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Non radial positive solutions for the Hénon equation with critical growth

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Abstract. We study the Dirichlet problem in a ball for the Hénon equation with critical growth and we establish, under some conditions, the existence of a positive, non radial solution. The solution is obtained as a minimizer of the quotient functional associated to the problem restricted to appropriate subspaces of H_0^1 invariant for the action of a subgroup of $\mathbf{O}(N)$. Analysis of compactness properties of minimizing sequences and careful level estimates are the main ingredients of the proof.

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1. Introduction

Let Ω be the unit ball in $\mathbf{R}^N$, with $N \geq 3$. In a 1973 paper, M. Hénon introduced the elliptic equation

$$-\Delta u = |x|^\alpha u^{p-1} \quad \text{in} \quad \Omega, \tag{1}$$

where $\alpha > 0$ and $p > 2$, in the context of spherically symmetric stellar clusters.

This equation is a good model for a series of problems of great mathematical interest, especially in the domain of nonlinear analysis and variational methods. So far the attention has been devoted to the Dirichlet problem for positive solutions, namely

$$\begin{cases} -\Delta u = |x|^\alpha u^{p-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_α}$$

for which existence, nonexistence, multiplicity and qualitative properties of solutions, such as radial symmetry, have been dealt with in various papers under different conditions on the parameters α and p.

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From the point of view of existence results for problem (P_α), the first observation that one can make is the fact that the exponent α affects the range of powers p for which (P_α) may possess solutions. Indeed the presence of the term $|x|^\alpha$ modifies, so to speak, the global homogeneity of the equation and shifts up the threshold between existence and nonexistence given by the application of the Pohozaev identity. The first to realize this was W.–M. Ni, who in 1982 proved the (existence part of the) following result, where, as usual, $2^* = \frac{2N}{N-2}$ is the critical exponent for the Sobolev embedding $H_0^1 \hookrightarrow L^p$.

Theorem 1.1 (Ni, [10]) *Problem (P_α) possesses a solution for all $p \in (2, 2^* + \frac{2\alpha}{N-2})$. There are no solutions if $p \geq 2^* + \frac{2\alpha}{N-2}$.*

At first sight this result is rather surprising since it provides existence also for critical and supercritical cases in a ball, while for $\alpha = 0$ it is well known that that no solutions can be present for $p \geq 2^*$. The solutions found by Ni are radial and arise via the application of the Mountain Pass Theorem in the space of radial functions.

The work [10] widened the range of exponents p for which problem (P_α) can be studied with respect to the standard subcritical growth. In spite of this fact, all the papers we are aware of that followed [10] concentrate on the subcritical cases $p < 2^*$.

A possible reason for this is that the structure of the problem allows the presence of very interesting symmetry breaking results. Indeed since the function $r \mapsto r^\alpha$ is increasing, the classical moving planes arguments of [6] cannot be applied to force radial symmetry of the solutions. And indeed non radial solutions appear in a natural way.

This was first proved in the elegant paper [12] by D. Smets, J. Su and M. Willem, the reading of which is the main motivation of the present work. In order to describe the main result of [12], let $Q_\alpha : H_0^1(\Omega) \setminus \{0\} \to \mathbf{R}$ be the functional

$$Q_\alpha(u) = \frac{\int_\Omega |\nabla u|^2 \, dx}{\left(\int_\Omega |x|^\alpha |u|^p \, dx\right)^{2/p}}.$$

Critical points of Q_α give rise, after scaling, to solutions of problem (P_α). In [12] the authors study, among other things, the ground states of Q_α, namely its minimizers, for subcritical p. They obtain

Theorem 1.2 (Smets, Su, Willem, [12]) *For every $p \in (2, 2^*)$, there exists $\alpha^* > 0$ such that no minimizer of Q_α is radial provided $\alpha > \alpha^*$.*

In particular this result shows that (in the subcritical case) the solutions found by Ni, which can be thought of as minimizers of Q_α on the space of $H_0^1(\Omega)$ radial functions, are not ground states for Q_α over the whole $H_0^1(\Omega)$, at least for α large.

In [12] the authors also provide interesting information about the behavior of α^* as a function of p. They show that the threshold α^* goes to zero when p tends to 2^*, namely that for nearly critical problems (P_α), virtually no ground state is radial, except if α is very very small. The reason why this occurs is clearly pointed out in [13] as follows: for p close to 2^* minimizers tend to "concentrate" about a point; in order to minimize Q_α this point should be as close as possible to $\partial\Omega$, where the effect of the weight $|x|^\alpha$ is minimal.

Further results on partial symmetry and asymptotic behavior of ground states as $p \to 2^*$ or as $\alpha \to \infty$ can be found in [13] and [5].

The results in the papers by Smets et al. cited above can in particular be seen as multiplicity results, in the sense that for α large problem (P_α) has at least two solutions: the radial one found by Ni (which survives when $p \geq 2^*$) and the non radial ground state found in [12], which disappears when $p = 2^*$.

In this paper we study the existence of non radial solutions to (P_α) when $p = 2^*$.

Clearly these solutions cannot arise as ground states of Q_α, so that a different variational procedure has to be applied. We will find them as critical points of Q_α restricted to appropriate subspaces of $H_0^1(\Omega)$ invariant under the action of some subgroup of $\mathbf{O}(N)$. This approach has already been used for example in [8] and [9] for problems on an annulus, and in [16] for a problem in $\mathbf{R}^N$.

Existence of a critical point is obtained through the two main ingredients that one expects in problems with critical growth: analysis of compactness properties for Palais–Smale sequences and careful level estimates to make sure to avoid levels where compactness is lost.

The same estimates will allow us to say that we are working below the minimal level of radial functions, thanks to a bound for such level obtained in [12].

Our main result is the following.

Theorem 1.3 *Let $N \geq 4$ and let Ω be the unit ball in $\mathbf{R}^N$. Then for every $\alpha > 0$ large enough, the problem*

$$\begin{cases} -\Delta u = |x|^\alpha u^{2^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (P_\alpha^*)$$

admits at least one non radial solution.

A further point of interest, specific of critical problems, that arises from the previous Theorem can be observed recalling that for critical problems domains with topology, such as domains with holes play an important role; in particular celebreted papers such as [1] demonstrated how domains with topology often carry solutions that cannot be present otherwise. Now in Theorem 1.3 to say that α is large means that $|x|^\alpha$ is very small in most of Ω. Roughly speaking, from the point of view of existence results, the coefficient $|x|^\alpha$ has an effect similar to the presence of a "hole" in Ω, which is possibly the real reason why, aside from the technical estimates, (P_α^*) admits a solution.

The paper is structured as follows: Sect. 2 contains the analysis of the compactness properties of Palais–Smale sequences for Q_α; Sect. 3 is devoted to the level estimates, in the spirit of [1], and the main results are proved in Sect. 4.

Notation. We denote by $H_0^1(\Omega)$ the usual Sobolev space, normed by $(\int_\Omega |\nabla u|^2)^{1/2}$. The space $\mathcal{D}^{1,2}(\mathbf{R}^N)$ is the closure of $\mathcal{C}_0^\infty(\mathbf{R}^N)$ for the norm $(\int_{\mathbf{R}^N} |\nabla u|^2)^{1/2}$. We denote by $B_r(x_0)$ the open ball $\{x \in \mathbf{R}^N \mid |x - x_0| < r\}$. The symbol $[\gamma]$ stands for the integer part of $\gamma \in \mathbf{R}$.

2. Compactness properties

From now on, for notational convenience, we set $p = 2^*$. This has to be kept in mind throughout the paper.

In this section we analyze the compactness properties enjoyed by minimizing sequences of the functional

$$Q_\alpha(u) = \frac{\int_\Omega |\nabla u|^2 \, dx}{\left(\int_\Omega |x|^\alpha |u|^p \, dx\right)^{2/p}}$$

restricted to appropriated subspaces of $H_0^1(\Omega)$ of symmetric functions.

To describe the symmetry that we are going to use we write $\mathbf{R}^N = \mathbf{R}^2 \times \mathbf{R}^{N-2} \simeq \mathbf{C} \times \mathbf{R}^{N-2}$ and $x = (z, y)$. For a given integer n let G_n be the group $\mathbf{Z}_n \times \mathbf{O}(N-2)$; we consider the action of G_n on $H_0^1(\Omega)$ given by

$$g(u)(x) = g(u)(z, y) = u(e^{j\frac{2\pi i}{n}} z, Ry),$$

where $j \in \{0, \dots, n-1\}$ and $R \in \mathbf{O}(N-2)$.

We denote by H_n the set of points in $H_0^1(\Omega)$ which are fixed by G_n, namely

$$H_n = \{u \in H_0^1(\Omega) \mid u(e^{\frac{2\pi i}{n}} z, Ry) = u(z, y) \ \forall R \in \mathbf{O}(N-2)\,\}. \tag{2}$$

In particular, functions in H_n are radial in y.

The functional Q_α is invariant under the action of G_n (actually both the numerator and the denominator are invariant), so that critical points of Q_α restricted to H_n are critical points of Q_α. These, after scaling, give rise to weak solutions of (P_α^*), which, by standard elliptic theory, are in fact classical solutions. We are going to study the lowest possible critical level of Q_α on H_n; to this aim we set

$$\Sigma_n = \inf_{u \in H_n \setminus \{0\}} Q_\alpha(u) = \inf_{u \in H_n \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{\left(\int_\Omega |x|^\alpha |u|^p \, dx\right)^{2/p}}. \tag{3}$$

Notice that since $|x|^\alpha \leq 1$ in Ω, we have $\Sigma_n \geq S$, the best Sobolev constant for the embedding $H_0^1 \hookrightarrow L^p$, for all n.

The main result in this section is the following.

Proposition 2.1 *Assume $N \geq 4$. If*

$$\Sigma_n < n^{\frac{2}{N}} S \tag{4}$$

then Σ_n is achieved.

The proof of Proposition 2.1 will result from the analysis of Palais–Smale sequences for Q_α and will take the rest of this section. It consists of a combination of arguments rather familiar when one deals with problems with critical growth.

We begin with a standard property.

Lemma 2.2 *Let $u_k \in H_n$ be a minimizing sequence for problem (3) converging weakly to u in $H_0^1(\Omega)$. If $u \not\equiv 0$, then u is a minimum and the convergence holds strongly in $H_0^1(\Omega)$.*

Proof. It uses arguments classical since [4] and we report it for completeness. Notice first that $u \in H_n$ since H_n is weakly closed in $H_0^1(\Omega)$. Next we have as $k \to \infty$, by weak convergence and the Brézis–Lieb Lemma ([3]),

$$\int_\Omega |\nabla u_k - \nabla u|^2\, dx = \int_\Omega |\nabla u_k|^2\, dx - \int_\Omega |\nabla u|^2\, dx + o(1)$$

and

$$\int_\Omega |x|^\alpha |u_k - u|^p\, dx = \int_\Omega |x|^\alpha |u_k|^p\, dx - \int_\Omega |x|^\alpha |u|^p\, dx + o(1).$$

Therefore

$$Q_\alpha(u) = \frac{\int_\Omega |\nabla u_k|^2\, dx - \int_\Omega |\nabla u_k - \nabla u|^2\, dx + o(1)}{\left(\int_\Omega |x|^\alpha |u_k|^p\, dx - \int_\Omega |x|^\alpha |u_k - u|^p\, dx + o(1)\right)^{2/p}}. \tag{5}$$

But the sequence u_k is minimizing, so that $\int_\Omega |\nabla u_k|^2\, dx = \Sigma_n \left(\int_\Omega |x|^\alpha |u_k|^p\, dx\right)^{2/p} + o(1)$, while $\int_\Omega |\nabla u_k - \nabla u|^2\, dx \geq \Sigma_n \left(\int_\Omega |x|^\alpha |u_k - u|^p\, dx\right)^{2/p}$, since $u_k - u \in H_n$. Inserting these relations into (5) we obtain

$$Q_\alpha(u) \leq \Sigma_n \frac{\int_\Omega |\nabla u_k|^2\, dx - \int_\Omega |\nabla u_k - \nabla u|^2\, dx + o(1)}{\left(\left(\int_\Omega |\nabla u_k|^2\, dx\right)^{p/2} - \left(\int_\Omega |\nabla u_k - \nabla u|^2\, dx\right)^{p/2} + o(1)\right)^{2/p}}. \tag{6}$$

Since $\int_\Omega |\nabla u_k - \nabla u|^2\, dx < \int_\Omega |\nabla u_k|^2\, dx$ for k large, it is easily seen that unless $\int_\Omega |\nabla u_k - \nabla u|^2\, dx$ tends to zero, the right–hand–side of (6) will be strictly less than Σ_n for k large, which would contradict the definition of Σ_n. Therefore it must be $u_k \to u$ strongly in $H_0^1(\Omega)$, which gives the desired conclusion. □

In the rest of this section we analyze what happens if a minimizing sequence in H_n tends weakly to zero in $H_0^1(\Omega)$.

Let then $u_k \in H_n$ be a minimizing sequence for problem (3) such that $u_k \rightharpoonup 0$ in $H_0^1(\Omega)$. By Ekeland's variational principle it is not restrictive to assume that the gradient $Q'_\alpha(u_k)$ tends to zero in H_n, and, since Q_α is invariant under the action of G_n, we see that $Q'_\alpha(u_k) \to 0$ in $H_0^1(\Omega)$. By homogeneity we can normalize u_k to obtain a sequence (still denoted u_k) such that as $k \to \infty$,

$$Q_\alpha(u_k) \to \Sigma_n, \quad Q'_\alpha(u_k) \to 0 \quad \text{in } H_0^1(\Omega), \quad u_k \rightharpoonup 0 \quad \text{in } H_0^1(\Omega),$$
$$\int_\Omega |x|^\alpha |u_k|^p\, dx = \Sigma_n^{N/2}.$$

Notice that in this way we also have $\int_\Omega |\nabla u_k|^2\, dx = \Sigma_n^{N/2} + o(1)$.

Remark 2.3 In this paper we are dealing with minimizing sequences for Q_α over H_n; the form of Q_α allows us to consider these sequences as made up of nonnegative functions. We will use this fact tacitly throughout the paper.

At this point it is more convenient for the computations to pass to the direct functional. An explicit computation of $Q'_\alpha(u_k)$, together with the properties just listed shows that u_k is a Palais–Smale sequence for the functional $f : H_0^1(\Omega) \to \mathbf{R}$ defined by

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p}\int_\Omega |x|^\alpha |u|^p\,dx$$

at level $\frac{1}{N}\Sigma_n^{N/2}$.

In what follows, when we need to, we will consider the u_k's (and other functions in $H_0^1(\Omega)$) as functions in $\mathcal{D}^{1,2}(\mathbf{R}^N)$ by extending them to zero outside Ω. In this case, of course, $u_k \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbf{R}^N)$.

As it is easy to imagine, the study of the behavior of u_k uses some version of the Concentration–Compactness Principle. To do this we first define rescalings of functions in $\mathcal{D}^{1,2}(\mathbf{R}^N)$.

Definition 2.4 *For any fixed $\lambda > 0$, $q \in \mathbf{R}^N$, the rescaling $T = T(\lambda, q)$ is the function*

$$T : \mathcal{D}^{1,2}(\mathbf{R}^N) \to \mathcal{D}^{1,2}(\mathbf{R}^N) \quad \textit{defined by} \quad Tv(x) = \lambda^{-\frac{N-2}{2}} v(\frac{x}{\lambda} + q).$$

Notice that if $T = T(\lambda, q)$ then $T^{-1} = T(1/\lambda, -\lambda q)$, that is,

$$T^{-1}v(x) = \lambda^{\frac{N-2}{2}} v(\lambda(x-q)).$$

The following theorem is a version of by now standard Concentration–Compactness type results (the form used here is taken from [14]).

Theorem 2.5 *Assume that v_k is a bounded sequence in $\mathcal{D}^{1,2}(\mathbf{R}^N)$. Then, up to a subsequence, one of the following alternatives holds as $k \to \infty$:*

(i) $v_k \to 0$ *strongly in $L^{2^*}(\mathbf{R}^N)$.*
(ii) *There is a sequence T_k of rescalings such that $T_k v_k \rightharpoonup v$ weakly in $L^{2^*}(\mathbf{R}^N)$ and $v \not\equiv 0$.*

We apply this result to the Palais–Smale sequence u_k, recalling that we have set $p = 2^*$. Notice that alternative (i) cannot occur since for all k

$$\int_{\mathbf{R}^N} |u_k|^p\,dx = \int_\Omega |u_k|^p\,dx \geq \int_\Omega |x|^\alpha |u_k|^p\,dx = \Sigma_n^{N/2} > 0.$$

Therefore there exists a sequence of rescalings $T_k = T(\lambda_k, q_k)$ such that (up to subsequences)

$$T_k u_k \rightharpoonup u \quad \text{weakly in} \quad \mathcal{D}^{1,2}(\mathbf{R}^N) \qquad \text{and} \qquad u \not\equiv 0.$$

Since the support of each u_k is in Ω it is easy to see that $\lambda_k \to \infty$ as $k \to \infty$ and $q_k \in \overline{\Omega}$ for all k. Up to subsequences we can also assume that $q_k \to q \in \overline{\Omega}$.

We first identify the equation solved by u.

Lemma 2.6 *Let $T_k = T(\lambda_k, q_k)$ be the above sequence of rescalings satisfying $\lambda_k \to \infty$ and $q_k \to q \in \overline{\Omega}$. Let $u \not\equiv 0$ be the weak $\mathcal{D}^{1,2}(\mathbf{R}^N)$ limit of $T_k u_k$. Then $\lambda_k \operatorname{dist}(q_k, \partial\Omega) \to \infty$, $q \neq 0$, and u satisfies*

$$-\Delta u = |q|^\alpha u^{p-1} \quad in \quad \mathbf{R}^N.$$

Proof. We first rule out the case $\lambda_k \operatorname{dist}(q_k, \partial\Omega) \le C$ for all k. Indeed, if this happens, after a rotation of coordinates one easily sees as for example in [15], Chapter III, Lemma 3.3, that taken any $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^N_+)$, where $\mathbf{R}^N_+ = \{x \in \mathbf{R}^N \mid x_1 > 0\}$, the support of the function ${T_k}^{-1}\varphi$ is contained in Ω for all large k. Set

$$\phi(x) = \begin{cases} |x|^\alpha & \text{if } x \in \overline{\Omega} \\ 0 & \text{if } x \notin \overline{\Omega} \end{cases}$$

The fact that u_k is a Palais–Smale sequence for f implies that, setting $y = \lambda_k(x - q_k)$ and $\phi_k(y) = \phi(\frac{y}{\lambda_k} + q_k)$,

$$\begin{aligned} o(1) = f'(u_k){T_k}^{-1}\varphi &= \int_\Omega \nabla u_k \nabla({T_k}^{-1}\varphi)\,dx - \int_\Omega |x|^\alpha {u_k}^{p-1}{T_k}^{-1}\varphi\,dx \\ &= \int_{\mathbf{R}^N} \nabla u_k \nabla({T_k}^{-1}\varphi)\,dx - \int_{\mathbf{R}^N} \phi(x){u_k}^{p-1}{T_k}^{-1}\varphi\,dx \\ &= \int_{\mathbf{R}^N} \nabla(T_k u_k)\nabla\varphi\,dy - \int_{\mathbf{R}^N} \phi_k(y)(T_k u_k)^{p-1}\varphi\,dy \\ &= \int_{\mathbf{R}^N} \nabla u \nabla\varphi\,dy - \int_{\mathbf{R}^N} \phi(q)u^{p-1}\varphi\,dy + o(1). \end{aligned}$$

We have used the fact that $T_k u_k \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbf{R}^N)$ and $\phi_k(y) \to \phi(q)$ for all y. Since this happens for all $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^N_+)$, we see that u satisfies $-\Delta u = |q|^\alpha u^{p-1}$ on $\mathbf{R}^N_+$ and $u = 0$ on $\{x_1 = 0\}$; this is impossible because $u \not\equiv 0$. As a remark we notice that in this case $q \in \partial\Omega$, so that $|q|^\alpha = 1$.

Therefore we must have (up to subsequences) $\lambda_k \operatorname{dist}(q_k, \partial\Omega) \to \infty$. In this case we can repeat the above argument, this time being allowed to take as a test function any $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^N)$; the above computations lead us to say that u is a nontrivial solution of $-\Delta u = |q|^\alpha u^{p-1}$ on $\mathbf{R}^N$, which also shows that $q \neq 0$. □

Remark 2.7 The function u is nothing else than a multiple of U, the unique radial positive solution (modulo rescalings) of $-\Delta U = U^{p-1}$ in $\mathbf{R}^N$. Precisely, $u = |q|^{\alpha\frac{2-N}{4}}U$.

We now go on to compare u_k to $T_k^{-1}u$ through the functional f; since $T_k^{-1}u$ is not supported in Ω we cut it off by means of the following procedure. Let $\chi : \mathbf{R} \to [0, 1]$ be a fixed piecewise linear function such that

$$\chi(t) = \begin{cases} 1 & \text{if } |t| \le 1 \\ 0 & \text{if } |t| \ge 2. \end{cases}$$

The fact that $\lambda_k \operatorname{dist}(q_k, \partial\Omega) \to \infty$ allows us to take a sequence $\overline{\lambda}_k \in \mathbf{R}^+$ such that $\overline{\lambda}_k \to \infty$, $\lambda_k/\overline{\lambda}_k \to \infty$ and $\overline{\lambda}_k \operatorname{dist}(q_k, \partial\Omega) \to \infty$ as $k \to \infty$.

We then set $\chi_k(x) = \chi(\frac{\overline{\lambda}_k}{\lambda_k}|x|)$; in this way the support of $T_k^{-1}\chi_k$ is contained in Ω for all k large enough and $u_k - T_k^{-1}(\chi_k u) \in H_0^1(\Omega)$.

For further use we also notice that $\chi_k u \to u$ strongly in $\mathcal{D}^{1,2}(\mathbf{R}^N)$, as one can readily check or look up as above in [15].

In the next proposition we shall use the functional $f_q : \mathcal{D}^{1,2}(\mathbf{R}^N) \to \mathbf{R}$, defined for $q \in \overline{\Omega}$ by

$$f_q(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2\, dx - \frac{1}{p}|q|^\alpha \int_{\mathbf{R}^N} |u|^p\, dx.$$

Lemma 2.8 *Let $T_k u_k$ be the sequence constructed above. Then as $k \to \infty$,*

$$f(u_k - T_k^{-1}(\chi_k u)) = f(u_k) - f_q(u) + o(1). \tag{7}$$

Proof. Set $w_k = u_k - T_k^{-1}(\chi_k u)$; then $w_k \in H_0^1(\Omega)$ and, putting $\lambda_k(x - q_k) = y$, we have

$$\begin{aligned}\int_\Omega |\nabla w_k|^2\, dx = \int_{\mathbf{R}^N} |\nabla w_k|^2\, dx &= \int_{\mathbf{R}^N} |\nabla(T_k u_k) - \nabla(\chi_k u)|^2\, dy \\ &= \int_{\mathbf{R}^N} |\nabla(T_k u_k)|^2\, dy + \int_{\mathbf{R}^N} |\nabla(\chi_k u)|^2\, dy \\ &\quad -2\int_{\mathbf{R}^N} \nabla(T_k u_k)\nabla(\chi_k u)\, dy \\ &= \int_\Omega |\nabla u_k|^2\, dy - \int_{\mathbf{R}^N} |\nabla u|^2\, dy + o(1)\end{aligned}$$

since $\chi_k u \to u$ strongly in $\mathcal{D}^{1,2}(\mathbf{R}^N)$ and $(T_k u_k) \rightharpoonup u$ weakly in the same space. This is how the integral of $|\nabla w_k|^2$ splits.

We turn to the second term in f. In the computations below we denote $\phi_k(y) = \phi(\frac{y}{\lambda_k} + q_k)$.

Changing variables as in the first part we have

$$\int_\Omega |x|^\alpha |w_k|^p\, dx = \int_{\mathbf{R}^N} \phi(x)|w_k|^p\, dx = \int_{\mathbf{R}^N} \phi_k(y)|T_k u_k - \chi_k u|^p\, dy.$$

Next we notice that as $k \to \infty$

$$\int_{\mathbf{R}^N} \phi_k(y)|T_k u_k - \chi_k u|^p\, dy = \int_{\mathbf{R}^N} \phi_k(y)|T_k u_k - u|^p\, dy + o(1)$$

since $\chi_k u \to u$ strongly in $\mathcal{D}^{1,2}(\mathbf{R}^N)$. Therefore, by the Brézis–Lieb lemma

$$\begin{aligned}\int_{\mathbf{R}^N} \phi_k(y)|T_k u_k - \chi_k u|^p\, dy &= \int_{\mathbf{R}^N} \phi_k(y)|T_k u_k - u|^p\, dy + o(1) \\ &= \int_{\mathbf{R}^N} \phi_k(y)|T_k u_k|^p\, dy - \int_{\mathbf{R}^N} \phi_k(y)|u|^p\, dy + o(1).\end{aligned} \tag{8}$$

Finally we observe that changing variables

$$\int_{\mathbf{R}^N} \phi_k(y)|T_k u_k|^p\, dy = \int_{\mathbf{R}^N} \phi(x)|u_k|^p\, dx = \int_\Omega |x|^\alpha |u_k|^p\, dx,$$

while

$$\int_{\mathbf{R}^N} \phi_k(y)|u|^p\,dy = \int_{\mathbf{R}^N} \phi(q)|u|^p\,dy + o(1) = |q|^\alpha \int_{\mathbf{R}^N} |u|^p\,dy + o(1)$$

since $\phi_k(y) \to \phi(q)$.

Inserting these into (8) we find that

$$\int_{\mathbf{R}^N} \phi_k(y)|T_k u_k - \chi_k u|^p\,dy = \int_\Omega |x|^\alpha |u_k|^p\,dx - |q|^\alpha \int_{\mathbf{R}^N} |u|^p\,dy + o(1)$$

which, combined with the splitting of the integral of $|\nabla w_k|^2$ yields

$$f(u_k - T_k^{-1}(\chi_k u)) = f(u_k) - f_q(u) + o(1),$$

as we wanted to prove. □

To complete the preliminary properties we need we state the analogue of the previous lemma concerning the gradients.

Lemma 2.9 *Let $T_k u_k$ be the sequence constructed above. Then as $k \to \infty$,*

$$f'(u_k - T_k^{-1}(\chi_k u)) = f'(u_k) + o(1) \qquad in \qquad H_0^1(\Omega). \tag{9}$$

Proof. For the sake of simplicity we work assuming that $u_k - T_k^{-1}(\chi_k u)$ is nonnegative; otherwise one replaces its $(p-1)$–th power by $|u_k - T_k^{-1}(\chi_k u)|^{p-2}(u_k - T_k^{-1}(\chi_k u))$.

If $\varphi \in \mathcal{C}_0^\infty(\Omega)$, then

$$f'(u_k - T_k^{-1}(\chi_k u))\varphi = \int_{\mathbf{R}^N} \nabla(u_k - T_k^{-1}(\chi_k u))\nabla\varphi\,dx - \int_{\mathbf{R}^N} \phi(x)(u_k - T_k^{-1}(\chi_k u))^{p-1}\varphi\,dx.$$

Plainly we have (recalling that $\chi_k u \to u$ strongly in $\mathcal{D}^{1,2}(\mathbf{R}^N)$),

$$\int_{\mathbf{R}^N} \nabla(u_k - T_k^{-1}(\chi_k u))\nabla\varphi\,dx = \int_{\mathbf{R}^N} \nabla u_k \nabla\varphi\,dx - \int_{\mathbf{R}^N} \nabla u \nabla(T_k\varphi)\,dy + o(1)||\varphi||, \tag{10}$$

$||\cdot||$ being the $\mathcal{D}^{1,2}(\mathbf{R}^N)$ norm.

For the second part we first notice that, still because $\chi_k u \to u$ strongly in $\mathcal{D}^{1,2}(\mathbf{R}^N)$,

$$\int_{\mathbf{R}^N} \phi(x)(u_k - T_k^{-1}(\chi_k u))^{p-1}\varphi\,dx = \int_{\mathbf{R}^N} \phi_k(y)(T_k u_k - u)^{p-1} T_k\varphi\,dy + o(1)||\varphi||,$$

where we have set, as above, $y = \lambda_k(x - q_k)$ and $\phi_k(y) = \phi(y/\lambda_k + q_k)$. Now a standard application of the Brézis–Lieb lemma shows that

$$\int_{\mathbf{R}^N} \phi_k(y)(T_k u_k - u)^{p-1} T_k\varphi\, dy = \int_{\mathbf{R}^N} \phi_k(y)(T_k u_k)^{p-1} T_k\varphi\, dy - \int_{\mathbf{R}^N} \phi_k u^{p-1} T_k\varphi\, dy + o(1)||\varphi||,$$

so that returning to the variable x in the first integral and using $\phi_k \to \phi(q) = |q|^\alpha$, we obtain

$$\phi(x)(u_k - T_k^{-1}(\chi_k u))^{p-1}\varphi\, dx = \int_{\mathbf{R}^N} \phi(x) {u_k}^{p-1}\varphi\, dx - |q|^\alpha \int_{\mathbf{R}^N} u^{p-1} T_k\varphi\, dy + o(1)||\varphi||. \quad (11)$$

Combining (10) and (11) we can write

$$f'(u_k - T_k^{-1}(\chi_k u))\varphi = f'(u_k)\varphi + f'_q(u)T_k\varphi + o(1)||\varphi|| = f'(u_k)\varphi + o(1)||\varphi||$$

since u is a critical point of f_q by Lemma 2.6. This holds for every $\varphi \in \mathcal{C}_0^\infty(\Omega)$ and therefore it is equivalent to the statement we wanted to prove. □

Recalling that u_k is a Palais–Smale sequence for f at level $\frac{1}{N}\Sigma_n^{N/2}$, the two above lemmas constitute the proof of the following proposition.

Proposition 2.10 *The sequence $u_k - T_k^{-1}(\chi_k u)$ satisfies*

i) $f(u_k - T_k^{-1}(\chi_k u)) = \frac{1}{N}\Sigma_n^{N/2} - f_q(u) + o(1)$,
ii) $f'(u_k - T_k^{-1}(\chi_k u)) = o(1)$ *in* $H_0^1(\Omega)$*, namely, it is a Palais–Smale sequence for f at level $\frac{1}{N}\Sigma_n^{N/2} - f_q(u)$.*

We are now ready to describe the behavior of Palais–Smale sequences for f.

Proposition 2.11 *Let u_k be a Palais–Smale sequence for f at level $\frac{1}{N}\Sigma_n^{N/2}$ converging weakly to zero in $H_0^1(\Omega)$.*

Then there is a positive integer m (depending only on Σ_n) such that for every $j = 1, \dots, m$ there exist sequences $\lambda_{jk} \in \mathbf{R}^+$ and $q_{jk} \in \overline{\Omega}$, with $\lambda_{jk} \to \infty$ and $q_{jk} \to q_j \in \overline{\Omega} \setminus \{0\}$ as $k \to \infty$, there exists a nontrivial critical point $u_j \in \mathcal{D}^{1,2}(\mathbf{R}^N)$ of f_{q_j} such that, setting $T_{jk} = T(\lambda_{jk}, q_{jk})$, there results (up to subsequences)

$$u_k = \sum_{j=1}^m T_{jk}^{-1}(u_j) + o(1) \qquad \textit{in} \qquad \mathcal{D}^{1,2}(\mathbf{R}^N), \quad (12)$$

$$f(u_k) = \sum_{j=1}^m f_{q_j}(u_j) + o(1). \quad (13)$$

Proof. If u_k is a Palais–Smale sequence as in the assumptions, we can apply to it the arguments that start with Theorem 2.5 and are concluded in Proposition 2.10. Explicitly, there exist a sequence of positive numbers $\lambda_{1k} \to \infty$, a sequence q_{1k} of points of $\overline{\Omega}$ with $q_{1k} \to q_1 \in \overline{\Omega} \setminus \{0\}$ and a nontrivial critical point u_1 of f_{q_1} such that, setting $T_{1k} = T(\lambda_{1k}, q_{1k})$, the sequence

$$w_{1k} := u_k - T_{1k}^{-1}(\chi_{1k} u_1)$$

is a Palais–Smale sequence for f at level $\frac{1}{N}\Sigma_n^{N/2} - f_{q_1}(u_1)$. Here χ_{1k} is defined according to the procedure that follows Remark 2.7.

We now iterate this scheme, still starting with the application of Theorem 2.5. If $w_{1k} \to 0$ strongly in $L^p(\mathbf{R}^N)$, then the fact that it is a Palais–Smale sequence implies that $w_{1k} \to 0$ strongly in $\mathcal{D}^{1,2}(\mathbf{R}^N)$. Since also $\chi_{1k} u_1 \to u_1$ strongly in $\mathcal{D}^{1,2}(\mathbf{R}^N)$ we can write

$$u_k = T_{1k}^{-1}(u_1) + o(1) \qquad \text{in} \qquad \mathcal{D}^{1,2}(\mathbf{R}^N),$$

and the proposition is proved with $m = 1$.

Otherwise $w_{1k} \to 0$ weakly in $L^p(\mathbf{R}^N)$ but not strongly. In this case, starting with Theorem 2.5 we can work on w_{1k} as we did for u_k: we can find sequences $\lambda_{2k} \to \infty$, $q_{2k} \to q_2 \in \overline{\Omega} \setminus \{0\}$, a nontrivial critical point u_2 of f_{q_2} such that the sequence

$$w_{2k} := w_{1k} - T_{2k}^{-1}(\chi_{2k} u_2)$$

is a Palais–Smale sequence for f at level $\frac{1}{N}\Sigma_n^{N/2} - f_{q_1}(u_1) - f_{q_2}(u_2)$. Here χ_{2k} and T_{2k} are defined similarly as we did above. Once again, if $w_{2k} \to 0$ strongly in $L^p(\mathbf{R}^N)$, then we see that

$$u_k = T_{1k}^{-1}(u_1) + w_{1k} = T_{1k}^{-1}(u_1) + T_{2k}^{-1}(u_2) + o(1) \qquad \text{in} \qquad \mathcal{D}^{1,2}(\mathbf{R}^N),$$

and the proposition is proved with $m = 2$. If, on the contrary, $w_{2k} \to 0$ weakly in $L^p(\mathbf{R}^N)$ but not strongly, we iterate the above argument; to check that the procedure ends after a finite number of steps, notice that by Remark 2.7, for all j,

$$f_{q_j}(u_j) = f_{q_j}(|q_j|^{\alpha \frac{2-N}{4}} U) = |q_j|^{\alpha \frac{2-N}{2}} \frac{1}{N} S^{N/2} \geq \frac{1}{N} S^{N/2},$$

by definition of U, so that after at most $m := [\Sigma_n / S]^{N/2}$ steps the remainder will be a Palais–Smale sequence at level zero, namely it will be $o(1)$ in $\mathcal{D}^{1,2}(\mathbf{R}^N)$, obtaining the requested representation for u_k and $f(u_k)$. □

Remark 2.12 In the statement of the previous proposition we have obtained a representation of u_k as a function in $\mathcal{D}^{1,2}(\mathbf{R}^N)$. This is the simplest way to express u_k. A representation in $H_0^1(\Omega)$ can be deduced even more directly from the proof of Proposition 2.11 if one does not suppress the cut–off functions χ_{jk}. Since we are going to use this version of the representation of u_k, we write it explicitly as

$$\begin{aligned} u_k &= \sum_{j=1}^{m} T_{jk}^{-1}(\chi_{jk} u_j) + o(1) \\ &= \sum_{j=1}^{m} \lambda_{jk}^{\frac{N-2}{2}} (\chi_{jk} u_j)(\lambda_{jk}(\cdot - q_{jk})) + o(1) \quad \text{in} \quad H_0^1(\Omega). \end{aligned} \tag{14}$$

Remark 2.13 In the introduction we have mentioned the fact that the weight $|x|^\alpha$ has an effect similar to that of a "hole" in Ω. This effect is manifest in the structure of the Palais–Smale sequences for f that, as Proposition 2.11 shows, cannot concentrate in zero.

The analysis of the Palais–Smale sequences for f allows us to obtain the proof of the main result of this section, Proposition 2.1. The conclusion is achieved by taking into account the symmetry properties of u_k; we borrow the argument from [16], where it is applied to a different problem.

Proof of Proposition 2.1. Let $u_k \in H_n$ be a (bounded) minimizing sequence for Q_α over H_n. Then u_k contains a subsequence (still denoted u_k) such that $u_k \rightharpoonup u$ in $H_0^1(\Omega)$.

If $u \not\equiv 0$, then by Lemma 2.2, u is a minimum point in H_n, and there is nothing left to prove.

If, on the contrary, $u_k \rightharpoonup 0$ in $H_0^1(\Omega)$, then, as we did above, we can assume without loss of generality that u_k is a Palais–Smale sequence for the functional f at level $\frac{1}{N}\Sigma_n^{N/2}$, normalized in such a way that $\int_\Omega |x|^\alpha |u_k|^p = \Sigma_n^{N/2}$. The behavior of such sequences is described in Proposition 2.11. In particular,

$$f(u_k)=\sum_{j=1}^m f_{q_j}(u_j)+o(1)=\sum_{j=1}^m |q_j|^{\alpha\frac{2-N}{2}}\frac{1}{N}S^{N/2}+o(1)\geq m\frac{1}{N}S^{N/2}+o(1). \tag{15}$$

Recall now that each u_j is a multiple of the radial function U and that the cut–off functions χ_{jk} are also radial. Furthermore, $\mathbf{O}(N-2)$ is a continuous group for $N \geq 4$. These two remarks imply that for every $j = 1, \ldots, m$, we must have $\lambda_{jk}\mathrm{dist}(q_{jk}, \mathbf{R}^2 \times \{0\}) \to 0$ when $k \to \infty$. Indeed, if this where not the case, the representation (14) would be incompatible with the symmetry properties of u_k. Therefore in (14) we can replace each q_{jk} with its projection on $\mathbf{R}^2 \times \{0\}$ so that we can assume that (14) holds in the space H_n.

This means that m must be a multiple of n, say $m = Kn$, for some integer $K \geq 1$. But in this case, from (15),

$$\frac{1}{N}\Sigma_n^{N/2} = f(u_k)+o(1) \geq Kn\frac{1}{N}S^{N/2}+o(1) \geq n\frac{1}{N}S^{N/2}+o(1),$$

namely $\Sigma_n \geq n^{\frac{2}{N}}S$, contradicting the assumption. □

3. Asymptotic expansions

This is a technical section in which we establish the main estimates we will need to prove that the assumption of Proposition 2.1, namely $\Sigma_n < n^{\frac{2}{N}}S$, is satisfied for suitable values of n and α.

The strategy we adopt is rather simple and well known: we will explicitly construct for each n a function $u \in H_n$ such that, for n large, $Q_\alpha(u) < n^{\frac{2}{N}}S$; this inequality will be proved in the next section. Presently we confine ourselves to the construction of u and we estimate carefully the numerator and the denominator of $Q_\alpha(u)$.

The idea we follow for the construction of u comes from the fact (already used in [1] and [2], for example) that the superposition of solutions to problems at infinity may “lower” the level of the functional. We will construct u as a $\mathbf{Z}_n \times \mathbf{O}(N-2)$–symmetric superposition of suitably rescaled solutions of $-\Delta U = U^{p-1}$ in $\mathbf{R}^N$; these functions will have to be projected onto $H_0^1(\Omega)$, and, in order to minimize the effect of the weight $|x|^\alpha$, everything will have to take place as close as possible to $\partial\Omega$.

The computations that follow are heavily based on the estimates obtained by Bahri and Coron in the celebrated paper [1], and especially in Proposition B5 of that paper. Some estimates we will need are exactly the same as corresponding ones in [1] and hence in these cases we will refer directly to computations carried out in that paper.

The main difference between our framework and the one in [1] is that while Bahri and Coron worked in a compact subdomain of Ω, we will be forced to work, for the reason described above, closer and closer to the boundary of Ω. Therefore some terms that were considered as constants in [1] will play a central role in our setting and will have to be treated with special care.

We begin with the construction of u: we pick $l \in (0,1)$ and for every $n \in \mathbf{N}$ we define n points x_j in $\mathbf{R}^N \simeq \mathbf{C} \times \mathbf{R}^{N-2}$ as

$$x_j = ((1-l)e^{j\frac{2\pi i}{n}}, 0), \qquad j = 0, \dots, n-1.$$

Notice that the points x_j are all in Ω. With the aid of these points we define for $\lambda > 0$ the functions

$$U_{\lambda,x_i}(x) = T(\lambda, x_i)^{-1}U(x) = C_N \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda^2|x-x_i|^2)^{\frac{N-2}{2}}}, \tag{16}$$

where $C_N = (N(N-2))^{(N-2)/4}$.

We recall that the functions U_{λ,x_i} are the unique positive solutions, radial about x_i, of the equation $-\Delta U = U^{p-1}$ in $\mathcal{D}^{1,2}(\mathbf{R}^N)$ and that $\int_{\mathbf{R}^N} |\nabla U_{\lambda,x_i}|^2 = \int_{\mathbf{R}^N} U^p_{\lambda,x_i} = S^{N/2}$ for all λ and all x_i, where S is the best constant in the embedding $H_0^1 \hookrightarrow L^p$.

To avoid heavy notation from now on we will write simply U_i for U_{λ,x_i}, and to fix ideas we anticipate that we will let $l \to 0$, $n \to \infty$ and $\lambda \to \infty$, with appropriate relations between l, n and λ.

The functions U_i are not in $H_0^1(\Omega)$, so that we will use instead their projections $\overline{U}_i$ on $H_0^1(\Omega)$ defined by

$$\begin{cases} \Delta\overline{U}_i = \Delta U_i & \text{in } \Omega \\ \overline{U}_i = 0 & \text{on } \partial\Omega. \end{cases}$$

If we set $\varphi_i = U_i - \overline{U}_i$, then $\Delta\varphi_i = 0$ in Ω, $\varphi_i = U_i$ on $\partial\Omega$ and $\varphi_i > 0$ in $\overline{\Omega}$ by the maximum principle.

Finally we define

$$u(x) = \sum_{i=0}^{n-1} \overline{U}_i(x) = \sum_{i=0}^{n-1}(U_i(x) - \varphi_i(x)), \tag{17}$$

and we observe that, due to the definition of the points x_i, we have $u \in H_n$. Notice that of course u depends on λ and on n and l through the choice of the points x_i.

We now turn to the estimates, starting from $\int_\Omega |\nabla u|^2$. To this aim notice that by definition of u and $\overline{U}_i$,

$$\int_\Omega |\nabla u|^2 = \sum_{i,j=0}^{n-1} \int_\Omega \nabla\overline{U}_i \nabla\overline{U}_j = -\sum_{i,j=0}^{n-1} \int_\Omega \Delta U_i \overline{U}_j = \sum_{i,j=0}^{n-1} \int_\Omega U_i^{p-1}(U_j - \varphi_j)$$

$$= \sum_{i=0}^{n-1} \left(\int_\Omega U_i^p - \int_\Omega U_i^{p-1}\varphi_i \right) + \sum_{\substack{i,j=0 \\ i \neq j}}^{n-1} \left(\int_\Omega U_i^{p-1} U_j - \int_\Omega U_i^{p-1}\varphi_j \right). \tag{18}$$

We will treat the four integrals separately. In what follows, the letter C will denote positive constants depending only on N. We will denote by $O(a)$ quantities such that $|O(a)| \leq C|a|$.

We also set

$$d_{ij} = x_j - x_i, \qquad \text{and} \qquad d = \frac{1}{2} \min_{i \neq j} |d_{ij}| = \frac{1}{2}|x_1 - x_0|,$$

and we assume that $2d \leq l$ and that $\lambda d \geq 1$ for all λ under consideration.

Remark also that due to our definition of the points x_i, we have

$$d = (1 - l) \sin \frac{\pi}{n} \sim \frac{C}{n}$$

for all l small.

Finally, we denote by $H : \Omega \times \Omega \to \mathbf{R}$ the function

$$H(x, y) = \frac{1}{\big||x|y - x/|x|\big|^{N-2}},$$

that is, the regular part of the Green function of the laplacian on the unit ball Ω, and by

$$G(x, y) = \frac{1}{|x - y|^{N-2}} - H(x, y)$$

the Green function itself. Recall that $H(x, \cdot)$ is harmonic in Ω for all x and equals $1/|x - \cdot|^{N-2}$ on $\partial\Omega$.

We want to estimate $Q_\alpha(u)$ in terms of G and H, in the spirit of the papers [1] and [11]. The behavior of the numerator is given in the following proposition.

Proposition 3.1 *As $\lambda d \to \infty$ (that is, $n/\lambda \to 0$) we have*

$$\int_\Omega |\nabla u|^2\,dx = nS^{N/2} - \frac{C_N c_0}{\lambda^{N-2}}\left\{\sum_{i=0}^{n-1} H(x_i,x_i) - \sum_{\substack{i,j=0\\ i\neq j}}^{n-1} G(x_i,x_j)\right\}$$
$$+n^N O\left(\frac{1}{\lambda^{N-1}}\right) + n^2 O\left(\frac{1}{(\lambda l)^N}\right),$$

where C_N is the constant introduced in (16) and $c_0 = \int_{\mathbf{R}^N} U^{p-1}\,dx$.

The proof of Proposition 3.1 consists in the evaluation of $\int_\Omega |\nabla u|^2\,dx$ in terms of the quantities in (18), which we treat in separate lemmas.

Lemma 3.2 $\displaystyle\int_\Omega U_i{}^p\,dx = S^{N/2} + O\left(\frac{1}{(\lambda l)^N}\right).$

Proof. Writing

$$S^{N/2} \geq \int_\Omega U_i{}^p \geq \int_{B_l(x_i)} U_i{}^p = \int_{\mathbf{R}^N} U_i{}^p - \int_{\mathbf{R}^N\setminus B_l(x_i)} U_i{}^p$$
$$= S^{N/2} - \int_{\mathbf{R}^N\setminus B_l(x_i)} U_i{}^p,$$

scaling $y = \lambda(x - x_i)$ and passing to spherical coordinates we have

$$\int_{\mathbf{R}^N\setminus B_l(x_i)} U_i{}^p \leq C\int_{\lambda l}^\infty \frac{r^{N-1}}{(1+r^2)^N}\,dr \leq \frac{C}{(\lambda l)^N},$$

which yields the desired expression. □

We turn to the second term in (18).

Lemma 3.3 $\displaystyle\int_\Omega U_i{}^{p-1}\varphi_i\,dx = \frac{C_N c_0}{\lambda^{N-2}} H(x_i,x_i) + O\left(\frac{1}{(\lambda l)^N}\right).$

Proof. If we set

$$\Gamma(x) = \varphi_i(x) - C_N\lambda^{\frac{2-N}{2}} H(x_i,x),$$

it is not difficult to check using harmonicity of H and φ_i (see also [1]) that for all $x \in \Omega$,

$$\Gamma(x) = O\left(\frac{1}{\lambda^{\frac{N+2}{2}} l^N}\right).$$

Therefore if we split the term to be estimated as

$$\int_\Omega U_i{}^{p-1}\varphi_i = \frac{C_N}{\lambda^{\frac{N-2}{2}}}\left[\left(\int_{B_{l/2}(x_i)} + \int_{\Omega\setminus B_{l/2}(x_i)}\right) U_i{}^{p-1}(x) H(x_i,x)\,dx\right]$$
$$+\int_\Omega U_i{}^{p-1}(x)\Gamma(x), \tag{19}$$

then for the last integral we have

$$\Big|\int_{\Omega} U_i^{p-1}(x)\Gamma(x)\,dx\Big| \le O\Big(\frac{1}{\lambda^{\frac{N+2}{2}} l^N}\Big)\int_{\mathbf{R}^N} U_i^{p-1}$$
$$\le O\Big(\frac{1}{\lambda^{\frac{N+2}{2}} l^N}\Big)\frac{C}{\lambda^{\frac{N-2}{2}}} = O\Big(\frac{1}{(\lambda l)^N}\Big), \qquad (20)$$

as one readily checks by scaling $y = \lambda(x - x_i)$.

Concerning the integral over $\Omega \setminus B_{l/2}(x_i)$ we first notice that by harmonicity,

$$H(x_i, x) \le \max_{\overline{\Omega}} H(x_i, \cdot) = \max_{\partial\Omega} H(x_i, \cdot) \le \frac{1}{l^{N-2}}.$$

Therefore

$$\frac{C_N}{\lambda^{\frac{N-2}{2}}}\int_{\Omega\setminus B_{l/2}(x_i)} U_i^{p-1}(x)H(x_i, x) \le \frac{C_N}{\lambda^{\frac{N-2}{2}} l^{N-2}}\int_{\Omega\setminus B_{l/2}(x_i)} U_i^{p-1}$$
$$= O\Big(\frac{1}{(\lambda l)^N}\Big), \qquad (21)$$

as one again checks via the usual scaling.

The first integral in the right–hand–side of (19) is the one that gives the relevant contribution. To evaluate it we first expand $H(x_i, \cdot)$ up to the third order near x_i as in [1], writing

$$H(x_i, x) = H(x_i, x_i) + H_1 + H_2 + H_3 + R,$$

where H_j denotes the j–th order term (e.g. $H_1 = \nabla H(x_i, x_i)(x - x_i)$). We notice that the integrals containing H_j are all zero ($j = 1, 3$ by symmetry and $j = 2$ by harmonicity). Up to here we have followed exactly the computations in [1]; however we need a slightly sharper estimate of the remainder R with respect to the one in [1] on account of the fact that we will have to let x_i tend to $\partial\Omega$, namely $l \to 0$.

Now since $|R| \le \sup_{B_{l/2}(x_i)} ||\nabla^4 H(x_i, \cdot)|||x - x_i|^4$, using the explicit form of H it is not difficult to check that

$$\sup_{B_{l/2}(x_i)} ||\nabla^4 H(x_i, \cdot)|| \le \frac{C}{l^{N+2}}$$

so that

$$\Big|\int_{B_{l/2}(x_i)} U_i^{p-1} R\,dx\Big| \le \frac{C}{l^{N+2}}\int_{B_{l/2}(x_i)} U_i^{p-1}|x - x_i|^4\,dx.$$

Computing via scaling and spherical coordinates

$$\int_{B_{l/2}(x_i)} U_i^{p-1}|x - x_i|^4\,dx = \frac{1}{\lambda^{\frac{N+6}{2}}}\int_{B_{l/2}(0)} U_i^{p-1}(y)|y|^4\,dy$$
$$= \frac{C}{\lambda^{\frac{N+6}{2}}}\int_0^{\lambda l/2} \frac{r^{N+3}}{(1 + r^2)^{\frac{N+2}{2}}}\,dr$$
$$\le \frac{C}{\lambda^{\frac{N+6}{2}}}(\lambda l)^2 = C\frac{l^2}{\lambda^{\frac{N+2}{2}}},$$

we can finally say that

$$\Big|\int_{B_{l/2}(x_i)} U_i^{p-1}(x)R\Big| \leq \frac{C}{\lambda^{\frac{N+2}{2}} l^N}.$$

To complete the proof we just have to notice that by the usual scaling arguments,

$$\begin{aligned}\frac{C_N}{\lambda^{\frac{N-2}{2}}}\int_{B_{l/2}(x_i)} U_i^{p-1}(x)H(x_i,x) &= \frac{C_N}{\lambda^{\frac{N-2}{2}}}\left(H(x_i,x_i)\int_{B_{l/2}(x_i)} U_i^{p-1}\right.\\ &\quad \left.+\int_{B_{l/2}(x_i)} U_i^{p-1}(x)R\right)\\ &= \frac{C_N c_0}{\lambda^{N-2}}H(x_i,x_i) + O\left(\frac{1}{(\lambda l)^N}\right). \qquad (22)\end{aligned}$$

Adding (20), (21) and (22) we obtain the desired expression. □

This concludes the analysis of the first two terms in (18). We now pass to the terms which mix i and j.

Lemma 3.4 $\displaystyle\int_\Omega U_i^{p-1}U_j\,dx = \frac{C_N c_0}{(\lambda|d_{ij}|)^{N-2}} + O\left(\frac{1}{(\lambda|d_{ij}|)^{N-1}}\right) + O\left(\frac{1}{(\lambda l)^N}\right).$

Proof. Writing

$$\int_\Omega U_i^{p-1}U_j\,dx = \int_{\mathbf{R}^N} U_i^{p-1}U_j\,dx - \int_{\mathbf{R}^N\setminus\Omega} U_i^{p-1}U_j\,dx, \qquad (23)$$

with the same type of calculation as in Lemma 3.2 we immediately see that

$$\int_{\mathbf{R}^N\setminus\Omega} U_i^{p-1}U_j\,dx \leq \int_{\mathbf{R}^N\setminus\Omega}\left(U_i^p + U_j^p\right)dx \leq \frac{C}{(\lambda l)^N}.$$

The integral over $\mathbf{R}^N$ in (23) has been estimated in [1], page 279–280, formula (B31); the only difference is that the indices i and j in [1] are permuted. Since the computations are rather involved and no changes are necessary, except for the normalization constant in the definition of U, we don't repeat them here and we just give the result in our notation:

$$\int_{\mathbf{R}^N} U_i^{p-1}U_j\,dx = \frac{C_N c_0}{(\lambda|d_{ij}|)^{N-2}} + O\left(\frac{1}{(\lambda|d_{ij}|)^{N-1}}\right). \qquad (24)$$

This, together with the previous estimate, proves the lemma. □

We conclude this set of estimates with the last term in (18).

Lemma 3.5 $\displaystyle\int_\Omega U_i^{p-1}\varphi_j\,dx = \frac{C_N c_0}{\lambda^{N-2}}H(x_i,x_j) + O\left(\frac{1}{(\lambda l)^N}\right).$

Proof. The computations are similar to the ones in the proof of Lemma 3.3, and we don't repeat them; the same approach can be found in [1]. □

We are now ready obtain Proposition 3.1.

Proof of Proposition 3.1. We collect the estimates of the terms in (18) provided by the preceding lemmas.

By Lemmas 3.2 and 3.3 we have

$$\sum_{i=0}^{n-1}\left(\int_\Omega {U_i}^p\,dx-\int_\Omega {U_i}^{p-1}\varphi_i\,dx\right)=nS^{N/2}-\frac{C_N c_0}{\lambda^{N-2}}\sum_{i=0}^{n-1}H(x_i,x_i)+nO\left(\frac{1}{(\lambda l)^N}\right), \tag{25}$$

while by Lemma 3.5 there results

$$\sum_{\substack{i,j=0\\ i\neq j}}^{n-1}\int_\Omega {U_i}^{p-1}\varphi_j\,dx=\frac{C_N c_0}{\lambda^{N-2}}\sum_{\substack{i,j=0\\ i\neq j}}^{n-1}H(x_i,x_j)+n^2O\left(\frac{1}{(\lambda l)^N}\right). \tag{26}$$

The remaining term requires some care (estimating $|d_{ij}|\geq d\sim C/n$ yields an error of the order $n^{N+1}O(\lambda^{1-N})$, which is not small enough for our purposes). Therefore we proceed as follows, using the symmetries of the points x_j:

$$\begin{aligned}\sum_{\substack{i,j=0\\ i\neq j}}^{n-1}\frac{1}{|d_{ij}|^{N-1}}&=n\sum_{j=1}^{n-1}\frac{1}{|x_j-x_0|^{N-1}}=n\sum_{j=1}^{n-1}\frac{1}{(2-2l)^{N-1}\sin^{N-1}(\pi j/n)}\\ &\sim\frac{2n}{(2-2l)^{N-1}}\sum_{j=1}^{[(n-1)/2]}\frac{C}{(j/n)^{N-1}}\sim Cn^N,\end{aligned} \tag{27}$$

since the series of j^{1-N} is convergent. Recall also that l will be taken small, so that we can always assume $l\leq 1/2$.

With this last estimate we obtain from Lemma 3.4

$$\sum_{\substack{i,j=0\\ i\neq j}}^{n-1}\int_\Omega {U_i}^{p-1}U_j\,dx=\frac{C_N c_0}{\lambda^{N-2}}\sum_{\substack{i,j=0\\ i\neq j}}^{n-1}\frac{1}{|x_i-x_j|^{N-2}}+n^N O\left(\frac{1}{\lambda^{N-1}}\right)+n^2O\left(\frac{1}{(\lambda l)^N}\right). \tag{28}$$

Adding (25), (26) and (28) as required and recalling the definition of G we obtain the estimate of Proposition 3.1. □

Having completed the estimate of the numerator of Q_α we now go on to estimate the denominator, namely $\int_\Omega |x|^\alpha u^p$. Also in this case we will split the computations in a series of lemmas.

Recall that we denote $d=\frac{1}{2}\min_{i\neq j}|x_i-x_j|$ and that we are assuming $2d\leq l$. We now set $B_i=B_d(x_i)$ for $i=0,\dots,n-1$. Then the B_i's are pairwise disjoint and they are all contained in the annulus $\Omega\setminus\overline{B}_{1-2l}(0)$.

Hence,

$$\int_\Omega |x|^\alpha u^p\,dx\geq(1-2l)^\alpha\int_{\Omega\setminus\overline{B}_{1-2l}(0)}u^p\,dx\geq(1-2l)^\alpha\sum_{i=0}^{n-1}\int_{B_i}u^p\,dx.$$

We first use the elementary convexity inequality $(a+b)^p \geq a^p + pa^{p-1}b$ which holds for $p \geq 1$, $a \geq 0$, $a+b \geq 0$ to write (for $x \in B_i$),

$$
\begin{aligned}
u^p &= \left(\sum_{i=0}^{n-1} \overline{U}_i\right)^p = \left(U_i + \sum_{\substack{j=0\\ j\neq i}}^{n-1} (U_j - \varphi_j) - \varphi_i\right)^p \\
&\geq U_i{}^p + pU_i{}^{p-1}\left(\sum_{\substack{j=0\\ j\neq i}}^{n-1} (U_j - \varphi_j) - \varphi_i\right).
\end{aligned}
$$

We obtain therefore

$$
\begin{aligned}
\sum_{i=0}^{n-1} \int_{B_i} u^p \, dx \geq \sum_{i=0}^{n-1} &\left(\int_{B_i} U_i{}^p - p\int_{B_i} U_i{}^{p-1}\varphi_i\right) \\
&+p \sum_{\substack{i,j=0\\ i\neq j}}^{n-1} \left(\int_{B_i} U_i{}^{p-1}U_j - \int_{B_i} U_i{}^{p-1}\varphi_j\right) \qquad (29)
\end{aligned}
$$

and, as above, we estimate the four integrals separately. The result we are aiming at is stated in the following proposition, which parallels Proposition 3.1.

Proposition 3.6 *As $\lambda d \to \infty$ (that is, $n/\lambda \to 0$) we have*

$$
\begin{aligned}
&\int_\Omega |x|^\alpha u^p \, dx \\
&\geq (1-2l)^\alpha \Bigg[nS^{N/2} - p\frac{C_N c_0}{\lambda^{N-2}} \Bigg\{ \sum_{i=0}^{n-1} H(x_i, x_i) - \sum_{\substack{i,j=0\\ i\neq j}}^{n-1} G(x_i, x_j) \Bigg\} \\
&+n^N O\left(\frac{1}{\lambda^{N-1}}\right) + n^2 O\left(\frac{1}{(\lambda l)^N}\right) \Bigg],
\end{aligned}
$$

where C_N is the constant introduced in (16) and $c_0 = \int_{\mathbf{R}^N} U^{p-1} \, dx$.

We begin with the first integral in the right–hand–side of (29).

Lemma 3.7 $\displaystyle\int_{B_i} U_i{}^p \, dx = S^{N/2} + O\left(\frac{1}{(\lambda d)^N}\right).$

Proof. The proof makes use of the same estimate as the one in the proof of Lemma 3.2, with l replaced by d this time. □

Lemma 3.8 $\displaystyle\int_{B_i} U_i{}^{p-1}\varphi_i \, dx \leq \frac{C_N c_0}{\lambda^{N-2}} H(x_i, x_i) + O\left(\frac{1}{(\lambda l)^N}\right).$

Proof. We work as in Lemma 3.3, with some further simplifications. Again we set

$$
\Gamma(x) = \varphi_i(x) - C_N \lambda^{\frac{2-N}{2}} H(x_i, x),
$$

but this time we notice that by harmonicity

$$\max_{\overline{\Omega}} \Gamma = \max_{\partial\Omega} \Gamma < \max_{x\in\partial\Omega} \frac{C_N}{\lambda^{\frac{N-2}{2}}} \left(\frac{1}{|x-x_i|^{N-2}} - \frac{1}{|x-x_i|^{N-2}} \right) = 0.$$

Therefore we can write

$$\begin{aligned}\int_{B_i} U_i^{p-1}\varphi_i &= \frac{C_N}{\lambda^{\frac{N-2}{2}}} \int_{B_i} U_i^{p-1} H(x_i,x) + \int_{B_i} U_i^{p-1}\Gamma \\ &\leq \frac{C_N}{\lambda^{\frac{N-2}{2}}} \int_{B_i} U_i^{p-1} H(x_i,x).\end{aligned}$$

Since $d \leq l/2$, the last term can be estimated as in (22), namely,

$$\begin{aligned}\frac{C_N}{\lambda^{\frac{N-2}{2}}} \int_{B_i} U_i^{p-1} H(x_i,x) &\leq \frac{C_N}{\lambda^{\frac{N-2}{2}}} \int_{B_{l/2}(x_i)} U_i^{p-1} H(x_i,x) \\ &= \frac{C_N c_0}{\lambda^{N-2}} H(x_i,x_i) + O\left(\frac{1}{(\lambda l)^N}\right),\end{aligned}$$

which gives the required expression. □

This concludes the analysis of the first part of (29); we now pass to the mixed terms.

Lemma 3.9

$$\int_{B_i} U_i^{p-1} U_j \, dx \geq \frac{C_N c_0}{(\lambda|d_{ij}|)^{N-2}} + O\left(\frac{1}{(\lambda|d_{ij}|)^{N-1}}\right) + O\left(\frac{1}{\lambda^N d^2 |d_{ij}|^{N-2}}\right).$$

Proof. Though the integral to be estimated is very similar to the one in Lemma 3.4, we cannot proceed exactly as in that proof because we would get an error of order $O((\lambda d)^{-N})$, which is too large for our aims.

We first write

$$\int_{B_i} U_i^{p-1} U_j \, dx = \int_{\mathbf{R}^N} U_i^{p-1} U_j \, dx - \int_{\mathbf{R}^N \setminus B_i} U_i^{p-1} U_j \, dx \tag{30}$$

and notice that the first integral in the right–hand–side has already been estimated in (24).

The treatment of the second integral is rather involved. To avoid unnecessary complications we will make great use of formulas already established in [1] to get estimate (24); in particular, after scaling $y = \lambda(x - x_i)$, the integral over $\mathbf{R}^N$ is split in three parts in [1] following the decomposition $\mathbf{R}^N = B_1 \cup B_2 \cup (\mathbf{R}^N \setminus (B_1 \cup B_2))$, where (in our notation),

$$B_1 = B_{\lambda|d_{ij}|/4}(\lambda d_{ij}) \quad \text{and} \quad B_2 = B_{\lambda|d_{ij}|/4}(0).$$

In our case we can use this procedure if we write (scaling as above)

$$
\begin{aligned}
\int_{\mathbf{R}^N\setminus B_i} U_i^{p-1}U_j\,dx &\le \int_{\mathbf{R}^N\setminus B_{d/2}(x_i)} U_i^{p-1}U_j\,dx \\
&= C_N^p \int_{\mathbf{R}^N\setminus B_{\lambda d/2}(0)} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,\frac{1}{(1+|y-\lambda d_{ij}|^2)^{\frac{N-2}{2}}}\,dy \\
&=: C_N^p \int_{\mathbf{R}^N\setminus B_{\lambda d/2}(0)} A(y)\,dy
\end{aligned}
$$

and if we use the decomposition $\mathbf{R}^N \setminus B_{\lambda d/2}(0) = B_1 \cup (B_2 \setminus B_{\lambda d/2}) \cup (\mathbf{R}^N \setminus (B_1 \cup B_2))$. Notice that $d/2 \le |d_{ij}|/4$ by definition.

In this way we have to evaluate three integrals, with the same integrand $A(y)$ as in [1]. Now

$$
\int_{\mathbf{R}^N\setminus(B_1\cup B_2))} A(y)\,dy = O\Big(\frac{1}{\lambda|d_{ij}|)^N}\Big) \tag{31}
$$

and

$$
\int_{B_1} A(y)\,dy = O\Big(\frac{1}{\lambda|d_{ij}|)^N}\Big), \tag{32}
$$

these being formulas (B29) and (B30) in [1], respectively.

Finally we deal with the integral over $B := B_2 \setminus B_{\lambda d/2}$. To this aim we make use of the argument in [1] (which we don't repeat) that reduces it to

$$
\begin{aligned}
\int_B A(y)\,dy \le \frac{1}{(\lambda|d_{ij}|)^{N-2}} \int_B \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy \\
+\frac{C}{(\lambda|d_{ij}|)^N}\int_B \frac{|y|^2}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy,
\end{aligned} \tag{33}
$$

see (B25) in [1].

Now with elementary computations we have for the first integral

$$
\int_B \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy \le \int_{\mathbf{R}^N\setminus B_{\lambda d/2}(0)} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy = O\Big(\frac{1}{(\lambda d)^2}\Big),
$$

while for the second one we obtain

$$
\int_B \frac{|y|^2}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy \le \int_{B_2} \frac{|y|^2}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy = O\big(\log(\lambda|d_{ij}|)\big).
$$

Inserting these in (33) we can say that

$$
\int_B A(y)\,dy \le O\Big(\frac{1}{\lambda^N d^2|d_{ij}|^{N-2}}\Big) + O\Big(\frac{1}{(\lambda|d_{ij}|)^{N-1}}\Big).
$$

Taking also into account (24), (31) and (32), the decomposition (30) yields the desired estimate. □

The following lemma completes the set of estimates needed to treat the denominator of Q_α.

Lemma 3.10 $\displaystyle\int_{B_i} U_i^{p-1}\varphi_j\,dx \le \frac{C_N c_0}{\lambda^{N-2}} H(x_i,x_j) + O\left(\frac{1}{(\lambda l)^N}\right).$

Proof. The computations can be adapted from the ones in the proof of Lemma 3.8. □

We are now ready obtain Proposition 3.6.

Proof of Proposition 3.6. We add the terms in (29) using their estimates provided by the preceding lemmas.

By Lemmas 3.7 and 3.8, and recalling that $l > d \sim C/n$,

$$\begin{aligned}&\sum_{i=0}^{n-1}\left(\int_{B_i} U_i^{p} - p\int_{B_i} U_i^{p-1}\varphi_i\right)\\&\ge nS^{N/2} - p\frac{C_N c_0}{\lambda^{N-2}}\sum_{i=0}^{n-1} H(x_i,x_i) + n^{N+1}O\left(\frac{1}{\lambda^N}\right)\end{aligned} \tag{34}$$

The remainders generated by Lemma 3.9 can be dealt with as in (27); notice that the argument works also for the sum of $|d_{ij}|^{2-N}$, the series of j^{2-N} being convergent since $N \ge 4$. We obtain

$$\sum_{\substack{i,j=0\\ i\neq j}}^{n-1}\frac{1}{|d_{ij}|^{N-1}} \sim Cn^N \qquad \text{and} \qquad \sum_{\substack{i,j=0\\ i\neq j}}^{n-1}\frac{1}{|d_{ij}|^{N-2}} \sim Cn^{N-1}. \tag{35}$$

Therefore

$$\begin{aligned}&p\sum_{\substack{i,j=0\\ i\neq j}}^{n-1}\int_{B_i} U_i^{p-1}U_j\,dx\\&\ge p\frac{C_N c_0}{\lambda^{N-2}}\sum_{\substack{i,j=0\\ i\neq j}}^{n-1}\frac{1}{|x_i-x_j|^{N-2}} + n^N O\left(\frac{1}{\lambda^{N-1}}\right) + n^{N+1}O\left(\frac{1}{\lambda^N}\right)\\&= p\frac{C_N c_0}{\lambda^{N-2}}\sum_{\substack{i,j=0\\ i\neq j}}^{n-1}\frac{1}{|x_i-x_j|^{N-2}} + n^N O\left(\frac{1}{\lambda^{N-1}}\right),\end{aligned} \tag{36}$$

since $n/\lambda \to 0$.

Finally, from Lemma 3.10,

$$p\sum_{\substack{i,j=0\\ i\neq j}}^{n-1}\int_{B_i} U_i^{p-1}\varphi_j\,dx \le p\frac{C_N c_0}{\lambda^{N-2}}\sum_{\substack{i,j=0\\ i\neq j}}^{n-1} H(x_i,x_j) + n^2 O\left(\frac{1}{(\lambda l)^N}\right). \tag{37}$$

Adding (34), (36) and (37) (all multiplied by $(1-2l)^\alpha$) and recalling the definition of G, we obtain the required estimate. □

4. The main result

This last section is devoted to the proof of the main result. This task is simplified by the fact that we have already established most of technical bounds.

We begin with the main level estimate; in its statement recall that the space H_n has been defined in (2).

Proposition 4.1 *Let $N \geq 4$. For every $\alpha > 0$, there exists $n_\alpha > 0$ such that for every integer $n \geq n_\alpha$,*

$$\inf_{H_n} Q_\alpha < n^{\frac{2}{N}} S.$$

Proof. The function u constructed in the preceding section depends on n, l and λ, and for each n it belongs to H_n. We show that for appropriate choices of these parameters, there results $Q_\alpha(u) < n^{\frac{2}{N}} S$. This will be accomplished by matching the estimates of Propositions 3.1 and 3.6.

For simplicity we set

$$A := \sum_{i=0}^{n-1} H(x_i, x_i) - \sum_{\substack{i,j=0 \\ i \neq j}}^{n-1} G(x_i, x_j)$$

and we begin with an estimate of A, noticing that we can write it as

$$A = \sum_{i,j=0}^{n-1} H(x_i, x_j) - \sum_{\substack{i,j=0 \\ i \neq j}}^{n-1} \frac{1}{|x_i - x_j|^{N-2}}.$$

By definition of H, and since $\left|x_i - x_j/|x_j|^2\right| \geq l$ for all i, j, we see that

$$\sum_{i,j=0}^{n-1} H(x_i, x_j) = \sum_{\substack{i,j=0 \\ i \neq j}}^{n-1} \frac{1}{|x_j|^{N-2} \left|x_i - x_j/|x_j|^2\right|^{N-2}} \leq C_1 \frac{n^2}{l^{N-2}}.$$

Moreover, as in (35),

$$\sum_{\substack{i,j=0 \\ i \neq j}}^{n-1} \frac{1}{|x_i - x_j|^{N-2}} \sim C_2 n^{N-1},$$

so that we obtain the bound

$$A \leq C_1 \frac{n^2}{l^{N-2}} - C_2 n^{N-1}.$$

If we put for simplicity $b := C_N c_0$, and we notice that $(1-2l)^{2\alpha/p} \geq (1 - 3\alpha l)$, for all $N \geq 4$, all $\alpha > 0$ and all l small enough, we see that from Proposition 3.6

$$\left(\int_\Omega |x|^\alpha u^p \, dx \right)^{2/p}$$

$$\geq (1 - 3\alpha l) n^{2/p} \left[S^{N/2} - \frac{pb}{n\lambda^{N-2}} A + n^{N-1} O\left(\frac{1}{\lambda^{N-1}}\right) + nO\left(\frac{1}{(\lambda l)^N}\right) \right]^{2/p}.$$

Later we will see that all the quantities depending on n in the square brackets tend to zero as $\lambda/n \to \infty$; therefore expanding about $S^{N/2}$ we obtain for our main estimate

$$Q_\alpha(u) \leq \frac{nS^{N/2} - \frac{b}{\lambda^{N-2}}A + n^N O\left(\frac{1}{\lambda^{N-1}}\right) + n^2 O\left(\frac{1}{(\lambda l)^N}\right)}{(1-3\alpha l)n^{2/p}\left[S^{\frac{N-2}{2}} - 2\frac{b}{nS\lambda^{N-2}}A + n^{N-1}O\left(\frac{1}{\lambda^{N-1}}\right) + nO\left(\frac{1}{(\lambda l)^N}\right)\right]},$$

and we must check that for suitable values of the parameters, the right–hand–side is strictly less than $n^{2/N}S$. A trivial computation shows that this amounts to prove that

$$R{:=}3\alpha l S^{N/2} + (1-6\alpha l)\frac{b}{n\lambda^{N-2}}A + 3\alpha l\left(n^{N-1}O\left(\frac{1}{\lambda^{N-1}}\right) + nO\left(\frac{1}{(\lambda l)^N}\right)\right) < 0.$$

In order to do this we choose

$$l = n^{-9/20}.$$

Notice that this choice is compatible with our assumption $2d \leq l$ and also that $\lambda l \to \infty$ as $\lambda/n \to \infty$.

Next we take n so large that

$$A \leq -\frac{C_2}{2}n^{N-1};$$

this is possible because

$$A \leq C_1\frac{n^2}{l^{N-2}} - C_2 n^{N-1} = C_1 n^{2+9(N-2)/20} - C_2 n^{N-1}$$

and $2 + 9(N-2)/20 < N-1$ for all $N \geq 4$, as one immediately checks.

Furthermore, noticing that $9/20 < \frac{N-1}{N}$, we see that

$$\frac{n}{(\lambda l)^N} = \frac{n^{1+9N/20}}{\lambda^N} \leq \frac{n^N}{\lambda^N} \leq \frac{n^{N-1}}{\lambda^{N-1}},$$

since $n/\lambda \to 0$. Therefore the second big O is unnecessary in the expression of R. We are thus led to

$$R \leq 3S^{N/2}\frac{\alpha}{n^{9/20}} - \left(1 - 6\frac{\alpha}{n^{9/20}}\right)\frac{bC_2}{2}\frac{n^{N-2}}{\lambda^{N-2}} + 3\frac{\alpha}{n^{9/20}}O\left(\frac{n^{N-1}}{\lambda^{N-1}}\right).$$

Finally we choose $\lambda = n^{1+\varepsilon}$, with $\varepsilon > 0$ and small; it is immediate to check that this is compatible with the assumptions. We obtain

$$R \leq 3S^{N/2}\frac{\alpha}{n^{9/20}} - \left(1 - 6\frac{\alpha}{n^{9/20}}\right)\frac{bC_2}{2}\frac{1}{n^{\varepsilon(N-2)}} + 3\frac{\alpha}{n^{9/20}}O\left(\frac{1}{n^{\varepsilon(N-1)}}\right). \quad (38)$$

Since α is fixed, this quantity will be negative for n large (depending on α) if we take ε small enough (essentially $\varepsilon < 9/(20(N-2))$), the leading term being negative. Now $R < 0$ means that $Q_\alpha(u) < n^{\frac{2}{N}}S$, as we wanted to prove. □

We are now ready for the main result of the paper.

Theorem 4.2 *Let $N \geq 4$ and let Ω be the unit ball in $\mathbf{R}^N$. Then for every $\alpha > 0$ large enough, the problem*

$$\begin{cases} -\Delta u = |x|^\alpha u^{2^*-1} & in \;\; \Omega \\ u > 0 & in \;\; \Omega \\ u = 0 & on \;\; \partial\Omega \end{cases} \qquad (P_\alpha^*)$$

admits at least one non radial solution.

Proof. For every $\alpha > 0$, Problem (P_α^*) has a solution in some H_n. Indeed, given $\alpha > 0$, let $n \in \mathbf{N}$ be larger than the number n_α provided by Proposition 4.1 and consider the space H_n; setting $\Sigma_n = \inf_{H_n} Q_\alpha$, by the same proposition there results

$$\Sigma_n < n^{\frac{2}{N}} S.$$

By Proposition 2.1, Σ_n is achieved by a function $u \in H_n$, which can be assumed nonnegative; by invariance u is a critical point of Q_α on $H_0^1(\Omega)$ which, after scaling, (weakly) satisfies the equation and the boundary condition in (P_α^*). Standard regularity theory shows that u is a classical solution. Moreover u is positive in Ω by usual maximum principle arguments.

We have to show that, at least for α large, u is not radial. To this aim we use a bound for the level of radial functions obtained in [12]. In Theorem 5 of that paper the authors proved a bound that for our purposes we can write as

$$\inf_{\substack{v \in H_0^1(\Omega)\setminus\{0\} \\ v \; radial}} Q_\alpha(v) \geq C\alpha^{\frac{2N-2}{N}} \quad \text{as} \quad \alpha \to \infty,$$

where the constant C depends only on N. We now show that the level Σ_n of the solution we find is strictly below this threshold for α large.

To this aim we must evaluate how large n_α of Proposition 4.1 has to be in terms of α. If we choose n of the order of $\alpha^{5/2}$, we see that (38) essentially becomes

$$R \leq \frac{3S^{N/2}}{\alpha^{1/8}} - (1 - \frac{6}{\alpha^{1/8}})\frac{bC_2}{2}\frac{1}{\alpha^{5\varepsilon(N-2)/2}} + \frac{3}{\alpha^{1/8}} O\left(\frac{1}{\alpha^{5\varepsilon(N-1)/2}}\right).$$

Therefore R will be negative for all α big enough when ε is sufficiently small, so that we find a solution to (P_α^*) in the corresponding H_n.

To achieve this we have taken $n \sim \alpha^{5/2}$; therefore we find a solution at level

$$\Sigma_n < n^{2/N} S \leq C'\alpha^{5/N} \leq C\alpha^{\frac{2N-2}{N}} \leq \inf_{\substack{v \in H_0^1(\Omega)\setminus\{0\} \\ v \; radial}} Q_\alpha(v)$$

for all α large enough since $5 < 2N - 2$ for all $N \geq 4$. Therefore our solution cannot be radial, and this completes the proof. □

Remark 4.3 In wiew of the result by Ni ([10]), Theorem 1.1 in the Introduction, we see that the preceding theorem can be interpreted as a multiplicity result: Problem (P_α^*) admits (for large α) at least two positive solutions, the radial one found by Ni and the non radial one given by Theorem 4.2.

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