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Local monotonicity formulas for some nonlinear diffusion equations

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Introduction

In this paper, we study the reaction-diffusion equation

$$\frac{\partial u}{\partial t} - \Delta u = |u|^{p-1}u$$

on $\mathbf{R}^n$, where $p > 1$, as well as the harmonic map heat flow which evolves maps u from $\mathbf{R}^n$ into a Riemannian submanifold $N \subset \mathbf{R}^\ell$ by means of

$$\frac{\partial u}{\partial t} - \Delta u \in T^\perp N,$$

that is $(\partial u/\partial t - \Delta u)(x,t)$ lies in the normal space of N at $u(x,t)$. Another closely related equation is the mean curvature flow of submanifolds. Common to all these equations is the fact that they arise as gradient flows for certain energy densities.

In [GK] and [St], monotonicity formulas involving the associated energy densities for the first two equations were derived. It is the main purpose of this paper to discuss these formulas in detail and establish localised versions. We will therefore devote this introduction to a description of the corresponding results for the mean curvature flow (see [Hu] and [E]). Monotonicity formulas for equations on more general domains were derived in [Ha].

Mean curvature flow is defined as a family of n-dimensional submanifolds $(M_t)_{t\in I}$ of $\mathbf{R}^{n+k}$ which satisfy the evolution equation

$$\frac{\partial x}{\partial t} = \mathbf{H}(x)$$

for $x \in M_t$, $t \in I$ where I is an interval. Here $x = F(p,t) \in M_t$ where $F_t = F(\cdot,t) : M^n \to \mathbf{R}^{n+k}$ is a family of immersions of an n-dimensional manifold M^n with $M_t = F_t(M^n)$ and $\mathbf{H}(x)$ denotes the mean curvature vector at $x \in M_t$. Mean curvature flow takes the alternative form

$$\frac{\partial x}{\partial t} = \Delta_{M_t} x$$

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where Δ_{M_t} is the Laplace-Beltrami operator on M_t. This more clearly emphasizes the formal connection with the other two evolution equations.

In [Hu], the monotonicity formula for mean curvature flow

$$\frac{d}{dt}\int_{M_t} \Phi \, d\mu_t = -\int_{M_t} |\mathbf{H} - \frac{x^\perp}{2t}|^2 \Phi \, d\mu_t \tag{0}$$

is proved which involves the backward heat kernel function

$$\Phi(x,t) = \frac{1}{(-4\pi t)^{\frac{n}{2}}} e^{\frac{|x|^2}{4t}}$$

defined for $t < 0$ and $x \in \mathbf{R}^{n+k}$. Here, $\perp$ denotes the component of a vector in $\mathbf{R}^{n+k}$ normal to M_t and μ_t is the surface measure on M_t. The identity

$$\mathbf{H} - \frac{x^\perp}{2t} = 0$$

or equivalently the vanishing of the right hand side of the monotonicity formula characterizes scale invariant (homothetic) solutions of mean curvature flow. In terms of the evolving submanifolds this can also be expressed by $\frac{1}{r} M_{r^2 t} = M_t$ for all $r > 0$ and $t < 0$, or alternatively by $M_t = \sqrt{-t}\, M_{-1}$ for all $t < 0$.

In the case where $M_t = \mathbf{R}^n$ for all $t < 0$, the monotonicity formula for mean curvature flow reduces to the statement

$$\frac{d}{dt}\int_{\mathbf{R}^n} \Phi(x,t)\, dx = 0$$

which describes the scaling behaviour of the n-dimensional heat kernel on $\mathbf{R}^n$. The monotonicity formulas in [GK] and [St] are analogous to the one for mean curvature flow in that they combine the appropriate energy density (the area element in the case of mean curvature flow) with a heat kernel function to produce a quantity which is non-increasing in time. We will describe the details in Sects. 1 and 2 below.

Note that in contrast to (0) the well-known monotonicity formula for minimal hypersurfaces (see for instance [S])

$$\frac{d}{dr}\left(\frac{\mathcal{H}^n(M \cap B_r)}{r^n}\right) = \frac{d}{dr}\int_{M \cap B_r} \frac{|x^\perp|^2}{|x|^{n+2}}\, d\mathcal{H}^n$$

is a local statement in balls $B_r \subset \mathbf{R}^{n+1}$. Since minimal surfaces are stationary solutions of mean curvature flow that is solutions of the corresponding elliptic equation it seems natural to look for an analogous local formula in the parabolic case.

Balls in the elliptic case should here be replaced by some natural sets in space-time. In [F, W] and [EG], the sets

$$E_r = \left\{(x,t) \in \mathbf{R}^n \times \mathbf{R},\, t < 0,\, \Phi(x,t) > \frac{1}{r^n}\right\},\ r > 0$$

termed 'heat balls', play an important role in the derivation of mean value formulas for solutions of the standard heat equation. In fact, it is shown in [W] that the formula

$$\frac{d}{dr}\Big(\frac{1}{r^n}\iint_{E_r} f(x,t)\frac{|x|^2}{4t^2}\,dx\,dt\Big) = 0$$

holds for any solution f of the standard heat equation. These sets also arise naturally in the identity

$$\int_{\mathbf{R}^n} \Phi(y,s)\,dy = \frac{1}{r^n}\iint_{E_r} \frac{n}{-2t}\,dx\,dt$$

which holds for all $r > 0$ and $s < 0$ and can be obtained as a straightforward consequence of Fubini's theorem. There is an analogous identity for scale invariant solutions of mean curvature flow. To state this let

$$\mathcal{M} = \bigcup_{t<0} M_t \times \{t\} \subset \mathbf{R}^{n+k} \times \mathbf{R}$$

denote the space-time track of the family (M_t). Let E_r be a heat ball defined as above except here $x \in \mathbf{R}^{n+k}$, that is $E_r \subset \mathbf{R}^{n+k} \times \mathbf{R}$. Therefore it makes sense to consider $\mathcal{M} \cap E_r$. It is then proved in [E] that for a scale invariant solution of mean curvature flow the identity

$$\frac{1}{r^n}\iint_{\mathcal{M}\cap E_r} \frac{n}{-2t}\,d\mu_t\,dt = \int_{M_s} \Phi\,d\mu_s$$

holds for all $s < 0$ and $r > 0$. In view of the monotonicity formula this identity suggests that differentiation of the locally defined expression on the left hand side with respect to r should provide useful information also in the case of a general solution. This led to the following local version of the monotonicity formula derived in [E]:

$$\frac{d}{dr}\left(\frac{1}{r^n}\iint_{\mathcal{M}\cap E_r}\Big(\frac{n}{-2t} + \frac{x^\perp}{2t}\cdot\Big(\mathbf{H} - \frac{x^\perp}{2t}\Big)\Big)\,d\mu_t\,dt\right)$$
$$= \frac{n}{r^{n+1}}\iint_{\mathcal{M}\cap E_r}\left|\mathbf{H} - \frac{x^\perp}{2t}\right|^2 d\mu_t\,dt.$$

In this paper, we establish analogues of this local monotonicity formula for a reaction-diffusion equation and for the harmonic map heat flow. For the convenience of the reader but also to ensure a common presentation and derivation for both equations we will also provide statements and proofs of the non-local formulas of [GK] and [St].

1. A reaction-diffusion equation

We consider solutions $u(x,t)$, $x \in \mathbf{R}^n$, $t < 0$ of the equation

$$\frac{\partial u}{\partial t} - \Delta u = |u|^{p-1}u \tag{1}$$

where $p > 1$. This equation arises as gradient flow of the energy density

$$e(u) = \frac{1}{2}|Du|^2 - \frac{1}{p+1}|u|^{p+1}.$$

For $r > 0$ and $\beta = \frac{1}{p-1}$ the rescaled function defined by

$$u_r(x,t) = r^{2\beta}u(rx, r^2t)$$

is again a solution of (1). One easily checks that the energy density scales like

$$e(u_r)(x,t) = r^{\gamma}e(u)(rx, r^2t) \tag{2}$$

where $\gamma = 4\beta + 2 = 2(p+1)/(p-1)$.

We call a function *scale invariant* if it satisfies $u_r(x,t) = u(x,t)$ for all $r > 0$. In particular then

$$0 = \frac{du_r}{dr}\Big|_{r=1}(x,t) = 2t\,\frac{\partial u}{\partial t}(x,t) + x \cdot Du(x,t) + 2\beta\, u(x,t). \tag{3}$$

One readily checks by setting $r = 1/\sqrt{-t}$ that a scale invariant solution of (1) satisfies

$$u(x,t) = \frac{1}{(-t)^{\beta}}w(\frac{x}{\sqrt{-t}})$$

where $w(z) = u(z,-1)$ solves the elliptic equation

$$\Delta w - \frac{z}{2} \cdot Dw - \beta w + |w|^{p-1}w = 0.$$

Let

$${}^{\gamma}\Phi(x,t) = \frac{1}{(-4\pi t)^{\frac{n-\gamma}{2}}}e^{\frac{|x|^2}{4t}}$$

for $t < 0$ and $x \in \mathbf{R}^n$. Note that ${}^{\gamma}\Phi(x,t) = (-4\pi t)^{\frac{\gamma}{2}}\,\Gamma(x,-t)$ where for $t > 0$

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$$

is the fundamental solution for the heat equation on $\mathbf{R}^n$. The function ${}^{\gamma}\Phi$ scales like

$${}^{\gamma}\Phi(rx, r^2t) = \frac{1}{r^{n-\gamma}}\,{}^{\gamma}\Phi(x,t). \tag{4}$$

For the remainder of this paper we consider only solutions of (1) which are in a suitable growth class at infinity (e.g. polynomial in $|x|$) so that for instance integrals over $\mathbf{R}^n$ with respect to ${}^{\gamma}\Phi(x,t)\,dx$ of u and its derivatives are finite and integration by parts is admissible.

1.1. Remark. In view of (4), we observe that for $x = ry$ and $t = r^2s$

$$\int_{\mathbf{R}^n} f(x,t)\,{}^{\gamma}\Phi(x,t)\,dx = \int_{\mathbf{R}^n} r^{\gamma}f(ry, r^2s)\,{}^{\gamma}\Phi(y,s)\,dy \tag{5}$$

for any function $f(x,t)$. Therefore, if f scales like $f(rx, r^2t) = r^{-\gamma} f(x,t)$ for all $r > 0$ then

$$\frac{d}{dt}\int_{\mathbf{R}^n} f(x,t)\,{}^\gamma\Phi(x,t)\,dx = 0.$$

Important examples of such functions are $f(x,t) = e(u)(x,t)$ and $f(x,t) = u^2(x,t)/t$ in the case u is scale invariant.

1.2 Theorem (Monotonicity formula [GK, St]). *Let u be a solution of (1). Then*

$$\frac{d}{dt}\int_{\mathbf{R}^n}\Big(e(u) - \frac{\beta}{2t}u^2\Big)\,{}^\gamma\Phi\,dx = -\int_{\mathbf{R}^n}\Big(\frac{\partial u}{\partial t} + \frac{x}{2t}\cdot Du + \frac{\beta}{t}u\Big)^2\,{}^\gamma\Phi\,dx.$$

1.3 Remark.

(i) The right hand side of this identity vanishes exactly when u is a scale invariant solutions of (1) that is of the form

$$u(x,t) = \frac{1}{(-t)^\beta} w\Big(\frac{x}{\sqrt{-t}}\Big)$$

where $w(z) = u(z,-1)$ solves the elliptic equation

$$\Delta w - \frac{z}{2}\cdot Dw - \beta w + |w|^{p-1}w = 0.$$

(ii) In [GK], this formula is not stated in this particular form. Instead, a solution of (1) is rescaled in a time-dependent fashion by setting $w(z,\sigma) = (-t)^\beta u(x,t)$ where $x = \sqrt{-t}\,z$ and $t = -\exp(-\sigma)$. This function satisfies the equation

$$\frac{\partial w}{\partial \sigma} = \Delta w - \frac{z}{2}\cdot Dw - \beta w + |w|^{p-1}w.$$

The monotonicity formula in [GK] then states that

$$\frac{d}{d\sigma}\int_{\mathbf{R}^n}\Big(\frac{1}{2}|Dw|^2 + \frac{\beta}{2}w^2 - \frac{1}{p+1}|w|^{p+1}\Big)\,e^{-\frac{|z|^2}{4}}\,dz = -\int_{\mathbf{R}^n}\Big(\frac{\partial w}{\partial\sigma}\Big)^2 e^{-\frac{|z|^2}{4}}\,dz.$$

Scaling back leads to the formula in Theorem 1.2. For the convenience of the reader we will give a separate proof of Theorem 1.2 by adapting the calculation in [St] used for the harmonic map heat flow (see Sect. 2 below).

Proof of Theorem. For $x = ry$ and $t = r^2 s$ the identity

$$\int_{\mathbf{R}^n} e(u)(x,t)\,{}^\gamma\Phi(x,t)\,dx = \int_{\mathbf{R}^n} e(u_r)(y,s)\,{}^\gamma\Phi(y,s)\,dy$$

holds in view of (2) and (5). Therefore,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbf{R}^n} e(u)\,{}^\gamma\Phi\,dx &= \frac{d}{dr}_{|r=1}\int_{\mathbf{R}^n} e(u_r)\,{}^\gamma\Phi\,dx\,\frac{dr}{dt}_{|r=1}\\ &= \frac{1}{2t}\int_{\mathbf{R}^n}\frac{d}{dr}_{|r=1} e(u_r)\,{}^\gamma\Phi\,dx\end{aligned}$$

$$= \frac{1}{2t} \int_{\mathbf{R}^n} \left(Du \cdot D \frac{du_r}{dr}\Big|_{r=1} - |u|^{p-1} u \frac{du_r}{dr}\Big|_{r=1} \right) {}^{\gamma}\Phi \, dx.$$

Integration by parts yields

$$\frac{d}{dt} \int_{\mathbf{R}^n} e(u)\, {}^{\gamma}\Phi \, dx = -\frac{1}{2t} \int_{\mathbf{R}^n} \left((\Delta u + |u|^{p-1} u)\, {}^{\gamma}\Phi + Du \cdot D\,{}^{\gamma}\Phi \right) \frac{du_r}{dr}\Big|_{r=1} \, dx.$$

On substituting (1) and since $D\,{}^{\gamma}\Phi(x,t) = \frac{x}{2t}\, {}^{\gamma}\Phi(x,t)$ this becomes

$$\frac{d}{dt} \int_{\mathbf{R}^n} e(u)\, {}^{\gamma}\Phi \, dx = -\frac{1}{2t} \int_{\mathbf{R}^n} \left(\frac{\partial u}{\partial t} + \frac{x}{2t} \cdot Du \right) \frac{du_r}{dr}\Big|_{r=1} \, {}^{\gamma}\Phi \, dx.$$

We now substitute (3) and rewrite the right hand side to obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbf{R}^n} e(u)\, {}^{\gamma}\Phi \, dx = &- \int_{\mathbf{R}^n} \left(\frac{\partial u}{\partial t} + \frac{x}{2t} \cdot Du + \frac{\beta}{t} u \right)^2 {}^{\gamma}\Phi \, dx \\ &+ \frac{1}{2t} \int_{\mathbf{R}^n} \frac{\beta}{t} u \frac{du_r}{dr}\Big|_{r=1} \, {}^{\gamma}\Phi \, dx. \end{aligned}$$

The last integral equals

$$\frac{1}{2t} \frac{d}{dr}\Big|_{r=1} \int_{\mathbf{R}^n} \frac{\beta}{2t} u_r^2 \, {}^{\gamma}\Phi \, dx.$$

Since for $x = ry$ and $t = r^2 s$ we have

$$\int_{\mathbf{R}^n} \frac{\beta}{2s} u_r^2(y,s)\, {}^{\gamma}\Phi(y,s) \, dy = \int_{\mathbf{R}^n} \frac{\beta}{2t} u^2(x,t)\, {}^{\gamma}\Phi(x,t) \, dx,$$

and for $r = 1$ the identities $\frac{dr}{dt} = 1/2t$, $x = y$ and $t = s$ hold we obtain

$$\frac{1}{2t} \frac{d}{dr}\Big|_{r=1} \int_{\mathbf{R}^n} \frac{\beta}{2t} u_r^2 \, {}^{\gamma}\Phi \, dx = \frac{d}{dt} \int_{\mathbf{R}^n} \frac{\beta}{2t} u^2 \, {}^{\gamma}\Phi \, dx.$$

Thus we finally arrive at

$$\frac{d}{dt} \int_{\mathbf{R}^n} e(u)\, {}^{\gamma}\Phi \, dx = - \int_{\mathbf{R}^n} \left(\frac{\partial u}{\partial t} + \frac{x}{2t} \cdot Du + \frac{\beta}{t} u \right)^2 {}^{\gamma}\Phi \, dx + \frac{d}{dt} \int_{\mathbf{R}^n} \frac{\beta}{2t} u^2 \, {}^{\gamma}\Phi \, dx$$

which is the desired formula. □

In the elliptic case, there are well-known monotonicity formulas which are local in contrast to the identity in Theorem 1.2. For instance for harmonic functions that is solutions of the equation

$$\Delta u = 0$$

with scaling exponents $\beta = 0$ for the solution and $\gamma = 2$ for the energy density $e(u) = \frac{1}{2}|Du|^2$, the identity

$$\frac{d}{dr} \left(\frac{1}{r^{n-2}} \int_{B_r} e(u) \, dx \right) = r \int_{\partial B_1} \left(\frac{d}{dr} u_r(\omega) \right)^2 d\omega$$

holds. The right hand side vanishes on scale invariant solutions that is when

$$u_r(x) \equiv u(rx) = u(x)$$

for all x and $r > 0$. Note that this formula does not contain any interesting information in the case $n = 2$.

It seems natural to look for local monotonicity formulas also in the parabolic case. Balls should here be replaced by bounded sets in space-time. As in [F, EG] and [W] we define for $r > 0$ the 'heat-ball' E_r^γ as the set

$$\begin{aligned} E_r^\gamma &= \left\{ (x,t) \in \mathbf{R}^n \times \mathbf{R},\, t < 0,\, {}^\gamma\Phi(x,t) > \frac{1}{r^{n-\gamma}} \right\} \\ &= \bigcup_{-\frac{r^2}{4\pi} < t < 0} B_{R_r^\gamma(t)} \times \{t\} \end{aligned}$$

where

$$R_r^\gamma(t) = \sqrt{2(n-\gamma)\, t \log\left(\frac{-4\pi t}{r^2}\right)}.$$

Here we assume of course that $n > \gamma$ since otherwise $E_r^\gamma = \emptyset$. We also note that integrals over heat-balls look like

$$\iint_{E_r^\gamma} f(x,t)\, dx\, dt = \int_{-\frac{r^2}{4\pi}}^0 \int_{B_{R_r^\gamma(t)}} f(x,t)\, dx\, dt.$$

1.4 Remark. For $x = ry$ and $t = r^2 s$ and arbitrary function f we note the rescaling identity

$$\frac{1}{r^{n-\gamma}} \iint_{E_r^\gamma} \frac{n-\gamma}{-2t} f(x,t)\, dx\, dt = \iint_{E_1^\gamma} \frac{n-\gamma}{-2s} r^\gamma f(ry, r^2 s)\, dy\, ds.$$

If f satisfies $f(rx, r^2 t) = r^{-\gamma} f(x,t)$ for all $r > 0$ this implies that

$$\frac{d}{dr}\left(\frac{1}{r^{n-\gamma}} \iint_{E_r^\gamma} \frac{n-\gamma}{-2t} f(x,t)\, dx\, dt \right) = 0.$$

1.5 Proposition. *If f satisfies $f(rx, r^2 t) = r^{-\gamma} f(x,t)$ for all $r > 0$ then for all $s < 0$ and $r > 0$*

$$\int_{\mathbf{R}^n} f(y,s)\, {}^\gamma\Phi(y,s)\, dy = \frac{1}{r^{n-\gamma}} \iint_{E_r^\gamma} \frac{n-\gamma}{-2t} f(x,t)\, dx\, dt.$$

Proof. In view of Remarks 1.1 and 1.4, the left hand side is independent of s and the right hand side is independent of r. It therefore suffices to prove the identity for $s = -1$ and $r = \sqrt{4\pi}$ which amounts to showing that

$$\int_{\mathbf{R}^n} f(y,-1)\, e^{-\frac{|y|^2}{4}}\, dy = \int_{-1}^0 \int_{B_{\sqrt{2(n-\gamma)t\log(-t)}}} \frac{n-\gamma}{-2t} f(x,t)\, dx\, dt.$$

By Fubini's theorem we have

$$\int_{\mathbf{R}^n} f(y,-1)\, e^{-\frac{|y|^2}{4}}\, dy = \int_0^\infty \int_{\{y\in\mathbf{R}^n,\, e^{-\frac{|y|^2}{4}}>\lambda\}} f(y,-1)\, dy\, d\lambda$$
$$= \int_0^1 \int_{B_{\sqrt{-4\log\lambda}}} f(y,-1)\, dy\, d\lambda.$$

For $\lambda = (-t)^{\frac{n-\gamma}{2}}$ the right hand side expression becomes

$$\int_{-1}^0 \frac{n-\gamma}{-2t} \int_{B_{\sqrt{-2(n-\gamma)\log(-t)}}} f(y,-1)\, dy\, dt$$

and since $f(x,t) = (-t)^{-\frac{\gamma}{2}} f(y,-1)$ for $x = \sqrt{-t}\, y$ we arrive at

$$\int_{-1}^0 \int_{B_{\sqrt{2(n-\gamma)t\log(-t)}}} \frac{n-\gamma}{-2t} f(x,t)\, dx\, dt.$$ □

The following integration by parts formulas on 'heat balls' will prove to be extremely useful:

1.6 Lemma. *For any C^1 vectorfield X we have*

$$(i) \qquad \iint_{E_r^\gamma} \mathrm{div} X\, dx\, dt = -\frac{r}{n-\gamma}\frac{d}{dr}\iint_{E_r^\gamma} X\cdot\frac{x}{2t}\, dx\, dt.$$

For $X = g\,Df$ where $g \in C^1$ and $f \in C^2$ this yields the integration by parts formula

$$(ii) \qquad \iint_{E_r^\gamma} (Df\cdot Dg + g\,\Delta f)\, dx\, dt = -\frac{r}{n-\gamma}\frac{d}{dr}\iint_{E_r^\gamma} g\,Df\cdot\frac{x}{2t}\, dx\, dt.$$

Proof. We abbreviate $\psi \equiv \log {}^\gamma\!\Phi$. Then $B_{R_r^\gamma(t)} = \{x\in\mathbf{R}^n,\ \psi(x,t) > -(n-\gamma)\log r\}$. The divergence theorem in $\mathbf{R}^n$ implies

(6)

$$\iint_{E_r^\gamma} \mathrm{div} X\, dx\, dt = \int_{-\frac{r^2}{4\pi}}^0 \int_{B_{R_r^\gamma(t)}} \mathrm{div} X\, dx\, dt = -\int_{-\frac{r^2}{4\pi}}^0 \int_{\partial B_{R_r^\gamma(t)}} \frac{X\cdot D\psi}{|D\psi|}\, dA\, dt.$$

where dA denotes the surface measure of the $(n-1)$-dimensional sphere. Since $B_{R_r^\gamma(-\frac{r^2}{4\pi})} = \emptyset$ we have

$$\frac{d}{dr}\iint_{E_r^\gamma} X\cdot D\psi\, dx\, dt = \int_{-\frac{r^2}{4\pi}}^0 \frac{d}{dr}\int_{B_{R_r^\gamma(t)}} X\cdot D\psi\, dx\, dt.$$

By the coarea formula the right hand side equals

$$-\int_{-\frac{r^2}{4\pi}}^0 \int_{B_{R_r^\gamma(t)}} \frac{X\cdot D\psi}{|D\psi|}\, dA\, dt \frac{d}{dr}\Big(-(n-\gamma)\log r\Big) = \frac{n-\gamma}{r}\int_{-\frac{r^2}{4\pi}}^0 \int_{\partial B_{R_r^\gamma(t)}} \frac{X\cdot D\psi}{|D\psi|}\, dA\, dt$$

so that we obtain the identity

$$\frac{d}{dr}\iint_{E_r^\gamma} X \cdot D\psi \, dx \, dt = \frac{n-\gamma}{r}\int_{-\frac{r^2}{4\pi}}^{0}\int_{\partial B_{R_r^\gamma(t)}} \frac{X \cdot D\psi}{|D\psi|}\, dA \, dt \tag{7}$$

Combining (6) and (7) and noting that $D\psi(x,t) = x/2t$ yields (i). □

1.7 Remark. In view of Proposition 1.5, the identity

$$\int_{\mathbf{R}^n}\Big(e(u) - \frac{\beta}{2s}u^2\Big)\,{}^\gamma\Phi\, dy = \frac{1}{r^{n-\gamma}}\iint_{E_r^\gamma}\frac{n-\gamma}{-2t}\Big(e(u) - \frac{\beta}{2t}u^2\Big)\, dx\, dt$$

holds for any $s < 0$ and $r > 0$ as long as u is scale invariant. The monotonicity formula, Theorem 1.2, states that for a general solution u of (1) the left hand side of this identity is a non-increasing function of s. This suggests that differentiating the right hand side with respect to r should lead to something useful.

1.8 Theorem (Local monotonicity). *Let u be a solution of (1). Then*

$$\frac{d}{dr}\bigg(\frac{1}{r^{n-\gamma}}\iint_{E_r^\gamma}\Big(\frac{n-\gamma}{-2t}\Big(e(u) - \frac{\beta}{2t}u^2\Big) - \frac{x}{2t}\cdot Du\Big(\frac{\partial u}{\partial t} + \frac{x}{2t}\cdot Du + \frac{\beta}{t}u\Big)\Big)\, dx\, dt\bigg)$$
$$= \frac{n-\gamma}{r^{n-\gamma+1}}\iint_{E_r^\gamma}\Big(\frac{\partial u}{\partial t} + \frac{x}{2t}\cdot Du + \frac{\beta}{t}u\Big)^2\, dx\, dt.$$

1.9 Remark.

(i) The above formula is only nontrivial in the case $n > \gamma$ since otherwise $E_r^\gamma = \emptyset$.

(ii) As in the monotonicity formula of Theorem 1.2 the right hand side of this local identity vanishes exactly when u is a scale invariant solutions of (1) that is of the form

$$u(x,t) = \frac{1}{(-t)^\beta} w\Big(\frac{x}{\sqrt{-t}}\Big)$$

where $w(z) = u(z,-1)$ solves the elliptic equation

$$\Delta w - \frac{z}{2}\cdot Dw - \beta w + |w|^{p-1}w = 0.$$

(iii) In the appendix we estimate the integrals in Theorem 1.8 in terms of more familiar looking expressions therefore also providing conditions on the solution u which ensure their finiteness.

(iv) It is envisaged to apply the above formula to the study of the asymptotic behaviour of solutions near blow-up points of the solution. This was done in [GK] and [St] using the non-local formula of Theorem 1.2 and similarity variables (see Remark 1.3 (ii) above). Our local formulas should look particularly natural when one writes the solution in terms of 'heat-ball' polar coordinates where r refers to the scaling parameter and ω to a point on the 'heat-unit-sphere' ∂E_1^γ. This way the set 'transverse' to the scaling direction is relatively compact.

Proof of Theorem. Setting $x = ry$ and $t = r^2 s$ we see from Remark 1.4 for $f = e(u) - \beta u^2/2t$ that

$$\begin{aligned}
&\frac{d}{dr}\left(\frac{1}{r^{n-\gamma}}\iint_{E_r^\gamma}\frac{n-\gamma}{-2t}\left(e(u)-\frac{\beta u^2}{2t}\right)dx\,dt\right)\\
&=\frac{d}{dr}\iint_{E_1^\gamma}\frac{n-\gamma}{-2s}\left(e(u_r)-\frac{\beta u_r^2}{2s}\right)dy\,ds.\\
&=\iint_{E_1^\gamma}\frac{n-\gamma}{-2s}\frac{d}{dr}\left(e(u_r)-\frac{\beta u_r^2}{2s}\right)dy\,ds.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{d}{dr}\left(\frac{1}{r^{n-\gamma}}\iint_{E_r^\gamma}\frac{n-\gamma}{-2t}\left(e(u)-\frac{\beta u^2}{2t}\right)dx\,dt\right)\\
&=\iint_{E_1^\gamma}\frac{n-\gamma}{-2s}\left(Du_r\cdot D\frac{du_r}{dr}-\left(|u_r|^{p-1}u_r+\frac{\beta}{t}u_r\right)\frac{du_r}{dr}\right)dy\,ds
\end{aligned}$$

and after scaling back on the right hand side

$$\begin{aligned}
&\frac{d}{dr}\left(\frac{1}{r^{n-\gamma}}\iint_{E_r^\gamma}\frac{n-\gamma}{-2t}\left(e(u)-\frac{\beta u^2}{2t}\right)dx\,dt\right)\\
&=\frac{n-\gamma}{r^{n-\gamma+1}}\iint_{E_r^\gamma}\frac{1}{-2t}\left(Du\cdot D\frac{du_r}{dr}\Big|_{r=1}-\left(|u|^{p-1}u+\frac{\beta}{t}u\right)\frac{du_r}{dr}\Big|_{r=1}\right)dx\,dt.
\end{aligned}$$

Integrating by parts, this becomes

$$\begin{aligned}
&\frac{d}{dr}\left(\frac{1}{r^{n-\gamma}}\iint_{E_r^\gamma}\frac{n-\gamma}{-2t}\left(e(u)-\frac{\beta u^2}{2t}\right)dx\,dt\right)\\
&=\frac{n-\gamma}{r^{n-\gamma+1}}\iint_{E_r^\gamma}\frac{1}{2t}\left(\Delta u+|u|^{p-1}u+\frac{\beta}{t}u\right)\frac{du_r}{dr}\Big|_{r=1}dx\,dt\\
&\quad+\frac{1}{r^{n-\gamma}}\frac{d}{dr}\iint_{E_r^\gamma}\frac{1}{2t}\frac{du_r}{dr}\Big|_{r=1}\frac{x}{2t}\cdot Du\,dx\,dt
\end{aligned}$$

in view of Lemma 1.6 (ii). We substitute (1) into the first integral on the right hand side to obtain

$$\begin{aligned}
&\frac{d}{dr}\left(\frac{1}{r^{n-\gamma}}\iint_{E_r^\gamma}\frac{n-\gamma}{-2t}\left(e(u)-\frac{\beta u^2}{2t}\right)dx\,dt\right)\\
&=\frac{n-\gamma}{r^{n-\gamma+1}}\iint_{E_r^\gamma}\frac{1}{2t}\left(\frac{\partial u}{\partial t}+\frac{\beta}{t}u\right)\frac{du_r}{dr}\Big|_{r=1}dx\,dt\\
&\quad+\frac{1}{r^{n-\gamma}}\frac{d}{dr}\iint_{E_r^\gamma}\frac{1}{2t}\frac{du_r}{dr}\Big|_{r=1}\frac{x}{2t}\cdot Du\,dx\,dt.
\end{aligned}$$

In view of (3) this can be rewritten as

$$\frac{d}{dr}\Big(\frac{1}{r^{n-\gamma}}\iint_{E_r^\gamma}\frac{n-\gamma}{-2t}\Big(e(u)-\frac{\beta u^2}{2t}\Big)\,dx\,dt\Big)$$
$$=\frac{n-\gamma}{r^{n-\gamma+1}}\iint_{E_r^\gamma}\Big(\Big(\frac{\partial u}{\partial t}+\frac{x}{2t}\cdot Du+\frac{\beta}{t}u\Big)^2-\frac{x}{2t}\cdot Du\Big(\frac{\partial u}{\partial t}+\frac{x}{2t}\cdot Du+\frac{\beta}{t}u\Big)\Big)dx\,dt$$
$$+\frac{1}{r^{n-\gamma}}\frac{d}{dr}\iint_{E_r^\gamma}\frac{x}{2t}\cdot Du\,\left(\frac{\partial u}{\partial t}+\frac{x}{2t}\cdot Du+\frac{\beta}{t}u\right)\,dx\,dt$$

so that we finally arrive at

$$\frac{d}{dr}\Big(\frac{1}{r^{n-\gamma}}\iint_{E_r^\gamma}\frac{n-\gamma}{-2t}\Big(e(u)-\frac{\beta u^2}{2t}\Big)\,dx\,dt\Big)$$
$$=\frac{n-\gamma}{r^{n-\gamma+1}}\iint_{E_r^\gamma}\Big(\frac{\partial u}{\partial t}+\frac{x}{2t}\cdot Du+\frac{\beta}{t}u\Big)^2\,dx\,dt$$
$$+\frac{d}{dr}\Big(\frac{1}{r^{n-\gamma}}\iint_{E_r^\gamma}\frac{x}{2t}\cdot Du\,\left(\frac{\partial u}{\partial t}+\frac{x}{2t}\cdot Du+\frac{\beta}{t}u\right)\,dx\,dt\Big). \qquad \square$$

2. The harmonic map heat flow

We consider maps $u(\cdot,t):\mathbf{R}^n\to\mathbf{R}^\ell$ into a Riemannian submanifold $N\subset\mathbf{R}^\ell$ which satisfy

$$\frac{\partial u}{\partial t}-\varDelta u\in T^\perp N \tag{8}$$

or more precisely, $(\partial u/\partial t-\varDelta u)(x,t)\in T^\perp_{u(x,t)}N$, the normal space of N at $u(x,t)$.

The associated energy density is given by

$$e(u)=\frac{1}{2}|Du|^2=D_iu^\alpha D_iu^\alpha$$

where we sum over repeated indices $1\le i\le n$ and $1\le\alpha\le\ell$. We adopt the same notation as in Sect. 1 but some expressions have to be suitably interpreted in the mapping case. For instance, for vectorfields u,v we write $u\cdot v=u^\alpha v^\alpha$ and $Du\cdot Dv=D_iu^\alpha D_iv^\alpha$. We also encounter the expressions $x\cdot Du=x_iD_iu$ and $x\cdot Du\cdot\eta=x_iD_iu^\alpha\eta^\alpha$.

All calculations in Sect. 1 carry over without change to solutions of (8). Indeed, in view of (8)

$$\int_{\mathbf{R}^n}\left(\frac{\partial u}{\partial t}-\varDelta u\right)\eta\,dx=0$$

for all smooth vectorfields $\eta:\mathbf{R}^n\to\mathbf{R}^\ell$ which are tangent to N. We apply this identity in our calculations with $\eta=\frac{du_r}{dr}{}_{|r=1}$. Note that $\frac{du_r}{dr}{}_{|r=1}$ is a tangent vectorfield to N since $r\to u_r(x,t)$ defines a path in N. For the harmonic map heat flow the scaling exponent β equals zero so $\gamma = 2$. In particular, the appropriate heat kernel is given by

$$ {}^2\Phi(x,t) = \frac{1}{(-4\pi t)^{\frac{n-2}{2}}} e^{\frac{|x|^2}{4t}} $$

in this case.

In view of the above remarks we will simply state the relevant monotonicity formulas for the harmonic map heat flow and refer to the previous section for the calculations. Struwe's monotonicity formula for the harmonic map heat flow says the following:

2.1 Theorem ([St]). *Let u be a solution of (8). Then*

$$ \frac{d}{dt}\int_{\mathbf{R}^n} e(u)\, {}^2\Phi\, dx = -\int_{\mathbf{R}^n} \left|\frac{\partial u}{\partial t} + \frac{x}{2t}\cdot Du\right|^2 {}^2\Phi\, dx. $$

The corresponding local monotonicity formula for the harmonic map heat flow then reads as follows:

2.2 Theorem. *Let u be a solution of (8). Then*

$$ \frac{d}{dr}\left(\frac{1}{r^{n-2}}\iint_{E_r^2} \frac{n-2}{-2t} e(u) - \frac{x}{2t}\cdot Du\cdot\left(\frac{\partial u}{\partial t} + \frac{x}{2t}\cdot Du\right) dx\, dt\right) $$
$$ = \frac{n-2}{r^{n-1}}\iint_{E_r^2} \left|\frac{\partial u}{\partial t} + \frac{x}{2t}\cdot Du\right|^2 dx\, dt. $$

2.3 Remark.

(i) Note that in both formulas, the right hand side vanishes on scale invariant solutions.
(ii) In the case $n = 2$ the statement of Theorem 2.2 is trivial since $E_r^2 = \emptyset$ for $n = 2$.
(iii) A localised version of Theorem 2.1 involving cut-off functions was established in [CS]. We shall make use of this in the appendix below.

Appendix

The local integrals featuring in Theorems 1.8 and 2.2 involve integrands which become singular as $t \to 0$. However, the spatial domain is the ball $B_{R_r^\gamma(t)}$ whose radius shrinks for $t \to 0$. As a result, our integrals can be bounded by more familiar looking expressions analogously to the situation in mean curvature flow, see [E, Sect. 1].

In the case $\beta = 0$ (that is $\gamma = 2$) one has an estimate of the form

$$ \frac{1}{r^{n-2}}\iint_{E_r^2} \left(\frac{n-2}{-2t}\, e(u) - \frac{x}{2t}\cdot Du\left(\frac{\partial u}{\partial t} + \frac{x}{2t}\cdot Du\right)\right) dx\, dt $$
$$ \le c(n)\Big(\int_{B_{c(n)r}} e(u)(x, -\frac{r^2}{4\pi})\, dx + \frac{1}{r^n}\int_{-\frac{r^2}{4\pi}}^{0}\int_{B_{c(n)r}} e(u)\, dx\, dt\Big). $$

For general β a similar bound holds with integrals involving $|u|^{p+1}$ appearing on the right hand side.

To establish this bound we essentially follow [E, Sect. 1] using a cut-off function argument from [CS, Sect. 4]. For ease of notation we shall omit the superscript 2 from ${}^2\Phi$, E_r^2 and $R_r^2(t)$ in the calculations to follow.

For a $C_0^2(\mathbf{R}^n)$ -function ϕ we calculate

$$\begin{aligned}\frac{d}{dt}\int_{\mathbf{R}^n} e(u)\Phi\,\phi^2\,dx &= \frac{1}{2t}\frac{d}{dr}\Big|_{r=1}\int_{\mathbf{R}^n} e(u_r)\Phi\,\phi^2(r\cdot)\,dx\\ &= -\int_{\mathbf{R}^n}\Big(\frac{\partial u}{\partial t}+\frac{x}{2t}\cdot Du\Big)^2\Phi\,\phi^2\,dx\\ &\quad+\int_{\mathbf{R}^n}\Big(e(u)\,\phi\,\frac{x}{t}\cdot D\phi-2\phi\,D\phi\cdot Du\Big(\frac{\partial u}{\partial t}+\frac{x}{2t}\cdot Du\Big)\Big)\Phi\,dx\end{aligned}$$

where $\phi(r\cdot)(x)=\phi(rx)$. Young's inequality implies

$$\begin{aligned}\frac{d}{dt}\int_{\mathbf{R}^n} e(u)\Phi\,\phi^2\,dx &\le -\frac{1}{2}\int_{\mathbf{R}^n}\Big(\frac{\partial u}{\partial t}+\frac{x}{2t}\cdot Du\Big)^2\Phi\,\phi^2\,dx\\ &\quad+4\int_{\mathbf{R}^n}\left(4|D\phi|^2+\frac{|x|}{|t|}\phi|D\phi|\right)e(u)\Phi\,dx.\end{aligned}$$

We can of course replace $\Phi(x,t)$ by $\Phi(x,t-t_0)$ for $t_0\ge 0$ and $t<t_0$. Choosing ϕ such that $\chi_{B_\rho}\le\phi\le\chi_{B_{2\rho}}$ with $|D\phi|\le c/\rho$ and then integrating over the interval (t_1,t_0) yields

$$\begin{aligned}&\int_{B_\rho} e(u)\Phi(x,t-t_0)\,dx+\int_{t_1}^{t}\int_{B_\rho}\Big(\frac{\partial u}{\partial t}+\frac{x}{2(\tau-t_0)}\cdot Du\Big)^2\Phi(x,\tau-t_0)\,dx\,d\tau\\ &\le c\int_{t_1}^{0}\int_{B_{2\rho}\backslash B_\rho}\Big(\frac{1}{\rho^2}+\frac{\rho}{|\tau-t_0|}\Big)e(u)\Phi(x,\tau-t_0)\,dx\,d\tau\\ &\quad+\int_{B_{2\rho}} e(u)(x,t_1)\Phi(x,t_1-t_0)\,dx\end{aligned}$$

for all $t<0$. On the set $B_{2\rho}\backslash B_\rho$ we observe that

$$\frac{1}{|\tau-t_0|}\Phi(x,\tau-t_0)\le c(n)\rho^{-n}$$

and hence

$$\begin{aligned}&\int_{B_\rho} e(u)\Phi(x,t-t_0)\,dx\\ &\quad+\int_{t_1}^{t}\int_{B_\rho}\Big(\frac{\partial u}{\partial t}+\frac{x}{2(\tau-t_0)}\cdot Du\Big)^2\Phi(x,\tau-t_0)\,dx\,d\tau\\ &\le c(n)\rho^{-n}\int_{t_1}^{0}\int_{B_{2\rho}} e(u)\,dx\,dt+\int_{B_{2\rho}} e(u)(x,t_1)\Phi(x,t_1-t_0)\,dx\end{aligned}$$

for all $t_0\ge 0$ and $t<0$.

We first set $t_0 = 0$ and $t_1 = -\frac{r^2}{4\,pi}$. One easily checks that $R_r(t) \leq c_0(n)r$ straight from the definition (see also [E]). We then let $\rho = c_0(n)r$ above. Since on $B_{R_r(t)}$ we have $\Phi(x,t) \geq \frac{1}{r^{n-2}}$ by definition we therefore obtain the estimate

$$\frac{1}{r^{n-2}} \int_{-\frac{r^2}{4\pi}}^{0} \int_{B_{R_r(t)}} \Big(\frac{\partial u}{\partial t} + \frac{x}{2t} \cdot Du\Big)^2 \, dx\, dt$$
$$\leq c(n) \frac{1}{r^n} \int_{-\frac{r^2}{4\pi}}^{0} \int_{B_{c'(n)r}} e(u) + \frac{1}{r^{n-2}} \int_{B_{c'(n)r}} e(u) \left(x, -\frac{r^2}{4\pi}\right) \, dx$$

where $c'(n) = 2c_0(n)$. We now proceed as in the proof of Lemma 1.1 in [E] by setting $t_1 = -\frac{r^2}{4\pi}$ for given $r > 0$ and $t_0 = R_r(t)^2 + t$. One checks as in [E] that $t_0 > 0$ for all $t \in (-\delta r^2, 0)$ when $\delta = \delta(n)$ is sufficiently small. Again setting $\rho = c_0(n)r$ above yields

$$\frac{1}{R_r(t)^{n-2}} \int_{B_{R_r(t)}} e(u)(x,t)\, dx$$
$$\leq c(n) \left(\frac{1}{r^{n-2}} \int_{B_{c'(n)r}} e(u) \left(x, -\frac{r^2}{4\pi}\right) dx + \frac{1}{r^n} \int_{-\frac{r^2}{4\pi}}^{0} \int_{B_{c'(n)r}} e(u)(x,t)\, dx\, dt \right)$$

for $t \in (-\delta r^2, 0)$. We then estimate

$$\frac{1}{r^{n-2}} \iint_{E_r} \Big(\frac{n-2}{-2t}\, e(u) - \frac{x}{2t} \cdot Du \Big(\frac{\partial u}{\partial t} + \frac{x}{2t} \cdot Du\Big)\Big) \, dx\, dt$$
$$\leq \int_{-\frac{r^2}{4\pi}}^{0} \int_{B_{R_r(t)}} \Big(\frac{n-2}{-2t} + \frac{|x|^2}{4t^2}\Big)\, e(u) + \frac{1}{2}\Big(\frac{\partial u}{\partial t} + \frac{x}{2t)} \cdot Du\Big)^2\Big) \, dx\, dt.$$

The right hand side can now be controlled using the two previous inequalities. The details are very similar to [E, p.509-510].

Unfortunately, the above argument does not work out quite as cleanly as in the mean curvature flow case since we were unable to find a time-dependent localisation function ϕ satisfying

$$\frac{d}{dt} \int_{\mathbf{R}^n} e(u)\, \Phi\, \phi \, dx \leq 0.$$

Given such a test-function ϕ (in mean curvature flow this was simply a shrinking sub-solution for the heat operator on the evolving hypersurfaces) we would be able to estimate our local monotonic quantity simply in terms of initial data, that is

$$\int_{B_{c(n)r}} e(u)(x, -\frac{r^2}{4\pi})\, dx.$$

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