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The relaxed Dirichlet energy of mappings into a manifold

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Abstract. Let $\mathcal{Y}$ be a smooth 1-connected compact oriented manifold without boundary, such that its 2-homology group has no torsion. We characterize in any dimension n the weak $W^{1,2}(B^n,\mathcal{Y})$ lower semicontinuous envelope of the Dirichlet integral of Sobolev maps in $W^{1,2}(B^n,\mathcal{Y})$.

Let $\mathcal{Y}$ be a smooth compact, connected, oriented Riemannian manifold of dimension $M\geq 2$, without boundary, and isometrically embedded in $\mathbb{R}^N$ for some $N\geq 3$. We shall assume that $\mathcal{Y}$ is 1-connected, i.e., $\pi_1(\mathcal{Y})=0$ and, moreover, that its integral 2-homology group $H_2(\mathcal{Y}):=H_2(\mathcal{Y};\mathbb{Z})$ has no torsion, so that $H_2(\mathcal{Y};X)=H_2(\mathcal{Y})\otimes X$ for $X=\mathbb{R},\mathbb{Q}$.

Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Define

$$W^{1,2}(\Omega,\mathcal{Y}):=\{u\in W^{1,2}(\Omega,\mathbb{R}^N)\mid u(x)\in\mathcal{Y}\quad\text{for a.e. } x\in\Omega\}$$

and for every $u\in W^{1,2}(\Omega,\mathcal{Y})$, and every Borel set $A\subset\Omega$, denote by

$$\mathbf{D}(u,A):=\frac{1}{2}\int_A |Du|^2\,dx\,,\qquad \mathbf{D}(u):=\mathbf{D}(u,\Omega)$$

the Dirichlet integral of u on A. Also, let B^n be the unit ball in $\mathbb{R}^n$, let $\widetilde{B}^n$ denote a bounded domain in $\mathbb{R}^n$ such that $B^n\subset\subset\widetilde{B}^n$, e.g. $\widetilde{B}^n:=B^n(0,2)$, and let $\varphi:\widetilde{B}^n\to\mathcal{Y}$ be a given smooth $W^{1,2}$ function. Finally, for $X=C^1$ or $W^{1,2}$ we define

$$X_\varphi(\widetilde{B}^n,\mathcal{Y}):=\{u\in X(\widetilde{B}^n,\mathcal{Y})\mid u=\varphi\quad\text{on } \widetilde{B}^n\setminus\overline{B}^n\}\,.$$

In this paper we consider the *relaxed Dirichlet energy* w.r.t. the weak $W^{1,2}$ convergence, defined for every $u\in W^{1,2}(\Omega,\mathcal{Y})$ by

$$\widetilde{\mathbf{D}}(u):=\inf\{\liminf_{k\to+\infty}\mathbf{D}(u_k)\mid\{u_k\}\subset C^1(\Omega,\mathcal{Y})\,,\quad u_k\rightharpoonup u\quad\text{weakly in } W^{1,2}(\Omega,\mathbb{R}^N)\}$$

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and for every $u \in W^{1,2}_{\varphi}(\widetilde{B}^n, \mathcal{Y})$ by

$$\widetilde{\mathbf{D}}_{\varphi}(u) := \inf\{\liminf_{k\to+\infty} \mathbf{D}(u_k) \mid \{u_k\} \subset C^1_{\varphi}(\widetilde{B}^n, \mathcal{Y}), \\ u_k \rightharpoonup u \quad \text{weakly in } W^{1,2}(\widetilde{B}^n, \mathbb{R}^N)\}.$$

By Schoen-Uhlenbeck density theorem [13], if $n = 2$ we have $\widetilde{\mathbf{D}}(u) = \mathbf{D}(u)$ and $\widetilde{\mathbf{D}}_{\varphi}(u) = \mathbf{D}_{\varphi}(u)$. Moreover, by the sequential weak density theorem of Pakzad-Rivière [12], $\widetilde{\mathbf{D}}(u) < +\infty$ and $\widetilde{\mathbf{D}}_{\varphi}(u) < +\infty$ if $n \geq 3$.

In this paper we characterize the relaxed Dirichlet energy of any $W^{1,2}$ map u in terms of the mass of the *minimal connections* between the *singularities* of u, see Theorems 1 and 2, extending this way the results of [3], for $n = 3$ and $\mathcal{Y} = S^2$, and [14] for $n \geq 3$ and $\mathcal{Y} = S^2$. Before stating our results, we recall some facts from [9], see also [8].

Singularities of Sobolev maps. Since $H_2(\mathcal{Y})$ is torsion-free, there are generators $[\gamma_1], \dots, [\gamma_{\overline{s}}]$, i.e., integral cycles in $\mathcal{Z}_2(\mathcal{Y})$, such that

$$H_2(\mathcal{Y}) = \left\{ \sum_{s=1}^{\overline{s}} n_s\, [\gamma_s] \mid n_s \in \mathbb{Z} \right\}. \tag{1}$$

By the de Rham theorem the second real homology group is in duality with the second cohomology group $H^2_{dR}(\mathcal{Y})$, the duality being given by the natural pairing

$$< [\gamma], [\omega] > := \gamma(\omega) = \int_{\gamma} \omega\,, \qquad [\gamma]_{\mathbb{R}} \in H_2(\mathcal{Y}; \mathbb{R})\,, \quad [\omega] \in H^2_{dR}(\mathcal{Y})\,.$$

We will then denote by $[\omega^1], \dots, [\omega^{\overline{s}}]$ a dual basis in $H^2_{dR}(\mathcal{Y})$ so that, δ_{sr} being the Kronecker symbols, $\gamma_s(\omega^r) = \delta_{sr}$. Also, we may and do assume that ω^s is the harmonic form in its cohomology class.

For every $u \in W^{1,2}_{\varphi}(\widetilde{B}^n, \mathcal{Y})$ it is well defined the current G_u carried by the graph of u, see [9]. Also, following [9], Vol. II, Sect. 5.4.2, for every $s = 1, \dots, \overline{s}$ we define the $(n-3)$-current $\mathbb{P}_s(u) \in \mathcal{D}_{n-3}(\widetilde{B}^n)$ by

$$\mathbb{P}_s(u) := \partial \pi_{\#}(G_u \llcorner \widehat{\pi}^{\#} \omega^s)\,, \tag{2}$$

where $\pi : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n$ and $\widehat{\pi} : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^N$ denote the orthogonal projection onto the first and second factor, respectively. More explicitly, since $d\omega^s = 0$,

$$\mathbb{P}_s(u)(\phi) = \partial G_u(\pi^{\#}\phi \wedge \widehat{\pi}^{\#}\omega^s) = \int_{\widetilde{B}^n} d\phi \wedge u^{\#}\omega^s \qquad \forall\, \phi \in \mathcal{D}^{n-3}(\widetilde{B}^n)\,.$$

Maps into the sphere. If $\mathcal{Y} = S^2$, the unit sphere in $\mathbb{R}^3$, setting

$$\mathbb{P}(u)(\phi) := \frac{1}{4\pi} \partial G_u(\pi^{\#}\phi \wedge \widehat{\pi}^{\#}\omega_{S^2}) = \frac{1}{4\pi} \int_{\widetilde{B}^n} d\phi \wedge u^{\#}\omega_{S^2}\,, \quad \phi \in \mathcal{D}^{n-3}(\widetilde{B}^n)\,, \tag{3}$$

ω_{S^2} being the volume 2-form on S^2, it turns out that

$$\mathbb{P}(u) = \partial\,\mathbb{D}(u)$$

for some $(n-2)$-dimensional current $\mathbb{D}(u)$ with $\operatorname{spt}\mathbb{D}(u) \subset \overline{B}^n$. More precisely, we let $D(u)$ denote the smooth $(n-2)$-vector field dual to $u^{\#}\omega_{S^2}$, the so called *D-field* of Brezis, Coron and Lieb [4], which is defined by

$$< D(u)(x), \phi > := u^{\#}\omega_{S^2} \wedge \phi \qquad \forall\,\phi \in \Lambda^{n-2}(\mathbb{R}^n)\,. \tag{4}$$

Then we have

$$\mathbb{D}(u)(\gamma) = \frac{1}{4\pi}\int_{\widetilde{B}^n} < \gamma, D(u) > \,dx\,, \qquad \gamma \in \mathcal{D}^{n-2}(\widetilde{B}^n)\,.$$

In the particular case $n = 3$, we have

$$D(u) := (u \cdot u_{x_2} \times u_{x_3},\, u \cdot u_{x_3} \times u_{x_1},\, u \cdot u_{x_1} \times u_{x_2})\,,$$

so that

$$\mathbb{P}(u) = 0 \qquad \Longleftrightarrow \qquad \operatorname{div} D(u) = 0 \qquad \text{on } \widetilde{B}^3\,.$$

In any dimension n, we have

$$\mathbb{P}(u) = 0 \iff \partial\,\mathbb{D}(u) = 0 \iff du^{\#}\omega_{S^2} = 0 \iff \partial G_u \llcorner \widetilde{B}^n \times S^2 = 0\,.$$

Spherical cycles. We shall need the following

Definition 1. We say that an integral 2-cycle $C \in \mathcal{Z}_2(\mathcal{Y})$ is *of spherical type* if its homology class in $H_2(\mathcal{Y})$ contains a Lipschitz image of the 2-sphere S^2. More precisely, if there exist $Z \in \mathcal{Z}_2(\mathcal{Y})$, $R \in \mathcal{R}_3(\mathcal{Y})$ and a Lipschitz function $\phi : S^2 \to \mathcal{Y}$ such that

$$C - Z = \partial R\,, \qquad \phi_{\#}[\![\, S^2 \,]\!] = Z\,.$$

Denoting then

$$H_2^{sph}(\mathcal{Y}) := \{[\gamma] \in H_2(\mathcal{Y}) \mid \exists\,\phi \in \operatorname{Lip}(S^2, \mathcal{Y}) \,:\, \phi_{\#}[\![\, S^2 \,]\!] \in [\gamma]\}\,,$$

we also may and do choose the γ_s's in (1) in such a way that $[\gamma_1], \ldots, [\gamma_{\widetilde{s}}]$ generate the spherical homology classes in $H_2^{sph}(\mathcal{Y})$ for some $\widetilde{s} \le \overline{s}$.

Minimal connections. We will finally make use of the following

Definition 2. *For every $n \ge 3$ and $\Gamma \in \mathcal{D}_{n-3}(\widetilde{B}^n)$ with $\operatorname{spt}\Gamma \subset \overline{B}^n$, we denote by*

$$m_i(\Gamma) := \inf\{\mathbf{M}(L) \mid L \in \mathcal{R}_{n-2}(\widetilde{B}^n)\,, \quad \operatorname{spt} L \subset \overline{B}^n\,, \quad \partial L = R\}$$

the integral mass *of Γ. Moreover, in case $m_i(\Gamma) < +\infty$, we say that an i.m. rectifiable current $L \in \mathcal{R}_{n-2}(\widetilde{B}^n)$ is an* integral minimal connection *of Γ if* $\operatorname{spt} L \subset \overline{B}^n$, $\partial L = \Gamma$ *and* $\mathbf{M}(L) = m_i(\Gamma)$.

The previous definitions are motivated by the following proposition proved below.

Proposition 1. *Let* $u \in W^{1,2}_{\varphi}(\widetilde{B}^n, \mathcal{Y})$. *Then the following facts hold:*

(i) $\mathbb{P}_s(u) = 0$ *for every* $s = \widetilde{s} + 1, \ldots, \overline{s}$;
(ii) $m_i(\mathbb{P}_s(u)) < +\infty$ *for every* $s = 1, \ldots, \widetilde{s}$.

In particular $\pi_{\#}(\partial G_u \llcorner \widehat{\pi}^{\#}\omega^s)$ *is an integral flat chain.*

Main result. In this paper we will prove the following

Theorem 1. *Let* $n \geq 3$ *and* $u \in W^{1,2}_{\varphi}(\widetilde{B}^n, \mathcal{Y})$. *Then*

$$\widetilde{\mathbf{D}}_{\varphi}(u) = \mathbf{D}(u, \widetilde{B}^n) + \sum_{s=1}^{\widetilde{s}} M_s \cdot m_i(\mathbb{P}_s(u)), \tag{5}$$

see Definition 2, *where for every* $s = 1, \ldots, \widetilde{s}$

$$M_s := \min\{\mathbf{M}(C) \mid C \in \mathcal{Z}^2(\mathcal{Y}), \quad C \in [\gamma_s]\} \tag{6}$$

is the mass of the mass minimizing integral spherical 2*-cycle in the homology class* $[\gamma_s]$.

Remark 1. If $n = 3$, from the proof below it readily follows that Theorem 1 is an immediate consequence of the strong density results for the class $\mathrm{cart}^{2,1}_{\varphi}(\widetilde{B}^3 \times \mathcal{Y})$ in [10]. Moreover, in the particular case $\mathcal{Y} = S^2$, equation (5) reads as

$$\widetilde{\mathbf{D}}_{\varphi}(u) = \mathbf{D}(u, \widetilde{B}^n) + 4\pi \cdot m_i(\mathbb{P}(u)),$$

where $\mathbb{P}(u)$ is given by (3).

No boundary data.
In a similar way, if Ω is a smooth bounded domain in $\mathbb{R}^n$, we set

Definition 3. *For every* $n \geq 3$ *and* $\Gamma \in \mathcal{D}_{n-3}(\Omega)$ *we denote by*

$$m_{i,\Omega}(\Gamma) := \inf\{\mathbf{M}(L) \mid L \in \mathcal{R}_{n-2}(\Omega), \quad \mathrm{spt}(\partial L - \Gamma) \subset \partial\Omega\}$$

the integral mass of Γ in Ω. *Moreover, in case* $m_{i,\Omega}(\Gamma) < +\infty$, *we say that an i.m. rectifiable current* $L \in \mathcal{R}_{n-2}(\Omega)$ *is an* integral minimal connection *of* Γ allowing connections to the boundary *if* $\mathrm{spt}(\partial L - R) \subset \partial\Omega$ *and* $\mathbf{M}(L) = m_{i,\Omega}(\Gamma)$.

Theorem 2. *Let* Ω *be a smooth bounded domain in* $\mathbb{R}^n$, *where* $n \geq 3$, *and* $u \in W^{1,2}(\Omega, \mathcal{Y})$. *Then*

$$\widetilde{\mathbf{D}}(u) = \mathbf{D}(u, \Omega) + \sum_{s=1}^{\widetilde{s}} M_s \cdot m_{i,\Omega}(\mathbb{P}_s(u)),$$

where M_s *is given by* (6). *In particular, if* $\mathcal{Y} = S^2$ *and* $\mathbb{P}(u)$ *is given by* (3), *we have*

$$\widetilde{\mathbf{D}}(u) = \mathbf{D}(u, \Omega) + 4\pi \cdot m_{i,\Omega}(\mathbb{P}(u)).$$

The rest of the paper is dedicated to the proof of Theorem 1. Theorem 2 is proved in a similar way. For the sake of brevity, we refer the reader to [9], [8] and [10] for the notation and details on the quoted results, if not otherwise stated.

Proof of Theorem 1. We first recall that the class of *Cartesian current* $\mathrm{cart}^{2,1}_\varphi(\widetilde{B}^n \times \mathcal{Y})$ is given by the currents $T \in \mathcal{D}_{n,2}(\widetilde{B}^n \times \mathcal{Y})$ which have no inner boundary,

$$\partial T = 0 \qquad \text{on } \mathcal{Z}^{n-1,2}(\widetilde{B}^n \times \mathcal{Y}),$$

and are of the type

$$T = G_{u_T} + \sum_{s=1}^{\widetilde{s}} \mathbb{L}_s(T) \times \gamma_s \tag{7}$$

on $\mathcal{Z}^{n,2}(\widetilde{B}^n \times \mathcal{Y})$, compare [9] and [8]. Here $u_T \in W^{1,2}_\varphi(\widetilde{B}^n, \mathcal{Y})$ and $\mathbb{L}_s(T)$ is an i.m. rectifiable current in $\mathcal{R}_{n-2}(\widetilde{B}^n)$, with support $\mathrm{spt}\,\mathbb{L}_s(T) \subset \overline{B}^n$, such that

$$\mathbb{P}_s(u_T) = -\partial\,\mathbb{L}_s(T) \qquad \forall\, s = 1, \dots, \widetilde{s}. \tag{8}$$

Since $\widetilde{\mathbf{D}}_\varphi(u) < +\infty$, by the closure-compactness of the class $\mathrm{cart}^{2,1}_\varphi(\widetilde{B}^n, \mathcal{Y})$, for every $\varepsilon > 0$ we find a smooth sequence $\{u_k\} \subset C^1_\varphi(\widetilde{B}^n, \mathcal{Y})$ such that $u_k \rightharpoonup u$ weakly in $W^{1,2}$, with energies $\mathbf{D}(u_k, \widetilde{B}^n) \le \widetilde{\mathbf{D}}_\varphi(u) + \varepsilon$, such that the graphs G_{u_k} converge weakly in the sense of the currents in $\mathcal{D}_{n,2}(\widetilde{B}^n \times \mathcal{Y})$ to a current $T \in \mathrm{cart}^{2,1}_\varphi(\widetilde{B}^n \times \mathcal{Y})$ for which (7) holds with $u_T = u$, see [8]. This clearly yields Proposition 1. Also, the Dirichlet energy of T

$$\mathbf{D}(T) := \mathbf{D}(u_T, \widetilde{B}^n) + \sum_{s=1}^{\widetilde{s}} \mathbf{M}(\mathbb{L}_s(T)) \cdot \mathbf{M}(\gamma_s)$$

is defined in such a way that it is lower semicontinuous, i.e.

$$\mathbf{D}(T) \le \liminf_{k \to +\infty} \mathbf{D}(G_{u_k}), \qquad \mathbf{D}(G_{u_k}) := \mathbf{D}(u_k, \widetilde{B}^n).$$

Taking into account Definition 2 and (8), we readily infer that inequality "$\ge$" holds true in (5).

To prove the opposite inequality, we first define a family $\{T_\varepsilon\}$ of minimizing currents in $\mathrm{cart}^{2,1}_\varphi$ which converge weakly and in energy to a current T_0 of the type in (7), where $u_{T_0} = u$ and $-\mathbb{L}_s(T_0)$ is an integral minimal connection of $\mathbb{P}_s(u)$. Secondly, since T_ε is a minimizer, by regularity theory and by arguments related to the ones of [14] [12] and [8], we infer that it can be approximated weakly by a sequence of smooth graphs with energies converging to the energy of T_ε. Finally, a diagonal argument yields the assertion.

Step 1: Definition of $\{T_\varepsilon\}$. Given $u \in W^{1,2}_\varphi(\widetilde{B}^n, \mathcal{Y})$ and $\varepsilon > 0$ we consider the minimum problem

$$\inf \left\{ \mathbf{D}_\varepsilon(T) \,:\, T \in \mathrm{cart}^{2,1}_\varphi(\widetilde{B}^n \times \mathcal{Y}) \right\}, \tag{9}$$

where

$$\mathbf{D}_\varepsilon(T) := \mathbf{D}(T) + \frac{1}{\varepsilon}\int_{B^n} |u - u_T|^2\, dx$$

and u_T is the function in $W^{1,2}_\varphi(\widetilde{B}^n, \mathcal{Y})$ for which the structure property (7) holds true. Clearly (9) has a solution $T_\varepsilon \in \operatorname{cart}^{2,1}_\varphi(\widetilde{B}^n \times \mathcal{Y})$ of the type

$$T_\varepsilon = G_{u_\varepsilon} + \sum_{s=1}^{\widetilde{s}} \mathbb{L}_s(T_\varepsilon) \times \gamma_s\,. \tag{10}$$

Possibly taking a sequence $\varepsilon_k \searrow 0$, we find that T_ε weakly converges to a current $T_0 \in \operatorname{cart}^{2,1}_\varphi(\widetilde{B}^n \times \mathcal{Y})$ which satisfies (7) with $u_{T_0} = u$, since $u_\varepsilon \to u$ in $L^2(\widetilde{B}^n, \mathbb{R}^N)$. Moreover, since

$$\mathbf{D}(T_\varepsilon) \le \mathbf{D}_\varepsilon(T_\varepsilon) \le \mathbf{D}_\varepsilon(T_0) = \mathbf{D}(T_0)\,,$$

by the lower semicontinuity of the Dirichlet energy we have

$$\mathbf{D}(T_0) \le \liminf_{\varepsilon\to 0^+} \mathbf{D}(T_\varepsilon) \le \limsup_{\varepsilon\to 0^+} \mathbf{D}(T_\varepsilon) \le \mathbf{D}(T_0)\,,$$

so that $\mathbf{D}(T_\varepsilon) \to \mathbf{D}(T_0)$. On the other side, if T is any Cartesian current in $\operatorname{cart}^{2,1}_\varphi(\widetilde{B}^n \times \mathcal{Y})$ of the type in (7) and such that $u_T = u$, we clearly have

$$\mathbf{D}(T_\varepsilon) \le \mathbf{D}_\varepsilon(T_\varepsilon) \le \mathbf{D}_\varepsilon(T) = \mathbf{D}(T)\,.$$

Letting $\varepsilon \to 0^+$ we obtain that $\mathbf{D}(T_0) \le \mathbf{D}(T)$, which yields

$$\sum_{s=1}^{\widetilde{s}} \mathbf{M}(\mathbb{L}_s(T_0)) \cdot \mathbf{M}(\gamma_s) \le \sum_{s=1}^{\widetilde{s}} \mathbf{M}(\mathbb{L}_s(T)) \cdot \mathbf{M}(\gamma_s)$$

and hence

$$\mathbf{M}(\mathbb{L}_s(T_0)) \le \mathbf{M}(\mathbb{L}_s(T)) \qquad \forall\, s = 1, \dots, \widetilde{s}\,.$$

Finally, since by (8)

$$-\partial\, \mathbb{L}_s(T_0) = \mathbb{P}_s(u) = -\partial\, \mathbb{L}_s(T)\,,$$

by the arbitrariness of T we infer that $-\,\mathbb{L}_s(T_0)$ is an integral minimal connection of $\mathbb{P}_s(u)$ for every s.

Step 2: Regularity of $\{T_\varepsilon\}$. Arguing similarly to when proving partial regularity results in [9, Vol. II, Sect. 4.2.9] or [8] to the minimum problem (9), since $\int_{B^n} |u - u_T|^2\, dx$ is a lower order term, it follows that the Sobolev maps $u_\varepsilon \in W^{1,2}_\varphi(\widetilde{B}^n, \mathcal{Y})$ in (10) satisfy the condition

$$\mathcal{L}^n(\operatorname{sing} u_\varepsilon) = 0\,,$$

where $\operatorname{sing} v$ denotes the closure of the complement of the discontinuity points of a $W^{1,2}$ map v. We are therefore reduced to prove the following density result.

Proposition 2. *Let* $T \in \mathrm{cart}^{2,1}_{\varphi}(\widetilde{B}^n \times \mathcal{Y})$ *be such that* (7) *holds on* $\mathcal{D}^{n,2}(\widetilde{B}^n \times \mathcal{Y})$. *Suppose that*

$$\mathcal{L}^n(\mathrm{sing}\, u_T) = 0\,. \tag{11}$$

Then, there exists a smooth sequence $\{u_k\} \subset C^1_{\varphi}(\widetilde{B}^n, \mathcal{Y})$ *such that* $G_{u_k} \rightharpoonup T$ *weakly in* $\mathcal{D}_{n,2}(\widetilde{B}^n \times \mathcal{Y})$, $u_k \rightharpoonup u_T$ *weakly in* $W^{1,2}(\widetilde{B}^n, \mathbb{R}^N)$ *and finally* $\mathbf{D}(u_k, \widetilde{B}^n) \to \mathbf{D}(T)$ *as* $k \to +\infty$.

In fact, by applying Proposition 2 to $T = T_\varepsilon$, and by means of a diagonal argument, we readily find a sequence $\{v_k\} \subset C^1_{\varphi}(\widetilde{B}^n, \mathcal{Y})$ such that $v_k \rightharpoonup u$ weakly in $W^{1,2}(\widetilde{B}^n, \mathbb{R}^N)$ and $\mathbf{D}(v_k, \widetilde{B}^n) \to \mathbf{D}(T_0)$, which yields inequality "$\leq$" in (5), taking into account the properties of T_0 from Step 1.

Step 3: Approximation of $\{T_\varepsilon\}$. It therefore remains to prove Proposition 2. We recall that this density result was already proved in [10] in the case of dimension $n = 3$ without assuming condition (11), and it is open in this generality, even in the case $\mathcal{Y} = S^2$, if $n \geq 4$. For our purposes, we recall the main steps of the proof from [10].

Let $R^{\infty}_{2,\varphi}(\widetilde{B}^n, \mathcal{Y})$ and $R^{0}_{2,\varphi}(\widetilde{B}^n, \mathcal{Y})$ denote the subsets of the Sobolev space $W^{1,2}_{\varphi}(\widetilde{B}^n, \mathcal{Y})$ given by all the maps u which are smooth, respectively continuous, except on a singular set $\Sigma(u)$ of the type

$$\Sigma(u) = \bigcup_{i=1}^{r} \Sigma_i\,, \qquad r \in \mathbb{N}\,, \tag{12}$$

where Σ_i is a smooth $(n-3)$-dimensional subset of B^n with smooth boundary, if $n \geq 4$, and Σ_i is a point if $n = 3$. The starting point is the following density result of Bethuel [2].

Theorem 3. $R^{\infty}_{2,\varphi}(\widetilde{B}^n, \mathcal{Y})$ *is strongly dense in* $W^{1,2}_{\varphi}(\widetilde{B}^n, \mathcal{Y})$.

If $\{u_k\} \subset R^{\infty}_{2,\varphi}(\widetilde{B}^n, \mathcal{Y})$ is such that $u_k \to u_T$ in $W^{1,2}(\widetilde{B}^n, \mathbb{R}^N)$, where u_T is given by (7), it then follows that

$$\lim_{k\to+\infty} m_r(\mathbb{P}_s(u_T) - \mathbb{P}_s(u_k)) = 0\,,$$

where m_r denotes the *real mass*

$$m_r(\Gamma) := \inf\{\mathbf{M}(D) \mid D \in \mathcal{D}_{n-2}(\widetilde{B}^n)\,, \quad \mathrm{spt}\, D \subset \overline{B}^n\,, \quad \partial D = R\}\,.$$

By Federer's theorem [6], if Γ has dimension zero we have $m_r(\Gamma) = m_i(\Gamma)$ and hence, in case $n = 3$, see [9], Vol. II, Sect. 4.2.5 and Sect. 5.4.2, it follows that

$$\lim_{k\to+\infty} m_i(\mathbb{P}_s(u_T) - \mathbb{P}_s(u_k)) = 0 \qquad \forall\, s = 1, \ldots, \widetilde{s}\,. \tag{13}$$

As a consequence, if $L^s_{u_k,u_T}$ denotes a 1-dimensional i.m. rectifiable current of least mass with support in $\overline{B}^3$ such that

$$\partial L^s_{u_k,u_T} = \mathbb{P}_s(u_T) - \mathbb{P}_s(u_k)\,,$$

then $\lim_{k\to+\infty} \mathbf{M}(L^s_{u_k,u_T}) = 0$ and hence, setting

$$T_k := G_{u_k} + \sum_{s=1}^{\widetilde{s}} (L^s_{u_k,u_T} + \mathbb{L}_s(T)) \times \gamma_s \,,$$

by (8) we infer that $\{T_k\} \subset \mathrm{cart}^{2,1}_\varphi(\widetilde{B}^3 \times \mathcal{Y})$, with $T_k \rightharpoonup T$ weakly in $\mathcal{D}_{3,2}(\widetilde{B}^3 \times \mathcal{Y})$ and $\mathbf{D}(T_k) \to \mathbf{D}(T)$ as $k \to +\infty$. Since $\mathbf{M}(\partial(L^s_{u_k,u_T} + \mathbb{L}_s(T))) < +\infty$ for every s and k, by applying Federer's strong polyhedral approximation theorem [5], we approximate T by a sequence of currents as in (7), where this time $u_T \in R^\infty_{2,\varphi}(\widetilde{B}^3, \mathcal{Y})$ and the $\mathbb{L}_s(T)$ are polyhedral chains. At the final step one reduces to approximate the dipoles $L \times \gamma_s$, where L is the current integration over a line segment, see [10].

To extend the density result to any dimension $n \geq 4$, the crucial point is to find an approximating sequence $\{u_k\} \subset R^\infty_{2,\varphi}(\widetilde{B}^n, \mathcal{Y})$ for which property (13) holds true. In fact, once we have proved that the minimal connection between the singularities of u_k and u_T is small as $k \to +\infty$, the rest of the proof follows similarly to the case $n = 3$. We refer to [11] for the details about the approximation of the dipoles $\Delta \times \gamma_s$, in the case $n \geq 4$ and Δ equal to the current integration over an $(n-2)$-simplex.

To obtain (13), we have to estimate the mass of the minimal connection between the singularities. To this aim, we recall the following result of Pakzad and Rivière [12].

Proposition 3. *Let $u \in R^\infty_{2,\varphi}(\widetilde{B}^n, \mathcal{Y})$. Then for every $s = 1, \dots, \widetilde{s}$ there exists an integral current $L_s \in \mathcal{R}_{n-2}(\widetilde{B}^n)$, with $\mathrm{spt}\, L_s \subset \overline{B}^n$, such that*

$$\partial L_s = \mathbb{P}_s(u) \qquad \text{and} \qquad \mathbf{M}(L_s) \leq C \int_{B^n} |Du|^2 \, dx$$

for some absolute constant $C > 0$ independent of u.

In case $\mathcal{Y} = S^2$ this property goes back to [4] and is proved in [1] by means of the coarea formula. In [12] the result is given in terms of polyhedral chains with coefficients in the homotopy group $\pi_2(\mathcal{Y})$. However, since $\mathcal{Y}$ is 1-connected, by the Hurewicz theorem $\pi_2(\mathcal{Y}) \approx H_2(\mathcal{Y}; \mathbb{R})$ and hence it can be re-stated in terms of currents in $\mathcal{D}_{n-2}(\widetilde{B}^n; H_2(\mathcal{Y}; \mathbb{R}))$. Moreover, from the construction we obtain the following local version, see [14] for the case $\mathcal{Y} = S^2$.

Proposition 4. *Let W be a relatively open subset of $\overline{B}^n$ such that $\mathcal{L}^n(\partial W) = 0$. Let $u, v \in R^\infty_{2,\varphi}(\widetilde{B}^n, \mathcal{Y})$ be such that $u = v$ a.e. on $\overline{B}^n \setminus W$. Then, for every $s = 1, \dots, \widetilde{s}$, there exists an i.m. rectifiable current $L_s \in \mathcal{R}_{n-2}(B^n)$ with $\mathrm{spt}\, L_s \subset \overline{W}$ such that*

$$\partial L_s = \mathbb{P}_s(u) - \mathbb{P}_s(v) \qquad \textit{and} \qquad \mathbf{M}(L_s) \leq C \big(\mathbf{D}(u, W) + \mathbf{D}(v, W)\big) \,.$$

We postpone the proof of Proposition 4 and we first conclude the proof of (13), and hence of Proposition 2. Using the same argument as in [14], there exists a sequence of relatively open sets $W_k \subset \overline{B}^n$ such that $\mathcal{L}^n(\partial W_k) = 0$, $\mathcal{L}^n(W_k) < 1/k$ and

$$\text{sing}(u_T) \subset \ldots \subset W_{k+1} \subset \overline{W}_{k+1} \subset W_k \subset \ldots \subset W_1 .$$

Setting $V_k := \overline{B}^n \setminus W_k$, then V_{k+2} is a neighborhood of $\overline{B}^n \setminus W_{k+1} = V_{k+1}$ and u is continuous on V_{k+2}. Therefore, applying a refined version of Bethuel's density result, Theorem 3, compare [14, Thm. 4], we find the existence of sequences $\{\widetilde{u}_k\} \subset R^0_{2,\varphi}(\widetilde{B}^n, \mathcal{Y})$ and $\{u_k\} \subset R^\infty_{2,\varphi}(\widetilde{B}^n, \mathcal{Y})$, both strongly converging to u_T in $W^{1,2}(\widetilde{B}^n, \mathbb{R}^N)$, such that for every k

$$\begin{aligned} &\widetilde{u}_k = u_T \quad \text{on } V_{k+1} , && \int_{B^n} (|\widetilde{u}_k - u_T|^2 + |D\widetilde{u}_k - Du_T|^2)\, dx < \frac{1}{k} , \\ &\text{sing}(\widetilde{u}_k) = \text{sing}(u_k) , && \int_{B^n} (|\widetilde{u}_k - u_k|^2 + |D\widetilde{u}_k - Du_k|^2)\, dx < \frac{1}{k} \end{aligned}$$

and finally

$$\mathbb{P}_s(\widetilde{u}_k) = \mathbb{P}_s(u_k) \qquad \forall\, s = 1, \ldots , \overline{s} .$$

By applying Proposition 4 with $u = u_k$, $v = u_{k+1}$ and $W = W_{k+1}$, for every s we find $L^{(k)}_s \in \mathcal{R}_{n-2}(B^n)$ with $\text{spt}\, L^{(k)}_s \subset \overline{W}_{k+1}$ such that

$$\begin{aligned} &\partial L^{(k)}_s = \mathbb{P}_s(u_k) - \mathbb{P}_s(u_{k+1}) \\ \text{and} \qquad &\mathbf{M}(L^{(k)}_s) \leq C \left(\mathbf{D}(u_k, W_{k+1}) + \mathbf{D}(u_{k+1}, W_{k+1})\right) . \end{aligned}$$

Since $\mathcal{L}^n(W_k) \to 0$, and both $\{\widetilde{u}_k\}$ and $\{u_k\}$ strongly converge to u_T, possibly passing to a subsequence we may and will suppose $\mathbf{M}(L^{(k)}_s) \leq 2^{-k}$ for every k and s. Setting then

$$L^s_{u_k,u_T} := -\sum_{j=k}^{+\infty} L^{(j)}_s ,$$

since $\mathbb{P}_s(u_k) \rightharpoonup \mathbb{P}_s(u_T)$, we have

$$\partial L^s_{u_k,u_T} = \mathbb{P}_s(u_T) - \mathbb{P}_s(u_k) \qquad \text{and} \qquad \lim_{k\to+\infty} \mathbf{M}(L^s_{u_k,u_T}) = 0 ,$$

so that (13) holds true, as required.

Step 4: Proof of Proposition 4. We recall from [9], Vol. II, Sect. 5.4, see also [10], that if $u \in W^{1,2}_\varphi(\widetilde{B}^n, \mathcal{Y})$, then $\partial G_u(\omega)$ depends only on the cohomology class of $\omega \in \mathcal{Z}^{n-1,2}(\widetilde{B}^n \times \mathcal{Y})$. As a consequence ∂G_u induces a functional $(\partial G_u)_\star$ on $\mathcal{H}^{n-1,2}(\widetilde{B}^n \times \mathcal{Y})$. Since $\mathcal{H}^{k,2}(\widetilde{B}^n \times \mathcal{Y}) \simeq \mathcal{D}^{k-2}(\widetilde{B}^n) \otimes H^2_{dR}(\mathcal{Y})$, the homology map $(\partial G_u)_\star$ is uniquely represented as an element of $\mathcal{D}_{n-3}(\widetilde{B}^n; H_2(\mathcal{Y}; \mathbb{R}))$. More explicitly, if $\phi \in \mathcal{D}^{n-3}(\widetilde{B}^n)$, we have $[(\partial G_u)_\star(\phi)] \in H_2(\mathcal{Y}; \mathbb{R})$ and for $s = 1, \ldots , \overline{s}$

$$< (\partial G_u)_\star(\phi), [\omega^s] >= \partial G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \omega^s) ,$$

$<,>$ denoting the de Rham duality between $H_2(\mathcal{Y};\mathbb{R})$ and $H^2_{dR}(\mathcal{Y})$. We now set

$$\mathbb{P}(u) := (\partial G_u)_\star \in \mathcal{D}_{n-3}(\widetilde{B}^n; H_2(\mathcal{Y};\mathbb{R})) \tag{14}$$

and, for each $\omega \in [\omega] \in H^2_{dR}(\mathcal{Y})$, we define the current $\mathbb{P}(u;\omega) \in \mathcal{D}_{n-3}(\widetilde{B}^n)$ by $\mathbb{P}(u;\omega) := \partial\pi_\#(G_u \llcorner \widehat{\pi}^\#\omega)$, so that

$$\mathbb{P}(u;\omega)(\phi) = \partial G_u(\pi^\#\phi \wedge \widehat{\pi}^\#\omega) \qquad \forall\, \phi \in \mathcal{D}^{n-3}(\widetilde{B}^n)\,.$$

The following facts hold:

(i) for $s = 1, \dots, \overline{s}$

$$\mathbb{P}(u;\omega^s)(\phi) =< \mathbb{P}(u)(\phi), [\omega^s] >\,,$$

i.e., $\mathbb{P}(u;\omega^s)$ does not depend on the representative in the cohomology class $[\omega^s]$ and hence we have $\mathbb{P}_s(u) = \mathbb{P}(u;\omega^s)$, compare (2);

(ii) $\partial\,\mathbb{P}(u) = 0$ and $\mathbb{P}(u) = \sum_{s=1}^{\overline{s}} \mathbb{P}(u;\omega^s) \otimes [\gamma_s]$, hence it does not depend on the choice of $[\gamma_1], \dots, [\gamma_{\overline{s}}]$;

(iii) if $\widetilde{u} \in R^\infty_{2,\varphi}(\widetilde{B}^n, \mathcal{Y})$, then $\mathbb{P}(\widetilde{u})$ is an i.m. rectifiable $(n-3)$-current with values in $H_2^{sph}(\mathcal{Y};\mathbb{Z})$ and is a finite combination

$$\mathbb{P}(\widetilde{u}) = \sum_{s=1}^{\widetilde{s}} R_s \otimes [\gamma_s]$$

where R_s is an i.m. rectifiable current in $\mathcal{R}_{n-3}(\widetilde{B}^n)$, with $\operatorname{spt} R_s \subset \overline{B}^n$; in particular, in case $n = 3$ we have

$$R_s = \sum_i d_{i,s}\delta_{a_i}\,,$$

where $d_{i,s} \in \mathbb{Z}$ are integer coefficients and the δ_{a_i}'s are Dirac unit measures at points $a_i \in \overline{B}^3$;

(iv) since the boundary data φ has a smooth extension from $\widetilde{B}^n$ into $\mathcal{Y}$, then each $\mathbb{P}_s(\widetilde{u})$ is the boundary of an i.m. rectifiable current.

If $u \in R^\infty_{2,\varphi}(\widetilde{B}^n, \mathcal{Y})$, its singular set $\Sigma(u)$, see (12), is contained in $B := \bigcup_{i=1}^{\mu} \sigma_i$, where the σ_i are non-overlapping $(n-3)$-dimensional polyhedra such that every $(n-4)$-face of B belongs to at least two σ_i and two different faces of B intersect only on their boundaries. The *topological singularity* of u is defined in [12] as the $\pi_2(\mathcal{Y})$-polyhedral chain

$$\mathbf{S}_u := \sum_{i=1}^{\mu} [u,\sigma_i]\,[\![\,\sigma_i\,]\!] \in \mathcal{P}_{n-3}(\widetilde{B}^n; \pi_2(\mathcal{Y}))\,.$$

The homotopic singularity $[u,\sigma_i]$ of u at σ_i is given, independently of the choice of $a \in \sigma_i$ and $\delta > 0$, by

$$[u,\sigma_i] := [u_{|\Sigma_{a,\delta}}]_{\pi_2(\mathcal{Y})}\,,$$

i.e., by the homotopy class of the restriction of u to the (suitably oriented) 2-sphere $\Sigma_{a,\delta} := \partial(B^n_\delta(a) \cap N_a)$, where N_a is the 3-dimensional affine space orthogonal to σ_i at a and $B^n_\delta(a)$ is the n-ball of radius δ centered at a. Therefore, $\mathbf{S}_u$ is a polyhedral $(n-3)$-dimensional chain in $\mathcal{P}_{n-3}(\widetilde{B}^n, \pi_2(\mathcal{Y}))$.

We now recall that the class of k-dimensional *flat chains* $\mathcal{F}_k(\widetilde{B}^n; \pi_2(\mathcal{Y}))$ is given by the completion w.r.t. the flat norm of the class of polyhedral k-chains in $\mathcal{P}_k(\widetilde{B}^n; \pi_2(\mathcal{Y}))$, compare [7], [15] and [12]. Now, since $\pi_2(\mathcal{Y}) \approx H_2(\mathcal{Y}; \mathbb{R})$, we readily infer that $\mathcal{F}_{n-3}(\widetilde{B}^n; \pi_2(\mathcal{Y}))$ coincides with the class of integral flat chains in $\mathcal{D}_{n-3}(\widetilde{B}^n; H_2(\mathcal{Y}; \mathbb{R}))$. Moreover, the masses of elements in $\mathcal{P}_{n-3}(\widetilde{B}^n; \pi_2(\mathcal{Y}))$ and $\mathcal{R}_{n-3}(\widetilde{B}^n; H_2(\mathcal{Y}))$ are defined in an equivalent way. Also, for every $u \in R^\infty_{2,\varphi}(\widetilde{B}^n, \mathcal{Y})$ we have

$$\mathbb{P}_s(u) = \tau(\Sigma(u), \theta_s, \overrightarrow{\Sigma}(u)),$$

where $\overrightarrow{\Sigma}(u)$ is the $(n-3)$-vector orienting $\Sigma(u)$ and $\theta_s := [u_\# [\![\Sigma_{a,\delta}]\!]] \cdot \omega^s$, see [9, Vol. II, Sect. 5.4.2]. In particular, we infer that

$$\mathbf{S}_u \approx \mathbb{P}(u), \tag{15}$$

compare (14).

Following [12], let $\mathcal{Y}^l$ be the l-skeleton of some triangulation of $\mathcal{Y}$, for $l = 2, \dots, M := \dim(\mathcal{Y})$. We have that $\mathcal{Y}^2$ is 1-connected and hence that $\pi_2(\mathcal{Y}^2)$ is finitely generated. We let $g_1, \dots, g_\beta$ be its generators. Also, the homomorphisms $\chi^{2,l} : \pi_2(\mathcal{Y}^2) \to \pi_2(\mathcal{Y}^l)$ induced by the injection maps $\mathcal{Y}^2 \hookrightarrow \mathcal{Y}^l$ are onto, whence $\pi_2(\mathcal{Y}^l)$ is finitely generated, too. Let $p_i : \mathcal{Y}^2 \to S^2$, $i = 1, \dots, \beta$, be smooth maps such that

$$[p_i(C)]_{\pi_2(S^2)} = \alpha_i([C]_{\pi_2(\mathcal{Y}^2)})$$

for any 2-cycle $C \in \mathcal{Z}^2(\mathcal{Y}^2)$, where, for every $a \in \pi_2(\mathcal{Y}^2)$,

$$a = \sum_{i=1}^{\beta} \alpha_i(a)\, g_i$$

is its unique decomposition. Let now $\widetilde{u} \in R^\infty_{2,\varphi}(\widetilde{B}^n, \mathcal{Y}^2)$. If the boundary datum φ is constant, since $p_i \circ \widetilde{u} \in R^\infty_{2,\varphi}(\widetilde{B}^n, \mathcal{Y})$, by [1] we find existence of $\mathbf{T}_i \in \mathcal{P}_{n-2}(\widetilde{B}^n; \mathbb{Z})$ such that

$$\partial \mathbf{T}_i = \mathbf{S}_{p_i \circ \widetilde{u}} \qquad \text{and} \qquad \mathbf{M}(\mathbf{T}_i) \leq C_i \int_{\widetilde{B}^n} |D(p_i \circ \widetilde{u})|^2 \, dx \,.$$

We now recall that if $u \in R^\infty_{2,\varphi}(\widetilde{B}^n, S^2)$, for any regular value $y \in S^2$ and every $x \in u^{-1}(y)$ the D-field $D(u)(x)$, see (4), is a tangent $(n-2)$-vector to the level surface $u^{-1}(y)$. Setting then

$$T^u_y := \tau\Big(u^{-1}(y), 1, \frac{D(u)}{|D(u)|}\Big),$$

we infer that $T^u_y \in \mathcal{R}_{n-2}(\widetilde{B}^n)$, with $\operatorname{spt} T^u_y \subset \overline{B}^n$, and $\mathbb{P}(u) = \partial(T^u_y - T^\varphi_y)$.

As a consequence, if $\widetilde{u}, \widetilde{v} \in R^\infty_{2,\varphi}(\widetilde{B}^n, \mathcal{Y}^2)$ are such that $\widetilde{u} = \widetilde{v}$ a.e. on $\overline{B}^n \setminus W$, we infer that for a.e. $y \in S^2$

$$\mathbb{P}(p_i \circ \widetilde{u}) - \mathbb{P}(p_i \circ \widetilde{v}) = \partial(T^{p_i \circ \widetilde{u}}_y - T^{p_i \circ \widetilde{v}}_y) \tag{16}$$

and therefore, by the coarea formula, compare [14] for the case $\mathcal{Y} = S^2$, since $\operatorname{spt}(T^{p_i \circ \widetilde{u}}_y - T^{p_i \circ \widetilde{v}}_y) \subset \overline{W}$, we find $y \in S^2$ such that (16) holds and

$$\mathbf{M}(T^{p_i \circ \widetilde{u}}_y - T^{p_i \circ \widetilde{v}}_y) \leq \frac{1}{4\pi} \left(\mathbf{D}(p_i \circ \widetilde{u}, W) + \mathbf{D}(p_i \circ \widetilde{v}, W) \right).$$

Consequently, as in [12] we find the existence of a polyhedral chain $T \in \mathcal{P}_{n-3}(\widetilde{B}^n, \pi_2(\mathcal{Y}^2))$ such that

$$\partial T = \mathbf{S}_{\widetilde{u}} - \mathbf{S}_{\widetilde{v}} \qquad \text{and} \qquad \mathbf{M}(T) \leq C \left(\mathbf{D}(\widetilde{u}, W) + \mathbf{D}(\widetilde{v}, W) \right).$$

Finally, arguing as in [12] we prove Proposition 4, taking into account (15). □

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