Calc. Var. (2005) 24(3): 261–281
DOI 10.1007/s00526-005-0308-y

Calculus of Variations

ORIGINAL ARTICLE

Peter W. Bates · Xinfu Chen ·
Adam J. J. Chmaj

Heteroclinic solutions of a van der Waals model with indefinite nonlocal interactions

Received: 1 March 2004 / Accepted: 13 September 2004 / Published online: 7 June 2005

Abstract We construct heteroclinic the global minimizers of a nonlocal free energy functional that van der Waals derived in 1893. We study the case where the nonlocality satisfies only a weakened type of ellipticity, which precludes the use of comparison methods. In the interesting case when the local part of the energy is nonconvex, we construct a classical the global minimizer by studying a relaxed functional corresponding to the convexification of the local part and exclude the possibility of minimizers of the relaxed functional having rapid oscillations. We also construct examples where the global minimizer is not monotonic.

1 Introduction

We construct a global minimizer of a nonlocal free energy functional

$$\mathcal{E}(u) := \frac{1}{4}\int\int_{\mathbb{R}^2} J(x-y)(u(x)-u(y))^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}} F(u(x))\,\mathrm{d}x, \tag{1.1}$$

in the space $u_0 + L^1 \cap L^\infty$, where $u_0 := \mathrm{sgn}(x) = \pm 1$ if $\pm x > 0$. Here F is a double-well potential with equal minima at ± 1, e.g., $F(s) = \frac{1}{4}(s^2-1)^2$. The

P. W. Bates (✉)
Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA
E-mail: bates@math.msu.edu

X. Chen
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA
E-mail: xinfu+@pitt.edu
URL: http://www.pitt.edu/~xinfu

A. J. J. Chmaj
Department of Statistics, University of Toronto, 100 St. George Street, Ontario M5S 3G3, Canada
E-mail: chmaj@math.utah.edu

double integral term in (1.1), with an even kernel J, such that $\int_{\mathbb{R}} J > 0$, replaces the more common $\frac{1}{2}\int_{\mathbb{R}} u'(x)^2\,\mathrm{d}x$ used, e.g., in the Ginzburg–Landau or Allen-Cahn [3] functional. The free energy (1.1) was postulated and studied (for the case $J \geq 0$) by Bates et al. [4]. Later it is derived as the Helmholtz free energy of a continuous spin system by Bates and Chmaj [6], where before rescaling to move the minima to ± 1, the double well potential is identified as

$$Tk\Big\{(1+u)\log(1+u) + (1-u)\log(1-u)\Big\} - q\,u^2, \quad u \in (-1,1) \tag{1.2}$$

where T is the absolute temperature, k is Boltzmann's constant and q is a positive constant.

Without loss of generality, let $\int_{\mathbb{R}} J = 1$. Then, with $f \equiv F'$, the Euler–Language equation for critical points of $\mathcal{E}(u)$ can be written as

$$J \star u = u + f(u). \tag{1.3}$$

Since the convolution $J \star u$ is continuous, monotonicity of $u + f(u)$, or equivalently, convexity of $\frac{1}{2}u^2$ the convolution $F(u)$, implies continuity of any minimizer. As demonstrated in [4, 5, 11], non-convexity leads to discontinuous global minimizers of (1.1).

For the particular potential (1.2), one sees that in the case $q > 1/2$ there are two critical temperatures $0 < T_1 < T_2$, such that (i) F is a double well potential if and only if $0 < T < T_2$, (ii) $\frac{1}{2}u^2 + F(u)$ is convex if and only if $T > T_1$. In this sense, phase transition can be divided into two types depending on the temperature.

At this moment, we would like to point out that van der Waals suggested a free energy of the form (1.1) in the study of capillarity in 1893 [31]. The feature of two critical temperatures was also speculated upon by van der Waals as he discussed the possibility of continuous and discontinuous phase transitions ([31, Sect. 15]). Van der Waals also expanded the interaction term in (1.1) in Taylor series and considered the first order truncation so the interaction energy (the double integral in (1.1)) is replaced by $c_1\int_{\mathbb{R}} u_x^2\,\mathrm{d}x$ where $c_1 = \frac{1}{4}\int_{\mathbb{R}} J(\eta)\eta^2\,\mathrm{d}\eta$, giving the Ginzburg–Landau energy. This type of expansion is quite common in the literature (e.g. [10, 34, 35]). van der Waals demonstrated that such truncated free energy (i.e. the Ginzburg–Landau energy) accurately measures some chemical and physical quantities [30, 32]. It is known that the Ginzburg–Landau energy admits only smooth solutions. Nonetheless, van der Waals observed and tried to explain phenomena where phases are transited in a discontinuous (abrupt) manner.

The interaction strength J, sometimes called the intermolecular potential (see [35]) is represented as $J = J^{AA} + J^{BB} - 2J^{AB}$ in the derivation in [6], where J^{ij}'s are Ising energies of interaction between spins i and j. All J^{ij}'s may be positive but the effective J could change sign. Moreover, it is not unthinkable that any of the J^{ij}'s also change sign themselves. Such a situation arises e.g., in some colloidal systems, where macromolecules self-attract within one spatial range, but repel at other distances [24]. Thus in general, J clearly can change sign, which is the case studied in this work. For example, J can have a "mexican-hat" shape, a situation that also arises in some biological systems [18].

There is a growing literature on (1.1) and various related evolution equations, including [1, 2, 4–6, 8, 9, 11, 12, 14, 16, 17, 19, 20, 23, 26, 27, 30, 36].

The problem on a torus with quite general periodic and integrable J was considered in the work of Comets et al. [16] which was brought to our attention by the referee. Assuming also that F is even and $\frac{1}{2}s^2 + F(s)$ is convex, they found periodic non-constant solutions, by studying the linearized operator around the trivial solution $u \equiv 0$ and using bifurcation methods.

If $J \geq 0$, a monotone global minimizer of (1.1) can be constructed using monotone rearrangements [1]; see also [4, 5, 11].

In this paper, we consider the case that J can change sign and that solutions are not small perturbations of trivial solutions. As none of the tools in [1, 16] seem to apply here, we shall assume a weakened type of ellipticity, namely, $\max \hat{J}(\xi) = J(0) = 1$, where $\hat{J}$ is the Fourier transform of J defined by

$$\hat{J}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{\mathbf{i}\xi z} J(z)\,\mathrm{d}z = \int_{\mathbb{R}} \cos(\xi z) J(z)\,\mathrm{d}z.$$

This weak ellipticity, although it does not imply a comparison principle, assures that (1.1) is bounded below by 0; see Lemma 3.1.

Our construction below uses convexity and duality arguments. Since $J * u$ is continuous, (1.3) implies that u is discontinuous when $\frac{1}{2}s^2 + F(s)$ is not convex in $(-1, 1)$. In this case, to ensure that minimizing sequences do not oscillate rapidly to give a Young measure type limit, we add an additional assumption that $J(0) > 0$. We remark that in the derivation given in [6], the absolute temperature T of the Helmholtz free energy is built into the function F. When this temperature is sufficiently low, $\frac{1}{2}s^2 + F(s)$ becomes non-convex. Indeed, we show in Sect. 8 that when T is sufficiently small, any global minimizer is not monotonic for certain type of $J's$.

For convenience, we work with the space $u_0 + L^1 \cap L^\infty$. However, the minimizer constructed is also a global minimizer of (1.1) in the space $u_0 + L^2$; see Theorem 4 in Sect. 6. On the other hand, we have no idea about the uniqueness of global minimizers.

The framework of our proof is similar in style to that of [7]. There we constructed a heteroclinic global minimizer of the lattice version of (1.1). However, the discrete nature of the problem provided some compactness of minimizing sequences, and made the construction simpler.

We would like to point out some interesting aspects of (1.1). Note that since there is no gradient term in (1.1), the underlying space is not restricted to differentiable functions and critical points are possibly discontinuous functions. In [11] and in more generality in [5], the authors found families of solutions of (1.3), discontinuous along arbitrarily prescribed interfaces, which are seemingly stable, since the formal second variation is positive. These states form continua in L^2 but the energy is not constant on these continua.

This phenomenon is reminiscent of Whitney's example [33] (which was brought to our attention by L.C. Evans). However, the majority of these solutions of (1.3) are not critical points of (1.1) in $u_0 + L^1$, which we believe to be a more natural space in which to consider variations.

Consequently, the L^2 gradient flow may not be the most interesting evolution for (1.1) when $\frac{1}{2}u^2 + F(u)$ is non-convex, since its mechanism is such that very often the solution of the Cauchy problem converges to a member of the continuum that is not a local minimum of the energy [5, 11, 20, 34]. We leave it here as an open question to find a local well-posed evolution which would in general lead to a critical point of (1.1) in L^1 (note that a critical point in the L^2 sense is not necessarily one in the L^1 sense).

2 Main result

We make the following assumptions.

(J) $J \in L^\infty(\mathbb{R})$, $J(z) = J(-z)$ for all $z > 0$, $\int_0^\infty z|J(z)|\,\mathrm{d}z < \infty$, and

$$\int_{\mathbb{R}} J(z)\,\mathrm{d}z = 1 \geq \int_{\mathbb{R}} \mathrm{e}^{\mathbf{i}\xi z} J(z)\,\mathrm{d}z \quad \forall \xi \in \mathbb{R}. \tag{2.1}$$

(F) $F \in C^1(\mathbb{R})$, $F(\pm 1) = 0 < F(s)$ for all $s \neq \pm 1$, and $\lim_{|s|\to\infty} \frac{F(s)}{s^2} = \infty$.
Let $f = F'$. One of the following holds:
(A1) $s \to g(s) := f(s) + s$ is nondecreasing, or
(A2) $J(0) > 0$ and $\int_{\mathbb{R}} |J'(z)|\,\mathrm{d}z < \infty$.

Theorem 1 *Assume* **(J), (F)**, *and* **(A1)** *or* **(A2)**. *Then (1.1) admits a global minimizer u in the space $u_0 + L^1 \cap L^\infty$ and in $u_0 + L^2$.*

If g is strictly increasing, u is continuous.

If g is not strictly increasing and **(A2)** *holds, then u is not continuous, but has only jump discontinuities. If we further assume that g has only a finite number of decreasing parts, then the total number of jumps of u is finite.*

Remark 2.1 The regularity assumption $F \in C^1$ can be weakened by the assumption that F is Lipschitz. The proof, which is omitted here, follows by taking the limit of a sequence of global minimizers of regularized problems.

Also, the regularity assumption $J' \in L^1(\mathbb{R})$ in **(A2)** can be weakened by the assumption that J has bounded variation and $J(0+) > 0$. That J has bounded variation implies that $v := J \star u$ is Lipschitz continuous, and hence differentiable almost everywhere allowing our proof of existence to proceed. This generalization allows piecewise constant J's.

Remark 2.2 As an example of an interaction satisfying (**J**) and in particular inequality (2.1) with J changing sign, let us take $J(x) = A(r_1\mathrm{e}^{-p_1|x|} - r_2\mathrm{e}^{-p_2|x|})$, where $r_1, p_1, r_2, p_2 > 0$ and $A = \frac{p_1 p_2}{2r_1 p_2 - 2r_2 p_1}$ (so that $\int J = 1$). Note that

$$\hat{J}(\xi) := \int_{\mathbb{R}} \mathrm{e}^{\mathbf{i}\xi z} J(z)\,\mathrm{d}z = A\left(\frac{2r_1 p_1}{p_1^2 + \xi^2} - \frac{2r_2 p_2}{p_2^2 + \xi^2}\right).$$

We calculate that

$$\hat{J}(0) - \hat{J}(\xi) = \frac{2A\xi^2}{p_1 p_2} \times \frac{\xi^2\left(\frac{r_2}{p_2} - \frac{r_1}{p_1}\right) + \frac{p_1^2 r_2}{p_2} - \frac{p_2^2 r_1}{p_1}}{(p_1^2 + \xi^2)(p_2^2 + \xi^2)}$$

thus (2.1) is satisfied if and only if $\frac{r_1}{r_2} > \frac{p_1}{p_2}$ and $\frac{r_1}{r_2} > \frac{p_1^3}{p_2^3}$. Note that J has a mexican hat shape if $p_1 > p_2$, in which case (2.1) is satisfied if and only if $\frac{r_1}{r_2} > \frac{p_1^3}{p_2^3}$. In a similar way, if we consider $J(x) = A(r_1 e^{-(p_1 x)^2} - r_2 e^{-(p_2 x)^2})$, then J has a mexican hat shape if $p_1 > p_2$ and it can be shown that (2.1) is satisfied if and only if $\frac{r_1}{r_2} > \frac{p_1^3}{p_2^3}$.

The proof of Theorem 1 is organized as follows. In Sect. 3 we discuss some properties of (1.1). In particular, **(J)** implies that the energy $\mathcal{E}$ in (1.1) is bounded below by 0. In Sect. 4 we study the convexified energy $\mathcal{E}^*$, defined by exchanging the local part $G(s) := \frac{s^2}{2} + F(s)$ of the energy density of $\mathcal{E}$ with its convexification G^*. As we show below, it turns out that $\mathcal{E}$ and the modified energy $\mathcal{E}^*$ have the same infimum. This suggests that it might suffice to restrict our attention to minimizing $\mathcal{E}^*$, which is a problem easier than minimizing $\mathcal{E}$. In Sect. 5 we prove Theorem 2 which states that there exists a minimizer of $\mathcal{E}^*$ in an appropriate space. Theorem 1 then follows from Theorem 2 if G is convex. In Sect. 6 we prove Theorem 3 which states that if G is nonconvex and **(A2)** holds then the minimizer of $\mathcal{E}^*$ constructed in Theorem 2 is also a minimizer of $\mathcal{E}$. This is accomplished by showing that the minimizer of has jump discontinuities at the convexification points of G^* (this corresponds to the result in [4]). In this case, Theorem 1 follows from Theorem 3. In Sect. 7 we consider the case where the wells of F are of unequal depth and in that case show that traveling waves exist when the interaction J has sufficiently large amplitude and has sufficiently short essential range. Finally in Sect. 8 we provide an example where global minimizers are non-monotonic.

3 The energy

For convenience, we introduce the bilinear form

$$\mathbf{E}(u, v) := \frac{1}{4} \int\int_{\mathbb{R}^2} J(x - y)(u(x) - u(y))(v(x) - v(y))\,\mathrm{d}x\,\mathrm{d}y.$$

Then

$$\mathcal{E}(u) = \int_{\mathbb{R}} F(u)\,\mathrm{d}x + \mathbf{E}(u, u) = \int_{\mathbb{R}} \left\{ G(u) - \frac{1}{2} u\, J \star u \right\} \mathrm{d}x, \tag{3.1}$$

where

$$G(s) := F(s) + \frac{1}{2} s^2.$$

We use $u_0 + L^1 \cap L^\infty$ to denote the affine space $\{u_0 + \phi | \phi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\}$ and $\mathbb{R}\, u_0 \oplus L^1 \cap L^\infty$ the space $\{cu_0 + \phi | c \in \mathbb{R}, \phi \in L^1 \cap L^\infty\}$ where

$$u_0(x) := \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x \leq 0. \end{cases}$$

Lemma 3.1 **(Positivity of energy)** *The bilinear form* $\mathbf{E}(\cdot, \cdot)$ *is a semi-positive definite symmetric form on* $\mathbb{R}u_0 \oplus L^2$*. Consequently,*

$$0 \leq \int_{\mathbb{R}} F(u)\,\mathrm{d}x \leq \mathcal{E}(u) < \infty \quad \forall u \in \mathbb{R}u_0 \oplus L^1 \cap L^\infty.$$

Lemma 3.2 (Energy decomposition) *Assume $u \in u_0 + L^1 \cap L^\infty$ and $x_0 \in \mathbb{R}$. For both the "+" and "−" sign, define*

$$u^r_\pm := \begin{cases} \pm 1 & \text{if } x \le x_0, \\ u(x) & \text{if } x > x_0, \end{cases} \qquad u^l_\pm := \begin{cases} u(x) & \text{if } x \le x_0, \\ \pm 1 & \text{if } x > x_0. \end{cases}$$

Then for all $\ell > 0$,

$$\mathcal{E}(u^r_\pm) + \mathcal{E}(u^l_\pm) \le \mathcal{E}(u) + \frac{\|J\|_{L^1}}{2} \int_{x_0-\ell}^{x_0+\ell} (u \mp 1)^2 \,\mathrm{d}x + \|u \mp 1\|^2_{L^\infty} \int_\ell^\infty (z - \ell)|J(z)| \,\mathrm{d}z.$$

Lemma 3.3 (Variation of energy) *For $u \in u_0 + L^1 \cap L^\infty$ and $\phi \in L^1 \cap L^\infty$,*

$$\mathcal{E}(u) - \mathcal{E}(u - \phi) = \int_{\mathbb{R}} \left\{ G(u) - G(u - \phi) - \phi \left[J \star u - \frac{1}{2} J \star \phi \right] \right\} \mathrm{d}x.$$

Lemma 3.4 (An L^∞ bound) *Let M_0 be a constant such that*

$$G(s) - G(0) > s^2 \|J\|_{L^1} \quad \text{for all } |s| \ge M_0.$$

Then for each $u \in u_0 + L^1 \cap L^\infty$,

$$\mathcal{E}(\tilde{u}) \le \mathcal{E}(u) \quad \textit{where} \quad \tilde{u} = u\chi_{\{|u| \le M_0\}}.$$

Here and in the sequel, χ_I represents the characteristic function of the set I; namely $\chi_I(x) = 1$ if $x \in I$ and $= 0$ otherwise.

Proof of Lemma 3.1 Since it is of independent interest, we first study $\mathbf{E}(u, v)$. Write $u = c_1 u_0 + \phi_1$ and $v = c_2 u_0 + \phi_2$, and denote by $\hat{\phi}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{\mathbf{i}\xi x} \phi(x) \,\mathrm{d}x$ the Fourier Transform of ϕ. Then, by Plancherel's identity,

$$\begin{aligned} \mathbf{E}(u, v) &= \frac{1}{4} \int_{\mathbb{R}} J(z) \int_{\mathbb{R}} (u(x - z) - u(x))(v(x - z) - v(x)) \,\mathrm{d}x \,\mathrm{d}z \\ &= \frac{\pi}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} J(z)[1 - \mathrm{e}^{\mathbf{i}\xi z}][1 - \mathrm{e}^{-\mathbf{i}\xi z}] \left\{ \frac{c_1}{\mathbf{i}\xi} + \hat{\phi}_1(\xi) \right\} \left\{ \frac{c_2}{-\mathbf{i}\xi} + \bar{\hat{\phi}}_2(\xi) \right\} \mathrm{d}\xi \,\mathrm{d}z \\ &= \pi \int_{\mathbb{R}} [1 - \hat{J}(\xi)] \left\{ \frac{c_1}{\mathbf{i}\xi} + \hat{\phi}_1(\xi) \right\} \left\{ \frac{c_2}{-\mathbf{i}\xi} + \bar{\hat{\phi}}_2(\xi) \right\} \mathrm{d}\xi. \end{aligned}$$

Here we remark that the integral is uniformly convergent since $\hat{J}(\xi) = \int_{\mathbb{R}} \cos(\xi z) J(z)$ and

$$\int_{\mathbb{R}} \frac{1 - \hat{J}(\xi)}{|\xi^2|} \mathrm{d}\xi = \int_{\mathbb{R}} \frac{1 - \cos \eta}{\eta^2} \mathrm{d}\eta \int_{\mathbb{R}} |z| J(z) \,\mathrm{d}z.$$

Since $\hat{J}(\xi) \le 1$ for all ξ, we see that $0 \le \mathbf{E}(u, u) < \infty$ for all $u \in \mathbb{R}u_0 \oplus L^2$. □

Remark 3.1 From the expression of $E(u, v)$, one sees that $\hat{J}(0) = \max \hat{J}$ is also a necessary condition for E to be semi-positive definite. Since if $\hat{J} > 1$ on (a, b), then $E(\phi, \phi) < 0$ for any ϕ being the Fourier inverse transform of a non-trivial function supported in (a, b).

Proof of Lemma 3.2 Clearly, $\int_{\mathbb{R}} F(u)\,dx = \int_{\mathbb{R}} F(u^r_\pm)\,dx + \int_{\mathbb{R}} F(u^l_\pm)\,dx$.

Denoting $v^r = u^r_\pm \mp 1$ and $v^l = u^l_\pm \mp 1$. Then $u = v^r + v^l \mp 1$. Hence,

$$\begin{aligned}\mathbf{E}(u, u) &= \mathbf{E}(v^r, v^r) + \mathbf{E}(v^l, v^l) + 2\mathbf{E}(v^r, v^l)\\ &= \mathbf{E}(u^r_\pm, u^r_\pm) + \mathbf{E}(u^l_\pm, u^l_\pm) + 2\mathbf{E}(v^r, v^l).\end{aligned}$$

Noting that $v^r v^l \equiv 0$, we obtain

$$\begin{aligned}-2\mathbf{E}(v^r, v^l) &= -\frac{1}{2}\int\int_{\mathbb{R}^2} J(x-y)(v^r(x) - v^r(y))(v^l(x) - v^l(y))\,dx\,dy\\ &= \int\int_{\mathbb{R}^2} J(x-y)v^r(x)v^l(y)\,dx\,dy\\ &\le \frac{1}{2}\int_{x_0}^{\infty} dx \int_{-\infty}^{x_0} |J(x-y)|\{v^r(x)^2 + v^l(y)^2\}\,dy\\ &\le \frac{1}{2}\|J\|_{L^1}\int_{x_0-\ell}^{x_0+\ell} (u \mp 1)^2\,dx + \|u \mp 1\|^2_{L^\infty}\int_{\ell}^{\infty} (z-\ell)|J(z)|\,dz\end{aligned}$$

for each $\ell > 0$, where we have used $\int_{x_0+\ell}^{\infty} dx \int_{-\infty}^{x_0} |J(x-y)|\,dy = \int_{x_0}^{\infty} dx \int_{-\infty}^{x_0-\ell} |J(x-y)|\,dy = \int_{\ell}^{\infty}(z-\ell)|J(z)|\,dz$. The assertion of the lemma thus follows. □

Proof of Lemma 3.3 The assertion follows from the second expression for $\mathcal{E}$ in (3.1). □

Proof of Lemma 3.4. Let $\varepsilon > 0$ be a small positive constant such that

$$G(s) - G(0) \ge |s|(|s| + \varepsilon)\|J\|_{L^1} \quad \forall\ |s| \ge M_0 - \varepsilon.$$

Set $M = \|u\|_{L^\infty}$ and assume that $M \ge M_0$. Define $u^1 = u\chi_{\{|u|<M-\varepsilon\}}$. Then

$$\mathcal{E}(u) - \mathcal{E}(u^1) = \int_{|u|\ge M-\varepsilon} \left\{ G(u) - G(0) - uJ \star \frac{u+u^1}{2} \right\} dx \ge 0$$

since $\|J \star \frac{u+u^1}{2}\|_{L^\infty} \le \|J\|_{L^1} M$. Replacing u by u^1 one can continue the process to show, for $u^k = u\chi_{\{|u|<M-k\varepsilon\}}$, that $\mathcal{E}(u^k) \le \mathcal{E}(u^{k-1})$ provided $M - k\varepsilon \ge M_0$. Hence, after a finite number of steps, we obtain the assertion of the lemma. □

4 Convexification

If G is nonconvex, we define its convexification G^* by

$$G^*(s) := \sup\{a + bs \mid G(u) \geq a + bu \quad \forall u \in \mathbb{R}\} \quad \forall s \in \mathbb{R}, \tag{4.1}$$

$$F^*(s) := G^*(s) - \frac{1}{2}s^2 \quad \forall s \in \mathbb{R}, \tag{4.2}$$

$$\mathcal{E}^*(u) := \int_{\mathbb{R}} F^*(u)\,\mathrm{d}x + \mathbf{E}(u,u) = \int_{\mathbb{R}} \left(G^*(u) - \frac{1}{2}u\, J \star u\right) \mathrm{d}x. \tag{4.3}$$

Lemma 4.1 (Properties of the convexification) *The function G^* is convex and has a continuous derivative $g^*(s) := \frac{\mathrm{d}}{\mathrm{d}s}G^*(s)$. In addition, the following holds:*

(i) For every $s \in \mathbb{R}$, there exist unique $\bar{s} \geq s$ and $\underline{s} \leq s$ such that

$$G(\underline{s}) = G^*(\underline{s}), \quad G(\bar{s}) = G^*(\bar{s}), \quad g^*(u) = g(\underline{s}) = g(\bar{s})\ \forall u \in [\underline{s}, \bar{s}],$$

$$\int_{\underline{s}}^{u} (g(s) - g(\underline{s}))\mathrm{d}s \begin{cases} \geq 0 & \text{if } u \in [\underline{s}, \bar{s}), \\ = 0 & \text{if } u = \bar{s}, \\ > 0 & \text{otherwise.} \end{cases}$$

(ii) $F^*(\pm 1) = 0 < F^*(u)$ for all $u \neq \pm 1$ and with $s = \pm 1$, $\underline{s} = \bar{s} = \pm 1$.

The following result, though it is not needed in our proof of the main theorem, provides us with the rationale to work with $\mathcal{E}^*$.

Lemma 4.2 (Equivalency of the minimization for the two energies) *Let*

$$e := \inf_{u \in u_0 + L^1 \cap L^\infty} \mathcal{E}(u), \quad e^* := \inf_{u \in u_0 + L^1 \cap L^\infty} \mathcal{E}^*(u).$$

Then $e = e^*$.

Proof of Lemma 4.1 For a super–linear function, G in our case, its dual (Legendre transform) is defined by

$$H(\beta) = \max\{\beta s - G(s) | s \in \mathbb{R}\} \quad \forall \beta \in \mathbb{R}.$$

The *Duality Principle*, a fundamental theorem in optimality theory, says that (i) the dual is convex, and (ii) the dual of the dual of a convex function is itself. We use the idea from the proof of this principle for our assertion.

Step 1 For each $\beta \in \mathbb{R}$, we define

$$\underline{s}(\beta) := \min\{s \mid \beta s - G(s) = H(\beta)\}, \quad \bar{s}(\beta) := \max\{s \mid \beta s - G(s) = H(\beta)\}.$$

Then $g(\underline{s}) = g(\bar{s}) = \beta$. Also,

$$\int_{\underline{s}}^{u} (g(s) - \beta)\,\mathrm{d}s = G(u) - [\beta u - H(\beta)] \begin{cases} > 0 & \text{if } u \in \mathbb{R} \setminus [\underline{s}, \bar{s}], \\ \geq 0 & \text{if } u \in [\underline{s}, \bar{s}], \\ = 0 & \text{if } u = \underline{s} \text{ or } u = \bar{s}. \end{cases} \tag{4.4}$$

Step 2 If $\beta_2 > \beta_1$, then $\underline{s}(\beta_2)\beta_2 - H(\beta_2) = G(\underline{s}(\beta_2)) > \underline{s}(\beta_2)\beta_1 - H(\beta_1)$ and $\bar{s}(\beta_1)\beta_1 - H(\beta_1) = G(\bar{s}(\beta_1)) > \bar{s}(\beta_1)\beta_2 - H(\beta_2)$. This implies that

$$\bar{s}(\beta_2) \geq \underline{s}(\beta_2) > \frac{H(\beta_2) - H(\beta_1)}{\beta_2 - \beta_1} > \bar{s}(\beta_1) \geq \underline{s}(\beta_1).$$

Therefore, $H(\cdot)$ is a Lipschitz continuous and strictly convex function, both $\underline{s}(\cdot)$ and $\bar{s}(\cdot)$ are strictly increasing functions, and $H'(\beta) = \underline{s}(\beta) = \bar{s}(\beta)$ for all $\beta \in \mathbb{R}$ except possibly a countable set where $\underline{s} < \bar{s}$.

Step 3 For each $s \in \mathbb{R}$, define $b(s) := \inf\{\beta | s \leq \bar{s}(\beta)\}$. Then with $\beta = b(s)$, $\underline{s} = \underline{s}(b(s))$ and $\bar{s} = \bar{s}(b(s))$, we have, in view of (4.4), that $G^*(u) = \beta u - H(\beta)$ for all $u \in [\underline{s}, \bar{s}] \ni s$. Since $G(u) \geq G^*(u) \geq \beta u - H(\beta)$ for all $u \in \mathbb{R}$, we see that G^* is differentiable at every point $u \in [\underline{s}, \bar{s}]$ and $g^*(s) = g^*(u) = \frac{d}{du}G^*(u) = \beta = b(s)$ for all $u \in [\underline{s}, \bar{s}]$.

Since $\bar{s}(\cdot)$ is strictly monotonic, $s \to b(s)$ is continuous and non-decreasing. Thus, g^* is continuous and non-decreasing, and G^* is convex. The rest of the first assertion follows from (4.4).

Step 4. Finally, using $F(\pm 1) = 0 < F(u)$ for all $u \neq \pm 1$, one concludes that $G(u) > \frac{1}{2}u^2$ for all $u \neq \pm 1$. Hence, $G^*(\pm 1) = \frac{1}{2}$ and $G^*(u) > \frac{1}{2}u^2$ for all $u \neq \pm 1$. That is, $F^*(\pm 1) = 0$ and $F^*(u) > 0$ for all $u \neq \pm 1$. This completes the proof (see [22, 25] for different proofs). □

Proof of Lemma 4.2 As $G^* \leq G$, $e^* \leq e$. We now show that $e^* \geq e$.

Let $\varepsilon > 0$ be arbitrarily fixed, and $u \in u_0 + L^1 \cap L^\infty$ be such that $\mathcal{E}^*(u) \leq e^* + \varepsilon$.

By Lemma 3.4, we can assume $\|u\|_{L^\infty} \leq M_0$. Also, by the energy decomposition Lemma 3.2, and the fact that $u - u_0 \in L^1$, we can find u_1 such that $u_1 - u_0$ has compact support and $\mathcal{E}^*(u_1) \leq \mathcal{E}^*(u) + \varepsilon$. We then can approximate u_1 by a piecewise constant function u_2 such that $\mathcal{E}^*(u_2) \leq \mathcal{E}^*(u_1) + \varepsilon$. We write

$$u_2 = -\chi_{(-\infty,a_0)]} + \sum_{i=0}^{n-1} s_i \chi_{(a_i,a_{i+1}]} + \chi_{(a_n,\infty)}.$$

For each $i = 0, 1, \ldots, n-1$, we define $\zeta_i(z)$ as follows.

(1) When $G(s_i) = G^*(s_i)$, we define $\zeta_i(z) \equiv 1$.
(2) Suppose $G(s_i) > G^*(s_i)$. Let $s_i \in [\underline{s}_i, \bar{s}_i]$, the maximal interval where G^* is linear. Then $G^*(\underline{s}_i) = G(\underline{s}_i)$, $G^*(\bar{s}_i) = G(\bar{s}_i)$ and there exist $\theta_i \in (0, 1)$ such that

$$s_i = \theta_i \underline{s}_i + (1 - \theta_i)\bar{s}_i, \quad G^*(s) = \theta_i G(\underline{s}_i) + (1 - \theta_i)G(\bar{s}_i).$$

We define $\zeta_i(z) = \underline{s}_i$ for $z \in (0, \theta_i(a_{i+1} - a_i)]$ and $\zeta_i(z) = \bar{s}_i$ for $z \in (\theta_i(a_{i+1} - a_i), a_{i+1} - a_i]$ and extend ζ_i as a periodic function on $\mathbb{R}$ with period $a_{i+1} - a_i$. For each positive integer k, we now define

$$u^k(x) = -\chi_{(-\infty,a_0]} + \sum_{i=0}^{n} \zeta_i(k[x - a_i])\chi_{(a_i,a_{i+1}]} + \chi_{(a_n,\infty)}.$$

Note that $\int_{\mathbb{R}}[G(u^k) - G^*(u_2)]\,\mathrm{d}x = 0$ for each integer k. Hence,

$$\mathcal{E}(u^k) - \mathcal{E}^*(u) = \frac{1}{2}\int_{\mathbb{R}}(u^k + u_2)J \star (u^k - u_2)\,\mathrm{d}x.$$

Note that as $k \to \infty$, $u^k \longrightarrow u_2$ weakly in $L^2(\mathbb{R})$, so that $J \star (u^k - u_2) \longrightarrow 0$ in $C([a_0, a_n])$. Thus, for some k large enough, $\mathcal{E}(u^k) \le \mathcal{E}^*(u_2) + \varepsilon \le e^* + 4\varepsilon$. This implies $e \le e^* + 4\varepsilon$. As ε is arbitrary, we conclude that $e = e^*$. □

5 The minimization

Lemma 5.1 (Compactness of a minimizing sequence) *There exists $u \in L^\infty$ and a sequence $\{u_j\}_{j=1}^\infty$ in $u_0 + L^1$ with $\|u_j\|_{L^\infty} \le M_0$ for all $j \ge 1$, such that as $j \to \infty$,*

(1) $\mathcal{E}^*(u_j) \longrightarrow e^*$,
(2) $u_j \longrightarrow u$ weakly in $L^2([-\ell, \ell])$ for all $\ell > 0$,
(3) $v_j := J \star u_j \longrightarrow v := J \star u$ in $C([-\ell, \ell])$ for all $\ell > 0$,
(4) $0 = v(0) = v_j(0)$ for all $j \ge 1$.

Lemma 5.2 (Strong convergence) *Denote $\{s : g^*(s) = v(x)\} := [\underline{u}(x), \bar{u}(x)]$ for all $x \in \mathbb{R}$.*

(1) For each $\ell > 0$, $\lim_{j\to\infty}\int_{-\ell}^{\ell}\mathrm{dist}(u_j(x), [\underline{u}(x), \bar{u}(x)])\,\mathrm{d}x = 0$,
(2) For a.e. $x \in \mathbb{R}$, $u(x) \in [\underline{u}(x), \bar{u}(x)]$ and $g^*(u(x)) = v(x) = J \star u(x)$,
(3) $\eta_0 := \liminf_{j\to\infty}\int_{-1}^{1} F^*(u_j(x))\,\mathrm{d}x > 0$,
(4) $\int_{\mathbb{R}} F^*(u(x))\,\mathrm{d}x \le e^*$.

Theorem 2 *After redefining u on a set of measure zero,*

$$g^*(u(x)) = J \star u(x) \quad \text{on } \mathbb{R}, \qquad \lim_{x\to\pm\infty} u(x) = \pm 1, \tag{5.1}$$

$$\int_{\mathbb{R}}\left\{G^*(u) - G^*(u-\phi) - \phi\left[J \star u - \frac{1}{2}J \star \phi\right]\right\}\mathrm{d}x \le 0 \tag{5.2}$$

for all $\phi \in L^1 \cap L^\infty$.

Note that if g is non-decreasing, then $G^* = G$ and Theorem 1 follows from Theorem 2.

Proof of Lemma 5.1 Since $e^* \ge 0$, there exists a minimizing sequence $\{u_j\}_{j=1}^\infty$. By Lemma 3.4, we may assume that $\|u_j\|_{L^\infty} \le M_0$ for all $j \ge 1$. Since $J \in L^1 \cap L^\infty$, we see that $\{v_j := J \star u_j\}_{j=1}^\infty$ is an equicontinuous family. As $u_j \in u_0 + L^1$, we see that $\lim_{x\to\pm\infty} v_j(x) = \pm 1$. Hence by shifting, we can assume that $v_j(0) = 0$. Up to a subsequence, the assertion of the lemma then follows from the weak compactness of balls in L^2 and Arzela-Ascoli's theorem. □

Proof of Lemma 5.2

(1) Let $\varepsilon \in (0,1)$ and $\ell > 0$ be arbitrarily fixed. Let $n \gg 1$ be an integer. Set

$$\begin{aligned}\rho(\varepsilon) &= \min\{H(\beta) - \beta s + G^*(s) : |\beta| \\ &\qquad \le M_0\|J\|_{L^1},\ s \notin [\underline{s}(\beta) - \varepsilon, \bar{s}(\beta) + \varepsilon]\} > 0,\\ I_j^\varepsilon &= \{x \in [-\ell, \ell] : \text{dist}\,(u_j(x), [\underline{u}(x), \bar{u}(x)]) > \varepsilon\},\\ I_j^k &= I_j^\varepsilon \cap \left[\frac{k\ell}{n}, \frac{(k+1)\ell}{n}\right] \quad \forall k = -n, \ldots, n-1,\\ u_j^k &= u_j \chi_{\mathbb{R}\setminus I_j^k} + \underline{u}(x)\chi_{I_j^k} \quad \forall k = -n, \ldots, n-1.\end{aligned}$$

Then

$$\begin{aligned}&\sum_{k=-n}^{n-1} \left\{\mathcal{E}^*(u_j) - \mathcal{E}^*\left(u_j^k\right)\right\} - \int_{I_j^\varepsilon} \{G^*(u_j) - G^*(\underline{u}) - v_j(u_j - \underline{u})\}\,\mathrm{d}x\\ &= \frac{1}{2}\sum_{k=-n}^{n-1} \int_{I_j^k}\int_{I_j^k} J(x-y)(u_j(x) - \underline{u}(x))(u_j(y) - \underline{u}(y))\,\mathrm{d}x\,\mathrm{d}y\\ &\ge -2\|J\|_{L^\infty} M_1 \sum_{k=-n}^{n-1} |I_j^k|^2 \ge -\frac{\ell^2 M_1 \|J\|_{L^\infty}}{n},\end{aligned}$$

where $M_1 = M_0^2 + 2M_0||\underline{u}||_{L^\infty} + ||\underline{u}||_{L^\infty}^2$. Note that $\mathcal{E}(u_j^k) \ge e^*$, and on the set I_j^ε, $G^*(u_j) - G^*(\underline{u}) \ge \rho(\varepsilon) + g^*(\underline{u})(u_j - \underline{u})$. Hence, using $g^*(\underline{u}) = v$ we have

$$2n\{\mathcal{E}^*(u_j) - e^*\} \ge \rho(\varepsilon)\left|I_j^\varepsilon\right| - \int_{I_j^\varepsilon} (v - v_j)(\underline{u} - u_j)\,\mathrm{d}x - \frac{2\ell^2 M_1 \|J\|_{L^\infty}}{n}.$$

Sending $j \to \infty$ we then obtain

$$\limsup_{j\to\infty} |I_j^\varepsilon| \le \frac{2\ell^2 M_1 \|J\|_{L^\infty}}{n\rho(\varepsilon)}.$$

Sending $n \to \infty$ we then obtain $\limsup_{j\to\infty} |I_j^\varepsilon| = 0$. As $\varepsilon > 0$ is arbitrary, the first assertion of the lemma thus follows.

(2) Since $u_j \to u$ weakly in $L^2_{loc}(\mathbb{R})$, we see from (1) that $u \in [\underline{u}, \bar{u}]$ a.e. on $\mathbb{R}$. As $g^*(s) = v(x)$ for all $s \in [\underline{u}(x), \bar{u}(x)]$, we have $g^*(u) = v = J \star u$ a.e. on $\mathbb{R}$.

(3) Since v is continuous and $v(0) = 0$, there exists $\varepsilon > 0$ such that $[\underline{u}(x), \bar{u}(x)] \subset [-1+\varepsilon, 1-\varepsilon]$ for all $x \in [-\varepsilon, \varepsilon]$. The first assertion (1) then implies that $\liminf_{j\to\infty} \int_{-\varepsilon}^{\varepsilon} F^*(u_j)\,\mathrm{d}x \ge 2\varepsilon \min_{s\in[-1+\varepsilon, 1-\varepsilon]}\{F^*(s)\} > 0$. Thus, $\eta_0 > 0$.

(4) For any bounded set I in $\mathbb{R}$, set $\tilde{u}_j = u_j\chi_{R\setminus I} + u\chi_I$. We then obtain

$$\begin{aligned}\int_I F^*(u)\,\mathrm{d}x &\le \int_{\mathbb{R}} F^*(\tilde{u}_j) \le \mathcal{E}^*(\tilde{u}_j)\\ &= \mathcal{E}^*(u_j) - \int_I \Big\{ G^*(u_j) - G^*(u)\\ &\quad - \Big[v_j - \frac{1}{2} J \star (u_j - \tilde{u}_j)\Big](u_j - u)\Big\}\,\mathrm{d}x\\ &\le \mathcal{E}^*(u_j) - \int_I \Big\{ g^*(u) - v_j + \frac{1}{2} J \star [(u_j - u)\chi_I]\Big\}(u_j - u)\,\mathrm{d}x\end{aligned}$$

since $G^*(u_j) - G^*(u) \ge g^*(u)(u_j - u)$. Note that $u_j - u \to 0$ weakly, $v_j \to v$ strongly, and $J \star [(u_j - u)\chi_I] \to 0$ strongly. Sending $j \to \infty$ we then see that $\int_I F^*(u)\,\mathrm{d}x \le e^*$. Since I is arbitrary, we then obtain $\int_{\mathbb{R}} F^*(u)\,\mathrm{d}x \le e^*$. This completes the proof. □

Proof of Theorem 2. We have shown $g^*(u) = J \star u$. Now we show $u(\pm\infty) = \pm 1$. Since (i) v is continuous, (ii) $u \in [\underline{u}, \bar{u}]$, (iii) $v = \pm 1$ if and only if $\underline{u} = \bar{u} = \pm 1$, (iv) $F^*(u(x)) = 0$ if and only if $u(x) = \pm 1$, and (v) $\int_0^\infty F^*(u) \le e^*$, we conclude that one of the following alternatives holds

$$\text{either } \lim_{x\to\infty} u(x) = 1 \quad \text{or } \lim_{x\to\infty} u(x) = -1.$$

We now show that the second alternative cannot happen, by a contradiction argument.

Let $\ell > 0$ and $x_0 > 1 + \ell$ be constants. Using the energy decomposition with the "minus" sign, we have

$$\begin{aligned}\mathcal{E}^*\big(u_j^r\big) + \mathcal{E}^*\big(u_j^l\big) \le \mathcal{E}^*(u_j) + \frac{\|J\|_{L^1}}{2}\int_{x_0-\ell}^{x_0+\ell} (u_j + 1)^2\,\mathrm{d}x\\ + (M_0 + 1)^2 \int_\ell^\infty (z - \ell)|J(z)|\,\mathrm{d}z.\end{aligned}$$

Note that $u_j^r \in u_0 + L^1 \cap L^\infty$ and $u_j^l = u_j$ for $x < x_0$. Hence $\mathcal{E}^*(u_j^r) \ge e^*$ and $\mathcal{E}^*(u_j^l) \ge \int_{-1}^1 F^*(u_j)\,\mathrm{d}x$. Sending $j \to \infty$ and using Lemma 5.2 we then obtain

$$\eta_0 \le \frac{\|J\|_{L^1}}{2}\int_{x_0-\ell}^{x_0+\ell} \max\{(\underline{u}+1)^2, (\bar{u}+1)^2\}\,\mathrm{d}x + (M_0+1)^2 \int_\ell^\infty (z-\ell)|J(z)|\,\mathrm{d}z.$$

Now if $\lim_{x\to\infty} u(x) = -1$, we can first send $x_0 \to \infty$ and then $\ell \to \infty$ to obtain $\eta_0 \le 0$, a contradiction. Hence, we must have $\lim_{x\to\infty} u(x) = 1$.

In a similar manner, we can show that $\lim_{x\to\infty} u(x) = -1$.

To prove the variational inequality (5.3), let $\phi \in L^1 \cap L^\infty$ be arbitrary. For any bounded set I, set $\tilde{u}_j = u_j \chi_{\mathbb{R}\setminus I} + (u - \phi)\chi_I$. Then

$$\begin{aligned}
\mathcal{E}^*(u_j) - e^* &\geq \mathcal{E}^*(u_j) - \mathcal{E}^*(\tilde{u}_j) \\
&= \int_I \left\{ G^*(u_j) - G^*(u-\phi) - (u_j - u + \phi)\left[v_j - \frac{1}{2} J \star (u_j - \tilde{u}_j)\right]\right\} \mathrm{d}x \\
&\geq \int_I \left\{ G^*(u) + v(u_j - u) - G^*(u-\phi) - (u_j - u + \phi) \right. \\
&\times \left.\left(v_j - \frac{1}{2} J \star ([u_j - u + \phi]\chi_I)\right)\right\} \mathrm{d}x
\end{aligned}$$

since $G^*(u_j) - G^*(u) \geq g^*(u)(u_j - u) = v(u - u_j)$. Sending $j \to \infty$ we then obtain

$$0 \geq \int_I \left\{ G^*(u) - G^*(u-\phi) - v\phi + \frac{1}{2}\phi J \star [\phi\chi_I]\right\} \mathrm{d}x.$$

Since $\phi \in L^1$ and G^* is C^1, the integrand is in L^1. Hence, we can let $I \to \mathbb{R}$ to conclude the assertion (5.3), which is equivalent to $\mathcal{E}^*(u) \leq \mathcal{E}^*(u - \phi)$. Thus u is a global minimizer of $\mathcal{E}^*$ over $u_0 + L^1 \cap L^\infty$. □

6 Passing from $\mathcal{E}^*$ to $\mathcal{E}$

In this section, we shall assume that g is not monotonic in $(-1, 1)$. Hence, we assume the alternative condition **(A2)**.

We denote

$$B := \{\beta | \underline{s}(\beta) < \bar{s}(\beta)\}.$$

Theorem 3 *Assume* **(A2)** *Let $x_0 \in \mathbb{R}$ be any point.*

(1) If $v(x_0) \notin B$, then u is continuous at x_0, i.e., $\lim_{x\to x_0} u(x) = u(x_0) = \underline{u}(x_0) = \bar{u}(x_0)$.
(2) If $v(x_0) \in B$, then $v'(x_0) \neq 0$ and one of the following holds:
 (i) $v'(x_0) > 0$, $\lim_{x\searrow x_0} u(x) = \bar{u}(x_0)$ and $\lim_{x\nearrow x_0} u(x) = \underline{u}(x_0)$,
 (ii) $v'(x_0) < 0$, $\lim_{x\searrow x_0} u(x) = \underline{u}(x_0)$ and $\lim_{x\nearrow x_0} u(x) = \bar{u}(x_0)$.
(3) u solves $J \star u = g(u)$ and $\mathcal{E}(u) = e = \inf_{w\in u_0 + L^1\cap L^\infty} \mathcal{E}(w)$.

Corollary 1 *If B has finitely many points, then u has finitely many jumps, occurring exactly at the set $\{x|v(x) \in B\}$. This concludes the proof of Theorem 1.*

Proof of Theorem 3.

(i) If $v(x_0) \notin B$, then $\underline{u}(x_0) = \bar{u}(x_0)$, which implies that $\lim_{x\to x_0} \bar{u}(x) = \lim_{x\to x_0} \underline{u}(x) = \underline{u}(x_0)$. Thus, u is continuous at x_0.
(ii) Assume $x_0 \in B$. Since $\int_{\mathbb{R}} |J'(z)|\,\mathrm{d}z < \infty$, we see that $v \in C^1(\mathbb{R})$. We now show that $v'(x_0) \neq 0$.

Let $\delta_0 > 0$ be small such that $J(z) = J(0)[1+o(1)] > \frac{1}{2}J(0)$ for all $|z| < 2\delta_0$. Let $\delta \in (0, \delta_0)$. Set $I = [x_0 - \delta, x_0 + \delta]$. We define

$$v^+ = \delta^2 + \max\{v(x)|x \in I\}, \quad v^- = \min\{v(x)|x \in I\} - \delta^2.$$

Using (5.3) with $\phi = (u - \underline{s}(v^-))\chi_I \geq 0$ we obtain

$$0 \geq \int_I \left\{ G^*(u) - G^*(\underline{s}(v^-)) - \phi v + \frac{1}{2}\phi J \star \phi \right\} \mathrm{d}x.$$

Since $\phi \geq 0$, we have

$$\frac{1}{2}\int_I \phi J \star \phi \, \mathrm{d}x \geq \frac{J(0)[1 - o(1)]}{2} \left(\int_I \phi \right)^2.$$

Also note that

$$G^*(u) - G^*(u - \phi) - v\phi \geq (g^*(u - \phi) - v)\phi \geq (v^- - v^+)\phi.$$

Hence, we obtain

$$0 \geq \frac{J(0)[1 - o(1)]}{2} \left(\int_I \phi \right)^2 + (v^- - v^+) \int_I \phi.$$

This implies that

$$v^+ - v^- \geq \frac{J(0)[1 - o(1)]}{2} \int_I \phi = \frac{J(0)[1 - o(1)]}{2} \int_I (u - \underline{s}(v^-)) \, \mathrm{d}x.$$

In a similar manner, if we take $\phi = (u - \bar{s}(v^+))\chi_I$ we can derive

$$v^+ - v^- \geq \frac{J(0)[1 - o(1)]}{2} \int_I (\bar{s}(v^+) - u) \, \mathrm{d}x.$$

Adding these two inequalities we then obtain

$$\frac{v^+ - v^-}{2\delta} \geq \frac{J(0)[1 - o(1)]}{4} (\bar{s}(v^+) - \underline{s}(v^-)).$$

Sending $\delta \to 0$ we then obtain the estimate

$$|v'(x_0)| \geq \frac{J(0)}{4} (\bar{u}(x_0) - \underline{u}(x_0)). \tag{6.1}$$

Once we know $v'(x_0) \neq 0$, the second assertion (2) then follows from the lower semicontinuity of $\underline{u}$ and $\bar{u}$.

(3) As long as we have (2), we then know that the set of discontinuities is at most countable. Hence, we have $g^*(u) = g(u)$ for all $x \in \mathbb{R}$ after we use the left or right limit of u as its definition at discontinuities. This completes the proof. □

Proof of Corollary 1 The assertion follows from the estimate (6.1) and an open covering theorem. □

Theorem 4 *The minimizer so constructed is also a global minimizer in* $u_0 + L^2$.

Proof Denote by $u = u_0 + \phi$ any one of the minimizers constructed. Here we provide a brief proof showing that $\phi \in L^1 \cap L^\infty$. From the construction of u, we see that $\phi \in L^\infty$ and

$$\lim_{|x|\to\infty} \phi(x) = 0.$$

For m large enough, set $\phi_m = \phi\chi_{[m,\infty)}$. The equation for u then can be written as

$$(f'(1) + 1)\phi_m - J \star \phi_m = R + N(\phi_m) \qquad \text{in } \mathbb{R}$$

where $N(\phi_m) = f(1+\phi_m) - f'(1)\phi_m = o(1)\phi_m$ and $R := \star(u_0+\phi-\phi_m) - u_0$ in $[m, \infty)$ and $R := -J \star \phi_m$ in $(-\infty, m]$. Since $\int_0^\infty |zJ(z)| < \infty$, one sees that $R \in L^1(\mathbb{R})$. Also, since $\max \hat{J} = J(0) = 1$, $f'(1) > 0$, and $J \hat{\star} \phi = \hat{J}\hat{\phi}$, the operator $\psi \to (f'(1) + 1)\psi - J \star \psi$ has a bounded inverse from L^1 to L^1. Hence, for large enough m, a contraction mapping theorem then shows that $\phi \in L^1(m, \infty)$. Similarly, one can show that $\phi \in L^1((-\infty, -m))$. Thus, $\phi \in L^1 \cap L^\infty$.

Next we show that u is a global minimizer in $u_0 + L^2$. Take an arbitrary $w \in u_0 + L^2$. Set $\psi = u - w$. Then $\psi \in L^2$ and $w = u - \psi$. As $\mathbf{E}(\cdot, \cdot)$ is a bounded bilinear form on $u_0 + L^2$, we need only consider the case when $\int_{\mathbb{R}} F(u-\psi) < \infty$. Using cut-off, we can also assume that $\psi \in L^\infty$. Also, by approximation, we can assume that ψ is smooth and having compact support. It then follows from $G^*(u) = G(u)$ a.e. and $0 \le G^*(u - \psi) \le G(u - \psi)$ that

$$\begin{aligned}\mathcal{E}(u) - \mathcal{E}(w) &\le \mathcal{E}^*(u) - \mathcal{E}^*(u - \psi)\\ &= \int_{\mathbb{R}} \Big\{[G^*(u) - G^*(u - \psi) - \psi J \star u]\\ &\quad + \frac{1}{2}\psi J \star \psi\Big\} \le 0.\end{aligned}$$

This completes the proof. □

7 A remark on the case $F(-1) \neq F(1)$

The discussion in this section follows similar lines as the one in [7]. We provide a short sketch of it.

If we assume $F(-1) \neq F(1)$, then one may look for heteroclinic traveling waves of the evolution equation

$$u_t = J \star u - u - f(u), \tag{7.1}$$

having the form $u(x + ct)$, with $u(\pm\infty) = \pm 1$. Such a problem does not have a variational formulation, however, a solution can be constructed using a singular perturbation approach. Namely, we consider a scaling which allows us to pass in the limit to the local bistable equation. More precisely, we seek traveling waves having the form $u(\varepsilon x + ct)$ and solving (7.1) with $\frac{1}{\varepsilon^2}(J_\varepsilon \star u - u)$ instead of $J \star u - u$, where $J_\varepsilon(x) := \frac{1}{\varepsilon}J(\frac{x}{\varepsilon})$. Let $z := \varepsilon x + ct$. We see that u solves

$$cu'(z) - \frac{1}{\varepsilon^2}(J_\varepsilon \star u(z) - u(z)) + f(u(z)) = 0, \quad u(\pm\infty) = \pm 1.$$

Below we show that $\Delta_\varepsilon \upsilon := \frac{1}{\varepsilon^2}(J_\varepsilon \star u - u)$ converges in an appropriate sense to du'', where $d := \frac{1}{2}\int_{\mathbb{R}} z^2 J(z)\,\mathrm{d}z > 0$. Thus formally, if a solution $(c_\varepsilon, u_\varepsilon)$ of

$$cu' - \Delta_\varepsilon u + f(u) = 0, \quad u(\pm\infty) = \pm 1 \tag{7.2}$$

exists, it is expected that it converges as $\varepsilon \to 0$ to a solution (c_0, u_0) of the bistable equation

$$cu' - du'' + f(u) = 0, \quad u(\pm\infty) = \pm 1,$$

which has been well studied, see e.g., [21] (in particular $c_0 \neq 0$ is unique and u_0 is unique up to translation). This observation motivates one to conjecture that it might be possible that (c_0, u_0) can be perturbed to a solution of (7.2). This is indeed the case, under the following additional assumptions on J.

(J1) $J \in L^\infty(\mathbb{R})$. $J(z) = J(-z)$, $\int_{\mathbb{R}} z^2 |J(z)|\,\mathrm{d}z < \infty$, $\int_{\mathbb{R}} z^2 J(z)\,\mathrm{d}z > 0$, and

$$\int_{\mathbb{R}} J(z)\,\mathrm{d}z = 1 \geq \int_{\mathbb{R}} \mathrm{e}^{\mathbf{i}\xi z} J(z)\,\mathrm{d}z \quad \forall \xi \in \mathbb{R}. \tag{7.3}$$

Remark 7.1 Note that (7.3) implies that $\hat{J}''(0) \leq 0$, where $\hat{J}(\xi) := \int_{\mathbb{R}} \mathrm{e}^{\mathbf{i}\xi z} J(z)\,\mathrm{d}z$. This is turn can be written as $\int_{\mathbb{R}} z^2 J(z)\,\mathrm{d}z \geq 0$.

Theorem 5 *Assume $\int_{-1}^{1} f(u)du \neq 0$, $f \in C^2(\mathbb{R})$ has exactly three zeros: -1, $q \in (-1, 1)$ and 1, with $f'(\pm 1) > 0$, and J satisfies* **(J1)**. *Then there exists a positive constant ε^* such that for every $\varepsilon \in (0, \varepsilon^*)$, problem (7.2) admits at least one solution, $(c_\varepsilon, u_\varepsilon)$, which is locally unique in $H^1(\mathbb{R})$ up to translation and which has the property that*

$$\lim_{\varepsilon \searrow 0} (c_\varepsilon, u_\varepsilon) = (c_0, u_0) \quad \text{in } \mathbb{R} \times H^1(\mathbb{R}).$$

The strategy of the proof is to construct an appropriate contraction. The construction in [7] can be used after we establish the following lemma.

Let, $(\phi, \psi) := \int_{\mathbb{R}} \phi\psi\,\mathrm{d}x$.

Lemma 7.1 *Let $\Delta_\varepsilon \phi = \varepsilon^{-2}(J_\varepsilon \star \phi - \phi)$, where J satisfies* **(J1)**. *Let $d = \frac{1}{2}\int_R z^2 J(z)\,\mathrm{d}z$. Then*

(1) for any $\phi \in H^2(\mathbb{R})$, $\|\Delta_\varepsilon \phi - d\phi''\|_{L^2} \to 0$ as $\varepsilon \searrow 0$;
(2) for any $\phi \in H^1(\mathbb{R})$, $(\Delta_\varepsilon \phi, \phi') = 0$;
(3) for any $\phi, \psi \in L^2(\mathbb{R})$, $(\Delta_\varepsilon \phi, \psi) = (\phi, \Delta_\varepsilon \psi)$ and $(\Delta_\varepsilon \phi, \phi) \leq 0$.

Proof (1) Using

$$\begin{aligned}\phi(x - \varepsilon y) - \phi(x) &= -\varepsilon \int_0^y \phi'(x - \varepsilon t)\,\mathrm{d}t \\ &= -\varepsilon y \phi'(x) + \varepsilon^2 \int_0^y (y - t)\phi''(x - \varepsilon t)\,\mathrm{d}t\end{aligned}$$

we have

$$\begin{aligned}\Delta_\varepsilon\phi(x) &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}} J(y)[\phi(x-\varepsilon y)-\phi(x)]\,\mathrm{d}y\\ &= \int_{\mathbb{R}} J(y)\int_0^y (y-t)\phi''(x-\varepsilon t)\,\mathrm{d}t\,\mathrm{d}y\\ &= d\phi''(x) + \int_{\mathbb{R}} J(y)\int_0^y (y-t)[\phi''(x-\varepsilon t)-\phi''(x)]\,\mathrm{d}t\,\mathrm{d}y.\end{aligned}$$

Then

$$\begin{aligned}&|\Delta_\varepsilon\phi(x) - d\phi''(x)|^2\\ &\le \left[\int_{\mathbb{R}} |J(y)|(\int_0^y |y-t|\,dt)^{\frac{1}{2}}(\int_0^y |y-t||\phi''(x-\varepsilon t)-\phi''(x)|^2\,\mathrm{d}t)^{\frac{1}{2}}\,\mathrm{d}y\right]^2\\ &\le \left(\int_{\mathbb{R}} \frac{y^2}{2}|J(y)|\,\mathrm{d}y\right)\left(\int_{\mathbb{R}} |J(y)|\int_0^y |y-t||\phi''(x-\varepsilon t)-\phi''(x)|^2\,\mathrm{d}t\,\mathrm{d}y\right).\end{aligned}$$

Thus, setting $C := \int_{\mathbb{R}} \frac{y^2}{2}|J(y)|\,\mathrm{d}y$ we get

$$\|\Delta_\varepsilon\phi - d\phi''\|_{L^2}^2 \le C\int_{\mathbb{R}} |J(y)|\int_0^y |y-t|\int_{\mathbb{R}} |\phi''(x-\varepsilon t)-\phi''(x)|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \to 0$$

as $\varepsilon \searrow 0$, by the Lebesgue's Dominated Convergence Theorem.

(2) $\varepsilon^2(\Delta_\varepsilon\phi, \phi') = \int_{\mathbb{R}}\int_{\mathbb{R}} J_\varepsilon(x-y)\phi(x)\phi'(y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}}\int_{\mathbb{R}} J_\varepsilon(z)[\phi(y-z)\phi(y)]'\,\mathrm{d}y\,\mathrm{d}z = 0.$

(3) $2\pi\varepsilon^2(\Delta_\varepsilon\phi, \phi) = 2\pi\int_{\mathbb{R}}(J_\varepsilon\star\phi - \phi)\phi = \int_{\mathbb{R}}(\hat{J}\hat{\phi} - \hat{\phi})\bar{\hat{\phi}} = \int_{\mathbb{R}}(\hat{J}-1)|\hat{\phi}|^2 \le 0.$ □

8 Non monotonic global minimizers

For a positive parameter ε, we consider global minimizers for $\mathcal{E}$ in (1.1) with

$$F(s) = \frac{W(s)}{\varepsilon} \tag{8.1}$$

where W is a smooth double-equal-well potential with zeros at ± 1, for example,

$$W(s) = (1-s^2)^2.$$

Theorem 6 *There exist an $\varepsilon_0 > 0$ and a J satisfying* **(J), (A2)** *such that for every $\varepsilon \in (0, \varepsilon_0]$, any global minimizer in $u_0 + L^1 \cap L^\infty$ to the energy in (1.1) with $F(u)$ in (8.1) is not monotonic.*

Proof In the sequel, $O(1)$ stands for a constant or a function that is bounded uniformly in $\varepsilon \in (0, 1]$. Also, $o(1)$ represents a constant or a function that approaches zero uniformly as $\varepsilon \searrow 0$.

For simplicity, we assume that

$$W''(1) = W''(-1) = A > 0.$$

Then $W''(u_0) = A$.

For any small $\varepsilon > 0$, let u^ε be any one of the global minimizers. Then u^ε satisfies the equation

$$W'(u^\varepsilon) = \varepsilon(J \star u^\varepsilon - u^\varepsilon) = O(1)\varepsilon \tag{8.2}$$

since u^ε is uniformly bounded.

Also, since the convexification of $G_\varepsilon(u) := \frac{1}{2}u^2 + \frac{1}{\varepsilon}W(u)$ approaches 1/2 for $u \in (-1, 1)$, we see that $G^*_\varepsilon < G_\varepsilon$ in $(-1+o(1), 1-o(1))$. As $G^*_\varepsilon(u^\varepsilon) = G_\varepsilon(u^\varepsilon)$, we see that the range of u^ε lies in a small neighborhood of $\{-1, 1\}$. Equation (8.2) then implies that $|u^\varepsilon| = 1 + O(1)\varepsilon$.

Now assume that u^ε is monotonic. By translation, we can assume that $u^\varepsilon \geq 0$ in $(0, \infty)$ and $u^\varepsilon < 0$ in $(-\infty, 0)$. It then follows that

$$u^\varepsilon = u_0 + \phi^\varepsilon, \quad \|\phi^\varepsilon\|_{L^\infty} = O(1)\varepsilon.$$

Writing $W'(u^\varepsilon) = W''(u_0)\phi^\varepsilon + O(1)\phi^{\varepsilon 2} = A\phi^\varepsilon + O(1)\phi^{\varepsilon 2}$ we obtain from Eq. (8.2) that

$$A\phi^\varepsilon + \varepsilon\phi^\varepsilon - \varepsilon J \star \phi^\varepsilon = \varepsilon(J \star u_0 - u_0) + O(1)\phi^{\varepsilon 2}.$$

It then follows that

$$\phi^\varepsilon = \frac{\varepsilon}{A}(J \star u_0 - u_0) + O(1)\varepsilon^2 \quad \text{in } L^1 \cap L^\infty.$$

Consequently,

$$\frac{1}{\varepsilon}\int_{\mathbb{R}} W(u^\varepsilon) = \frac{1}{\varepsilon}\int_{\mathbb{R}} \left\{\frac{A}{2} + O(1)\varepsilon\right\}\phi^{\varepsilon 2} = O(\varepsilon).$$

Hence,

$$\mathcal{E}(u^\varepsilon) = \mathcal{E}(u_0) + O(1)\varepsilon. \tag{8.3}$$

We now provide an upper bound for the minimum energy.

Define

$$w = \begin{cases} 1 & \text{in } (-1, 0) \cup (1, \infty), \\ -1 & \text{in } (-\infty, -1) \cup (0, 1). \end{cases}$$

We shall use the formula

$$\mathbf{E}(u, u) := \frac{1}{4}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}y\, J(x-y)(u(x)-u(y))^2 = \frac{1}{4}\int_{-\infty}^{\infty} J(z)\|u(\cdot+z)-u(\cdot)\|^2_{L^2}\, \mathrm{d}z$$

to calculate $\mathbf{E}(u_0, u_0)$ and $\mathbf{E}(w, w)$.

Direct calculation show that $\|u_0(\cdot + z) - u_0(\cdot)\|^2_{L^2} = 4|z|$ and

$$\|w(\cdot + z) - w(\cdot)\|^2_{L^2} = \begin{cases} 12z & \text{if } z \in [0, 1], \\ 4(4 - z) & \text{if } z \in (1, 2], \\ 4z & \text{if } z \in (2, \infty]. \end{cases}$$

It then follows that

$$\mathcal{E}(u_0) - \mathcal{E}(w) = -4 \int_0^\infty J(z)\left\{z\chi_{[0,1]} + (2-z)\chi_{[1,2]}\right\} \mathrm{d}z$$
$$= \frac{8}{\pi} \int_0^\infty \frac{(\cos\xi - 1)\ \hat{J}(\xi)\ \cos\xi}{\xi^2} \mathrm{d}\xi =: B(\hat{J}).$$

It is easy to find a function $\hat{J}$ such that (i) $B(\hat{J}) > 0$ (ii) $\hat{J}(0) = 1 \geq \hat{J}(\xi) = \hat{J}(-\xi)$ for all ξ, and (iii) $J \in C^\infty(\mathbb{R})$ and has compact support. The inverse transform J of $\hat{J}$ then satisfies **(J)**, (**A2**).

With such J, we see that any global minimizer of $\mathcal{E}$ in $u_0 + L^1 \cap L^\infty$ will satisfy

$$\mathcal{E}(u^\varepsilon) \leq \mathcal{E}(w) = \mathcal{E}(u_0) - B(\hat{J}).$$

Clearly, (8.3) is impossible for all small positive ε. Thus, for all small positive ε, u^ε is not monotonic. □

Remark 8.1 As u_0 is a monotonic rearrangement of w, that $B(\hat{J}) > 0$ implies that rearrangement does not always decrease energy; namely, the method of [1] does not apply here.

Remark 8.2 Making a translation such that $\inf\{x \mid u^\varepsilon > 0\} = 0$. Then formally, as $\varepsilon \searrow 0$,

$$u^\varepsilon \to u_0 - 2\chi_S$$

where S is a subset of $[0, \infty)$ having bounded measure. In this case, we can imagine that S is the minimum of the functional

$$\Phi(S) := \int_{S^c} \mathrm{d}x \int_S J(x-y)\,\mathrm{d}y = |S| - \int\int_{S\times S} J(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

In would be interesting to investigate the minimizers of the functional $\Phi(S)$, in the space consisting sets of bounded measures.

In the case when J is non-negative, a trivial solution is that S is an empty set.

Remark 8.3 The process of passing to the limit, as $\varepsilon \to 0$, used in this section can be put in the framework of Γ-convergence, studied earlier in a more general setting by Chmaj and Ren [14, 15]. There, local and global minimizers with multiple layers were constructed, based on information obtained from the Γ-limit functional and using matched asymptotics.

Acknowledgements PWB and XC were partially supported by the National Science Foundation grants DMS-9974340 and DMS–0203991, respectively.

References

1. Alberti, G., Bellettini, G.: A nonlocal anisotropic model for phase transitions. Part I: The optimal profile problem. Math. Ann. **310**, 527–560 (1998)
2. Alberti, G., Bellettini, G.: A non-local anisotropic model for phase transitions: Asymptotic behaviour of rescaled energies. Eur. J. Appl. Math. **9**, 261–284 (1998)

3. Allen, S., Cahn, J.W.: A microscopic theory for antiphase boundary motion and its application to antiphast domain coarsening. Acta Metall. **27**, 1084–1095 (1979)
4. Bates, P.W., Fife, P.C., Ren, X., Wang, X.: Traveling waves in a convolution model for phase transitions. Arch. Ration. Mech. Anal. **138**, 105–136 (1997)
5. Bates, P.W., Chmaj, A.: An integrodifferential model for phase transitions: Stationary solutions in higher space dimensions. J. Stat. Phys. **95**, 1119–1139 (1999)
6. Bates, P.W., Chmaj, A.: On a discrete convolution model for phase transitions. Archive Rat. Mech. Anal. **150**, 281–305 (1999)
7. Bates, P.W., Chen, X., Chmaj, A.: Traveling waves on a lattice. SIAM J. Math. Anal. **35**, 520–546 (2003)
8. Bellettini, G., Buttà, P., Presutti, E.: Sharp interface limits for non-local anisotropic interactions. Arch. Ration. Mech. Anal. **159**, 109–135 (2001)
9. Brandon, D., Rogers, R.C.: The coercivity paradox and nonlocal ferromagnetism. Contin. Mech. Thermodyn. **4**, 1–21 (1992)
10. Cahn, J.W., Hilliard, J.E.: Free energy of a nonuniform system I. Interfacial free energy. J. Chem. Phys. **28**, 258–267.
11. Chen, X.: Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations. Adv. Diff. Eqs. (**2**), 125–160 (1997)
12. Chmaj, A., Ren, X.: Homoclinic solutions of an integral equation: existence and stability. J. Diff. Eqs. **155**, 17–43 (1999)
13. Chmaj, A., Ren, X.: Multiple solutions of the nonlocal bistable equation. Phys. D **147**, 135–154 (2000)
14. Chmaj A.J.J., Ren, X.: The nonlocal bistable equation: stationary solutions on a bounded interval. Electron. J. Diff. Eqs. **2002**, 1–12 (2002)
15. Chmaj, A.J.J., Ren, X.: Pattern formtion in the nonlocal bistable equation. Methods Appl. Anal. **8**, 369–386 (2001)
16. Comets, F., Eisele, T., Schatzman, M.: Bifurcations for some nonlinear convolution equations. Trans. Am. Math. Soc. **296**, 661–702 (1986)
17. de Masi, A., Gobron, T., Presutti, E.: Traveling fronts in non-local evolution equations. Arch. Rational Mech. Anal. **132**, 143–205 (1995)
18. Ermentrout, G.B.: Neural networks as spatio-temporal pattern-forming systems. Rep. Prog. Phys. **61**, 353–430 (1998)
19. Ermentrout, G.B., McLeod, J.B.: Existence and uniqueness of traveling waves for a neural network. Proc. Roy. Soc. Edinburgh Sect. A **123**, 461–478 (1993)
20. Fife, P.C.: Well-posedness issues for models of phase transitions with weak interaction. Nonlinearity **14**, 221–238 (2001)
21. Fife, P.C., McLeod, J.B.: The approach of solutions of nonlinear diffusion equations to traveling front solutions. Arch. Ration. Mech. Anal. **65**, 335–361 (1977)
22. Griewank, A., Rabier, P.J.: On the smoothness of convex envelopes. Trans. Am. Math. Soc. **322**, 691–709 (1990)
23. Fosdick, R.L., Mason, D.E.: Single phase energy minimizers for materials with nonlocal spatial dependence. Quart. Appl. Math. **54**, 161–195 (1996)
24. Jones, R.: Soft condensed matter. Oxford University Press (2002)
25. Kirchheim, B., Kristensen, J.: Differentiability of convex envelopes. C. R. Acad. Sci. Paris Sér. I Math. **333**, 725–728 (2001)
26. Lebowitz, J.L., Mazel, A., Presutti, E.: Liquid–vapor phase transitions for systems with finite-range interactions. J. Stat. Phys. **94**, 955–1025 (1999)
27. Ren, X., Truskinovsky, L.: Finite scale microstructures in nonlocal elasticity. J. Elasticity **59**, 319–355 (2000)
30. Ren, X., Wei, J.: On the spectra of 3-D lamellar solutions of the diblock copolymer problem. SIAM J. Math. Anal. **35**, 1–32 (2003)
29. Rowlinson, J.S.: Translation of J.D. van der Waals' "The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density". J. Stat. Phys. **20**, 197–244 (1979)
30. Rowlinson, J.S., Widom, B.: Molecular Theory of Capillarity. Dover Publications (2003)
31. van der Waals, J.D.: The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density (Translation by J.S. Rowlinson, J. Stat. Phys. **20**, 197–244 (1979)). Verhandel Konink. Akad. Weten. Amsterdam **8** (1893). (in Dutch)

32. van der Waals, J.D.: The equation of state for gases and liquids. Nobel Lecture, December 12, 1910.
33. Whitney, H.: A function not constant on a connected set of critical points. Duke Math. J. **1**, 514–517 (1935)
34. Widom, B.: Surface tension of fluids. In: Domb, C. and Green, M.S. (eds.) Phase Transitions and Critical Phenomena, pp. 79–100. Academic Press, London, New York (1972)
35. Widom, B.: What do we know that van der Waals did not know? STATPHYS 20 (Paris 1998) Phys. A **263**, 500–515 (1999)
36. Wang, X.: Metastability and stability of patterns in a convolution model for phase transitions. J. Diff. Eqs. **183**, 434–461 (2002)