Calc. Var. 22, 375–389 (2005) **Calculus of Variations**
DOI: 10.1007/s00526-004-0281-x

Ryo Ikota · Eiji Yanagida

Stability of stationary interfaces of binary-tree type

Received: 8 December 2003/ Accepted: 5 April 2004
Published online: 16 July 2004 –

Abstract. We consider the curvature-driven motion of an interface on a bounded domain that contacts with the boundary at the right angle and has triple junctions with prescribed angles. We derive a linearized system at a stationary interface, and obtain a characteristic function whose zeros correspond to the eigenvalues of the linearized operator. From the characteristic function, it is shown that the unstable dimension is not relevant to the topology of the stationary interface but depends mainly on the curvature of the boundary.

1. Introduction

In various nonlinear phenomena such as annealing pure metal (Mullins [12]) and segregation between biological species (Ei et al. [4]), we can observe that the medium is separated into subregions by interfaces with triple junctions. In some situation, these interfaces evolve in time depending on their curvatures with prescribed angles at triple junctions.

In this paper, we consider the curvature-driven motion of curves in a two-dimensional bounded domain under the situation where the curves form a network with triple junctions. Our purpose is to study an eigenvalue problem derived by formal linearization of a model equation around stationary interfaces of the motion. Though a part of the results also holds for more general networks, we restrict ourselves to networks that are topologically equivalent to binary trees (see Fig. 1). Here by binary trees, we mean connected graphs without any cycles in which every vertex has either one edge or three edges (see [1] for the terminologies).

This situation can be formulated mathematically as follows. Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, and consider a network of curves with triple junctions in Ω. We assume that the network $\Gamma = \Gamma(t)$ consists of n curves denoted by $\gamma_i = \gamma_i(t), i = 1, 2, \ldots, n$, and contacts with $\partial\Omega$ at endpoints (see Fig. 1). We often regard Γ as the set of the curves $\{\gamma_i\}$, and denote by B the subset of Γ that consists of curves touching $\partial\Omega$. We denote by $V = \{x_l\}$ the set of triple junctions. Each curve γ_i is driven to the center of curvature at the normal speed V_i

R. Ikota: Faculty of Mathematics, Kyushu University, Fukuoka 812-8581, Japan
(e-mail: ikota@math.kyushu-u.ac.jp)

E. Yanagida: Mathematical Institute, Tohoku University, Sendai 980-8578, Japan
(e-mail: yanagida@math.tohoku.ac.jp)

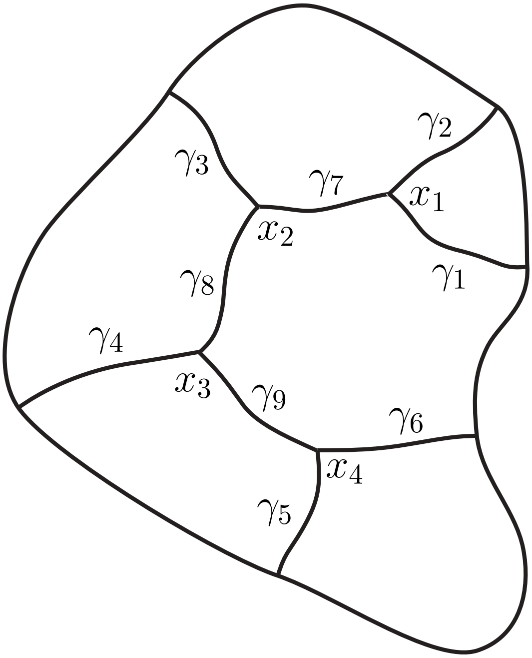

Fig. 1 An interface Γ with triple junctions. The set B is $\{\gamma_i | i = 1, 2, \dots, 6\}$

that is equal to the curvature of γ_i at each point. At each triple junction x_l, three curves meet with prescribed angles, and each $\gamma_i \in B$ contacts with $\partial\Omega$ at the right angle.

At first we express every curve $\gamma_i(t)$ by using a position vector $p_i(s, t)$, where s is an arclength parameter measured from one end of γ_i. Later we will use another expression. Assuming that p_i is sufficiently smooth, the motion of $\Gamma(t)$ is described as follows:

(M1) The normal velocity $V_i = \frac{\partial p_i}{\partial t} \cdot N_i$ satisfies $V_i = \kappa_i$, where N_i is the unit normal vector to γ_i pointing the left of the unit tangent vector $\frac{\partial p_i}{\partial s}$, and κ_i represents the signed curvature of γ_i given by $\kappa_i = \frac{\partial^2 p_i}{\partial s^2} \cdot N_i$.

(M2) If three curves γ_i, γ_j, γ_k meet at a triple junction $x_l(t)$, then the contact angles among them satisfy Young's law, that is, for some positive constants σ_i, σ_j and σ_k, it holds

$$\frac{\sin\theta_{l,i}}{\sigma_i} = \frac{\sin\theta_{l,j}}{\sigma_j} = \frac{\sin\theta_{l,k}}{\sigma_k}, \ (0 < \theta_{l,i}, \theta_{l,j}, \theta_{l,k} < \pi, \theta_{l,i} + \theta_{l,j} + \theta_{l,k} = 2\pi)$$

where $\theta_{l,i}$ is the contact angle between γ_j and γ_k, and so on.

(M3) If $\gamma_i(t)$ contacts with $\partial\Omega$, then the tangent vectors of $\gamma_i(t)$ and $\partial\Omega$ at the point of contact are orthogonal to each other.

We note that each curve of a stationary interface is a line segment. Although it is not clear whether or not there exists a stationary interface for a given domain, we can always construct a domain that admits the existence of a given set of line segments satisfying Young's law as a stationary interface. In this paper, assuming the existence of stationary interfaces, we study their linearized stability. Suppose that perturbations to a stationary interface are represented as graphs on the line segments of the stationary interface. Then we obtain a system of linear elliptic equations by formal linearization in the same manner as in our preceding paper [10]. We denote by $\mathcal{L}$ the resulting linear operator, and by N_{U} the number of positive eigenvalues of $\mathcal{L}$. Our goal is to determine the unstable dimension N_{U}. (More precise description of $\mathcal{L}$ and N_{U} will be given in the next section.)

The unstable dimension can be interpreted as follows from a variational viewpoint. Let $L_i(t)$ denote the length of γ_i, and define an energy functional by

$$E[\Gamma] := \sum_{\gamma_i \in \Gamma} \sigma_i L_i.$$

Then the energy $E[\Gamma(t)]$ is decreasing in t, because

$$\frac{d}{dt} E[\Gamma(t)] = -\sum_{\gamma_i \in \Gamma} \sigma_i \int_0^{L_i} V_i \kappa_i ds = -\sum_{\gamma_i \in \Gamma} \sigma_i \int_0^{L_i} \kappa_i^2 ds \leq 0.$$

In particular, this implies that any stationary interface corresponds to a critical point of this energy functional. The second variation of $E[\Gamma]$ at a stationary interface is associated with the linearized operator $\mathcal{L}$, and the unstable dimension N_{U} corresponds to the Morse index.

In order to determine the unstable dimension N_{U}, we encounter the following difficulties:

(i) Direct computations are extremely complicated. In fact, the computation is complicated enough even for a stationary interface with only one triple junction (see [10]).
(ii) There are infinitely many kinds of topologically different networks. Moreover, there are topologically different networks with the same number of triple junctions (see Fig. 2).

Thus, in a general setting, we need a systematic approach to determine N_{U}. Specifically, in order to overcome the above difficulties, we will construct characteristic functions for eigenvalues inductively and combine them with variational methods.

Now we are in a position to state our main result.

Theorem 1.1 *Let $\Gamma = \{\gamma_i\}$ be a stationary interface that is homeomorphic to a binary tree. Define a characteristic index D by*

$$D = \sum_{\gamma_i \in \Gamma} \sigma_i L_i \times \prod_{\gamma_i \in B} h_i + \sum_{\gamma_i \in B} \left\{ \sigma_i \prod_{\gamma_j \in B \setminus \{\gamma_i\}} h_j \right\},$$

where h_i denotes the curvature of $\partial\Omega$ at the point of contact with $\gamma_i \in B$. (Note that h_i is taken to be nonpositive if Ω is convex.)

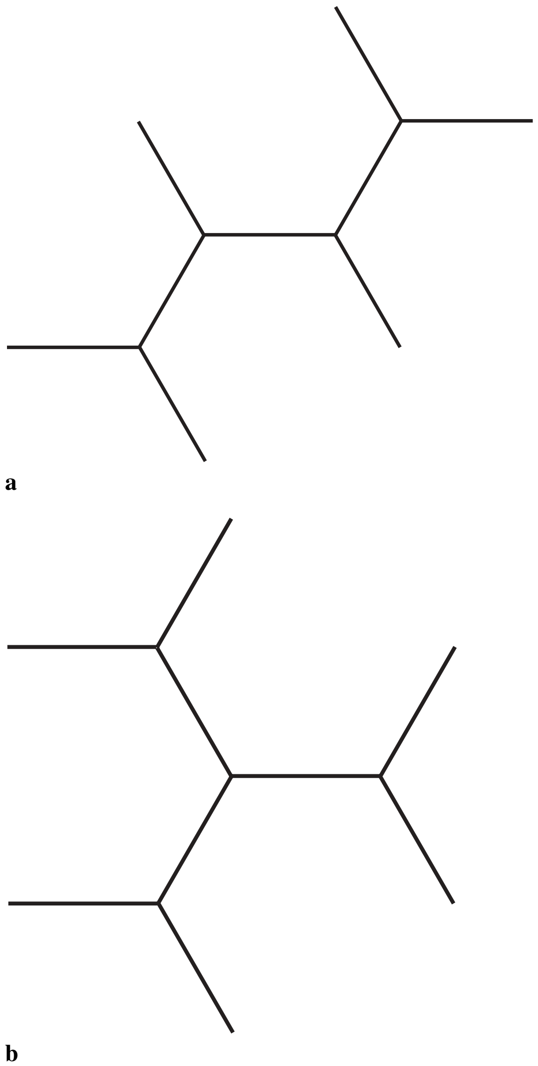

Fig. 2a,b. Interfaces with four triple junctions; they are topologically different

(i) *The unstable dimension* N_{U} *is given by*

$$N_{\mathrm{U}} = \begin{cases} m-1 & \textit{for} \;\; (-1)^m D \leq 0, \\ m & \textit{for} \;\; (-1)^m D > 0, \end{cases}$$

where $m = \#\{h_i < 0\}$.

(ii) *The stationary interface is degenerate (i.e., there exists a zero eigenvalue) if and only if* $D = 0$.

We say that a stationary interface is *linearly stable* if there is no nonnegative eigenvalue, and is *linearly unstable* if there is at least one positive eigenvalue. As a direct consequence of Theorem 1.1, we have the following result.

Corollary 1.1

(i) *If all of h_i are positive, then any stationary interface is linearly stable.*
(ii) *If at least two of h_i are negative, then any stationary interface is linearly unstable.*

Here are some remarks about this result. First, for an interface with one triple junction, the characteristic index is given by

$$D = (\sigma_1 L_1 + \sigma_2 L_2 + \sigma_3 L_3)h_1 h_2 h_3 + (\sigma_1 h_2 h_3 + \sigma_2 h_1 h_3 + \sigma_3 h_1 h_2), \tag{1}$$

which was obtained in our previous paper [10]. For an interface with two triple junctions as in Fig. 3, the curve γ_3 does not contact with the boundary. In this case, the characteristic index is written as

$$D = (\sigma_1 L_1 + \sigma_2 L_2 + \sigma_3 L_3 + \sigma_4 L_4 + \sigma_5 L_5)h_1 h_2 h_4 h_5 \\ +(\sigma_1 h_2 h_4 h_5 + \sigma_2 h_1 h_4 h_5 + \sigma_4 h_1 h_2 h_5 + \sigma_5 h_1 h_2 h_4).$$

Thus, D is symmetric with respect to $\gamma_1, \gamma_2, \gamma_4, \gamma_5$, but γ_3 is different from others. For an interface with four triple junctions, there are two possible configurations that are topologically different (see Fig. 2). It is interesting to note that the characteristic indices for these two interfaces are the same. In fact, the characteristic index which we will construct later is independent of such topological difference.

This paper is organized as follows. In Sect. 2, we formulate the linearized operator. The procedure is similar to that in our former study [10]. Section 3 introduces a variational formulation for eigenvalues. Section 4 deals with characteristic indices. In Sect. 5 we give a proof of Theorem 1.1.

In the following sections, Γ is supposed to be a stationary interface that is topologically equivalent to a binary tree.

2. Linearization

In this section we derive a system of linearized equations that approximates the motion of interfaces near a stationary interface. We consider perturbations that can be represented as graphs of functions on Γ, and describe the motion of nearby interfaces by using some nonlinear partial differential equations with moving boundaries.

Suppose that $\gamma_i, \gamma_j, \gamma_k$ meet at a triple junction $x_l \in V$. We take x_l as the origin of ξ-η coordinate system. For γ_i, the ξ-axis is taken along γ_i, and the η-axis is taken by rotating the ξ-axis by $\pi/2$ radian counter-clockwise. In this coordinate system we consider a perturbation which can be represented as a graph of $\eta = w_i(\xi)$. The function w_i is defined on some interval of ξ with moving boundaries. Approximating the time evolution of w_i, we obtain a linear operator $\mathcal{L}$ at γ_i. Notice that the resulting

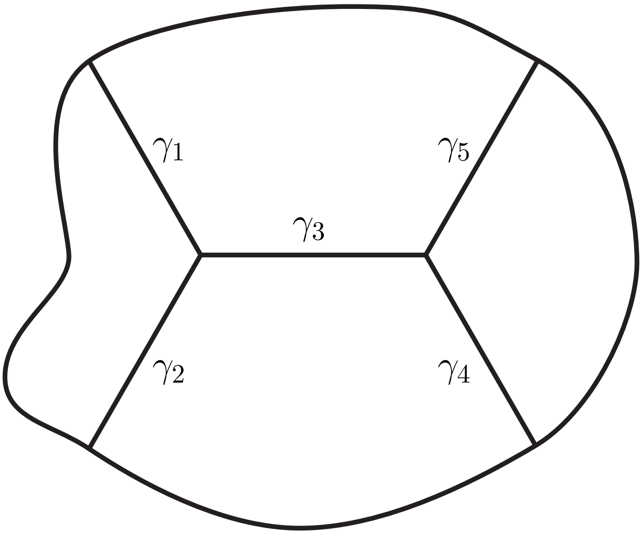

Fig. 3 A stationary interface with two triple junctions

equation is defined on the fixed domain $[0, L_i]$. We take coordinate systems for γ_j and γ_k in the same way, and describe perturbations by using some functions $w_j(t,\xi)$ and $w_k(t,\xi)$. The boundary conditions and matching conditions on the interface can be transformed into boundary conditions on $w_i(t,\xi)$, $w_j(t,\xi)$ and $w_k(t,\xi)$. For details of this procedure, we refer to our previous paper [10].

For $\gamma_i \in B$ we chose a coordinate system in which $\xi = L_i$ corresponds to the point of contact with $\partial\Omega$. As for $\gamma_i \in \Gamma \setminus B$, both endpoints are triple junctions. Hence there are two ways of introducing the coordinate system on γ_i. We will choose one of these coordinate systems according to situations in order to make the presentation simple. We remark that if we take the other end of γ_i as the origin, we will obtain the function $\eta = \tilde{w}_i(\xi) = -w_i(L_i - \xi)$ for the same perturbation.

Now let us describe the linear operator $\mathcal{L}$ more concretely. Put $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, where u_i is defined on γ_i. Then $\mathcal{L}$ is written as

(2) $$\mathcal{L}[\boldsymbol{u}] = \frac{\partial^2 \boldsymbol{u}}{\partial \xi^2}.$$

The associated boundary conditions are given as follows.

1. For $\gamma_i \in B$,

(3) $$\frac{\partial u_i}{\partial \xi}(L_i) + h_i u_i(L_i) = 0.$$

2. If $\gamma_i, \gamma_j, \gamma_k$ meet at $x_l \in V$,

(4) $$\sigma_i u_i(0) + \sigma_j u_j(0) + \sigma_k u_k(0) = 0,$$

(5) $$\frac{\partial u_i}{\partial \xi}(0) = \frac{\partial u_j}{\partial \xi}(0) = \frac{\partial u_k}{\partial \xi}(0).$$

We set

$$\boldsymbol{H} := \bigoplus_{\gamma_i \in \Gamma} L^2(0, L_i),$$

and treat $\mathcal{L}$ as an operator from $\boldsymbol{H}$ to $\boldsymbol{H}$ with a domain of definition

$$\mathcal{D}(\mathcal{L}) = \left\{ \boldsymbol{u} \in \bigoplus_{\gamma_i \in \Gamma} H^2(0, L_i) \ \mid \ \boldsymbol{u} \text{ satisfies conditions (3)–(5)} \right\}.$$

The inner product $(\cdot, \cdot)_{\boldsymbol{H}}$ of $\boldsymbol{H}$ is given by

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathbf{H}} := \sum_{\gamma_i \in \Gamma} \left\{ \sigma_i \int_0^{L_i} u_i v_i d\xi \right\}.$$

3. Variational methods

The operator $\mathcal{L}$ introduced in Sect. 2 naturally leads to a bilinear form.

Definition 3.1 *A bilinear form $J : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ is defined by*

$$J(\boldsymbol{u}, \boldsymbol{v}) := \sum_{\gamma_i \in B} h_i u_i(L_i) v_i(L_i) + \sum_{\gamma_i \in \Gamma} \sigma_i \int_0^{L_i} \partial_\xi u_i(\xi) \partial_\xi v_i(\xi) d\xi,$$

where

$$\boldsymbol{V} := \{ \boldsymbol{u} \in \bigoplus_{\gamma_i \in \Gamma} H^1(0, L_i) \ \mid \ \boldsymbol{u} \textit{ satisfies the condition } (4) \}.$$

The inner product $(\cdot, \cdot)_{\boldsymbol{V}}$ is given by

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathbf{V}} := \sum_{\gamma_i \in \Gamma} \left\{ \sigma_i \int_0^{L_i} (u_i v_i + \partial_\xi u_i \, \partial_\xi v_i) d\xi \right\}.$$

In addition we introduce a functional $I : \boldsymbol{V} \setminus \{\mathbf{0}\} \to \mathbb{R}$ defined by

$$I(\boldsymbol{u}) := \frac{J(\boldsymbol{u}, \boldsymbol{u})}{(\boldsymbol{u}, \boldsymbol{u})_{\mathbf{H}}}.$$

We can characterize the eigenvalues of $\mathcal{L}$ in terms of I. Here we describe some useful properties of $\mathcal{L}$, J and I. Discussions similar to [10] yield the following result:

Proposition 3.1 *There exist positive numbers c and d such that*

$$\|\boldsymbol{u}\|_{\mathbf{V}}^2 \leq c(\boldsymbol{u}, \boldsymbol{u})_{\mathbf{H}} + dJ(\boldsymbol{u}, \boldsymbol{u}) \qquad \textit{for all } \boldsymbol{u} \in \boldsymbol{V}.$$

From this we deduce that the operator $\mathcal{L}$ is self-adjoint.

Let $\mathfrak{H}$ be the family of all finite dimensional subspaces of $\boldsymbol{H}$. Denote by λ_j the jth eigenvalue of $\mathcal{L}$. Then we have $\lambda_j \geq \lambda_{j+1}$. The eigenvalues λ_j are characterized by the sup-inf principle:

$$-\lambda_j = \sup_{\substack{K \in \mathfrak{H} \\ \dim K \leq j-1}} \inf_{v \in K^{\perp} \setminus \{\mathbf{0}\}} I(\boldsymbol{v}), \tag{6}$$

where

$$K^{\perp} := \{\boldsymbol{u} \in \boldsymbol{V} \mid (\boldsymbol{u}, \boldsymbol{v})_{\mathbf{H}} = 0 \text{ for all } \boldsymbol{v} \in K\}.$$

For the proof, see Sect. 1, Chapter 13 of [13].

If we take $\{h_i\}$ as parameters, each eigenvalue is a continuous and monotone decreasing function of h_i. See Theorems 6 and 9 in Chapter 6 of [3].

Proposition 3.2 *Put* $m = \#\{h_i < 0 \mid \gamma_i \in B\}$. *Then* $N_{\mathrm{U}} \geq m - 1$.

Proof. We can assume $B = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ by renumbering the elements of Γ if necessary. Suppose that $h_1, h_2, \ldots, h_m$ $(m \geq 2)$ are negative. Since Γ is of binary-tree type, for each $i = 1, 2, \ldots, m-1$, there is a unique path on Γ which connects γ_i and γ_m. More precisely, for each $i = 1, 2, \ldots, m-1$, there is a unique subset, say Γ^i, of Γ such that Γ^i is homeomorphic to a line segment and $\gamma_i, \gamma_m \in \Gamma^i$. Then we can choose a function $\boldsymbol{\varphi}^i$ on Γ such that

(i) $\boldsymbol{\varphi}^i$ is constant and nonzero on each γ_i,
(ii) $\boldsymbol{\varphi}^i$ is equal to zero on $\Gamma \setminus \Gamma^i$, and
(iii) $\boldsymbol{\varphi}^i$ satisfies (4).

Now, for any $v \in M := \mathrm{Span}[\varphi^1, \varphi^2, \ldots, \varphi^{m-1}]$, we have $I(v) < 0$. Moreover,

$$\sup_{\substack{v \in M \\ v \neq 0}} I(v) = \sup_{\substack{\|v\|_{\boldsymbol{H}} = 1 \\ v \in M}} I(v) < 0,$$

because $\{v \in M \mid \|v\|_{\boldsymbol{H}} = 1\}$ is compact.

When $m \geq 3$, for any $\psi_1, \psi_2, \ldots, \psi_{m-2} \in \boldsymbol{H}$, we can choose $\varphi \in M$, $\varphi \neq 0$ such that $(\varphi, \psi_i)_{\boldsymbol{H}} = 0$ $(i = 1, 2, \ldots, m-2)$. Therefore from the sup-inf principle, we obtain $\lambda_{m-1} > 0$. The case $m = 2$ is similar and the case $m = 1$ is trivial. □

Remark 3.1. Proposition 3.2 holds also for stationary interfaces that are not necessarily of binary-tree type. The proof is the same except that the path connecting γ_i and γ_m may not be unique.

4. Characteristic functions

In this section we define a characteristic function whose zeros correspond to the eigenvalues of $\mathcal{L}$. When Γ has only one triple junction, such a characteristic function was obtained in Sect. 4 of [10]. We first sketch the outline briefly.

Let Γ be a stationary interface which consists of three line segments γ_i, $i = 1, 2, 3$. We denote the eigenfunctions of $\mathcal{L}$ associated with an eigenvalue $\lambda = \mu^2 \neq 0$ by $(U_1(\xi), U_2(\xi), U_3(\xi))$. Then, by (2), we have

$$\frac{d^2}{d\xi^2}U_i(\xi) = \mu^2 U_i(\xi), \quad 0 < \xi < L_i \qquad (i = 1, 2, 3)$$

and hence we can express $U_i(\xi)$ as

$$U_i(\xi) = a_i \sinh(\mu\xi) + b_i \cosh(\mu\xi) \qquad (i = 1, 2, 3)$$

with some constants a_i, b_i. By (3)$\sim$(5), we have

$$\text{(7)} \qquad \begin{cases} a_i\varphi(\mu; h_i, L_i) + b_i\psi(\mu; h_i, L_i) = 0 \qquad (i = 1, 2, 3), \\ a_1 = a_2 = a_3, \\ \sigma_1 b_1 + \sigma_2 b_2 + \sigma_3 b_3 = 0, \end{cases}$$

where φ and ψ are defined by

$$\varphi(\mu; h, L) := h \sinh(\mu L) + \mu \cosh(\mu L),$$
$$\psi(\mu; h, L) := h \cosh(\mu L) + \mu \sinh(\mu L).$$

We regard (7) as a system of linear homogeneous equations for unknowns a_i, b_i $(i = 1, 2, 3)$. Then the determinant of the coefficient matrix is computed as

$$\text{(8)} \qquad \begin{aligned} F(\mu) = \sigma_1\varphi(\mu; h_1, L_1)\psi(\mu; h_2, L_2)\psi(\mu; h_3, L_3) \\ +\sigma_2\psi(\mu; h_1, L_1)\varphi(\mu; h_2, L_2)\psi(\mu; h_3, L_3) \\ +\sigma_3\psi(\mu; h_1, L_1)\psi(\mu; h_2, L_2)\varphi(\mu; h_3, L_3), \end{aligned}$$

and the system of linear equations has a nontrivial solution (and hence $\lambda = \mu^2$ is a non-zero eigenvalue of $\mathcal{L}$) if and only if $F(\mu) = 0$. Thus, $F(\mu)$ with $\lambda = \mu^2$ is a characteristic function for the stationary interface with one triple junction.

For a stationary interface with two or more triple junctions, it is extremely complicated to compute a characteristic function directly in the same way as in the case of one triple junction. Our idea to overcome this difficulty is to decompose the stationary interface into two stationary interfaces with less triple junctions, and define a characteristic function inductively.

Let Γ be a stationary interface with two or more triple junctions. We divide the interface Γ into two parts Γ^α and Γ^β by considering a virtual boundary which separates the domain into two subdomains (see Fig. 4). More precisely, we first choose $\gamma_k \in \Gamma \setminus B$ arbitrarily. Since each eigenfunction is analytic on $\gamma_k \in \Gamma$

and has at most finite number of zeros on γ_k, we can take a point $\xi = \xi_0$ such that any eigenfunction does not vanish at $\xi = \xi_0$. We divide the domain Ω by a smooth curve C which intersects γ_k orthogonally at $\xi = \xi_0$ and does not intersect the other γ_i ($\neq \gamma_k$). By this, we assume that γ_k is divided into two parts γ_k^α $(0 < \xi < \xi_0)$ and γ_k^β $(\xi_0 < \xi < L_k)$. We regard Γ^α and Γ^β as stationary interfaces on the domains Ω^α and Ω^β, respectively. If the curvature of C at the point of contact with Γ^α is $+h$, then the curvature of C at the point of contact with Γ^β is given by $-h$.

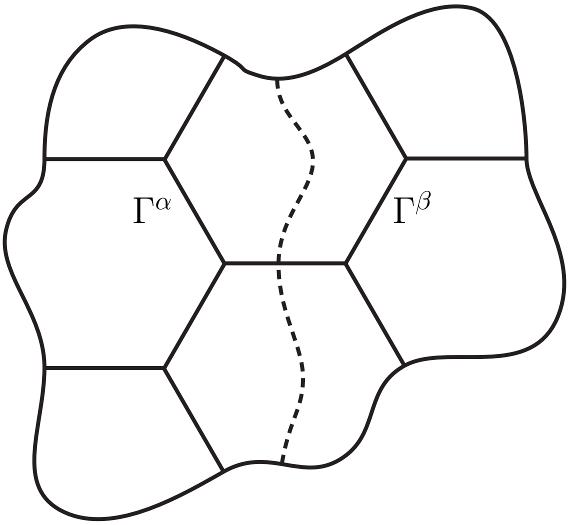

Fig. 4 The interface divided into two parts. The dotted line stands for the virtual boundary

Assuming that characteristic functions, say F^α and F^β, are obtained for Γ^α and Γ^β, respectively, we define a characteristic function F for Γ by using F^α and F^β.

Proposition 4.1 *For any stationary interface Γ, there exists a complex-valued characteristic function $F = F(\mu)$ of a complex variable μ with parameters σ_i, L_i, $(\gamma_i \in \Gamma)$ and h_i $(\gamma_i \in B)$ satisfying the following properties:*

(i) *For $\mu \neq 0$, $F(\mu) = 0$ if and only if $\lambda = \mu^2$ is an eigenvalue of $\mathcal{L}$.*
(ii) *$F(\mu)$ is analytic in μ, σ_i, L_i, and h_i. Further, $F(\mu)$ is real-valued if μ is restricted to real numbers.*
(iii) *$F(\mu)$ is odd with respect to μ. In particular, $F(0) = 0$ for any σ_i, L_i, h_i.*
(iv) *Any zero of $F(\mu)$ lies on the real axis or imaginary axis, and it depends on σ_i, L_i, h_i continuously.*
(v) *$F(\mu) \to +\infty$ as $\mu \to +\infty$.*
(vi) *For each $\gamma_i \in B$, F is written as $F = P(\mu)h_i + Q(\mu)$, where P and Q are independent of h_i.*

Proof. We prove this by induction. First, if Γ has only one triple junction, the assertion follows immediately from the explicit formula (8).

Next, let Γ be a stationary interface with two or more triple junctions. We divide Γ into two parts Γ^α and Γ^β as above by introducing a virtual boundary C which intersects γ_k orthogonally. Suppose that the assertion is true for Γ^α and Γ^β, and denote by F^α and F^β the characteristic functions for Γ^α and Γ^β, respectively, satisfying the properties (i)$\sim$(vi). By (vi), we can write them as

$$F^\alpha(\mu) = P^\alpha(\mu)h + Q^\alpha(\mu),$$
$$F^\beta(\mu) = -P^\beta(\mu)h + Q^\beta(\mu), \tag{9}$$

where $P^\alpha, Q^\alpha, P^\beta, Q^\beta$ are independent of h. From $F^\alpha(\mu) = 0$ and $F^\beta(\mu) = 0$, we can eliminate h to define a function $F^{\alpha+\beta}(\mu)$ by

$$F^{\alpha+\beta}(\mu) := \frac{P^\alpha(\mu)Q^\beta(\mu) + Q^\alpha(\mu)P^\beta(\mu)}{\sigma_k\mu} \qquad \text{for } \mu \neq 0 \tag{10}$$

and $F^{\alpha+\beta}(0) = 0$.

We will show that this function satisfies the desired properties. First, let $\lambda_0 = \mu_0^2$ $(\neq 0)$ be any nonzero eigenvalue of $\mathcal{L}$ and denote the corresponding eigenfunction by $(U_i)_{\gamma_i\in\Gamma}$. Setting $h = -\partial_\xi U_i(\xi_0)/U_i(\xi_0)$, we can regard $(U_i)_{\gamma_i\in\Gamma^\alpha}$ and $(U_i)_{\gamma_i\in\Gamma^\beta}$ as eigenfunctions for Γ^α and Γ^β, respectively, associated with the eigenvalue λ_0. We note that λ_0 is a real number and that μ_0 is a real or purely imaginary number such that

$$\begin{cases} P^\alpha(\mu_0)h + Q^\alpha(\mu_0) = 0, \\ -P^\beta(\mu_0)h + Q^\beta(\mu_0) = 0. \end{cases} \tag{11}$$

Hence, if λ_0 is an eigenvalue, we obtain $F^{\alpha+\beta}(\mu_0) = 0$.

Conversely, we show that if $F^{\alpha+\beta}(\mu_0) = 0$ for some $\mu_0 \neq 0$, then $\lambda_0 = {\mu_0}^2$ is an eigenvalue of $\mathcal{L}$ for Γ. If

$$F^{\alpha+\beta}(\mu_0) = P^\alpha(\mu_0)Q^\beta(\mu_0) + Q^\alpha(\mu_0)P^\beta(\mu_0) = 0,$$

there exists a real number h such that (11) holds. Then $\lambda_0 = \mu_0^2$ is an eigenvalue for both Γ^α and Γ^β with such h. Let $(U_i)_{\gamma_i\in\Gamma^\alpha}$ and $(U_i)_{\gamma_i\in\Gamma^\beta}$ denote associated eigenfunctions for the eigenvalue $\lambda_0 = \mu_0^2$. In particular, let U_k^α and U_k^β denote eigenfunctions on γ_k. Since these functions must be given by linear combinations of two hyperbolic functions, we can extend the domain of definition to $(0, L_k)$ (or γ_k). On γ_k, these eigenfunctions satisfy the same equation

$$\frac{d^2}{d\xi^2}U_k(\xi) = \lambda_0 U_k(\xi), \qquad 0 < \xi < L_k$$

and boundary conditions

$$\partial_\xi U_k^\alpha(\xi_0) + hU_k^\alpha(\xi_0) = 0,$$
$$\partial_\xi U_k^\beta(\xi_0) + hU_k^\beta(\xi_0) = 0.$$

These equalities imply that U_k^α and U_k^β are not linearly independent on γ_k so that

$$C_1 U_k^\alpha(\xi) \equiv C_2 U_k^\alpha(\xi), \qquad 0 < \xi < L_k$$

for some $(C_1, C_2) \neq (0,0)$. Then $(C_1 U_i)_{\gamma_i \in \Gamma^\alpha} \cup (C_2 U_i)_{\gamma_i \in \Gamma^\beta}$ become an eigenfunction of $\mathcal{L}$ for Γ. Thus we have shown that $F(\mu) = F^{\alpha+\beta}(\mu)$ with $\lambda = \mu^2$ is a characteristic function for Γ.

From the definition (10) we see that $F(\mu)$ is a polynomial of $\sigma_i, \mu, h_i, \sinh(a_j\mu)$ and $\cosh(a_j\mu)$, where a_j are some positive constants. Looking at the leading order term we obtain the property (v). The other properties are easily shown. The proof is now complete by induction. □

Proposition 4.2 *Let F be the characteristic function constructed as above. Then the derivative of $F(\mu)$ at $\mu = 0$ is given by*

$$D := \left.\frac{dF}{d\mu}\right|_{\mu=0} = \sum_{\gamma_i \in \Gamma} \sigma_i L_i \times \prod_{\gamma_i \in B} h_i + \sum_{\gamma_i \in B} \left\{ \sigma_i \prod_{\gamma_j \in B \setminus \{\gamma_i\}} h_j \right\}.$$

Proof. We prove this by induction. First, for a stationary interface with one triple junction, D is computed directly from (8) as (1).

Next, let Γ be a stationary interface with two or more triple junctions. We divide Γ into two parts Γ^α and Γ^β as above, and define B^α and B^β by

$$B^\alpha := B \cap \Gamma^\alpha, \qquad B^\beta := B \cap \Gamma^\beta.$$

Note that B^α and B^β do not contain γ_k^α or γ_k^β. Suppose that the assertion is true for Γ^α and Γ^β. Then, by (9), we have

$$\frac{d}{d\mu} F^\alpha(0) = P_\mu^\alpha(0) h + Q_\mu^\alpha(0)$$

with

$$P_\mu^\alpha(0) = \left\{ \sigma_k \xi_0 + \sum_{\gamma_i \in \Gamma^\alpha \setminus \{\gamma_k^\alpha\}} \sigma_i L_i \right\} \prod_{\gamma_i \in B^\alpha} h_i + \sum_{\gamma_i \in B^\alpha} \left\{ \sigma_i \prod_{\gamma_j \in B^\alpha \setminus \{\gamma_i\}} h_j \right\},$$

$$Q_\mu^\alpha(0) = \sigma_k \prod_{\gamma_i \in B^\alpha} h_i.$$

Here the subscript μ means differentiation with respect to μ. Similarly, we have

$$\frac{d}{d\mu} F^\beta(0) = -P_\mu^\beta(0) h + Q_\mu^\beta(0)$$

with

$$P_\mu^\beta(0) = \left\{ \sigma_k (L_k - \xi_0) + \sum_{\gamma_i \in \Gamma^\beta \setminus \{\gamma_k^\beta\}} \sigma_i L_i \right\} \prod_{\gamma_i \in B^\beta} h_i + \sum_{\gamma_i \in B^\beta} \left\{ \sigma_i \prod_{\gamma_j \in B^\beta \setminus \{\gamma_i\}} h_j \right\},$$

$$Q_\mu^\beta(0) = \sigma_k \prod_{\gamma_i \in B^\beta} h_i.$$

Noting that $P^\alpha(0) = Q^\alpha(0) = P^\beta(0) = Q^\beta(0) = 0$, we obtain

$$
\begin{aligned}
\frac{dF^{\alpha+\beta}}{d\mu}\bigg|_{\mu=0} &= \frac{1}{\sigma_k}\Big\{P^\alpha_\mu(0)Q^\beta_\mu(0) + Q^\alpha_\mu(0)P^\beta_\mu(0)\Big\} \\
&= \left\{\sigma_k\xi_0 + \sum_{\gamma_i\in\Gamma^\alpha\setminus\{\gamma^\alpha_k\}} \sigma_i L_i\right\} \prod_{\gamma_i\in B^\alpha} h_i \prod_{\gamma_i\in B^\beta} h_i \\
&\quad + \sum_{\gamma_i\in B^\alpha}\left\{\sigma_i \prod_{\gamma_j\in B^\alpha\setminus\{\gamma_i\}} h_j\right\} \prod_{\gamma_i\in B^\beta} h_i \\
&\quad + \left\{\sigma_k(L_k-\xi_0) + \sum_{\gamma_i\in\Gamma^\beta\setminus\{\gamma^\beta_k\}} \sigma_i L_i\right\} \prod_{\gamma_i\in B^\alpha} h_i \prod_{\gamma_i\in B^\beta} h_i \\
&\quad + \sum_{\gamma_i\in B^\beta}\left\{\sigma_i \prod_{\gamma_j\in B^\beta\setminus\{\gamma_i\}} h_j\right\} \prod_{\gamma_i\in B^\alpha} h_i \\
&= \left\{\sigma_k L_k + \sum_{\gamma_i\in\Gamma\setminus\{\gamma_k\}} \sigma_i L_i\right\} \prod_{\gamma_i\in B} h_i + \sum_{\gamma_i\in B}\left\{\sigma_i \prod_{\gamma_j\in B\setminus\{\gamma_i\}} h_j\right\} \\
&= \sum_{\gamma_i\in\Gamma} \sigma_i L_i \times \prod_{\gamma_i\in B} h_i + \sum_{\gamma_i\in B}\left\{\sigma_i \prod_{\gamma_j\in B\setminus\{\gamma_i\}} h_j\right\}.
\end{aligned}
$$

Thus the assertion is true for Γ. The proof is completed by induction. □

5. Proof of Theorem 1.1

We prepare the following lemma on the nondegeneracy of zero eigenvalues.

Lemma 5.1 *If at most one of h_i $(\gamma_i \in B)$ is zero, then any zero eigenvalue is simple.*

Proof. Assume that there exists a zero eigenvalue, and denote an associated eigenfunction by $(U_i)_{\gamma_i\in\Gamma}$. Then each U_i is a linear combination of 1 and s:

$$U_i = a_i + b_i s.$$

Take an edge $\gamma_j \in \Gamma$. If we fix the value of b_j, the other b_k $(\gamma_k \in \Gamma \setminus \{\gamma_j\})$ are determined uniquely from (5). Then, for every $\gamma_i \in B$ with $h_i \neq 0$, the value of a_i is given by (3) . Hence other a_i $(\gamma_i \in \Gamma \setminus B)$ are determined by (4) successively. As for $\gamma_i \in B$ with $h_i = 0$, even if it exists, a_i is obtained by (4) because the other a_k $(\gamma_k \in \Gamma, k \neq i)$ has been already determined. Therefore the degree of freedom of zero eigenfunctions is at most one. This implies the simplicity of the zero eigenvalue. □

Now let us complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We note first that we can deform Ω without changing the shape of a given stationary interface. Hence we may regard h_i ($\gamma_i \in B$) as variable parameters. Without loss of generality, we put $B = \{\gamma_1, \ldots, \gamma_k\}$, and write D as $D(h_1, h_2, \ldots, h_k)$.

Take an edge $\gamma_i \in B$. Let h_j ($\gamma_j \in B \setminus \{\gamma_i\}$) be nonzero and fixed, and let h_i vary on $\mathbb{R}$. Then D changes its sign at some value of h_i, because D is a linear function of h_i. When h_i decreases and the sign of D changes, zero points of F transfer from the imaginary axis to the real axis. At this moment, by Lemma 5.1, exactly one negative eigenvalue becomes positive so that N_{U} increases by one.

Bearing the above observation in mind, we count the number of positive eigenvalues as follows. Assume first that $h_1, h_2, \ldots, h_k > 0$. Then (6) implies $N_{\mathrm{U}} = 0$. Next, we decrease the values of $h_1, h_2, \ldots, h_m$ one by one to negative values. By this procedure, the index D can change its sign at most m times and hence $N_{\mathrm{U}} \leq m$. On the other hand, Proposition 3.2 shows $N_{\mathrm{U}} \geq m - 1$. Hence $N_{\mathrm{U}} = m - 1$ or m. Since $D > 0$ if $h_1, h_2, \ldots, h_k > 0$, N_{U} is even if $D > 0$ and is odd if $D < 0$. Thus (i) is proved.

Finally, let us consider the existence of zero eigenvalues. If at most one of h_i ($\gamma_i \in B$) is zero, then Lemma 5.1 and the above argument imply that a zero eigenvalue appears if and only if $D = 0$. Suppose that $h_j = h_k = 0$ $(j \neq k)$. Then we have $D = 0$ by Proposition 4.2. Since Γ is of binary-tree type, there is a unique path on Γ which connects γ_j and γ_k. Then we can take a function $(U_i)_{\gamma_i \in \Gamma}$ such that

(i) U_i is a nonzero constant if γ_i is on the path,
(ii) U_i is identically equal to zero if γ_i is not on the path, and
(iii) $(U_i)_{\gamma_i \in \Gamma}$ satisfies (3)$\sim$(5).

Then $(U_i)_{\gamma_i \in \Gamma}$ becomes an eigenfunction associated with the zero eigenvalue. Thus the proof is complete. □

*Acknowledgement.*The first author is partially supported by Kyushu University 21st Century COE Program, Development of Dynamic Mathematics with High Functionality, of the Ministry of Education, Culture, Sports, Science and Technology of Japan. The second author is partially supported by the Grant-in-Aid for Exploratory Research (No. 13874014).

References

1. Bollobás, B.: Modern graph theory. Number 184 in Graduate Texts in Mathematics. Berlin Heidelberg New York: Springer 1983
2. Bronsard, L., Reitich, F.: On three-phase boundary motion and the singular limit of a vector-valued Ginzburg-Landau equation. Arch. Rat. Mech. **124**: 355–379 (1993)
3. Courant, R., Hilbert, D.: Methods of mathematical physics. Vol. 1. Interscience publishers 1953
4. Ei, S.-I., Ikota, R., Mimura, M.: Segregating partition problem in competition-diffusion systems. Interfaces and Free Boundaries **1**(1): 57–80 (1999)
5. Ei, S.-I., Sato, M.-H., Yanagida, E.: Stability of stationary interfaces with contact angle in a generalized mean curvature flow. Amer. J. Math. **118**: 653–687 (1996)

6. Ei, S.-I., Yanagida, E.: Stability of stationary interfaces in a generalized mean curvature flow. J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **40**: 651–661 (1993)
7. Ei, S.-I., Yanagida, E.: Instability of stationary solutions for equations of curvature-driven motion of curves. J. Dyn. Diff. Eqs. **7**: 423–435 (1995)
8. Garcke, H., Novick-Cohen, A.: A singular limit for a system of degenerate Cahn-Hilliard equations. Advances in Differential Equations **5**: 401–434 (2000)
9. Grayson, M.A.: The heat equation shrinks embedded plane curves to round points. J. Differential Geom. **26**: 285–314 (1987)
10. Ikota, R., Yanagida, E.: A stability criterion for stationary curves to the curvature-driven motion with a triple junction. Differential Integral Equations **16**(6), 707–726 (2003)
11. Ito, K., Kohsaka, Y.: Three phase boundary motion by surface diffusion in triangular domain. Adv. Math. Sci. Appl. **11**(2), 753–779 (2001)
12. Mullins, W.W.: Two-dimensional motion of idealized grain boundaries. J. Appl. Phys. **27**(8): 900–904 (1956)
13. Reed, M., Simon, B.: Methods of modern mathematical physics IV. Analysis of operators. New York London: Academic Press 1978
14. Sternberg, P., Ziemer, W.P.: Local minimizers of a three phase partition problem with triple junctions. Proc. Royal Soc. Edin. **124A**: 1059–1073 (1994)