon$ be the solution of

$$\Delta \overline{w}_\varepsilon = 0 \quad \text{in } \mathbb{R}^2_+$$
$$\frac{\partial \overline{w}_\varepsilon}{\partial \nu} = -\frac{1}{2\varepsilon} \sin 2\overline{w}_\varepsilon \quad \text{on } \mathbb{R} = \partial \mathbb{R}^2_+$$

that satisfies $\overline{w}_\varepsilon(x, 0) \to k_\pm$ as $x \to \pm\infty$ and $w_\varepsilon(0, 0) = \frac{k_- + k_+}{2}$. Without loss of generality, we assume $k_+ = k_- + 1$. By Toland's uniqueness result Theorem 6.1, $\overline{w}_\varepsilon$ is given by

$$\overline{w}_\varepsilon(z) = k_- + W_0\left(\frac{z}{\varepsilon}\right) \tag{8.1}$$

where W_0 is the base solution

$$W_0(z) = \frac{\pi}{2} + \arctan \frac{x}{y+1}. \tag{8.2}$$

Proposition 8.1 *For $R < \frac{R_0}{2}$, there holds*

$$\int_{B_R^+} |\nabla w_\varepsilon - \nabla \overline{w}_\varepsilon|^2 + \frac{1}{\varepsilon} \int_{\Gamma_R} |w_\varepsilon - \overline{w}_\varepsilon|^2 \leq C. \tag{8.3}$$

Proof The function $\overline{w}_\varepsilon$ is a solution of

$$\int_{B_R^+} \delta_{ij} \partial_i \overline{w}_\varepsilon \partial_j \varphi + \frac{1}{2\varepsilon} \int_{\Gamma_R} \sin 2\overline{w}_\varepsilon \varphi = 0 \quad \text{for all } \varphi \in C_0^\infty\left(B_R^+\right). \tag{8.4}$$

With the notation used already in Proposition 5.6 and setting $g_j = (a_{ij} - \delta_{ij})\partial_i \overline{w}_\varepsilon$ and $H = bh + (b-1)\frac{1}{2\varepsilon} \sin 2\overline{w}_\varepsilon$, we can rewrite (8.4) as

$$\int_{B_R^+} a_{ij} \partial_i \overline{w}_\varepsilon \partial_j \varphi + \frac{1}{2\varepsilon} \int_{\Gamma_R} \sin 2\overline{w}_\varepsilon b\varphi = \int_{B_R^+} g_j \partial_j \varphi + \int_{\Gamma_R} H\varphi, \tag{8.5}$$

where H and g_j satisfy by the definition of a_{ij} and the explicit form of $\overline{w}_\varepsilon$ the estimates $|g_j| \leq Cr \frac{1}{\sqrt{r^2+\varepsilon^2}} \leq C$ and $|H| \leq C + C\frac{r}{2\varepsilon}(1 \wedge \frac{\varepsilon}{r}) \leq C$. Subtracting

(8.5) from (5.4) and testing with $\eta^2(w_\varepsilon - \overline{w}_\varepsilon)$ leads using ellipticity and Young's inequality for any $\delta > 0$ to the estimate

$$
\begin{aligned}
&c\int_{B_R^+}\eta^2|\nabla w_\varepsilon - \nabla\overline{w}_\varepsilon|^2 + \frac{1}{2\varepsilon}\int_{\Gamma_R}\eta^2(\sin 2w_\varepsilon - \sin 2\overline{w}_\varepsilon)(w_\varepsilon - \overline{w}_\varepsilon)\\
&\quad\le C\int_{B_R^+}|\nabla\eta|^2(w_\varepsilon - \overline{w}_\varepsilon)^2 + C\int_{B_R^+}\eta^2\big(g_1^2 + g_2^2\big) + \frac{C\varepsilon}{\delta}\int_{\Gamma_R}\eta^2 H^2\\
&\qquad+\frac{\delta}{\varepsilon}\int_{\Gamma_R}\eta^2(w_\varepsilon - \overline{w}_\varepsilon)^2.
\end{aligned}
$$

On $\Gamma_R \setminus \Gamma_{\sigma\varepsilon}$, we have by assumption that $(\sin 2w_\varepsilon - \sin 2\overline{w}_\varepsilon)(w_\varepsilon - \overline{w}_\varepsilon) \ge c|w_\varepsilon - \overline{w}_\varepsilon|^2$ and so we can choose $\delta > 0$ small enough and use $\eta \le 1$ and the bounds on g_j and H to obtain

$$
\begin{aligned}
&c\int_{B_R^+}\eta^2|\nabla w_\varepsilon - \nabla\overline{w}_\varepsilon|^2 + \frac{c}{\varepsilon}\int_{\Gamma_R}\eta^2|w_\varepsilon - \overline{w}_\varepsilon|^2 \le C\int_{B_R^+}|\nabla\eta|^2(w_\varepsilon - \overline{w}_\varepsilon)^2\\
&\quad+CR^2 + CR\varepsilon + \frac{1}{2\varepsilon}\int_{\Gamma_{\sigma\varepsilon}}|(\sin 2w_\varepsilon - \sin 2\overline{w}_\varepsilon)(w_\varepsilon - \overline{w}_\varepsilon)| + |w_\varepsilon - \overline{w}_\varepsilon|^2.
\end{aligned}
\tag{8.6}
$$

Choosing for η a smooth function that is equal to 1 on $B_{R/2}^+$, equal to 0 outside B_R^+ and satisfying $|\nabla\eta| \le \frac{C}{R}$, the right hand side is seen to be bounded by a constant depending on R. □

Proposition 8.2 *For all $\varepsilon > 0$ there exists $a_\varepsilon \in \mathbb{R}$ such that for all $C_1 > 0$ there holds*

$$\frac{1}{2\varepsilon}\int_{\Gamma_{C_1\varepsilon}}|w_\varepsilon(x,0) - \overline{w}_\varepsilon(x - a_\varepsilon\varepsilon, 0)|^2 \to 0 \quad \textit{as } \varepsilon \to 0. \tag{8.7}$$

The shifts a_ε are uniformly bounded: $|a_\varepsilon| \le C_0$.

Proof Rescaling by ε, we obtain the functions $W_\varepsilon(z) = w_\varepsilon(\varepsilon z)$ and $\overline{W}_\varepsilon(z) = \overline{w}_\varepsilon(\varepsilon z) = \overline{W}(z)$. If the assertion were false, then there exists a subsequence $\varepsilon \to 0$ and a $\delta > 0$ such that

$$\frac{1}{2}\int_{\Gamma_{C_1}}|W_\varepsilon(x) - \overline{W}(x - a)|^2 \ge \delta > 0 \tag{8.8}$$

for all a with $|a| \le C_0$. Repeating up to rescaling the proof of Proposition 6.2, we obtain that $W_\varepsilon \rightharpoonup W_*$ in $H^1(B_R^+)$ for all $R > 0$, for some $W_* \in H^1_{\mathrm{loc}}(\mathbb{R}^2_+)$. W_* must be a solution of the half-space problem, and by Rellich-Kondrachov embedding on the boundary, we obtain

$$\int_{\Gamma_R}|W_* - \overline{W}|^2 = \lim_{\varepsilon\to 0}\int_{\Gamma_R}|W_\varepsilon - \overline{W}|^2 \le \int_{\mathbb{R}}|W_\varepsilon - \overline{W}|^2 \le C$$

by (8.3), in particular the difference $W_* - \overline{W}$ is in $L^2(\mathbb{R})$. From Toland's theorem 6.1 we obtain that W_* can only be a translation of $\overline{W}$, i.e. $W_*(z) = \overline{W}(z-a)$. From

$$\int_{\mathbb{R}} |W_* - \overline{W}|^2 \leq C$$

and the explicit form of the solution we deduce that $|a| \leq C_0$ for some C_0. The convergence $W_\varepsilon \rightharpoonup W_*$ also implies

$$\int_{\Gamma_{C_1}} |W_\varepsilon - W_*|^2 \to 0$$

as $\varepsilon \to 0$, contradicting (8.8). □

Proposition 8.3 *If we redefine $\overline{w}_\varepsilon$ by choosing the shifts a_ε as in Proposition 8.2, then the energies of w_ε and $\overline{w}_\varepsilon$ are asymptotically close:*

$$\limsup_{\rho\to 0}\limsup_{\varepsilon\to 0}\int_{B_\rho^+} |\nabla w_\varepsilon - \nabla \overline{w}_\varepsilon|^2 = 0 \tag{8.9}$$

and

$$\limsup_{\rho\to 0}\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Gamma_\rho} |w_\varepsilon - \overline{w}_\varepsilon|^2 = 0 \tag{8.10}$$

Proof Let $R \leq \frac{R_0}{4}$. Then by a suitable Poincaré inequality

$$\int_{B_R^+\setminus B_{R/2}^+} |w_\varepsilon - \overline{w}_\varepsilon|^2 \leq CR^2 \int_{B_R^+\setminus B_{R/2}^+} |\nabla w_\varepsilon - \nabla\overline{w}_\varepsilon|^2 + CR\int_{\Gamma_R\setminus\Gamma_{R/2}} (w_\varepsilon - \overline{w}_\varepsilon)^2. \tag{8.11}$$

This and (8.6) show together with Proposition 8.2 that

$$\int_{B_{R/2}^+} |\nabla w_\varepsilon - \nabla\overline{w}_\varepsilon|^2 \leq C\int_{B_R^+\setminus B_{R/2}^+} |\nabla w_\varepsilon - \nabla\overline{w}_\varepsilon|^2 + C\frac{\varepsilon}{R} + CR^2 + \omega(\varepsilon), \tag{8.12}$$

where $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. Adding the integral over $B_{R/2}^+$ to both sides ("filling the hole") leads with $\vartheta = \frac{C}{C+1} < 1$ to

$$\int_{B_{R/2}^+} |\nabla w_\varepsilon - \nabla\overline{w}_\varepsilon|^2 \leq \vartheta\int_{B_R^+} |\nabla w_\varepsilon - \nabla\overline{w}_\varepsilon|^2 + C\frac{\varepsilon}{R} + CR^2 + \omega(\varepsilon), \tag{8.13}$$

from which we conclude the first claim: By (8.3) the limit is finite, and if it is not zero then letting $\varepsilon \to 0$ and $R \to 0$ in (8.13) leads to a contradiction. The second follows from the first, (8.6) and Proposition 8.2. □

Proposition 8.4 *There holds*

$$\limsup_{\rho\to 0}\limsup_{\varepsilon\to 0}\left|\int_{B_\rho^+} a_{ij}\partial_i w_\varepsilon \partial_j w_\varepsilon - \int_{B_\rho^+} |\nabla\overline{w}_\varepsilon|^2\right| = 0 \tag{8.14}$$

Proof We have

$$\int_{B_\rho^+} a_{ij}\partial_i w_\varepsilon \partial_j w_\varepsilon = \int_{B_\rho^+} a_{ij}\partial_i \overline{w}_\varepsilon \partial_j \overline{w}_\varepsilon + 2\int_{B_\rho^+} a_{ij}\partial_i \overline{w}_\varepsilon \partial_j (w_\varepsilon - \overline{w}_\varepsilon) + \int_{B_\rho^+} a_{ij}\partial_i (w_\varepsilon - \overline{w}_\varepsilon)\partial_j (w_\varepsilon - \overline{w}_\varepsilon). \tag{8.15}$$

The last term tends to 0 by Proposition 8.3. For the other terms, we have

$$\left| \int_{B_\rho^+} a_{ij}\partial_i \overline{w}_\varepsilon \partial_j \overline{w}_\varepsilon - \int_{B_\rho^+} |\nabla \overline{w}_\varepsilon|^2| \right| \le \int_{B_\rho^+} Cr\frac{1}{r^2} \le C\rho$$

and

$$\left| \int_{B_\rho^+} a_{ij}\partial_i \overline{w}_\varepsilon \partial_j (w_\varepsilon - \overline{w}_\varepsilon) \right| \le \int_{B_\rho^+} Cr\frac{1}{r}|\nabla w_\varepsilon - \nabla \overline{w}_\varepsilon|,$$

which goes to 0 by Hölder's inequality and (8.9). For the final term, we use the harmonicity of $\overline{w}_\varepsilon$ and integrate by parts. This shows

$$\int_{B_\rho^+} \nabla \overline{w}_\varepsilon \cdot \nabla (w_\varepsilon - \overline{w}_\varepsilon) = \int_{\Gamma_\rho} \frac{\partial \overline{w}_\varepsilon}{\partial \nu}(w_\varepsilon - \overline{w}_\varepsilon) + \int_{\partial B_\rho \cap \mathbb{R}^2_+} \frac{\partial \overline{w}_\varepsilon}{\partial \nu}(w_\varepsilon - \overline{w}_\varepsilon)$$

The integral over $\partial B_\rho \cap \mathbb{R}^2_+$ can be estimated since the integrand is bounded by $\frac{C\varepsilon}{\rho^2}$, which tends to 0 under the convergence considered. The other is via Hölder's inequality bounded by $(\frac{C}{\varepsilon}\int_{\Gamma_R} |w_\varepsilon - \overline{w}_\varepsilon|^2)^{1/2}$, which tends to 0 by (8.10). □

Proposition 8.5 *The energy of* $\overline{w}_\varepsilon$ *on* B_ρ^+ *satisfies*

$$\lim_{\varepsilon\to 0}\left(\frac{1}{2}\int_{B_\rho^+} |\nabla \overline{w}_\varepsilon|^2 + \frac{1}{2\varepsilon}\int_{\Gamma_\rho} \sin^2 \overline{w}_\varepsilon - \frac{\pi}{2}\left(\log\frac{\rho}{\varepsilon} + 1 - \log 2\right)\right) = 0. \tag{8.16}$$

Proof This follows from an explicit calculation. □

Theorem 8.6 *Assume that* u_ε *are stationary points of* E_ε *with* $E_\varepsilon(u_\varepsilon) \le M\log\frac{1}{\varepsilon}$ *and* $u_\varepsilon \to u_*$ *in* $H^1_{\mathrm{loc}} \cap W^{1,p}(\Omega)$, *where* u_* *is the harmonic function corresponding to* (a_i, d_i) *as in Proposition 7.1. Assume furthermore that the vortices are isolated, i.e. the centers of the balls covering the approximate vortex set* S_ε *converge to distinct points. Then as* $\varepsilon \to 0$, *there holds*

$$E_\varepsilon(u_\varepsilon) = \pi D\log\frac{1}{\varepsilon} + W(a_i, d_i) + \pi D(1 - \log 2) + \omega(\varepsilon), \tag{8.17}$$

where $D = \frac{1}{2}\sum d_i^2$, W *is the renormalized energy of Proposition 7.1, and* $\omega(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

The configuration (a_i) *is a stationary point for* W *with fixed* d_i, *and (locally) minimizing if* u_ε *has been (locally, i.e. w.r.t. variations of small support) minimizing.*

Proof By Proposition 5.8, we have $u_\varepsilon \to u_*$ in $H^1(\Omega_\rho)$, and in particular for any $\rho > 0$

$$\lim_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon; \Omega_\rho) = \frac{1}{2}\int_{\Omega_\rho} |\nabla u_*|^2.$$

Inside $\omega_\rho = B_\rho(a_j)\cap\Omega$, we use again a harmonic extension G of g. With $v_\varepsilon = u_\varepsilon - G$ there holds

$$\int_{\omega_\rho} |\nabla u_\varepsilon|^2 = \int_{\omega_\rho} |\nabla v_\varepsilon|^2 + 2\int_{\partial\omega_\rho} v_\varepsilon \frac{\partial G}{\partial \nu} + \int_{\omega_\rho} |\nabla G|^2 = \int_{\omega_\rho} |\nabla u_\varepsilon|^2 + O(\rho).$$

In the limit $\lim_{\rho\to 0}\lim_{\varepsilon\to 0}$, we can thus work with v_ε instead of u_ε. From Proposition 8.4 we already know that the energy of v_ε on $\Psi(B_\rho^+)$ is close to that of $\overline{w}_\varepsilon$ on B_ρ^+. The symmetric difference $\Delta_\rho := \Psi(B_\rho^+)\triangle(B_\rho\cap\Omega)$ does not play a role here since

$$\begin{aligned}
&\limsup_{\rho\to 0}\limsup_{\varepsilon\to 0}\int_{\Delta_\rho} |\nabla v_\varepsilon|^2 \\
&\quad\le C\limsup_{\rho\to 0}\limsup_{\varepsilon\to 0}\int_{\Psi^{-1}(\Delta_\rho)} |\nabla w_\varepsilon|^2 \\
&\quad= C\limsup_{\rho\to 0}\limsup_{\varepsilon\to 0}\int_{\Psi^{-1}(\Delta_\rho)} |\nabla \overline{w}_\varepsilon|^2 = 0
\end{aligned}$$

by (8.14) and the explicit form of $\overline{w}_\varepsilon$. Similarly, there also holds (using (8.10))

$$\limsup_{\rho\to 0}\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\partial\Delta_\rho\cap\partial\Omega} \sin^2(u_\varepsilon - g) = 0.$$

Since

$$\limsup_{\rho\to 0}\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\left|\int_{\Gamma_\rho} \sin^2 w_\varepsilon b - \int_{\Gamma_\rho} \sin^2 \overline{w}_\varepsilon\right| = 0$$

as can again be deduced form (8.10) and

$$\lim_{\rho\to 0}\sup_{|t|<\rho} b = 1,$$

the energy of u_ε in $B_\rho\cap\Omega$ is thus asymptotically that of $\overline{w}_\varepsilon$ in B_ρ^+. We thus obtain the claim from Proposition 7.1 and Proposition 8.5.

To show that (a_j) is a stationary point of the renormalized energy, we use Proposition 7.2 so we need to show that $\frac{\partial}{\partial\nu}(u_* - d_j\vartheta)$ is zero at a_j, where $\vartheta = \arg(z - a_j)$. To this end, we calculate using harmonicity of u_ε and u_* and setting $h = u_* - d_j\vartheta$

$$\begin{aligned}
\int_{\partial\Omega\cap B_\rho} \frac{\partial u_\varepsilon}{\partial\nu} &= -\int_{\partial B_\rho\cap\Omega} \frac{\partial u_\varepsilon}{\partial\nu} \\
&\xrightarrow{\varepsilon\to 0} -\int_{\partial B_\rho\cap\Omega} \frac{\partial u_*}{\partial\nu} = \int_{\partial\Omega\cap B_\rho} \frac{\partial u_*}{\partial\nu} = \int_{\partial\Omega\cap B_\rho} \frac{\partial h}{\partial\nu} \pm \frac{\partial\vartheta}{\partial\nu}.
\end{aligned}$$

Using the PDE, we have

$$\int_{\partial\Omega\cap B_\rho} \frac{\partial u_\varepsilon}{\partial\nu} = \frac{1}{2\varepsilon}\int_{\partial\Omega\cap B_\rho} \sin 2(u_\varepsilon - g) = \frac{1}{2\varepsilon}\int_{\Psi^{-1}(\partial\Omega\cap B_\rho)} \sin 2w_\varepsilon b.$$

Using estimates from above, we can again replace $\Psi^{-1}(\partial\Omega\cap B_\rho)$ by Γ_ρ up to an error that is $O(\rho^2)$. Similarly, we can estimate

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\left|\frac{1}{2\varepsilon}\int_{\Gamma_\rho}\sin 2w_\varepsilon b - \int_{\Gamma_\rho}\sin 2\overline{w}_\varepsilon\right| \\
&\quad\le \limsup_{\varepsilon\to 0}\left(\int_{\Gamma_\rho}\sin^2 2\overline{w}_\varepsilon\right)^{1/2}\left(\int_{\Gamma_\rho}(b-1)^2\right)^{1/2} \\
&\qquad + C\limsup_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{\Gamma_\rho}(\sin 2w_\varepsilon - \sin 2\overline{w}_\varepsilon) \\
&\quad = O(\rho^{3/2}) + 0
\end{aligned}$$

by (8.10) and $|b(s)-1| \le Cs$. Since $\int_{\partial\Omega\cap B_\rho}\frac{\partial\vartheta}{\partial\nu} = O(\rho^2)$ we obtain that, as $\rho\to 0$, we also have $\frac{1}{\rho}\int_{\partial\Omega\cap B_\rho}\frac{\partial h}{\partial\nu}\to 0$, hence $\frac{\partial h}{\partial\nu} = 0$ at a_j.

To show that (a_j) is (locally) minimizing if we started with (local) minimizers, we can construct for any (a'_j) a test function v_ε similar to that of Proposition 3.1 by interpolating linearly in the radial variable between $G + \overline{w}_\varepsilon \circ \Psi^{-1}$ inside B_ρ and u_* in $\Omega_{2\rho}$. It is not hard to show that resulting function v_ε then has an energy whose $O(1)$ part is given up to a constant by $W(a'_j, d_j)$, and by minimality we obtain $W(a_j, d_j) \le W(a'_j, d_j)$. □

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