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Mixed volume preserving curvature flows

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1. Introduction

Let M_0 be a compact, strictly convex hypersurface of dimension $n \geq 2$, without boundary, smoothly embedded in $\mathbb{R}^{n+1}$ and represented locally by some diffeomorphism $\varphi_0 : \mathbb{R}^n \supset U \to \varphi_0(U) \subset M_0 \subset \mathbb{R}^{n+1}$. We consider the family of maps $\varphi_t = \varphi(\cdot, t)$ evolving according to

$$\frac{\partial}{\partial t}\varphi(x,t) = \{h(t) - F(\mathcal{W}(x,t))\}\nu(x,t) \;\; x \in U, \; 0 < t \leq T \leq \infty \qquad (1)$$
$$\varphi(\cdot, 0) = \varphi_0,$$

where $\mathcal{W}(x,t)$ is the matrix of the Weingarten map of $M_t = \varphi_t(U)$ at the point $\varphi_t(x)$, $\nu(x,t)$ is the outer unit normal to M_t at $\varphi_t(x)$ and $h(t)$ is a global term to be specified. The function F should have the following properties:

Conditions 1.1

i) *$F(\mathcal{W}) = f(\kappa(\mathcal{W}))$ where $\kappa(\mathcal{W})$ gives the eigenvalues of $\mathcal{W}$ and f is a smooth, symmetric function defined on the positive cone*

$$\Gamma = \{\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n : \kappa_i > 0 \textit{ for all } i = 1, 2, \ldots, n\}.$$

ii) *f is strictly increasing in each argument: $\frac{\partial f}{\partial \kappa_i} > 0$ for each $i = 1, \ldots, n$ at every point of Γ.*
iii) *f is homogeneous of degree one: $f(k\kappa) = kf(\kappa)$ for any $k > 0$.*
iv) *f is strictly positive on Γ and $f(1, \ldots, 1) = 1$.*
v) *Either:*
- *$n = 2$,*
- *f is convex, or*
- *f is concave and one of the following hold*
  a) *f approaches zero on the boundary of Γ,*

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b) $\sup_{M_0}\left(\frac{H}{F}\right) < \liminf_{\kappa\to\partial\Gamma}\left(\frac{\sum_i \kappa_i}{f(\kappa)}\right)$, *where* H *denotes mean curvature,*

c) f *is* inverse concave*, that is, the function* $\tilde{f}(x_1,\dots,x_n) = -f\left(x_1^{-1},\dots,x_n^{-1}\right)$ *is concave.*

For $n=1$, in [Ga], Gage considered the flow of curves in the plane subject to the constraint that the enclosed area remains fixed, while in [P], Pihan considered the length preserving evolution of curves. In each case it was shown that a unique solution to the flow problem exists for all time and the solution exponentially approaches a circle enclosing the same area or of the same length as the initial curve.

For $n\geq 2$, flows with similar F, but with $h\equiv 0$ were considered by Gerhardt ([Ge1]), Urbas ([U1,U2]) and Andrews ([A1,A2,A4,A6]). Various flows with nonzero $h(t)$ have been considered, particularly for practical applications, and short-time existence of solutions is well known ([GG]). However, there are few results to date where $h(t)$ involves curvature integrals over the evolving hypersurface. The main such results have

$$nF(\mathcal{W}) = \text{trace}(\mathcal{W}) = H,$$

that is, globally constrained mean curvature flows. In 1987, Huisken studied in [Hu2] the volume preserving mean curvature flow, taking

$$h(t) = \frac{\int_{M_t} H d\mu}{\int_{M_t} d\mu},$$

where $d\mu = \mu(x,t)\,dx$ denotes the surface area element of M_t. He showed that under the flow, the evolving hypersurface converges exponentially to a sphere enclosing the same volume as M_0. Later, in [M1], the author obtained a similar result for the surface area preserving mean curvature flow, while in [M2] he generalised these results, taking

$$h_k(t) = \frac{\int_{M_t} HE_{k+1}d\mu}{\int_{M_t} E_{k+1}d\mu},$$

for each $k=-1,0,1,\dots,n-1$, obtaining globally constrained mean curvature flows which preserve each of the $n+1$ mixed volumes of M_t. The mixed volumes of a convex hypersurface M can be written as

$$V_{n-k}(\Phi) = \begin{cases} \text{Vol}(\Phi) & k=-1 \\ \left\{(n+1)\binom{n}{k}\right\}^{-1}\int_M E_k d\mu & k=0,1,\dots,n-1, \end{cases}$$

where Φ is the $(n+1)$-dimensional region contained inside M, $\partial\Phi = M$. Here, for any $l=0,\dots,n$, E_l is the lth elementary symmetric function of $\kappa_1,\dots,\kappa_n$, the principal curvatures of M,

$$E_l = \begin{cases} 1 & l=0 \\ \sum_{1\leq i_1<\dots<i_l\leq n} \kappa_{i_1}\kappa_{i_2}\cdots\kappa_{i_l} & l=1,\dots,n. \end{cases}$$

The cases $k = -1$ and $k = 0$ correspond respectively to the volume preserving mean curvature flow and the surface area preserving mean curvature flow. For notational convenience we generally omit the argument Φ of the mixed volumes.

In this paper we investigate more general mixed volume preserving curvature flows, proving the following theorem:

Theorem 1.2 *Let M_0 be as stated earlier and suppose F satisfies Conditions 1.1. Then the evolution equation* (1)*, with*

$$h(t) = h_k(t) := \frac{\int_{M_t} F(\mathcal{W})\, E_{k+1} d\mu}{\int_{M_t} E_{k+1} d\mu}, \tag{2}$$

has a smooth solution M_t for all times $0 \le t < \infty$, and the M_t's converge, as $t \to \infty$, in the C^∞-topology, to a sphere with the same value of V_{n-k} as M_0.

We remark that in practical applications it would be interesting to consider similar flows with different global functions h. Curvature evolutions are often used to model moving interfaces where the term h represents the bulk free energy difference between two materials. This quantity is temperate dependent, so in a controlled environment it could be written as a function of time. The interested reader is referred to the works of Giga and Goto ([GG]) and Gurtin and Jabbour ([GJ]) for practical applications.

Our analysis follows the framework of [M2]; we make modifications to allow for the more general F. In particular, we need to use more sophisticated results for fully nonlinear elliptic and parabolic partial differential equations to establish long-time existence of solutions to the flow equation. We also give an application of our new flows: using specific choices of F and h we give a new proof of the Minkowski inequalities of convex geometry. We hope that in the future other useful results can be obtained using flows which fit our conditions.

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2. Notation

We will use similar notation as in [Hu2, U2, A4] and [M1]. In particular, $g = \{g_{ij}\}$, $A = \{h_{ij}\}$ and $\mathcal{W} = \left\{h^i_{\ j}\right\}$ denote respectively the metric, second fundamental form and Weingarten map of M_t. The mean curvature of M_t is

$$H = g^{ij} h_{ij} = h^i_{\ i}$$

and the norm of the second fundamental form is

$$|A|^2 = g^{ij} g^{lm} h_{il} h_{jm} = h^j_{\ l} h_j^{\ l}$$

where g^{ij} is the (i, j)-entry of the inverse of the matrix (g_{ij}). Throughout this paper we sum over repeated indices from 1 to n unless otherwise indicated. Raised indices indicate contraction with the metric.

We will denote by $\left(\dot{F}^{kl}\right)$ the matrix of first partial derivatives of F with respect to the components of its argument:

$$\frac{\partial}{\partial s}F\left(A+sB\right)\Big|_{s=0}=\dot{F}^{kl}\left(A\right)B_{kl}.$$

Similarly for the second partial derivatives of F we write

$$\frac{\partial^2}{\partial s^2}F\left(A+sB\right)\Big|_{s=0}=\ddot{F}^{kl,rs}\left(A\right)B_{kl}B_{rs}.$$

We will also use the notation

$$\dot{f}_i\left(\kappa\right)=\frac{\partial f}{\partial \kappa_i}\left(\kappa\right) \text{ and } \ddot{f}_{ij}\left(\kappa\right)=\frac{\partial^2 f}{\partial \kappa_i \kappa_j}\left(\kappa\right).$$

Unless otherwise indicated, throughout this paper we will always evaluate partial derivatives of F at $\mathcal{W}$ and partial derivatives of f at $\kappa\left(\mathcal{W}\right)$.

Several function spaces and associated norms on $\mathbb{S}^n$ and on $\mathbb{S}^n\times[0,T)$ will be needed. These are as used, for example, by Urbas in [U1] and [U2]. For $k\in\mathbb{N}$, $C^k\left(\mathbb{S}^n\right)$ is the Banach space of real valued functions on $\mathbb{S}^n$ which are k-times continuously differentiable, equipped with the norm

$$\|u\|_{C^k(\mathbb{S}^n)}=\sum_{|\beta|\le k}\sup_{\mathbb{S}^n}\left|\nabla^\beta u\right|.$$

Here β is a standard multi-index for partial derivatives and ∇ is the derivative on $\mathbb{S}^n$. We further define, for $\alpha\in(0,1]$, $C^{k,\alpha}$ to be the space of functions $u\in C^k\left(\mathbb{S}^n\right)$ such that the norm

$$\|u\|_{C^{k,\alpha}(\mathbb{S}^n)}=\|u\|_{C^k(\mathbb{S}^n)}+\sup_{|\beta|=k}\sup_{\substack{x,y\in\mathbb{S}^n\\ x\neq y}}\frac{\left|\nabla^\beta u\left(x\right)-\nabla^\beta u\left(y\right)\right|}{|x-y|^\alpha}$$

is finite. Here $|x-y|$ is the distance between x and y in $\mathbb{S}^n$.

On the space-time $\mathbb{S}^n\times I$, $I=[a,b]\subset\mathbb{R}$, we denote by $C^k\left(\mathbb{S}^n\times I\right)$ the space of real valued functions u which are k-times continuously differentiable with respect to x and $\left[\frac{k}{2}\right]$-times continuously differentiable with respect to t such that the norm

$$\|u\|_{C^k(\mathbb{S}^n\times I)}=\sum_{|\beta|+2r\le k}\sup_{\mathbb{S}^n\times I}\left|\nabla^\beta D_t^r u\right|$$

is finite. Here $\left[\frac{k}{2}\right]$ is the largest integer not greater than $\frac{k}{2}$. We also denote by $C^{k,\alpha}\left(\mathbb{S}^n\times I\right)$ the space of functions in $C^k\left(\mathbb{S}^n\times I\right)$ such that the norm

$$\|u\|_{C^{k,\alpha}(\mathbb{S}^n\times I)}=\|u\|_{C^k(\mathbb{S}^n\times I)}+\sup_{|\beta|+2r=k}\sup_{\substack{(x,s),(y,t)\in\mathbb{S}^n\times I\\ (x,s)\neq(y,t)}}\frac{\left|\nabla^\beta D_t^r u\left(x,s\right)-\nabla^\beta D_t^r u\left(y,t\right)\right|}{\left(|x-y|^2+|s-t|\right)^{\frac{\alpha}{2}}}$$

is finite.

3. Some time independent facts

We first state some results for symmetric functions of the eigenvalues of matrices. The first theorem, proofs of which appear in [A4] and in [Ge2], allows us to seamlessly swap between derivatives of F and derivatives of f. We denote by $\mathrm{Sym}(n)$ the set of all $n \times n$ real symmetric matrices.

Theorem 3.1 ([Ge2, A3]) *Let f be a C^2 symmetric function defined on a symmetric region Ω in $\mathbb{R}^n$. Let $\tilde{\Omega} = \{A \in \mathrm{Sym}(n) : \kappa(A) \in \Omega\}$ and define $F : \tilde{\Omega} \to \mathbb{R}$ by $F(A) = f(\kappa(A))$. Then at any diagonal $A \in \tilde{\Omega}$ with distinct eigenvalues, the second derivative of F in direction $B \in \mathrm{Sym}(n)$ is given by*

$$\ddot{F}^{kl,rs} B_{kl} B_{rs} = \sum_{k,l} \ddot{f}_{kl} B_{kk} B_{ll} + 2 \sum_{k<l} \frac{\dot{f}_k - \dot{f}_l}{\kappa_k - \kappa_l} B_{kl}^2.$$

We immediately have the following Corollary.

Corollary 3.2 *Suppose $\mathcal{W}$ has distinct eigenvalues κ_i. Then F is convex (concave) at $\mathcal{W}$ if and only if f is convex (concave) at $\kappa(\mathcal{W})$ and*

$$\frac{\dot{f}_i - \dot{f}_j}{\kappa_i - \kappa_j} \geq (\leq)\, 0 \text{ for all } i \neq j.$$

Corollary 3.3

i) *If F satisfies Conditions 1.1 and F is convex (concave) at $\mathcal{W}$, then at this $\mathcal{W}$,*

$$F |A|^2 - H \dot{F}^{kl} h_{km} h^m_{\ l} \leq (\geq)\, 0.$$

ii) *If F satisfies Conditions 1.1 and F is concave at $\mathcal{W}$, then at this $\mathcal{W}$,*

$$FC - |A|^2 \dot{F}^{kl} h_{km} h^m_{\ l} \geq 0.$$

Remarks

1. Above we are using the notation $H = \sum_i \kappa_i$, $|A|^2 = \sum_i \kappa_i^2$ and $C = \sum_i \kappa_i^3$ because of course later we wish to apply this inequality where $\mathcal{W}$ is the Weingarten map of our evolving hypersurface.
2. The inequality of part ii) above plays the same role as $HC - \left(|A|^2\right)^2 \geq 0$ in the analyses of mean curvature flows in [Hu1, Hu2, M1] and [M2]. The inequality of part i) does not say anything for mean curvature flow.

Proof of Corollary 3.3.

i) We compute using the Euler relation

$$F |A|^2 - H \dot{F}^{kl} h_{km} h^m_{\ l} = \sum_{i,j} \left(\dot{f}_i \kappa_i \kappa_j^2 - \kappa_j \dot{f}_i \kappa_i^2 \right)$$
$$= \frac{1}{2} \sum_{i,j} \kappa_i \kappa_j \left(\dot{f}_i - \dot{f}_j \right) (\kappa_j - \kappa_i) = -\frac{1}{2} \sum_{i \neq j} \kappa_i \kappa_j \left(\frac{\dot{f}_i - \dot{f}_j}{\kappa_i - \kappa_j} \right) (\kappa_i - \kappa_j)^2,$$

where we have symmetrised in i and j. The inequality now follows using Corollary 3.2.

ii) We compute similarly

$$FC - |A|^2 \dot{F}^{kl} h_{km} h^m_{\ l} = \frac{1}{2} \sum_{i \neq j} \left(\dot{f}_i \kappa_i \kappa_j^3 + \dot{f}_j \kappa_j \kappa_i^3 - \dot{f}_i \kappa_i^2 \kappa_j^2 - \dot{f}_j \kappa_j^2 \kappa_i^2 \right)$$
$$= \frac{1}{2} \sum_{i \neq j} \kappa_i \kappa_j \left(\dot{f}_i \kappa_j - \dot{f}_j \kappa_i \right) (\kappa_j - \kappa_i) \geq 0,$$

again using Corollary 3.2. □

The inequalities of the following Lemma are derived in [U2] for concave F. For convex F we get the opposite signs.

Lemma 3.4 *For any convex (concave) F satisfying Conditions 1.1, for all $\kappa \in \Gamma$,*

i) $f(\kappa) \geq (\leq) \frac{1}{n} H$,
ii) $\sum_k \dot{f}_k = \text{trace} \left(\dot{F}^{kl} \right) \leq (\geq) 1$.

Next we state a result of Andrews, concerning compact, convex manifolds M with suitably pinched curvatures.

Theorem 3.5 (Andrews, [A1]) *If a smooth, compact, strictly convex manifold M satisfies everywhere the pointwise curvature pinching estimate*

$$\kappa_{max} \leq C_1 \kappa_{min}$$

for some constant $C_1 < \infty$, then the outer radius (circumradius) ρ_+ of M satisfies

$$\rho_+ \leq C_2 \rho_-$$

where $C_2 = \left(\frac{n+2}{\sqrt{2}} \right) C_1$. Here ρ_- denotes the inner radius (inradius) of M.

Using this theorem we can obtain a lower bound on the inradius of M in terms of any of the mixed volumes of M.

Corollary 3.6 *Let M be as in Theorem 3.5. Then for any $l = 1, \ldots, n+1$, the inradius and circumradius of M satisfy*

$$\rho_- \geq \frac{1}{C_2} \left(\frac{V_l}{\omega_{n+1}} \right)^{\frac{1}{l}} \text{ and } \rho_+ \leq C_2 \left(\frac{V_l}{\omega_{n+1}} \right)^{\frac{1}{l}},$$

where ω_{n+1} is the volume of the $(n+1)$-dimensional unit ball.

Proof. We prove the inradius case; the circumradius case is similar. A sphere with the same value of V_l as M has radius ρ_∞ given by

$$\rho_\infty = \left(\frac{V_l}{\omega_{n+1}} \right)^{\frac{1}{l}}. \tag{3}$$

(The notation ρ_∞ is used to be consistent with Sect. 9.) By the monotonicity of mixed volumes, the smallest sphere containing M must have radius $\rho_+ \geq \rho_\infty$, so using Theorem 3.5,

$$\rho_- \geq \frac{1}{C_2} \rho_+ \geq \frac{1}{C_2} \rho_\infty = \frac{1}{C_2} \left(\frac{V_l}{\omega_{n+1}} \right)^{\frac{1}{l}}.$$ □

4. Evolution equations

We compute the evolution equations for various geometric quantities associated with M_t evolving under (1).

These equations are easily derived, similarly as in [Hu1] and [A1]. In Sects. 4 to 6 inclusive, unless otherwise indicated ∇ denotes the gradient on the evolving hypersurface M_t.

Lemma 4.1 *Under the flow* (1)*,*

i) $\frac{\partial}{\partial t} g_{ij} = 2\,(h - F)\,h_{ij}$, iv) $\frac{\partial}{\partial t}\nu = \nabla F$,

ii) $\frac{\partial}{\partial t} g^{ij} = -2\,(h - F)\,h^{ij}$, v) $\frac{\partial}{\partial t} h_{ij} = \nabla_i \nabla_j F + (h - F)\,h_{im} h^m_{\ j}$,

iii) $\frac{\partial}{\partial t}\mu = (h - F)\,H\mu$, vi) $\frac{\partial}{\partial t} h^i_{\ j} = \nabla^i \nabla_j F - (h - F)\,h^i_{\ m} h^m_{\ j}$.

We will denote by $\mathcal{L}$ the operator given by $\mathcal{L}\psi = \dot{F}^{kl}\nabla_k \nabla_l \psi$. Condition 1.1, ii) ensures $\mathcal{L}$ is an elliptic operator. In the case of mean curvature flow, $\mathcal{L}$ is just the Laplace-Beltrami operator on M_t.

The following evolution equations are also easily computed, similarly as, for example, in [A1]. We use the Codazzi equations, the Gauss equations and interchange second covariant derivatives in parts ii) and iii).

Lemma 4.2 *Under the flow* (1)*,*

i) $\frac{\partial}{\partial t} F = \mathcal{L}F - (h - F)\,\dot{F}^{kl} h_{km} h^m_{\ l}$,
ii) $\frac{\partial}{\partial t} h_{ij} = \mathcal{L}h_{ij} + \ddot{F}^{kl,rs}\nabla_i h_{kl} \nabla_j h_{rs} + \dot{F}^{kl} h_{km} h^m_{\ l} h_{ij} + (h - 2F)\,h_{im} h^m_{\ j}$,
iii) $\frac{\partial}{\partial t} h^i_{\ j} = \mathcal{L}h^i_{\ j} + \ddot{F}^{kl,rs}\nabla^i h_{kl} \nabla_j h_{rs} + \dot{F}^{kl} h_{km} h^m_{\ l} h^i_{\ j} - h h^i_{\ m} h^m_{\ j}$,
iv) $\frac{\partial}{\partial t} H = \mathcal{L}H + \ddot{F}^{kl,rs}\nabla^i h_{kl} \nabla_i h_{rs} + \dot{F}^{kl} h_{km} h^m_{\ l} H - h\,|A|^2$,
v) $\frac{\partial}{\partial t}\left(\frac{H}{F}\right) = \mathcal{L}\left(\frac{H}{F}\right) + \frac{1}{F}\ddot{F}^{kl,rs}\nabla^i h_{kl} \nabla_i h_{rs} + \frac{2}{F}\dot{F}^{kl}\nabla_k F \nabla_l \left(\frac{H}{F}\right)$
$-\frac{h}{F^2}\left(F\,|A|^2 - H\dot{F}^{kl} h_{km} h^m_{\ l}\right)$.

The following lemma is a straightforward consequence of the definition of mixed volumes, as, for example, in [A3], along with (1).

Lemma 4.3 *For any flow of the form* (1)*,*

$$\frac{d}{dt}\int_{M_t} E_l \, d\mu = \begin{cases} (l+1)\int_{M_t} (h - F)\,E_{l+1}\,d\mu & l = 0, 1, \dots, n-1 \\ 0 & l = n. \end{cases}$$

We then see immediately:

Corollary 4.4 *If we set* $h = h_k$ *as given by* (2)*, then the flow* (1) *preserves* $\int_{M_t} E_k \, d\mu$.

It is often beneficial to reduce (1) to a scalar parabolic equation on $\mathbb{S}^n$ (up to tangential diffeomorphisms). One appropriate scalar quantity to consider is the support function of M_t.

Definition The *support function* of M_t, $u : \mathbb{S}^n \times [0,T) \to \mathbb{R}$, is defined by

$$u(x,t) = \langle \varphi(x,t), \nu(x,t) \rangle .$$

The support function gives the perpendicular distance to the origin of the tangent plane to M_t at $\varphi(x,t)$. We complete this section by computing the evolution equation for u.

Lemma 4.5 *Under the flow* (1)*, the support function of* M_t *evolves according to*

$$\frac{\partial}{\partial t} u = \mathcal{L} u + \dot{F}^{kl} h_{km} h^m_{\ l} u + (h - 2F) . \tag{4}$$

Proof. Using (1) and Lemma 4.1, iv),

$$\frac{\partial}{\partial t} u = \left\langle \frac{\partial}{\partial t}\varphi, \nu \right\rangle + \left\langle \varphi, \frac{\partial}{\partial t}\nu \right\rangle = (h - F) + \langle \varphi, \nabla F \rangle . \tag{5}$$

We compute similarly as in [Hu3],

$$\begin{aligned}
\mathcal{L} u &= \dot{F}^{kl} \nabla_k \left(\langle \nabla_l \varphi, \nu \rangle + \langle \varphi, \nabla_l \nu \rangle \right) \\
&= \dot{F}^{kl} \nabla_k \langle \varphi, h^m_{\ l} \nabla_m \varphi \rangle \\
&= \dot{F}^{kl} \left\{ \langle \nabla_k \varphi, h^m_{\ l} \nabla_m \varphi \rangle + \langle \varphi, (\nabla_k h^m_{\ l}) \nabla_m \varphi \rangle + \langle \varphi, h^m_{\ l} \nabla_k \nabla_m \varphi \rangle \right\} \\
&= \dot{F}^{kl} h_{kl} + \left\langle \varphi, \dot{F}^{kl} \nabla^m h_{kl} \nabla_m \varphi \right\rangle - \dot{F}^{kl} \langle \varphi, h_{km} h^m_{\ l} \nu \rangle \\
&= F + \langle \varphi, \nabla F \rangle - \dot{F}^{kl} h_{km} h^m_{\ l} u.
\end{aligned}$$

Substituting into (5) for $\langle \varphi, \nabla F \rangle$ gives the result. □

5. Preservation of convexity

To show long-time existence and investigate the long time behaviour of the evolving M_t we will require the *a priori* estimates of the following two sections.

We require the following generalisation by Andrews of Hamilton's maximum principle for tensors from [Ha1].

Theorem 5.1 (Andrews, [A4]) *Let* S_{ij} *be a smooth time-varying symmetric tensor field on a compact manifold* M *(possibly with boundary), satisfying*

$$\frac{\partial}{\partial t} S_{ij} = a^{kl} \nabla_k \nabla_l S_{ij} + u^k \nabla_k S_{ij} + N_{ij}$$

where a^{kl} *and* u *are smooth,* ∇ *is a (possibly time-dependent) smooth symmetric connection, and* a^{kl} *is positive definite everywhere. Suppose that*

$$N_{ij} v^i v^j + \sup_w 2a^{kl} \left(2 w^p_k \nabla_l S_{ip} v^i - w^p_k w^q_l S_{pq} \right) \geq 0$$

whenever $S_{ij} \geq 0$ and $S_{ij}v^j = 0$. (Here the supremum is over all matrices w.) If S_{ij} is positive definite everywhere on M at time $t = 0$ and on ∂M for $0 \leq t \leq T$, then S_{ij} is positive definite on $M \times [0, T)$.

Theorem 5.2 *If initially $h^i_{\ j} \geq \tilde{\varepsilon} F g^i_{\ j}$ for some $\tilde{\varepsilon} > 0$, then $h^i_{\ j} \geq \varepsilon F g^i_{\ j}$, on $[0, T)$, where $\varepsilon = \varepsilon(n, \tilde{\varepsilon}, F) > 0$.*

Proof. First consider the case of $n = 2$. As in [A6], we can obtain the pinching result without requiring F to be convex or concave. Consider the quantity

$$Q(\mathcal{W}) = \frac{2|A|^2 - H^2}{H^2},$$

a function of the eigenvalues of $\mathcal{W}$ with corresponding

$$q(\kappa_1, \kappa_2) = \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2}.$$

Q is symmetric, homogeneous of degree zero and evolves according to

$$\frac{\partial}{\partial t} Q = \mathcal{L} Q + \left(\dot{Q}^{ij} \ddot{F}^{kl,rs} - \dot{F}^{ij} \ddot{Q}^{kl,rs}\right) \nabla_i h_{rs} \nabla_j h_{kl} - h \dot{Q}^{ij} h_{im} h^m_{\ j}. \tag{6}$$

Since $\dot{Q}^{ij} h_{im} h^m_{\ j} \geq 0$, the same maximum principle argument as in [A6] gives that the supremum of Q is nonincreasing in time, so the pinching ratio

$$\frac{\kappa_2}{\kappa_1} = \frac{2}{1 - \sqrt{q}} - 1$$

is also nonincreasing. The result follows.

Now suppose $n \geq 3$. If F is convex, set $S^i_{\ j} = h^i_{\ j} - \tilde{\varepsilon} F g^i_{\ j}$. Using Lemma 4.2 we compute

$$\frac{\partial}{\partial t} S^i_{\ j} = \mathcal{L} S^i_{\ j} + N^i_{\ j},$$

where

$$N^i_{\ j} = \ddot{F}^{kl,rs} \nabla^i h_{kl} \nabla_j h_{rs} + \dot{F}^{kl} h_{km} h^m_{\ l} S^i_{\ j} + h\left(\tilde{\varepsilon} \dot{F}^{kl} h_{km} h^m_{\ l} g^i_{\ j} - h^i_{\ m} h^m_{\ j}\right).$$

In this case we don't need the full force of Theorem 5.1; we just make the obvious generalisation to Hamilton's maximum principle from [Ha1] to accommodate the elliptic operator $\mathcal{L}$. Now suppose v is a null eigenvector of $S^i_{\ j}$ at (x_0, t_0) for some first time $t_0 > 0$. Then, choosing coordinates such that $\left(h^i_{\ j}\right) = \operatorname{diag}(\kappa_1, \ldots, \kappa_n)$ at this point, using the convexity of F and the Euler relation we see that

$$N^i_{\ j} v_i v^j \geq \tilde{\varepsilon} h |v|^2 \left\{\left(\sum_i \dot{f}_i \kappa_i^2\right) - \tilde{\varepsilon} F^2\right\} = \tilde{\varepsilon} h |v|^2 \sum_i \dot{f}_i \kappa_i (\kappa_i - \tilde{\varepsilon} F) \geq 0,$$

since at (x_0, t_0), $\tilde{\varepsilon} F$ is the smallest eigenvalue of $\left(h^i_{\ j}\right)$, with corresponding eigenvector v. The result follows by Hamilton's maximum principle and so in the case of convex F we can actually take $\varepsilon = \tilde{\varepsilon}$.

If $n \geq 3$, F is concave and F satisfies Condition 1.1 v), c), then we proceed similarly as above, except that we need to use Theorem 5.1. To fit Andrews' framework in [A4] it is more convenient to set $S_{ij} = h_{ij} - \tilde{\varepsilon} H g_{ij}$. We compute

$$\frac{\partial}{\partial t} S_{ij} = \mathcal{L} S_{ij} + N_{ij},$$

where this time

$$N_{ij} = \ddot{F}^{kl,rs} \nabla_i h_{kl} \nabla_j h_{rs} - \tilde{\varepsilon} g_{ij} \ddot{F}^{kl,rs} \nabla^p h_{kl} \nabla_p h_{rs} + \dot{F}^{kl} h_{km} h^m_{\ l} S_{ij} + (h - 2F) h_i^{\ m} S_{mj} + \tilde{\varepsilon} h \left(g_{ij} |A|^2 - H h_{ij} \right). \quad (7)$$

This is exactly the same as in [A4], except of course for the h terms. At a null eigenvector of S_{ij} the first of these terms vanishes, while for the second we estimate

$$|A|^2 = \kappa_1^2 + \ldots + \kappa_n^2 \geq \kappa_{\text{min}} (\kappa_1 + \ldots + \kappa_n) = \tilde{\varepsilon} H^2,$$

so the last term of (7) in the brackets is nonnegative. Since F is inverse concave, Theorem 4.1 from [A4] gives that at the null eigenvector,

$$\left(\ddot{F}^{kl,rs} \nabla_i h_{kl} \nabla_j h_{rs} - \tilde{\varepsilon} g_{ij} \ddot{F}^{kl,rs} \nabla^p h_{kl} \nabla_p h_{rs} \right) v^i v^j + 2 \sup_w \dot{F}^{kl} \left(2 w_k^p \nabla_l S_{ip} v^i - w_k^p w_l^q S_{pq} \right) \geq 0.$$

Hence Theorem 5.1 gives that $h_{ij} \geq \tilde{\varepsilon} H g_{ij}$ on $[0, T)$. Finally Lemma 3.4, i) shows that we may take $\varepsilon = n \tilde{\varepsilon}$ to give the required result.

If $n \geq 3$, F is concave and satisfies one of the other conditions, then, as in [A1], we consider the evolution of the quantity $\frac{H}{F}$. From Lemma 4.2, v) we have

$$\frac{\partial}{\partial t} \left(\frac{H}{F} \right) \leq \mathcal{L} \left(\frac{H}{F} \right) + \frac{2}{F} \dot{F}^{kl} \nabla_k F \nabla_l \left(\frac{H}{F} \right),$$

where we have used concavity of F and Corollary 3.3, i). It follows by the same maximum principle argument as in [A1] that there is a $c = c(F, M_0)$ such that for all i and j,

$$\frac{\kappa_i}{\kappa_j} \leq c \quad (8)$$

at every point of M_t. In particular, this means that for all j,

$$\kappa_j \geq \frac{1}{c} \kappa_{\text{max}},$$

from which we infer using Conditions 1.1, i) and ii) that at every point of M_t,

$$\kappa_j \geq \frac{1}{c} f(\kappa),$$

hence the result. □

We now list several important corollaries. Applying Corollary 3.6 to M_t and recalling Corollary 4.4, we see that

Corollary 5.3 *For $t \in [0,T)$, the inradius ρ_- of M_t satisfies*

$$\rho_- \geq r_i(n,F,M_0) > 0.$$

Remarks

1. In our earlier paper, [M2], we used pinching together with the Minkowski inequalities to obtain a lower bound on ρ_-. Here we use Theorem 3.5 instead so that later we may use specific cases of our flow to *prove* the Minkowski inequalities.
2. Of course we can also get an upper bound on the circumradius of M_t using the other result of Corollary 3.6, however we will not need this in view of Corollary 5.6.

Later we will need the following estimate, as used in [A1].

Corollary 5.4 *There is a constant $c = c(n,F,M_0)$ such that, for $t \in [0,T)$, we have the estimate*

$$\dot{F}^{kl} h_{km} h^m_{\ l} \geq cF^2.$$

Proof. Using Theorem 5.2 and the Euler relation, we compute at any point of M_t

$$\dot{F}^{kl} h_{km} h^m_{\ l} = \dot{f}_i \kappa_i^2 \geq cF \dot{f}_i \kappa_i = cF^2.$$

□

Remark For convex F satisfying Conditions 1.1, Theorem 5.2 is not needed to establish such an inequality with $c = 1$. We will use this later in the proof of Corollary 9.3.

Also as in [A1], curvature pinching implies control on the first derivatives of F:

Corollary 5.5 *There are constants $0 < \underline{C} \leq \overline{C}$, depending only on n, F and M_0, such that, for $t \in [0,T)$,*

$$\underline{C}\, Id \leq \dot{F}(\mathcal{W}(x,t)) \leq \overline{C}\, Id.$$

Our final corollary of this section concerns the support function of the evolving hypersurface M_t. A result of Chow and Gulliver from [CG] concerning parabolic equations on $\mathbb{S}^n$, applied to (4) in a very similar way as in [M1] and [M2], gives

Corollary 5.6 *For $t \in [0,T)$,*

i) *there is a $c = c(M_0,F)$ such that $\left|\nabla^{\mathbb{S}^n} u\right| \leq c$,*
ii) *there is a $d = d(M_0,F)$ such that $M_t \subset B_d(O)$.*

6. Upper bound on F

By Corollary 5.3, for any $t \in [0,T)$, there is a ball of radius $r_i(n,F,M_0)$ contained inside Φ_t. However, for an upper bound on F we need to show that a ball with fixed centre remains inside the evolving Φ_t for a short time.

Lemma 6.1 *If $B_{4\delta}(p_0) \subset \Phi_{t_0}$ for some $t_0 \in [0,T)$, then $B_{2\delta}(p_0) \subset \Phi_t$ for $t \in \left[t_0, \min\left(t_0 + \frac{6\delta^2}{n\overline{C}}, T\right)\right)$, where $\overline{C} = \overline{C}(n,F,M_0)$ is the constant of Corollary 5.5.*

Proof. We proceed similarly as in [M2] for the mean curvature bound, comparing M_t evolving by (1) with a sphere shrinking under mean curvature flow. As we are now working with the fully nonlinear operator $\mathcal{L}$, we scale the speed of the shrinking sphere appropriately so we may use Corollary 5.5.

For convenience, let $p_0 = O$, the origin in $\mathbb{R}^{n+1}$ and assume M_{t_0} encloses O. The radius of the evolving sphere $\partial B_{r_B(t)}(O)$ satisfies

$$\frac{dr_B}{dt} = -\frac{v}{r_B},$$

with the constant scaling factor $v > 0$ to be chosen. With initial condition $r_B(t_0) = 4\delta$, this ODE has solution

$$r_B(t) = \sqrt{16\delta^2 - 2v(t-t_0)}.$$

The sphere shrinks to $B_{2\delta}(O)$ by time $t = t_0 + \frac{6\delta^2}{v}$. Set $g(x,t) = |\varphi(x,t)|^2 - r_B^2(t)$. We compute

$$\mathcal{L}g = 2\dot{F}^{kl}\langle\nabla_k\nabla_l\varphi,\varphi\rangle + 2\dot{F}^{kl}\langle\nabla_k\varphi,\nabla_l\varphi\rangle = 2\langle\mathcal{L}\varphi,\varphi\rangle + 2\dot{F}^k_k.$$

Now

$$\mathcal{L}\varphi = \dot{F}^{kl}\nabla_k\nabla_l\varphi = \dot{F}^{kl}h_{kl}\nu = -F\nu$$

by the Euler relation, so therefore

$$\mathcal{L}g = -2Fu + 2\dot{F}^k_k$$

and we compute

$$\begin{aligned}\frac{\partial g}{\partial t} &= 2\langle(h-F)\nu,\varphi\rangle + 2v = 2(h-F)u + 2v\\ &= \mathcal{L}g - 2\dot{F}^k_k + 2hu + 2v.\end{aligned}$$

Using now Corollary 5.5, we observe that at any point of M_t,

$$\dot{F}^k_k = \text{trace}\left(\dot{F}_{kl}\right) \le n\overline{C}$$

and so if we take $v = n\overline{C}$, then

$$\frac{\partial g}{\partial t} \ge \mathcal{L}g,$$

where we have also used that $h, u \geq 0$. Since $g(x, t_0) \geq 0$, it follows by the maximum principle that $g(x,t) \geq 0$, and hence $B_{2\delta} \subset \Phi_t$, for $t \in [t_0, \min(t_0 + \frac{6\delta^2}{n\overline{C}}, T))$ as required. □

Remark Although it may at first appear in the above proof that we are comparing a fully nonlinear flow with a quasilinear flow, Conditions 1.1, iii) and iv) show that, for a sphere,

$$F = \frac{1}{n} H,$$

that is, for a sphere, our fully nonlinear F degenerates to a multiple of the mean curvature.

Now as in [T,A1,A3] and [M2], we consider the function $Z = \frac{F}{u-\delta}$, for a constant δ to be chosen later. Its evolution equation is straightforward to compute using Lemma 4.2, i) and Lemma 4.5.

Lemma 6.2 *For $t \in [0, T)$,*

$$\frac{\partial}{\partial t} Z = \mathcal{L} Z + \frac{2}{(u-\delta)} \dot{F}^{kl} \nabla_k u \nabla_l Z - \frac{h}{(u-\delta)} \dot{F}^{kl} h_{km} h^m_l \\ - \frac{h}{(u-\delta)} Z + 2Z^2 - \frac{\delta}{(u-\delta)} \dot{F}^{kl} h_{km} h^m_l Z.$$

Corollary 6.3 *As long as $t \in [0, T)$ and $u > 2\delta$,*

$$\frac{\partial}{\partial t} Z \leq \mathcal{L} Z + \frac{2}{(u-\delta)} \dot{F}^{kl} \nabla_k u \nabla_l Z + \left(2 - c\delta^2 Z\right) Z^2.$$

where $c = c(n, F, M_0)$ is the constant of Corollary 5.4.

Proof. We neglect the h terms in Corollary 6.2 and on the last term we use Corollary 5.4 and our assumption on u. □

By a very similar maximum principle argument as in [M2], using Corollary 5.3, Lemma 6.1 and Theorem 5.2, applying Lemma 3.5 from [Ha2] to Corollary 6.3 yields

Theorem 6.4 *For $t \in [0, T)$,*

$$F(\mathcal{W}(x,t)) \leq \max\left(\frac{1}{3\delta} \max_{M_0} F, \frac{3}{c\delta^2} d\right),$$

where $c = c(n, F, M_0)$ and $d(n, F, M_0)$ are the constants of Corollaries 5.4 and 5.6 respectively and $\delta = \frac{r_i}{4}$, where $r_i = r_i(n, F, M_0)$ is the constant of Corollary 5.3.

Inserting this estimate into (2), we immediately obtain

Corollary 6.5 $h(t) \leq c(n, F, M_0)$ *for $t \in [0, T)$.*

Hence the speed of the evolving hypersurfaces is bounded.

Corollary 6.6 *For* $t \in [0, T)$,

$$\left|\frac{\partial \varphi}{\partial t}\right| \leq c\,(n, F, M_0)$$

We also see that the curvature of M_t remains bounded.

Corollary 6.7 *For* $t \in [0, T)$,

$$|A|^2 \leq c\,(n, F, M_0)\,.$$

Proof. In the case of convex F, this follows immediately from Lemma 3.4, i). For concave F, we instead use the curvature pinching result. Indeed, choosing $\kappa_j = \kappa_{\min}$ in (8), we have for any i,

$$\kappa_i \leq c\kappa_{\min} \leq cF$$

by Conditions 1.1, i) and ii). Hence the result. □

7. Short-time existence of a solution to the flow equation

We may parametrise M_0 in terms of its support function u_0 by $\varphi_0 : \mathbb{S}^n \to \mathbb{R}^{n+1}$, where

$$\varphi_0\,(x) = u_0\,(x)\,x + \nabla u_0\,(x)\,.$$

In this section ∇ denotes the gradient on $\mathbb{S}^n$. By adding a tangential diffeomorphism to the flow (1), we can ensure that the parametrisation of M_t,

$$\varphi\,(x,t) = u\,(x,t)\,x + \nabla u\,(x,t)\,,$$

is preserved under the flow and x is the normal vector to M_t at $\varphi\,(x,t)$ at all times. The corresponding initial value problem for u is then

$$\begin{aligned} \frac{\partial u}{\partial t} &= h\,(t) - F\,(\mathcal{W}) \\ u\,(\cdot, 0) &= u_0, \end{aligned} \tag{9}$$

where $\mathcal{W} = \left(\nabla^2 u + Iu\right)^{-1}$. This expression for the Weingarten map in terms of the support function is easily derived, as, for example, in [U2]. Replacing h, given by (2), by any $C^{1,\frac{\alpha}{2}}$ function $g : [0, T) \to \mathbb{R}$ satisfying $g\,(0) = h\,(0) > 0$, we consider the family of modified initial value problems

$$\begin{aligned} \frac{\partial}{\partial t} u_g &= g\,(t) - F\,(\mathcal{W}_g) =: \mathcal{F}\left(t, \nabla u_g, \nabla^2 u_g\right) \\ u_g\,(\cdot, 0) &= u_0, \end{aligned} \tag{10}$$

where $\mathcal{W}_g$ denotes the Weingarten map of the associated hypersurface. In view of Condition 1.1, ii), (10) is a parabolic equation, for which short-time existence via the implicit function theorem is well known. In particular, if $u_0 \in C^{4,\alpha}\,(\mathbb{S}^n)$ then we have a unique solution $u_g \in C^{2,\alpha}(\mathbb{S}^n \times [0,T))$ (see, for example [GG]). Note that uniform parabolicity of the modified problem follows by the same arguments as in Sect. 5; in particular, we get an analogue of Corollary 5.5.

A routine fixed point argument then yields a solution $u \in C^{4,\alpha}(\mathbb{S}^n \times [0,T))$ to (9), with $h \in C^{1,\frac{\alpha}{2}}([0,T))$ given by (2). We remark that the smoothness of the solution u_g to the modified problem is just sufficient for the second derivative of h_g to be bounded for a short time. This in turn is sufficient in the fixed point argument to show that the operator P given by

$$P(g) = h_g := \frac{\int F(\mathcal{W}) E_{k+1} d\mu}{\int E_{k+1} d\mu}$$

maps a suitable closed convex set in $C^{1,\frac{\alpha}{2}}([0,T))$ into a precompact subset of itself. Again note that all the quantities in the definition of P are associated with the modified problem.

Uniqueness of the solution u follows by a small modification of the standard argument for nonlinear PDEs to incorporate the global term as a ratio of integrals.

8. Long-time existence

Since we have F in place of H, we cannot use the induction argument of Hamilton as in [Ha1,Hu1,Hu2,M1,M2], etc, to obtain uniform estimates on all orders of curvature derivatives and hence smoothness and long-time existence of the solution to (1). Instead we use a more PDE theoretic approach. Some similar ideas were used by Urbas ([U1,U2]) but, to avoid differentiating $h(t)$ until late in the argument, we use a perturbation result of Caffarelli for a class of fully nonlinear elliptic PDEs ([Ca]) and some recent results of Andrews on time continuity for the parabolic problem ([A5]). The results of this section are first obtained locally in space and easily extended to the whole $\mathbb{S}^n$.

For concreteness and ease of computation, in this section we will adopt a local graph representation of the solution hypersurface. Locally, let $\varphi : U \subset \mathbb{R}^n \times [0,T) \to \mathbb{R}^{n+1}$ be given by

$$\varphi(x,t) = (x, z(x,t)).$$

Incorporating a tangential diffeomorphism into the flow (1) to ensure that this parametrisation is preserved, we find

$$\frac{\partial z}{\partial t} = \sqrt{1 + |Dz|^2}\,\{h(t) - F(W)\}, \tag{11}$$

where D denotes the derivative on $\mathbb{R}^n$.

In graphical coordinates, the matrices of the inverse metric and Weingarten map are given by

$$g^{ij} = \delta^{ij} - \frac{D^i z D^j z}{1 + |Dz|^2}$$

and

$$h^i_{\ j} = -\frac{1}{\sqrt{1+|Dz|^2}}\left(\delta^{ik} - \frac{D^i z D^k z}{1+|Dz|^2}\right) D_k D_j z$$

respectively. It is easy to see that under the flow (11), the corresponding evolution equations for z and F can be written as

$$\frac{\partial z}{\partial t} = g^{ik}\dot{F}_{ij}\left(\mathcal{W}\right) D_k D_j z + \sqrt{1+|Dz|^2}\, h\left(t\right) \tag{12}$$

and

$$\frac{\partial F}{\partial t} = g^{ik}\dot{F}_{ij}\left(\mathcal{W}\right) D_k D_j F - g^{ik}\dot{F}_{ij}\left(\mathcal{W}\right) \Gamma^l_{kj} D_l F - \left(h - F\right) g^{ik} g^{lm} \dot{F}_{ij}\left(\mathcal{W}\right) h_{il} h_{jm}, \tag{13}$$

where Γ^l_{kj} are the Christoffel symbols of the connection on M_t.

Noting as in [U1] that the matrix product $g^{-1}\dot{F} = \tilde{g}\dot{F}\tilde{g}$, where

$$\tilde{g}^{ij} = \delta^{ij} - \frac{D^i z D^j z}{\sqrt{1+|Dz|^2}\left(1+\sqrt{1+|Dz|^2}\right)},$$

we see in view of Corollaries 5.5 and 5.6 and convexity, equations (12) and (13) are uniformly parabolic. Furthermore, by the same results, together with Theorem 6.4 and Corollaries 6.5 and 6.7, equations (12) and (13) have bounded, measurable coefficients. A result of Krylov and Safonov from [KS] therefore gives $z \in C^{0,\alpha}\left(U \times [\delta, T)\right)$ and $F \circ \mathcal{W} \in C^{0,\alpha}\left(U \times [\delta, T)\right)$, for any $\delta > 0$, where $\alpha = \alpha\left(n, M_0\right)$. Actually, α also depends on δ^{-1}, but we take a small fixed δ here. Importantly, the bounds on the corresponding Hölder norms are independent of T.

Now for any $t_0 \in [\delta, T)$ let $v_{t_0} : U \to \mathbb{R}$ be given by

$$v_{t_0}\left(x\right) = \frac{1}{\sqrt{1+|Dz\left(x,t_0\right)|^2}} \frac{\partial}{\partial t} z\left(x, t_0\right).$$

Recalling (11), the Hölder estimate for F implies $v_{t_0} \in C^{0,\alpha}\left(U\right)$. We consider the corresponding elliptic PDE

$$G\left(D^2 z\left(x,t_0\right), z\left(x,t_0\right)\right) = v_{t_0}\left(x\right)$$

where

$$G\left(D^2 z\left(x,t_0\right), z\left(x,t_0\right)\right) = h\left(t_0\right) - F\left(\mathcal{W}\left(x,t_0\right)\right).$$

Set

$$\mathcal{G}\left(N, x\right) = G\left(N, z\left(x, t_0\right)\right),$$

where G incorporates the Bellman extension of F.[1] This extension allows all symmetric matrices N in the argument of G, not just positive definite matrices. Explicitly, for a given symmetric matrix $N = (N_{ij})$, set

$$\tilde{h}^i_{\ j} = -g^{ik} N_{kj}.$$

Then in the case of concave F, the Bellman extension $\tilde{F}$ of F is given by

$$\tilde{F}\left(\tilde{h}^i_{\ j}\right) := \inf_{\left(h^i_{\ j}\right) \in S_+} \left[F\left(h^i_{\ j}\right) + \dot{F}^{kl}\left(h^i_{\ j}\right)\left(\tilde{h}_{kl} - h_{kl}\right)\right],$$

while in the case of convex F, $\tilde{F}$ takes the form

$$\tilde{F}\left(\tilde{h}^i_{\ j}\right) := \sup_{\left(h^i_{\ j}\right) \in S_+} \left[F\left(h^i_{\ j}\right) + \dot{F}^{kl}\left(h^i_{\ j}\right)\left(\tilde{h}_{kl} - h_{kl}\right)\right].$$

Here S_+ is the set of all positive definite matrices. Notice that since F is homogeneous of degree one, these extensions simplify to

$$\tilde{F} = \begin{cases} \inf_{\left(h^i_{\ j}\right) \in S_+} \dot{F}^{kl}\left(h^i_{\ j}\right)\tilde{h}_{kl} & \text{for } F \text{ concave,} \\ \sup_{\left(h^i_{\ j}\right) \in S_+} \dot{F}^{kl}\left(h^i_{\ j}\right)\tilde{h}_{kl} & \text{for } F \text{ convex.} \end{cases}$$

The Bellman extension preserves convexity or concavity and importantly, $\tilde{F}$ is uniformly elliptic, in view of Corollary 5.5. It is now straightforward to check that, using the smoothness of F and Corollary 5.6, i), the elliptic equation

$$\mathcal{G}\left(D^2 z\left(x, t_0\right), x\right) = v_{t_0}(x)$$

satisfies the conditions of Theorem 3 from [Ca]. This theorem gives that $z(\cdot, t_0) \in C^{2,\alpha}(U)$.

Spatial regularity of z at each $t_0 \in [\delta, T)$ now implies time regularity of first and second spatial derivatives of z, by the parabolic maximum principle argument of Andrews in [A5]. Once again, bounds on the corresponding Hölder norms are independent of T. Together with the Hölder continuity of z, we therefore have that $\mathcal{W}$, E_{k+1}, g_{ij} and μ are Hölder continuous in time. Using these results and again the Hölder continuity of $F \circ \mathcal{W}$ we see therefore that $h \in C^{0,\frac{\alpha}{2}}([\delta, T))$. Thus $\frac{\partial z}{\partial t} \in C^{0,\alpha}(U \times [\delta, T))$. Hence $z \in C^{2,\alpha}(U \times [\delta, T))$.

For higher regularity, first observe that the Krylov-Safonov Harnack inequality ([KS]) applied to (13) together with Theorem 6.4 yield

$$F \geq C^*(n, M_0) > 0$$

on $U \times [\delta, T)$. Theorem 5.2 then gives

$$\kappa_i \geq \tilde{\varepsilon} C^*,$$

so, together with Corollary 6.7 we see that under the flow, after time δ, the principle curvatures remain within a compact subset of the positive cone Γ. Hence second and

[1] I would like to thank Professor Neil Trudinger for suggesting the Bellman extension.

higher order derivatives of F remain bounded under the flow, so standard Schauder estimates (for example, Theorem 4.9 from [Li]), yield $z \in C^{k,\alpha}(U \times [\delta, T))$.

Extending these local results to the whole of $\mathbb{S}^n$ and combining with our short-time existence result we have therefore back in terms of the embedding φ of M_t,

$$\varphi \in C^{k,\alpha}\left(\mathbb{S}^n \times [0,T)\right).$$

The independence of these results on T implies that our solution can be extended smoothly up to time T, then the short-time existence result gives existence on a slightly longer time interval. Hence $T = \infty$. Uniqueness is clear via uniqueness of the short-time solution.

9. Exponential convergence to the sphere

We first show that the solution of (1) is converging to a sphere, using a suitable geometric quantity which is monotone under the flow. In view of Corollary 4.4, the radius ρ_∞ of the final sphere can be written in terms of $V_{n-k}(\Phi_0)$ using (3).

In the case $n = 2$, again similarly as in [A6], we can apply the strong maximum principle to (6) to see that the pinching ratio is strictly decreasing unless M_t is a sphere.

We need to use different monotone quantities for $n \geq 3$. For the case of convex F it is useful to consider the quantity $\frac{K}{F^n}$, as in [Ch1], where $K = \det \mathcal{W}$ is the Gauss curvature of M_t. It is easy to compute the following evolution equations using Lemma 4.2, where from now on ∇ denotes the gradient on M_t.

Lemma 9.1 *Under the flow* (1),

i) $\frac{\partial}{\partial t}K = \mathcal{L}K - \frac{1}{K}\dot{F}^{kl}\nabla_k K \nabla_l K - K\dot{F}^{kl}\nabla_k h^{-1}_{im}\nabla_l h^{im} + nK\dot{F}^{kl}h_{km}h^m_l$
$+K\ddot{F}^{kl,rs}h^{-1}_{ij}\nabla^i h_{kl}\nabla^j h_{rs} - hHK,$

ii) $\frac{\partial}{\partial t}F^n = \mathcal{L}F^n - n(n-1)F^{n-2}\dot{F}^{kl}\nabla_k F \nabla_l F$
$-n(h-F)F^{n-1}\dot{F}^{kl}h_{km}h^m_l,$

where h^{-1} denotes the inverse of the second fundamental form.

Corollary 9.2

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{K}{F^n}\right) &= \mathcal{L}\left(\frac{K}{F^n}\right) + w^k\nabla_k\left(\frac{K}{F^n}\right) + \frac{K}{F^n}h^{-1}_{ij}\ddot{F}^{kl,rs}\nabla^i h_{kl}\nabla^j h_{rs} \\ &\quad + \frac{K}{F^{n+2}}\dot{F}^{kl}h^{-1}_{pq}h^{-1}_{rs}\left(F\nabla_k h^{pr} - h^{pr}\nabla_k F\right)\left(F\nabla_l h^{qs} - h^{qs}\nabla_l F\right) \\ &\quad + \frac{hK}{F^{n+1}}\left(n\dot{F}^{kl}h_{km}h^m_l - HF\right)\end{aligned} \tag{14}$$

where $w^k = \dot{F}^{kl}\left\{\frac{n+2}{F}\nabla_l F - \frac{1}{K}\nabla_l K\right\}$.

Proof. The result follows from the previous lemma and the identities

$$\frac{1}{F^2} h^{-1}_{pq} h^{-1}_{rs} \left(F \nabla_k h^{pr} - h^{pr} \nabla_k F\right) \left(F \nabla_l h^{qs} - h^{qs} \nabla_l F\right)$$
$$= \frac{F^{2n}}{nK^2} \nabla_k \left(\frac{K}{F^n}\right) \nabla_l \left(\frac{K}{F^n}\right) - \frac{1}{nK^2} \nabla_k K \nabla_l K - \nabla_k h^{-1}_{im} \nabla_l h^{im}$$

and

$$\frac{1}{F^n} \dot{F}^{kl} \nabla_k K \nabla_l K = 2 \dot{F}^{kl} \nabla_k \left(\frac{K}{F^n}\right) \nabla_l K - F^n \dot{F}^{kl} \nabla_k \left(\frac{K}{F^n}\right) \nabla_l \left(\frac{K}{F^n}\right)$$
$$+ \frac{n^2 K^2}{F^{n+2}} \dot{F}^{kl} \nabla_k F \nabla_l F. \quad \square$$

Remark The above identities are generalisations of those used by Chow in [Ch1].

Corollary 9.3 *In the case of convex F, we have, with w^k as above,*

$$\frac{\partial}{\partial t} \left(\frac{K}{F^n}\right) \geq \mathcal{L} \left(\frac{K}{F^n}\right) + w^k \nabla_k \left(\frac{K}{F^n}\right). \tag{15}$$

Proof. By strict convexity of M_t, $K > 0$ and $\left(h^{-1}_{ij}\right)$ is positive definite. By Condition 1.1, iv), $F > 0$ and so by Condition 1.1, ii), we may neglect the norm-like term of (14). By convexity of F we also neglect the $\ddot{F}$ term. Finally for the other term we first compute using the Euler relation and the Cauchy-Schwarz inequality

$$f^2 = \left(\sum_{i=1}^n \frac{\partial f}{\partial \kappa_i} \kappa_i\right)^2 \leq \left(\sum_{i=1}^n \frac{\partial f}{\partial \kappa_i} \kappa_i^2\right) \left(\sum_{j=1}^n \frac{\partial f}{\partial \kappa_j}\right),$$

so therefore by Lemma 3.4, ii),

$$F^2 \leq \dot{F}^{kl} h_{km} h^m_{\ l}.$$

Then in view of Lemma 3.4, i), we have

$$n \dot{F}^{kl} h_{km} h^m_{\ l} - HF \geq nF^2 - nF^2 = 0,$$

so the corresponding term from (14) can also be discarded. $\square$

Corollary 9.4 *In the case of convex F, the function $\frac{K}{F^n}$ is strictly increasing unless M_t is a sphere.*

Proof. Applying the weak maximum principle to (15),

$$\min_{M_t} \left(\frac{K}{F^n}\right) \geq \min_{M_0} \left(\frac{K}{F^n}\right).$$

Furthermore, by the strong maximum principle, if the minimum is attained at some (x_0, t_0), $t_0 > 0$, then $\frac{K}{F^n}$ is identically constant. If this is the case, then substituting into (14) yields

$$0 \equiv K\dot{F}^{kl} h^{-1}_{pq} h^{-1}_{rs} \left(F\nabla_k h^{pr} - h^{pr}\nabla_k F\right)\left(F\nabla_l h^{qs} - h^{qs}\nabla_l F\right) + KF^2 h^{-1}_{ij}\ddot{F}^{kl,rs}\nabla_i h_{kl}\nabla_j h_{rs} + hKF\left(n\dot{F}^{kl}h_{km}h^m_{\ l} - HF\right).$$

Since each of these three expressions is nonnegative, each must be identically equal to zero. That the last expression in brackets is equal to zero means that at any point of M_t,

$$0 = \sum_{i,j}\left(\dot{f}_i\kappa_i^2 - \kappa_i\dot{f}_j\kappa_j\right) = \sum_{i\neq j}\left(\dot{f}_i\kappa_i - \dot{f}_j\kappa_j\right)(\kappa_i - \kappa_j).$$

Now for any pair $i \neq j$ with $\kappa_i \neq \kappa_j$, the corresponding term in the sum is strictly positive, by Corollary 3.2. But a sum of such terms cannot be equal to zero, so we conclude that we must have had $\kappa_i = \kappa_j$ for all i and j. Hence the hypersurface M_t is umbillic and therefore a sphere. □

Next we consider the case of concave F. A useful quantity to consider now is $\frac{|A|^2}{F^2}$ due to the convexity of the numerator in the principal curvatures. It is straight forward to compute the following evolution equations using Lemma 4.2, i) and ii) and Lemma 4.1, ii).

Lemma 9.5 *Under the flow* (1)*,*

i) $\frac{\partial}{\partial t}|A|^2 = \mathcal{L}|A|^2 - 2\dot{F}^{kl}\nabla_k h^{ij}\nabla_l h_{ij} + 2h^{ij}\ddot{F}^{kl,rs}\nabla_i h_{kl}\nabla_j h_{rs}$ $+2\dot{F}^{kl}h_{km}h^m_{\ l}|A|^2 - 2hC,$

ii) $\frac{\partial}{\partial t}F^2 = \mathcal{L}F^2 - 2\dot{F}^{kl}\nabla_k F\nabla_l F - 2F(h-F)\dot{F}^{kl}h_{km}h^m_{\ l}.$

Corollary 9.6

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|A|^2}{F^2}\right) = \mathcal{L}\left(\frac{|A|^2}{F^2}\right) + \frac{2}{F}\dot{F}^{kl}\nabla_k\left(\frac{|A|^2}{F^2}\right)\nabla_l F + \frac{2}{F^2}h^{ij}\ddot{F}^{kl,rs}\nabla_i h_{kl}\nabla_j h_{rs} \\ - \frac{2}{F^4}\dot{F}^{kl}g^{pq}g^{rs}\left(F\nabla_k h_{pr} - h_{pr}\nabla_k F\right)\left(F\nabla_l h_{qs} - h_{qs}\nabla_l F\right) \\ - \frac{2h}{F^3}\left(FC - |A|^2\dot{F}^{kl}h_{km}h^m_{\ l}\right).\end{aligned} \tag{16}$$

Remark The norm-like term appears as a generalisation of the corresponding identity used by Chow in [Ch2].

Corollary 9.7

$$\frac{\partial}{\partial t}\left(\frac{|A|^2}{F^2}\right) \le \mathcal{L}\left(\frac{|A|^2}{F^2}\right) + \frac{2}{F}\dot{F}^{kl}\nabla_k\left(\frac{|A|^2}{F^2}\right)\nabla_l F. \tag{17}$$

Proof. Clearly we can neglect the norm-like term, and the $\ddot{F}$ term by concavity. We can also drop the last term by Corollary 3.3, ii). □

Corollary 9.8 *In the case of concave F, the function $\frac{|A|^2}{F^2}$ is strictly decreasing unless M_t is a sphere.*

Proof. Applying the weak maximum principle to (17),

$$\max_{M_t}\left(\frac{|A|^2}{F^2}\right) \leq \max_{M_0}\left(\frac{|A|^2}{F^2}\right).$$

Furthermore, by the strong maximum principle, if the maximum is attained at some (x_0, t_0), $t_0 > 0$, then $\frac{|A|^2}{F^2}$ is identically constant. If this is the case, then substituting into (16) yields

$$\begin{aligned} 0 \equiv F^2 h^{ij} \ddot{F}^{kl,rs} \nabla_i h_{kl} \nabla_j h_{rs} - hF\left(FC - |A|^2 \dot{F}^{kl} h_{km} h^m_{\ l}\right) \\ - \dot{F}^{kl} g^{pq} g^{rs} \left(F \nabla_k h_{pr} - h_{pr} \nabla_k F\right)\left(F \nabla_l h_{qs} - h_{qs} \nabla_l F\right). \end{aligned}$$

Since each of these three expressions is nonpositive, each must be identically equal to zero. That the second expression in brackets is equal to zero means, using Corollary 3.2 in a very similar argument as in the case of convex F, that at any point of M_t, all the principal curvatures are equal, so we again conclude that M_t is a sphere. □

Now we show exponential convergence of the solution of (1) to the sphere, whether F be convex or concave. As in [M2], we now write M_t as a radial graph, setting

$$\varphi(x, t) = \rho(x, t)\, x$$

for $x \in \mathbb{S}^n$. Incorporating a tangential diffeomorphism to the flow (1) to ensure that this parametrisation is preserved for all time, we find

$$\frac{\partial \rho}{\partial t} = \frac{1}{\rho}\left(\rho^2 + |\nabla \rho|^2\right)^{\frac{1}{2}}(h - F),$$

where ∇ is the gradient on $\mathbb{S}^n$. Setting $\rho = \rho_\infty(1 + \varepsilon \eta)$, the linearised equation about the stationary sphere solution with radius ρ_∞ is

$$\frac{\partial \eta}{\partial t} = \frac{1}{n \rho_\infty^2}\left(\Delta \eta + n\eta - \frac{n}{|\mathbb{S}^n|}\int_{\mathbb{S}^n} \eta\, d\sigma\right),$$

where Δ is the Laplacian on $\mathbb{S}^n$. This equation is the same as that for the mixed volume preserving mean curvature flow in [M2], except for the factor $\frac{1}{n}$ due to Condition 1.1, iv). The same argument as in [M2] and [A3], using Theorem 9.1.2 of [Lu], gives that the M_t's converge exponentially to the sphere with the same value of V_{n-k} as M_0. This completes the proof of Theorem 1.2.

10. A proof of the Minkowski inequalities

The Minkowski inequalities are special cases of the Aleksandrov-Fenchel inequality for mixed volumes: for a convex set $\Phi \subset \mathbb{R}^{n+1}$ and $0 < k < l \leq n+1$,

$$V_l^k\left(\Phi\right) \leq \omega_{n+1}^{k-l} V_k^l\left(\Phi\right), \tag{18}$$

where ω_{n+1} is the volume of the $(n+1)$-dimensional unit ball. We can prove these inequalities using particular flows in the class considered in this paper.

Set $F(\mathcal{W}) = \beta(\frac{E_{n+1-k}}{E_{n+1-l}})^{\alpha}$, where $1 \leq k < l \leq n$ are fixed and $\alpha = \frac{1}{l-k}$ and $\beta = (\binom{n}{k-1}/\binom{n}{l-1})^{\frac{1}{k-l}}$ are chosen so that F satisfies Conditions 1.1 iii) and iv) respectively. It is shown in [A4] that such F satisfy Conditions 1.1. By Lemma 4.3,

$$\frac{d}{dt}\int_{M_t} E_{n-k} d\mu = (n+1-k)\left(h\left(t\right)\int_{M_t} E_{n+1-k} d\mu - \int_{M_t} E_{n+1-l}^{-\alpha} E_{n+1-k}^{1+\alpha} d\mu\right) \tag{19}$$

and

$$\frac{d}{dt}\int_{M_t} E_{n-l} d\mu = (n+1-l)\left(h\left(t\right)\int_{M_t} E_{n+1-l} d\mu - \int_{M_t} E_{n+1-l}^{1-\alpha} E_{n+1-k}^{\alpha} d\mu\right). \tag{20}$$

In view of (20), for a flow which preserves V_l, we take

$$h\left(t\right) = \frac{\int_{M_t} E_{n+1-l}^{1-\alpha} E_{n+1-k}^{\alpha} d\mu}{\int_{M_t} E_{n+1-l} d\mu}.$$

Equation (19) then becomes

$$\frac{\int_{M_t} E_{n+1-l} d\mu}{(n+1-k)} \frac{d}{dt}\int_{M_t} E_{n-k} d\mu = \int_{M_t} E_{n+1-k} d\mu \int_{M_t} E_{n+1-l}^{1-\alpha} E_{n+1-k}^{\alpha} d\mu - \int_{M_t} E_{n+1-l} d\mu \int_{M_t} E_{n+1-l}^{-\alpha} E_{n+1-k}^{1+\alpha} d\mu. \tag{21}$$

This is nonpositive; by the Hölder inequality

$$\int_{M_t} E_{n+1-k} d\mu \leq \left(\int_{M_t} E_{n+1-k}^{\alpha+1} E_{n+1-l}^{-\alpha} d\mu\right)^{\frac{1}{\alpha+1}} \left(\int_{M_t} E_{n+1-l} d\mu\right)^{\frac{\alpha}{\alpha+1}}$$

and

$$\int_{M_t} E_{n+1-l}^{1-\alpha} E_{n+1-k}^{\alpha} d\mu \leq \left(\int_{M_t} E_{n+1-k}^{\alpha+1} E_{n+1-l}^{-\alpha} d\mu\right)^{\frac{\alpha}{\alpha+1}} \left(\int_{M_t} E_{n+1-l} d\mu\right)^{\frac{1}{\alpha+1}}.$$

So our choices of F and h give a flow which preserves V_l while V_k does not increase. We now compute

$$V_k\left(\Phi_0\right) \geq V_k\left(\Phi_\infty\right) = \omega_{n+1}\rho_\infty^k = \omega_{n+1}\left(\frac{V_l\left(\Phi_0\right)}{\omega_{n+1}}\right)^{\frac{k}{l}}$$

using Theorem 1.2 and (3). Hence we have proved inequality (18) for any $1 \leq k < l \leq n$.

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