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Lipschitz continuity of state functions in some optimal shaping

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Abstract. We prove local Lipschitz continuity of the solution to the state equation in two kinds of shape optimization problems with constraint on the volume: the minimal shaping for the Dirichlet energy, with no sign condition on the state function, and the minimal shaping for the first eigenvalue of the Laplacian. This is a main first step for proving regularity of the optimal shapes themselves.

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1 Introduction

Our goal is to prove Lipschitz continuity results for the solution of the state equation in two kinds of shape optimization problems with constraint on the volume: one is the minimal shaping for the Dirichlet energy, without any sign condition for the state function, the other one the minimal shaping for the first eigenvalue of the Laplacian. This is a main first step in proving regularity of the optimal shapes themselves.

The first shape functional considered here is the following (we refer for instance to [19, 8, 9] for details on the origin of the corresponding minimization question). Let D be a fixed open subset of $\mathbb{R}^d, d \geq 2$ (non necessarily bounded) and let $f \in L^2(D)$. To each open subset Ω of $\mathbb{R}^d$ with finite measure, we associate the "Dirichlet energy" $J(u_\Omega)$ where J is the functional defined on $H_0^1(\Omega)$ by

$$J(v) := \frac{1}{2}\int_D |\nabla v|^2\,dx - \int_D fv\,dx, \tag{1}$$

and u_Ω is the solution of the Dirichlet problem

$$u_\Omega \in H_0^1(\Omega), \;\; -\Delta u_\Omega = f \text{ in } \Omega. \tag{2}$$

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As usual, we denote by $H_0^1(\Omega)$ the closure for the H^1-norm of the space $C_0^\infty(D)$ of the infinitely differentiable functions with compact support in D ($\|v\|_{H^1}^2 = \int_D v^2 + |\nabla v|^2$). We will denote by $|E|$ the Lebesgue measure of any measurable subset E of $\mathbb{R}^d$ and, for $v \in H_0^1(\Omega)$, we set $\Omega_v = [v \neq 0] = \{x \in D; v(x) \neq 0\}$. Given $m \in (0, |D|)$, we introduce $\mathcal{O}_m = \{\Omega \textit{ open}, \ \ \Omega \subset D, \ |\Omega| = m\}$. We are interested in the solutions of the following shape optimization problem with measure constraint:

$$\Omega^* \in \mathcal{O}_m, \ \ J(u_{\Omega^*}) = \min\{J(u_\Omega); \ \Omega \in \mathcal{O}_m\}. \tag{3}$$

This problem does not always have a solution Ω^* in $\mathcal{O}_m$. However, the infimum is always reached in the family of *quasi-open* subsets of D (see the beginning of Sect. 2). In general, we cannot say more about the regularity of the optimal quasi-open set Ω^*: as shown in [15], it may indeed not be open. However, if f is bounded, or at least in some L^q for q large enough, we expect u_{Ω^*} to be at least continuous. Then, if moreover its support fills in Ω^* (that is in the "saturated" case where $[u_{\Omega^*} \neq 0] = \Omega^*$), then Ω^* is at least open.

Actually, our main goal is to establish Lipschitz continuity properties for the optimal function u_{Ω^*}. Here is a main result of this paper.

Theorem 1.1. *Assume $f \in L^\infty(D) \cap L^2(D)$. Then any solution $u \in H_0^1(D)$ of*

$$|\Omega_u| \leq m, \ \forall\, \Omega \in \mathcal{O}_m, \ J(u) \leq J(u_\Omega), \tag{4}$$

is locally Lipschitz continuous on D.

As indicated in the beginning of Sect. 2, the problem (4) has always a solution. If moreover $|\Omega_u| = m$, then the open set Ω_u is a solution of the minimization problem (3). In any case, we have

Corollary 1.2. *Under assumptions of Theorem 1.1, the problem (3) has at least a solution and any solution is such that u_{Ω^*} is locally Lipschitz continuous.*

Recall that, for an open subset Ω of D, the solution u_Ω of the Dirichlet problem is also characterized by the minimization property

$$u_\Omega \in H_0^1(\Omega), \ \ J(u_\Omega) = \min\{J(v), \ v \in H_0^1(\Omega)\}.$$

As a consequence, since $\Omega_1 \subset \Omega_2 \Rightarrow H_0^1(\Omega_1) \subset H_0^1(\Omega_2)$, then $\Omega \to J(u_\Omega)$ is nonincreasing with respect to the inclusion. Therefore, the solution u of (4) satisfies also

$$|\Omega_u| \leq m, \ J(u) \leq J(u_\Omega), \ \forall\, \Omega \textit{ open with } \Omega \subset D \textit{ and } |\Omega| \leq m,$$

and even (see Lemma 2.1)

$$|\Omega_u| \leq m, \ J(u) \leq J(v), \ \forall\, v \in H_0^1(D) \textit{ with } |\Omega_v| \leq m. \tag{5}$$

Actually, we will rather work with this new variational formulation of our problem which does not involve subsets of D but only functions of $H_0^1(D)$. The needed details for the equivalence with the previous ones will be given at the beginning of Sect. 2.

Even if the formulation (5) widens the family of "test" functions v that may be used to study the local properties of u, one still needs to widen it more, since we will have to use "perturbations" v for which $|\Omega_v|''$ is larger than m. A first step in the proof will precisely be to prove that u satisfies a *unilateral penalized* version of (5) (see Theorem 2.4) which says exactly how (5) should be corrected for functions $v \in H_0^1(D)$ such that $|\Omega_v| > m$, namely

$$\forall v \in H_0^1(D), J(u) \leq J(v) + \lambda\big(|\Omega_v| - m\big)^+,$$

for some $\lambda \geq 0$. But, this problem is now very close to those studied in the pioneering papers [2,3] by W. Alt, L. Caffarelli and A. Friedman, where main tools have been developed to prove regularity of free boundaries and, as a first step, to prove Lipschitz continuity of the corresponding state function. Our problem here is different by the presence of the nonhomogeneity f and by the fact that, for instance, a "saturation hypothesis is needed" to expect regularity of the boundary (see [4]). Moreover, even with this natural hypothesis, cusps may occur at the boundary here as soon as the dimension is equal to 2, while the boundary is proved to be regular in the two-dimensional problems considered in [2,3]. To be more precise, we have to emphasize the main difference between the case when $u \geq 0$ and the case when u has positive and negative values. It is well known that positivity helps a lot for regularity in these problems. Besides [2] where positivity plays a main role, we may refer to [1] and [14] where tools of [2] are used to get regularity for penalized problems of type (7) or constrained problems of type (5) with positivity. Our approach here is similar, although slightly different since it is mainly based, even in the positive case, on the estimate of Lemma 3.9 rather than on estimates of type (29).

In the general case, without sign for u, the main extra tool is the celebrated *monotonicity lemma*, introduced for the first time in [3], and extended to the nonhomogeneous case in [6] (see Lemma 3.12 here).

Let us mention that the Lipschitz continuity of the solution u of (5) was already proved in dimension 2 by M. Crouzeix in [7] (using a very two-dimensional approach). It was also proved in any dimension for solutions of (5) obtained as limits of penalized solutions, first in [15] for the positive case (for general elliptic operators), then in [3] for the case without sign (as a particular case of a more general family of penalized problems, see Theorem 5.1 in [6] coupled with the approximation described in [15]). Our approach here is different and valid for all solutions.

It turns out that this approach also works to prove the local Lipschitz continuity of the optimal eigenfunction, in the minimal shaping, with measure and inclusion constraints, for *the first eigenvalue of the Laplacian operator* with zero Dirichlet boundary conditions (see Theorem 4.1). Here too, the constrained problem may be proved to be equivalent to a penalized version to which the above tools may be applied. This extends the results in [16] where the case of limits of penalized solutions was considered.

Finally we refer to [4] where the study of the regularity of the boundary is made assuming local Lipschitz continuity and nonnegativity of the state function.

2 Equivalence with the penalized version

Let us first give an existence result for our main problem (5) and indicate its relationship with the other problems mentioned in the introduction.

Lemma 2.1. *Assume $f \in L^2(D)$. Then*

- *the problem (5) has at least one solution u;*
- *the problems (4) and (5) are equivalent;*
- *if a solution u of (5) is continuous, and if $|\Omega_u| = m$, then Ω_u is a solution of (3);*
- *if a solution u of (5) satisfies $|\Omega_u| < m$, then there exists Ω^* containing Ω_u and solution of (3);*
- *the infimum in (3) is always reached by a quasi-open set Ω^*.*

Remark 2.2. According to the equivalence between (4) and (5), Theorem 1.1 will directly follow from Theorem 3.1 claiming that the solution of (5) is locally Lipschitz continuous. Corollary 1.2 follows from Theorem 1.1 and Lemma 2.1.

Proof of Lemma 2.1. Let v_n be a minimizing sequence in the problem (5). To prove the existence of u solution of (5), it is sufficient to prove that v_n is bounded in $H^1(D)$. Indeed, up to a subsequence, we can then assume that v_n converges weakly in $H^1(D)$ and a.e. to some function $u \in H_0^1(D)$. At the limit

$$|\Omega_u| \le \liminf |\Omega_{v_n}| \le m;\ \lim \int_D f v_n = \int_D f u;\ \int_D |\nabla u|^2 \le \liminf \int_D |\nabla v_n|^2.$$

Using that $J(v_n)$ is bounded from above and that $-\int_D f v_n \ge -\|f\|_{L^2(D)}\|v\|_{L^2(D)}$, the H^1-bound on v_n is a direct consequence of the following Poincaré inequality

$$\forall\, v \in H_0^1(D) \quad \int_D v^2 \le C(d)|\Omega_v|^{2/d} \int_D |\nabla v|^2. \tag{6}$$

This inequality may be deduced from the same (obvious) inequality for radial functions using radial symmetrization (see e.g. [7]).

A solution of (5) is obviously solution of (4). According to the remarks in the introduction (in particular (5)), a solution of (4) is solution of (5). To reach (5), we use that $J(v) = \lim J(v_n)$ where v_n is regular and $|\Omega_{v_n}| \le |\Omega_v|$: for instance, if $v \ge 0$, we choose $v_n = [(v - 1/n)^+] * \rho_{\epsilon_n}$ where ρ_{ϵ_n} is a regularizing sequence and ϵ_n converges to zero fast enough; in general, we use the decomposition $v = v^+ - v^-$.

The 3rd point of the lemma is obvious. If now, $|\Omega_u| < m$, then, as in [15], we consider the open sets $\omega_\eta = \cup_{x \in \Omega_u}\{y \in D; |y - x| < \eta\}$. Then, we may choose $\eta^* > 0$ such that $|\omega_{\eta^*}| = m$. Since $\forall v \in H_0^1(\omega_{\eta^*})$, $J(u) \le J(v)$, then $u = u_{\omega_{\eta^*}}$. Therefore, $\Omega^* = \omega_{\eta^*}$ has the required properties for the 4th point of the lemma.

Finally, if u is a solution of (5),

- either $|\Omega_u| = m$ and (see remark below)
  $u = u_{\Omega_u}, J(u_{\Omega_u}) = J(u) = \inf\{J(u_\Omega); \Omega \in \mathcal{O}_m\}$ and Ω_u is a quasi-open reaching the infimum in (3);
- either, $|\Omega_u| < m$ and, by the previous point, the minimum is reached in (3).

Remark 2.3. Recall that a quasi-open set is a set of the form $[w > a]$ where $w \in H^1(D)$ and $a \in \mathbb{R}$. We refer, for instance, to [17] or [18] for the definitions of $H_0^1(\Omega)$ and of u_Ω when Ω is only a quasi-open set. Note that the existence of an optimal quasi-open set may be also deduced from the general existence result proved in [5] for nondecreasing functionals.

We now may state the first result of this section.

Theorem 2.4. *Let u be a solution of (5). Then, there exists $\lambda > 0$ such that*

$$\forall v \in H_0^1(D), J(u) \leq J(v) + \lambda\big(|\Omega_v| - m\big)^+. \tag{7}$$

Proof: The method is the following (see e.g. [11,21] for the same approach): We introduce $J_\lambda(v) = J(v) + \lambda\big(|\Omega_v| - m\big)^+$ and we consider the problem

$$u_\lambda \in H_0^1(D), \;\; \text{and} \;\; \forall v \in H_0^1(D), J_\lambda(u_\lambda) \leq J_\lambda(v). \tag{8}$$

We will show that the solution of this problem satisfies $|\Omega_{u_\lambda}| \leq m$ for λ large enough, so that we have,

$$J(u_\lambda) = J_\lambda(u_\lambda) \leq J_\lambda(u) = J(u).$$

But, by the definition of u, we also have $J(u) \leq J(u_\lambda)$. Therefore, $J(u) = J(u_\lambda)$ so that, by (8), u satisfies (7).

Note first that the problem (8) has a solution u_λ, at least for λ greater than some $\lambda^* = \lambda^*(d, f)$. Indeed, if $x^2 = \int_D |\nabla v|^2, y = |\Omega_v|$, using (6), if $\lambda \geq \lambda^* = 2C(d) \int_D f^2$, we have

$$J_\lambda(v) \geq \frac{1}{2}x^2 - \sqrt{\lambda^*/2}\, x\, y^{1/d} + \lambda^*(y - m)^+ \geq \frac{1}{4}x^2 + \lambda^*[y - y^{2/d}/2 - m].$$

This proves that $J_\lambda(v)$ is bounded from below since $d \geq 2$ (note it is the case as soon as $\lambda > 0$ when $d \geq 3$; see also the remark below if $d = 1$). The existence of u_λ easily follows. Moreover, since $J_\lambda(0) = 0 \geq J_\lambda(u_\lambda)$, it follows from the above inequality that $|\Omega_{u_\lambda}|$ is bounded from above by a constant depending on λ^*, m so that we may write, for future reference, that

$$\int_{\Omega_{u_\lambda}} |f| \leq |\Omega_{u_\lambda}|^{1/2} \|f\|_{L^2(D)} \leq K = K(d, m, f). \tag{9}$$

Assume that $|\Omega_{u_\lambda}| > m$. Then, for $t > 0$ small enough, the function $u^t = (u_\lambda - t)^+ - (u_\lambda + t)^-$ also satisfies $|\Omega_{u^t}| > m$. Therefore, we may write

$$J(u_\lambda) + \lambda\big(|\Omega_{u_\lambda}| - m\big) \leq J(u^t) + \lambda\big(|\Omega_{u^t}| - m\big).$$

It follows (using also (9)) that

$$\int_{[0<|u_\lambda|<t]} |\nabla u_\lambda|^2 + 2\lambda\big|[0<|u_\lambda|<t]\big| \le 2\,t\int_{\Omega_{u_\lambda}} |f| \le 2\,t\,K.$$

Using the coarea formula (see e.g. [10, 13]), we may rewrite this as

$$\int_0^t ds \int_{[|u_\lambda|=s]} \left[|\nabla u\lambda| + \frac{2\lambda}{|\nabla u_\lambda|}\right] d\mathcal{H}_{d-1} \le 2\,t\,K, \tag{10}$$

where $\mathcal{H}_{d-1}$ denotes the $(d-1)$-Hausdorff measure. But the function $x \to x + 2\lambda x^{-1}$ is bounded from below on $(0,\infty)$ by $2\sqrt{2\lambda}$. It follows that

$$\sqrt{2\lambda}\int_0^t \int_{[|u_\lambda|=s]} d\mathcal{H}_{d-1} \le t\,K.$$

Next, we plug the isoperimetric inequality

$$\int_{[|u_\lambda|=s]} d\mathcal{H}_{d-1} \ge C(d)\big|[|u_\lambda|>s]\big|^{\frac{d-1}{d}}.$$

Dividing by t and letting t decrease to zero, we finally obtain

$$C(d)\sqrt{2\lambda}\,m^{\frac{d-1}{d}} \le K.$$

Thus "$|\Omega_{u_t}| > m$" is impossible when $\lambda > \max\{\lambda^*, \lambda_0\}$ where $C(d)\sqrt{2\lambda_0}\,m^{\frac{d-1}{d}} = K$.

Remark 2.5. Throughout this paper, we assume $d \ge 2$. Actually, if $d = 1$, the solution u of (5) exists and is at least continuous by embedding of H^1 in $C^{1/2}$. Moreover, we easily check that, on the open set $\omega = [u \ne 0]$, we have $-u'' = f$. On any maximal subinterval I of ω, since $u \in H_0^1(I)$, there exists $\xi \in I$ such that $u'(\xi) = 0$. It follows that for $x \in I$, ${u'}^2(x) = -2\int_\xi^x u'f \le 2\|u'\|_{L^\infty(I)}\|f\|_{L^1(I)}$. Therefore, $\|u'\|_{L^\infty(\omega)} \le 2\|f\|_{L^1(\omega)}$.

Remark 2.6. This proof of Theorem 2.4 is very general and carries over to general nonlinear elliptic variational problems in $W_0^{1,p}(D)$ associated with $v \to \int_D G(v) - f\,v$ where, for $p > 1$, and some $k > 0$,

$$\forall\, v \in W_0^{1,p}(D),\ G(v) \ge k\,|\nabla v|^p.$$

In this case, the inequality (10) is to be replaced by

$$\int_0^t ds \int_{[|u_\lambda|=s]} \left[k|\nabla u_\lambda|^{p-1} + \frac{\lambda}{|\nabla u_\lambda|}\right] d\mathcal{H}_{d-1} \le tK.$$

Then we use that the function $x \to kx^{p-1} + \lambda x^{-1}$ is bounded from below by $C(p,k)\lambda^{1-\frac{1}{p}}$.

Remark 2.7. Note that a "local" version of this equivalence with a penalized problem has been proved in [4]: roughly speaking, if $J(u) \leq J(v)$ for all $v \in H_0^1(D)$ such that $m - \eta \leq |\Omega_v| \leq m$, then (7) holds for $|\Omega_v|$ close to m. Moreover, a more precise estimate is proved for the v's such that $|\Omega_v| \leq m$.

Remark 2.8. This approach also works with several constraints (see e.g. [21]). We prove it here for the first eigenvalue of the Laplacian operator with volume, L^2-norm and inclusion constraints.

Theorem 2.9. *Assume D is bounded. Let $m \in (0, |D|)$ and let $u \in H_0^1(D)$ be a solution of*

$$\lambda_m = \int_D |\nabla u|^2 = \min\left\{\int_D |\nabla v|^2;\, v \in H_0^1(D),\, \int_D v^2 = 1,\, |\Omega_v| \leq m\right\}. \quad (11)$$

Then, there exists $\lambda > 0$ such that

$$\forall v \in H_0^1(D),\ \int_D |\nabla u|^2 \leq \int_D |\nabla v|^2 + \lambda_m \left[1 - \int_D v^2\right]^+ + \lambda[|\Omega_v| - m]^+. \quad (12)$$

Remark 2.10. The existence of u solution of (11) is obvious. Moreover, $u \geq 0$ up to changing u into $-u$ in some connected components of D. Indeed, first we may always change u into $-u$ so that $|\Omega_{u^+}| > 0$. Next, either $|\Omega_{u^+}| = |\Omega_u|$ (in which case $u > 0$ on Ω_u) or $0 < |\Omega_{u^+}| < |\Omega_u| \leq m$. Then denote

$$w_+ = u^+/\|u^+\|_{L^2}, \quad w_- = u^-/\|u^-\|_{L^2}.$$

By (11), we have

$$\int_D |\nabla u|^2 \leq \int_D |\nabla w_+|^2, \quad \int_D |\nabla u|^2 \leq \int_D |\nabla w_-|^2,$$

and these inequalities are actually equalities, otherwise

$$\int_D |\nabla u|^2 = \left[\int_D (u^+)^2 + \int_D (u^-)^2\right] \int_D |\nabla u|^2 < \int_D |\nabla u^+|^2 + |\nabla u^-|^2,$$

which is false. Now, since $\int_D |\nabla u|^2 = \int_D |\nabla w_+|^2 = \lambda_m$ and $|\Omega_{w_+}| < m$, then, for all small ball $B \subset D$ with measure less than $m - |\Omega_{w_+}|$, and for all $\varphi \in H_0^1(B)$, we have

$$\forall t > 0,\ \int_D |\nabla w_+|^2 \leq \int_D |\nabla (w_+ + t\varphi)|^2 / \int_D (w_+ + t\varphi)^2.$$

Differentiating with respect to t at $t = 0$ implies $-\Delta w_+ = \lambda_m w_+ \geq 0$ *on* D. It follows that, on each component of D, either $w^+ \equiv 0$, or $w^+ > 0$. This proves that $u \geq 0$ on D up to changing u into $-u$ on some of the components.

It follows also from this analysis that, if D is connected, then $|\Omega_u| = |\Omega_{u^+}| = m$. If D is not connected, it may happen that $|\Omega_u| < m$; but, in this case, u is strictly positive on some of the components of D and identically equal to zero on the others. As an example of this situation, we may take $D := D_1 \cup D_2$ where D_1, D_2 are disjoint disks in $\mathbb{R}^2$ of radius R_1, R_2 with $R_1 > R_2$ and $m = \pi R_1^2 + \epsilon < \pi(R_1^2 + R_2^2)$. Then, u coincides with the first eigenfunction of D_1 and is identically 0 on D_2.

Remark 2.11. We easily check that $\Omega^* = \Omega_u$ is solution of the shape optimization problem

$$\lambda_1(\Omega^*) = \min\{\lambda_1(\Omega); \Omega\ quasi-open\ \subset D,\ |\Omega| \leq m\}, \tag{13}$$

where $\lambda_1(\Omega)$ denotes the first eigenvalue of the Laplacian operator $-\Delta$ in

$$H_0^1(\Omega) = \{v \in H_0^1(D); v = 0\ quasi-everywhere\ on\ D \setminus \Omega\},$$

so that

$$\lambda_1(\Omega) = \min\left\{\int_\Omega |\nabla v|^2;\ v \in H_0^1(\Omega),\ \int_\Omega v^2 = 1\right\}. \tag{14}$$

Indeed, by the definitions (11) and (14), since $|\Omega_u| \leq m$ and $\int u^2 = 1$, we have $\lambda_m = \int_D |\nabla u|^2 \leq \lambda_1(\Omega_u)$ and the equality holds since $u \in H_0^1(\Omega_u)$. Now, if Ω is a quasi-open set with $|\Omega| \leq m$, by (11) again, $\lambda_m \leq \lambda_1(\Omega)$. Thus, Ω_u is solution of (13).

By Remark 2.10, if D is connected, then $|\Omega_u| = m$. If D is not connected, it may happen that $|\Omega_u| < m$, as shown by an example. In this case, any quasi-open set Ω^* such that $\Omega_u \subset \Omega^* \subset D, |\Omega^*| = m$, is also solution of (13). Note that such an Ω^* may be as irregular as a quasi-open set may be: for instance, in the example given in Remark 2.10, let ω be *any quasi-open subset* of D_2 with $|\omega| = m - \pi R_1^2$; then $\Omega^* := \omega \cup D_1$ is a solution of (13).

Remark 2.12. Conversely, it is immediate to check that, if Ω^* is solution of (13) and if $u^* \in H_0^1(\Omega^*)$ is such that $\lambda_1(\Omega^*) = \int_{\Omega^*} |\nabla u^*|^2, \int_{\Omega^*} {u^*}^2 = 1$, then u^* is solution of (11).

Proof of Theorem 2.9. Note first that, by definition of λ_m, for all $v \in H_0^1(D)$ with $|\Omega_v| \leq m$, we have $\int_D |\nabla v|^2 - \lambda_m \int_D v^2 \geq 0$ or also

$$\int_D |\nabla u|^2 \leq \int_D |\nabla v|^2 + \lambda_m \left[1 - \int_D v^2\right]. \tag{15}$$

Next the proof is similar to the proof of Theorem 2.4. Denote by $J_\lambda(v)$ the right-hand side of (12) and let u_λ minimize J_λ over $H_0^1(D)$ (its existence is obvious). Up to replacing u_λ by $|u_\lambda|$, one may assume $u_\lambda \geq 0$. It is sufficient to prove $|\Omega_{u_\lambda}| \leq m$ since then

$$J_\lambda(u_\lambda) \leq J_\lambda(u) = \int_D |\nabla u|^2 \leq J_\lambda(u_\lambda),$$

the last inequality coming from (15). Assume $|\Omega_{u_\lambda}| > m$ and introduce $u^t = (u_\lambda - t)^+$ as in the proof of Theorem 2.4. Then $J_\lambda(u_\lambda) \leq J_\lambda(u^t)$ leads to

$$\begin{aligned}\int_{[0<u_\lambda<t]} |\nabla u_\lambda|^2 + \lambda\big|[0 < u_\lambda < t]\big| &\leq \lambda_m \left|\int_D u_\lambda^2 - (u^t)^2\right| \\ &\leq \lambda_m \int_{[0<u_\lambda<t]} u_\lambda^2 + 2\,t\,\lambda_m \int_D u_\lambda.\end{aligned}$$

But $\int_D u_\lambda \leq |D|^{1/2}[\int_D u_\lambda^2]^{1/2}$, and using that $J_\lambda(u_\lambda) \leq J_\lambda\big(u_\lambda/[\int_D u_\lambda^2]^{1/2}\big)$ if $\int_D u_\lambda^2 \geq 1$, we check that actually $\int_D u_\lambda^2 \leq 1$. Finally, we obtain via the coarea formula

$$\int_0^t ds \int_{[|u_\lambda|=s]} \left[|\nabla u\lambda| + \frac{\lambda - \lambda_m s^2}{|\nabla u_\lambda|}\right] d\mathcal{H}_{d-1} \leq 2\,t\,\lambda_m |D|^{1/2}.$$

But the function $x \to x + (\lambda - \lambda_m s^2)x^{-1}$ is bounded from below by $2\sqrt{\lambda - s^2\lambda_m}$ and also by $\sqrt{2\lambda}$ as soon as $s^2 \leq t^2 \leq \lambda/2\lambda_m$. Then, using the isoperimetric inequality and letting t tend to zero as in the proof of Theorem 2.4, we obtain

$$C(d)\sqrt{\lambda} m^{\frac{d-1}{d}} \leq 2\,\lambda_m\,|D|^{1/2},$$

and this finishes the proof of the proposition.

3 Lipschitz continuity

Theorem 3.1. *Let u be a solution of (5) with $f \in L^1(D) \cap L^\infty(D)$. For $\delta > 0$, set $D_\delta = \{\xi \in D; d(\xi, \partial D) \geq \delta\}$. Then u is Lipschitz continuous on D_δ. If moreover, $u \geq 0$ on D_δ and $f \in L^1(D) \cap L^q(D)$ with $q > d$ only, then the same conclusion holds.*

Remark 3.2. Note that if $D = \mathbb{R}^d$, then $D_\delta = \mathbb{R}^d$. In this case, Theorem 3.1 says that u is globally Lipschitz on $\mathbb{R}^d$.

Remark 3.3. The monotonicity lemma, which is required for the proof in the case the sign of u is not constant, is proved in the case when the nonhomogeneity f is bounded (see [6]), but not for $f \in L^q, q > d$ (and seems not to be true, at least in the usual versions). This explains the assumption $f \in L^\infty$ in the general case.

Let us first collect some main properties of the solution u of our problem

Lemma 3.4. *Under the assumptions of Theorem 3.1, the solution u satisfies the following properties:*
For all balls $B \subset D$, and for v solution of

$$-\Delta v = f \text{ in } B, \quad v - u \in H_0^1(B),$$

one has

$$\int_B |\nabla(u - v)|^2 \leq 2\,\lambda\,|[u = 0] \cap B|, \tag{16}$$

with λ as in Theorem 2.4. For all $\varphi \in C_0^\infty(D)$

$$| < \Delta u + f, \varphi > | \leq \sqrt{2\lambda}\left\{\int |\nabla\varphi|^2\right\}^{1/2} \left\{|\Omega_\varphi|\right\}^{1/2}. \tag{17}$$

$$\Delta u^+ + f\chi_{[u>0]} = \mu_1 \geq 0, \quad \Delta u^- - f\chi_{[u<0]} = \mu_2 \geq 0, \tag{18}$$

$$\mu_1([u \neq 0]) = \mu_2([u \neq 0]) = 0. \tag{19}$$

$$u \in L^\infty(D). \tag{20}$$

Proof of Lemma 3.4. For (16), we apply Theorem 2.4 with v equal to u outside B and defined as in the lemma on B. This gives

$$\int_B \frac{1}{2}|\nabla u|^2 - f\,u \leq \int_B \frac{1}{2}|\nabla v|^2 - f\,v + \lambda\big(|\Omega_v \cap B| - |\Omega_u \cap B|\big)^+. \quad (21)$$

Using that $\big(|\Omega_v \cap B| - |\Omega_u \cap B|\big)^+ \leq |B \cap [u=0]|$ yields (16).

For (17), we apply Theorem 2.4 with $v = u + t\varphi, t > 0$: this gives

$$< \Delta u + f, \varphi > \leq \frac{t}{2} \int |\nabla \varphi|^2 + \frac{\lambda}{t}|\Omega_\varphi|.$$

Minimizing over $t > 0$ and changing φ into $-\varphi$ yields (17).

The proof of the last two points may be found in [4]. We recall here the main ingredients. We define $p_n : \mathbb{R} \to \mathbb{R}$ by

$$\forall r \leq 0, p_n(r) = 0;\ \forall r \in [0, 1/n], p_n(r) = nr;\ \forall r \geq 1/n, p_n(r) = 1,$$

and $q_n(r) = \int_0^r p_n(s)\,ds$. Let $\psi \in C_0^\infty(D)$. We apply the definition of u in (5) with $v = u + t\psi p_n(u)$ (note that $|\Omega_v| \leq |\Omega_u|$). Dividing by t and letting t tend to 0 give

$$0 = \int_D p_n(u)\nabla\psi\nabla u + \psi p_n'(u)|\nabla u|^2 - f\psi p_n(u),$$

that is

$$n|\nabla u|^2\chi_{[0<u<1/n]} - \Delta(q_n(u)) - f p_n(u) = 0 \ in\ D,$$

in the sense of distributions. As n tends to ∞, $p_n(u)$ converges a.e. to $\chi_{[u>0]}$ and $q_n(u)$ converges to u^+ in $L^2(D)$. This proves that the sequence of nonnegative functions $\mu_1^n = n|\nabla u|^2\chi_{[0<u<1/n]}$ converges in the sense of measures to a nonnegative measure $\mu_1 = \Delta(u^+) + f\chi_{[u>0]}$. Moreover, for all $\eta > 0$ and n large enough $\mu_1^n([u > \eta] \cap [u < 0]) = 0$. The property (19) follows for μ_1 (see Remark 3.5 below). The proof is the same for μ_2.

For the last point, we use that $-\Delta|u| \leq |f|\ on\ D$. We can find an open set ω such that $\Omega_u \subset \omega \subset D$ with $|\omega| \leq 2|\Omega_u| = 2m$. Then, we can use classical L^∞-estimates (see for instance [12], Theorem 8.16), to deduce that

$$\|u\|_{L^\infty(D)} = \|u\|_{L^\infty(\omega)} \leq C(d,m)\|f\|_{L^r(\omega)},$$

for some $r \in (d/2, q)$. Then we use $\|f\|_{L^r(\omega)} \leq C(m,q)\|f\|_{L^q(D)}$.

Remark 3.5. Since μ_1, μ_2 are measures which do not charge the sets of H^1-capacity zero, it makes sense to say that they do not charge the set $[u \neq 0]$ since it is defined up to a set of capacity zero. Actually, we will use the point (19) only after showing that u is continuous. Therefore, we could have stated and proved (19) only in this simpler case where u is continuous and where only open sets are involved. In particular, we used

$$\forall \eta > 0, \mu_1([u > \eta] \cap [u < 0]) \leq \liminf_{n\to+\infty} \mu_1^n([u > \eta] \cap [u < 0]),$$

which is valid since μ_1^n converges to μ_1 in the sense of measures. This remains true in the quasi-continuous case, but one needs to use the framework of quasi-open sets, see e.g. [17, 18].

The proof of Theorem 3.1 will require the following two general lemmas. They are more or less classical (see e.g. [12, 14]). We recall the main ingredients in the Appendix. We use the notation $\fint_{\partial B(x_0,r)} U$ to denote the average of U over $\partial B(x_0, r)$.

Lemma 3.6. *Let $B(x_0, r_0) \subset D$ and $U \in C^2(B(x_0, r_0))$. Then, for all $r \in (0, r_0)$*

$$\fint_{\partial B(x_0,r)} U - U(x_0) = (d\omega_d)^{-1} \int_0^r s^{1-d} \left[\int_{B(x_0,s)} \Delta U \right] ds. \tag{22}$$

This remains valid for all $U \in H^1(B(x_0, r_0))$ such that ΔU is a measure, satisfying

$$\int_0^r s^{1-d} \left[\int_{B(x_0,s)} d(|\Delta U|) \right] ds < +\infty, \tag{23}$$

and U is then pointwise defined by

$$U(x_0) = \lim_{\rho \to 0} \fint_{\partial B(x_0,\rho)} U.$$

The estimate (23) is satisfied if, moreover, $U \in L^\infty(B(x_0, r_0))$ and there exists $g \in L^q(B(x_0, r_0)$ with $q > d/2$ such that $\Delta U^+ \geq -g, \Delta U^- \geq -g$.

Lemma 3.7. *Let $B(x_0, r_0) \subset D, r_0 \leq 1, F \in L^q(B(x_0, r_0)), q > d$. Then, there exists $C = C(\|F\|_{L^q(B(x_0,r_0))}, d)$ such that, for $r \in (0, r_0)$*

- *if $\Delta U = F$ on $B(x_0, r_0)$, then*

$$\|\nabla U\|_{L^\infty(B(x_0,r/2))} \leq C[1 + r^{-1}\|U\|_{L^\infty(B(x_0,r)}], \tag{24}$$

- *if $\Delta U \geq F$ and $U \geq 0$ on $B(x_0, r_0)$, then*

$$\|U\|_{L^\infty(B(x_0,2r/3)} \leq C \left[r + \fint_{\partial B(x_0,r)} U \right]. \tag{25}$$

Now, the proof of the Theorem will rely on the two following properties of the solution u of (5).

Lemma 3.8. *Under the assumptions of Theorem 3.1, the function u is continuous on D.*

Lemma 3.9. *Under the assumptions of Theorem 3.1, there exists C such that if $x_0 \in D_{\delta/2}$ and $u(x_0) = 0$, then*

$$\forall\, r \in (0, \delta/16), \ \ |\Delta|u||(\bar{B}(x_0, r)) \leq Cr^{d-1}.$$

Proof of Theorem 3.1. We denote by C any constant depending only on $\|F\|_{L^q(D)}, \|u\|_{L^\infty(D)}, d, \delta$. We may assume $\delta \le 1$.

By Lemma 3.8, u is continuous on D. Let $\omega = [u \neq 0]$. Then, by (16), $-\Delta u = f$ on the open set ω. For $x \in \omega \cap D_\delta$, let $X_x \in \partial\omega$ be such that $d_x = d(x, \bar{D} \setminus \omega) = |x - X_x|$.

If $d_x \ge \delta/48$, then $-\Delta u = f$ on $B(x, \delta/48) \subset D$ and, by (24), $|\nabla u(x)| \le C$.

If $d_x < \delta/48$, then $u(X_x) = 0, X_x \in D_{\delta/2}, 3d_x \le \delta/16$. By (24) applied on $B(x, d_x) \subset \omega \subset D$ with $U = u, F = -f$, we have

$$|\nabla u(x)| \le C[1 + d_x^{-1}\|u\|_{L^\infty(B(x,d_x))}] \le C[1 + d_x^{-1}\|u\|_{L^\infty(B(X_x,2d_x))}].$$

By (25) applied on $B(X_x, 3d_x) \subset D$ with $U = |u|, F = -|f|$,

$$\|u\|_{L^\infty(B(X_x,2d_x))} \le C\left[d_x + \fint_{\partial B(X_x,3d_x)} |u|\right].$$

But, by (22) and Lemma 3.9 applied on $B(X_x, 3d_x)$,

$$\fint_{\partial B(X_x,3d_x)} |u| \le C d_x.$$

We deduce from all these inequalities that $\nabla u(x)$ is bounded by C on $\omega \setminus D_\delta$. Since $\nabla u = 0$ a.e. on $D \setminus \omega$, this completes the proof of the theorem.

Proof of Lemma 3.8. Let x_n converge to $x_\infty \in D$. Set $\delta_n = |x_\infty - x_n|$. If, for some n, $|B(x_\infty, \delta_n) \cap [u = 0]| = 0$, then, by (16), $-\Delta u = f$ on $B(x_\infty, \delta_n)$ and, in particular u is continuous at x_∞.

Assume now that for all n, $|B(x_\infty, \delta_n) \cap [u = 0]| \neq 0$. Consider the function $u_n(\xi) = u(x_\infty + \delta_n \xi)$. Since it is uniformly bounded, up to a subsequence, we may assume that u_n converges to some function u_∞, at least $*$-weakly in $L^\infty(\mathbb{R}^d)$. We will prove that $u_\infty = 0$ and that the convergence holds uniformly on B_1. It will follow that u is continuous at x_∞ (and $u(x_\infty) = 0$: note that, by Lemma 3.6 and (18) in Proposition 3.4), we may assume that u is everywhere defined).

For all $R \ge 1$, let us introduce the solution v_R of

$$-\Delta v_R = f \;\; on \;\; B(x_\infty, \delta_n R), \;\; v_R - u \in H_0^1(B(x_\infty, \delta_n R)),$$

and set $v_n(\xi) = v_R(x_\infty + \delta_n \xi)$. By (16), we have

$$\int_{B_R} |\nabla(u_n - v_n)|^2 \le C(\lambda, R)\, \delta_n^2, \; -\Delta v_n = \delta_n^2 f(x_\infty + \delta_n \xi). \tag{26}$$

In particular, $v_n - u_n$ tends to 0 in $H^1(B_R)$. Since v_n is bounded and Δv_n converges to 0 in L^q, it converges uniformly on compact subsets of B_R. The limit, which is necessarily equal to u_∞, is a harmonic function on B_R, that is on $\mathbb{R}^d$ since R is arbitrary. As it is also globally bounded, it is a constant. Moreover, the convergence of u_n holds in $H^1_{loc}(B_R)$.

Let us prove that

$$u_\infty \equiv 0. \tag{27}$$

Then, thanks to the inequality

$$-\Delta(|u_n|)(\xi) \leq \delta_n^2 |f(x_\infty + \delta_n \xi)|,$$

where the right-hand side tends to zero in L^q, the convergence will hold uniformly on B_1.

Assume $u_\infty > 0$. Then, u_n^- tends to 0 in H^1_{loc} and, thanks to the inequality $-\Delta u_n^- \leq \delta_n^2 f$, the convergence holds uniformly on balls. Let $y_n = x_\infty + \delta_n \xi_n,\ \xi_n \in B_1$ be such that $u(y_n) = 0$ (it does exist since $|B(x_\infty, \delta_n) \cap [u = 0]| \neq 0$). We denote $B_s = B(y_n, s)$. We use (17) with

$$\varphi \in C_0^\infty(B_{2s}), 0 \leq \varphi \leq 1, \varphi \equiv 1 \textit{ on } B_s, \|\nabla \varphi\|_{L^\infty(B_{2s})} \leq C/s.$$

Then, by (17),(18) and $f \in L^q$, $| < \mu_1 - \mu_2, \varphi > | \leq C\, s^{d-1}$. We deduce

$$\mu_1(B_s) \leq < \mu_1, \varphi > = < \mu_1 - \mu_2, \varphi > + < \mu_2, \varphi > \leq C s^{d-1} + \mu_2(B_{2s}),$$

and also, using again $f \in L^q$, (18) and $s \leq 1$,

$$\Delta u^+(B_s) \leq \Delta u^-(B_{2s}) + C s^{d-1}.$$

We multiply by s^{1-d} and integrate from 0 to δ_n to obtain, by using (22)

$$\fint_{\partial B_{\delta_n}} u^+ \leq C \fint_{\partial B_{2\delta_n}} u^- + C\,\delta_n \textit{ or } \fint_{\partial B_1} u_n^+(\xi_n + \xi) \leq \fint_{\partial B_2} u_n^-(\xi_n + \xi) + C\,\delta_n.$$

Since the right-hand side tends to 0, so does the left-hand side. But, up to a subsequence, we may assume that $\xi_n \to \xi_\infty \in B_1$ and $u_n(\xi_n + \cdot)$ converges to $u_\infty(\xi_\infty + \cdot) \equiv u_\infty$ in H^1 so that $\fint_{\partial B_1} u_n^+(\xi_n + \cdot) \to u_\infty$. We deduce (27).

Remark 3.10. In order to apply the "monotonicity lemma" 3.12, we need to know first that u is continuous. To prove it, we could have used classical results (see e.g. [20]) stating even the C^α-regularity for functions u satisfying a weakened version of (16), namely: For all balls $B \subset D$, and for v solution of $-\Delta v = f\chi_{\Omega_u} \textit{ in } B,\ \ v - u \in H^1_0(B)$, then,

$$\int_B |\nabla(u - v)|^2 \leq C|B|.$$

Here, we chose to give an independent proof based on a preliminary easy proof of the continuity.

Remark 3.11. In the case when u does not change sign, we do not need the "monotonicity lemma". Thererefore, we could avoid proving first that u is continuous. However, we still use "slightly" the continuity of u at the end of the proof of Theorem 3.1 when we claim that $u = 0\ a.e.$ (and therefore $\nabla u = 0\ a.e.$) outside ω. Actually, if we define ω as being instead the union of the balls B where $|B \cap [u = 0]| = 0$, then everything else remains valid. Therefore, it only remains to prove that

$$u = 0\ a.e.\ \ outside\ this\ new\ \omega. \tag{28}$$

For this, we can instead use the method in [2] based on a clever bound from below for $\int_B |\nabla(u-v)|^2$, which, in our nonhomogeneous case, has the following extension: Let B_r be a ball of radius $r \in (0, r_0)$ in $\mathbb{R}^d$, $U \in H^1(B_r), U \geq 0$ and $F \in L^q(B_{r_0})$ with $q > d$. Let V be the solution of

$$V - U \in H_0^1(B_r),\ -\Delta V = F\ in\ B_r.$$

Then, there exists $C_1 = C_1(d), C_2 = C_2(d, \|F\|_{L^q(B_{r_0})})$ such that, $\forall r \in (0, r_0)$

$$\int_{B_r} |\nabla(V-U)|^2 \geq C_1 \{ [\frac{1}{r} \fint_{\partial B_r} U - C_2\, r^{1-\frac{d}{q}}]^+ \}^2 |[U=0] \cap B_r|. \quad (29)$$

This result may be found in [2] for $F \equiv 0$. It is also given in [14] for bounded F. For the general case, using the change of function $u_r(x) = r^{(\frac{d}{q}-2)} u(rx)$, we reduce the proof of the lemma to the case $r = r_0 = 1$. We then introduce the solution of $W \in H_0^1(B_1),\ -\Delta W = F\ in\ B_1$ and we adapt the proof of [2]. The main point is that ∇W is uniformly bounded since $q > d$.

Now, to prove (28), let $x_0 \in D \setminus \omega$, that is such that,

$$\forall r > 0,\ |B(x_0, r) \cap [u = 0]| > 0. \quad (30)$$

Assume by contradiction that $u(x_0) = \lim_{\rho \to 0} \fint_{\partial B(x_0,\rho)} u > 0$. Then,

$$\lim_{\rho \to 0} \frac{1}{r} \fint_{\partial B(x_0,\rho)} u > 0 = +\infty.$$

Applying (29) with $U = u$, (16) and (30), we get a contradiction and (28) is proved.

Remark. A localized version of the same proof would directly show that u is locally Lipschitz continuous on each region where it does not change sign.

Proof of Lemma 3.9 in the case $0 \leq u$. We apply (17) for a test function φ with support in $B(x_0, 2r)$ and

$$\varphi \equiv 1\ on\ B(x_0, r),\ \ \|\nabla \varphi\|_\infty \leq C r^{-1},\ 0 \leq \varphi \leq 1.$$

Since $\Delta u + f \chi_{\Omega_u} \geq 0$ and $f \in L^q(D)$, the estimate of Lemma 3.9 follows. (Note that here we did not assume $u(x_0) = 0$).

Proof of Lemma 3.9 for any sign and f bounded. here, we apply the ad hoc "monotonicity Lemma" of [3] in its non homogeneous version given in [6].

Lemma 3.12. *[3,6] Let* $U \in H^1(B_{r_0})$*, continuous on* $\bar{B}_{r_0}$ *with* $U(0) = 0$ *and such that, for some* $a \geq 0$*,*

$$\Delta U^+ \geq -a,\ \ \Delta U^- \geq -a\ on\ B_{r_0}.$$

Set

$$\Phi(r) = \left(\frac{1}{r^2} \int_{B_r} \frac{|\nabla U^+|^2}{|x|^{d-2}} \right) \left(\frac{1}{r^2} \int_{B_r} \frac{|\nabla U^-|^2}{|x|^{d-2}} \right).$$

Then, if $a = 0$, $r \to \Phi(r)$ is nondecreasing on $(0, r_0)$. In all cases, there exists C such that

$$\forall r \in (0, r_0/2),\ \Phi(r) \le C\left[1 + \int_{B_{r_0}} U^2\right].$$

Since $f \in L^\infty, u(x_0) = 0$ and u is continuous, we may apply Lemma 3.12 to $U = u(x_0 + \cdot)$ on $B(0, \delta/2)$ (by assumption $B(x_0, \delta/2) \subset D$). Thus $\Phi(r)$ is uniformly bounded for $r \in (0, \delta/4)$ by $C(\delta)$ and if $B^r = B(x_0, r)$

$$r^{-2d}\left(\int_{B^r} |\nabla u^+|^2\right)\left(\int_{B^r} |\nabla u^-|^2\right) \le \Phi(r) \le C(\delta). \tag{31}$$

For each r, we introduce $v^r = v^r_+ - v^r_-, w^r = w^r_+ - w^r_-$ where $v^r_+, v^r_-, w^r_+, w^r_-$ are the solutions of

$$-\Delta v^r_+ = f^+,\ -\Delta v^r_- = f^- \ \mathit{on}\ \ B^r,\ \ v^r_+ - u^+, v^r_- - u^- \in H^1_0(B^r),$$

$$-\Delta w^r_+ = f^+,\ -\Delta w^r_- = f^- \ \mathit{on}\ \ B^r,\ \ w^r_+, w^r_- \in H^1_0(B^r).$$

Since, for $i = +, -$, $v^r_i - w^r_i$ is harmonic on B^r and equal to u^i on ∂B^r, we have

$$\int_{B^r} |\nabla(v^r_i - w^r_i)|^2 \le \int_{B^r} |\nabla u^i|^2,$$

and also (using that $v^r_i - w^r_i$ is harmonic and $u_i - v^r_i + w^r_i = 0$ on ∂B^r)

$$\int_{B^r} |\nabla(u^i - v^r_i + w^r_i)|^2 = \int_{B^r} \nabla(u^i - v^r_i + w^r_i)\nabla u^i \le 2\int_{B^r} |\nabla u^i|^2.$$

Therefore, from (31), we deduce

$$\left(r^{-d}\int_{B^r} |\nabla(u^+ - v^r_+ + w^r_+)|^2\right)\left(r^{-d}\int_{B^r} |\nabla(u^+ - v^r_- + w^r_-)|^2\right) \le C(\delta) \tag{32}$$

On the other hand

$$\int_{B^r} |\nabla(u - v^r + w^r)|^2 \le C\int_{B^r} |\nabla(v^r - u)|^2 + |\nabla w^r|^2,$$

and these last two integrals are bounded by Cr^d, the first thanks to (16), the second since $f \in L^q, q > d$ (we have $\|\nabla w\|_{L^\infty(B^r)} \le Cr^{1-\frac{d}{q}} \le C$). It follows that *each of the parentheses in (32) is bounded independently of r small*. We deduce (using also that $r^{-d}\int_{B^r} |\nabla w^r_i|^2, i = +, -$, are bounded)

$$\int_{B^r} |\nabla(u^+ - v^r_+)|^2 + \int_{B^r} |\nabla(u^- - v^r_-)|^2 \le Cr^d. \tag{33}$$

By the definition of v^r_+, v^r_- and (18–19), we have

$$\Delta(u^+ - v^r_+) = \mu_1 + f^+(1 - \chi_{u>0}) \ge \mu_1,\ \Delta(u^- - v^r_-) \ge \mu_2.$$

Therefore, integrating by parts in (33) and using also (19), we deduce

$$\int_{B^r} v_+^r \, d\mu_1 + \int_{B^r} v_-^r \, d\mu_2 \le C r^d. \tag{34}$$

But, for instance, by the formula (22) applied with $U = u^+ - v_+^r$ (so that $\Delta U \ge \mu_1$ and $U \le 0$), we may write for all $z \in B^{r/4}$ (so that $B(z, 3r/4) \subset B(x_0, r)$)

$$v_+^r(z) \ge \fint_{\partial B(z,3r/4)} U - U(z) \ge C(d) \int_0^{3r/4} s^{1-d} \int_{B(z,s)} d\mu_1. \tag{35}$$

But (35), integrated with respect to μ_1 for $z \in B^{r/4}$, together with (34) leads to

$$Cr^d \ge C \int_{B^{r/4}} d\mu_1(z) \int_0^{3r/4} ds \, s^{1-d} \int_{B(z,s)} d\mu_1 \ \ge C r^{2-d} \left[\int_{B^{r/4}} d\mu_1 \right]^2,$$

where we used $B^{r/4} \subset B(z, r/2)$ and

$$\begin{aligned} \int_0^{3r/4} ds \, s^{1-d} \, \mu_1(B(z,s)) &\ge (3r/4)^{1-d} \int_{r/2}^{3r/4} ds \, \mu_1(B(z,s)) \\ &\ge C r^{2-d} \mu_1(B^{r/4}). \end{aligned}$$

We do the same for μ_2. Lemma 3.9 follows by using also (18).

4 Lipschitz continuity for the eigenvalue problem

Theorem 4.1. *Let u be a solution of the constrained eigenvalue problem (11). Then u is Lipschitz continuous on D_δ. It follows that the shape minimization problem (13) has a solution Ω^* with $|\Omega^*| = m$ and which is at least an open subset of D whose corresponding eigenfunction is locally Lipschitz continuous.*

The proof of this theorem will be a consequence of the following properties of u.

Lemma 4.2. *Under the assumptions of Theorem 4.1, the solution u of (11) satisfies:*

$$u \ge 0, \quad \Delta u + \lambda_m u \ge 0, \quad u \in L^\infty(D). \tag{36}$$

There exists $C \ge 0$ such that, for all ball $B(x_0, r)$ such that $B(x_0, 2r) \subset D$,

$$|\Delta u|\big(B(x_0, r)\big) \le C \, r^{d-1}. \tag{37}$$

Finally, u is continuous on D and $-\Delta u = \lambda_m u$ on the open set $[u \ne 0]$.

Proof of Theorem 4.1. The proof of the Lipschitz continuity is the same as the proof of Theorem 3.1 where f is to be replaced by $\lambda_m u$. We use Lemma 4.2 for the necessary properties.

For the last part, note that Ω_u is open since u is continuous. If $|\Omega_u| = m$, then $\Omega^* = \Omega_u$ is an open solution of (13) (recall that it is the case if D is connected). If $|\Omega_u| < m$, as proved in Remark 2.10, u is either strictly positive or zero on each component of D. But there exists $R > 0$ such that $\Omega^* := \Omega_u \cup B(0,R) \cap D$ satisfies $|\Omega^*| = m$. By monotonicity, $\lambda_1(\Omega^*) \le \lambda_1(\Omega_u) = \lambda_m$ and, by minimality, equality holds. Moreover, u is also an eigenfunction for Ω^*. This completes the proof of the theorem.

Proof of Lemma 4.2. By Remark 2.10, we already know that $u \ge 0$. Then, we use the inequality (12) with $v = u - t\psi p_n(u)$ where $t > 0, \psi \in C_0^\infty(D), \psi \ge 0$ and p_n is defined as in the proof of Lemma 3.4. Note that $|\Omega_v| \le m$ and $\int_D v^2 \le 1$ for $t > 0$ small. Differentiating at $t = 0$, we obtain

$$-n|\nabla u|^2 \chi_{[0<u<1/n]} + \Delta(q_n(u)) + \lambda_m u\, p_n(u) \ge 0 \ in\ D.$$

As in the proof of Lemma 4.2, we prove that $n|\nabla u|^2\chi_{[0<u<1/n]}$ converges to a nonnegative measure μ_1, and at the limit $\Delta u + \lambda_m u \ge \mu_1$.

In particular $-\Delta u \le \lambda_m u$. Since $u \ge 0$, this implies

$$\|u\|_{L^{d/(d-2k)}} \le C\|u\|_{L^k}\ if\ 1 < k < d/2;\ \|u\|_\infty \le C\|u\|_{L^k}\ if\ k > d/2.$$

By a finite bootstrap over k, starting at $k = 2$, we obtain $u \in L^\infty$.

Next, we apply (12) to $v = u + t\varphi$ with $\varphi \ge 0$ and we are led to

$$2 < \Delta u + \lambda_m u, \varphi > \le \int_D [2\lambda_m u\varphi + t|\nabla\varphi|^2] + \frac{\lambda}{t}|\Omega_\varphi|.$$

We choose $\varphi \in C_0^\infty(B(x_0, 2r))$ such that

$$0 \le \varphi \le 1,\ \ \varphi \equiv 1\ on\ B(x_0, r),\ \ \|\nabla\varphi\|_{L^\infty} \le C/r.$$

Minimizing over $t > 0$ as in the proof of Lemma 4.2 and using that $u \in L^\infty$ and $\Delta u + \lambda_m u \ge 0$, we deduce the estimate (37).

Now, let us prove that

$$|B(x_0, r) \cap [u = 0]| = 0 \Rightarrow -\Delta u = \lambda_m u\ \ in\ B(x_0, r). \tag{38}$$

Let $v = u$ outside $B(x_0, r)$ and equal on $B(x_0, r)$ to the solution of

$$-\Delta v = \lambda_m u,\ \ v - u \in H_0^1(B(x_0, r)).$$

Since $u > 0$ a.e. on $B = B(x_0, r)$, then $|\Omega_v| \le |\Omega_u| = m$. Therefore, by (15),

$$\int_B |\nabla u|^2 - |\nabla v|^2 + \lambda_m(v^2 - u^2) \le 0,$$

or also

$$\int_B |\nabla(u - v)|^2 + \lambda_m(u - v)^2 \le 0,$$

so that $u = v$, which proves (38).

Finally, for the continuity of u, we argue like in the proof of Lemma 3.8: the only changes are the following

- The datum f is to be replaced by $\lambda_m u$.
- The inequality (26) is obtained via (12) applied to $v = v_R$, solution on $B(x_\infty, R\delta_n)$ of $-\Delta v_R = \lambda_m v_R$, and $v = u$ outside, which gives
$$\int_{B(x_\infty,R\delta_n)} |\nabla u|^2 - |\nabla v_R|^2 \leq \lambda (R\delta_n)^d,$$
or
$$\int_{B(x_\infty,R\delta_n)} |\nabla(u - v_R)|^2 \leq \lambda (R\delta_n)^d + 2\lambda_m \int_{B(x_\infty,R\delta_n)} u(v_R - u).$$
Then, we rewrite this last inequality in terms of u_n, v_n and we use the fact that $\|v_n\|_{L^\infty(B_R)}$ is bounded (in terms of $\|u_n\|_{L^\infty(B_R)}$) to deduce (26).
- And there is no need to involve u_n^- since $u \geq 0$.

5 Appendix

We recall here the main ingredients in the proof of the more or less classical Lemmas 3.6 and 3.7.

Proof of Lemma 3.6. The relation
$$\fint_{\partial B(x_0,r)} U - \fint_{\partial B(x_0,\rho)} U = (d\omega_d)^{-1} \int_\rho^r ds\, s^{1-d} \int_{B(x_0,s)} d(\Delta U), \qquad (39)$$
may be obtained for regular U by integrating from ρ to r the identity
$$\frac{d}{ds} \fint_{\partial B_1} U(x_0 + s\xi) = \fint_{\partial B_1} \nabla U(x_0 + s\xi) \cdot \xi = (d\omega_d)^{-1} s^{1-d} \int_{B(x_0,s)} \Delta U.$$
For more general functions $U \in (L^\infty \cap H^1)(B(x_0, r_0))$, such that ΔU is a finite measure on $B(x_0, r_0)$, we use an approximation by mollifiers $U_p = U * \rho_p$ to obtain (39).

If the estimate (23) is satisfied, then $\lim_{\rho\to 0} \fint_{\partial B(x_0,\rho)} U$ exists and we can pass to the limit as ρ tends to zero in (39).

Now, for $g \in L^q(B(x_0, r_0))$ with $q > d/2$, $\int_{B(x_0,s)} |g| \leq C(d) s^{d(1-\frac{1}{q})}$ so that $[s \to \tilde{g}(s) = s^{1-d} \int_{B(x_0,s)} |g|] \in L^1(0, s_0)$. If $U \in L^\infty \cap H^1$ with ΔU finite measure and $\Delta U^+ + g \geq 0$ *on* $B(x_0, r_0)$, then, using (39)
$$\begin{aligned}\int_\rho^r ds\, s^{1-d} \int_{B(x_0,s)} d(|\Delta U^+|) &\leq \int_\rho^r ds \left[\tilde{g}(s) + s^{1-d} \int_{B(x_0,s)} d(\Delta U^+ + g)\right] \\ &\leq 2\|U\|_\infty + 2 \int_\rho^r \tilde{g}.\end{aligned}$$
The same is true for U^- so that (23) follows.

Proof of Lemma 3.7. Recall that for the solution of
$$W \in H_0^1(B_1),\ -\Delta W = G \ \ on \ \ B_1,$$

since $q > d$, we have with $C = C(d, q)$

$$\|W\|_{C^1(B_1)} \leq C\|G\|_{L^q(B_1)} \leq Cr^{2-\frac{d}{q}}\|F\|_{L^q(B(x_0,r_0))} \leq Cr. \tag{40}$$

We apply this to the rescaled functions

$$\forall \xi \in B_1, \;\; V(\xi) = U(x_0 + r\xi), \; G(\xi) = r^2 F(x_0 + r\xi).$$

For (24), we notice that $\Delta(V - W) = 0$ *on* B_1 so that

$$\|\nabla(W - V)\|_{L^\infty(B_{1/2})} \leq C(d)\|V\|_{L^\infty(\partial B_1)}.$$

Together with (40), this inequality gives

$$\|\nabla V\|_{L^\infty(B_{1/2})} \leq C[r + \|V\|_{L^\infty(B_1)}].$$

Going back to U gives (24) by change of variable. For (25), we first notice that $-\Delta(V - W) \leq 0$, so that $(V - W)(x) \leq \int_{\partial B_1} P_x(z)V(z)\, d\sigma(z)$ where $P_x(\cdot)$ denotes the Poisson kernel at x. Using (40) again and $V \geq 0$, we deduce that

$$\|V\|_{L^\infty(B_{2/3})} \leq C[r + \fint_{\partial B_1} V(z)\, d\sigma(z)].$$

The relation (25) follows by change of variable.

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