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Mixed states for an Allen-Cahn type equation, II

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Abstract. This paper continues the recent study of an Allen-Cahn model PDE [1] by eliminating a strong spatial reversibility condition and by weakening certain nondegeneracy conditions on families of basic heteroclinic solutions, enabling us to obtain multibump solutions in a much more general setting. As in [1], novel minimization arguments play a key role in finding solutions.

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1 Introduction

In a recent paper, [1], the authors studied the existence of solutions for an Allen-Cahn model equation:

$$\text{(PDE)} \qquad -\Delta u + F_u(x, y, u) = 0.$$

Here $(x, y) \in \mathbb{R}^2$ and F satisfies

(F_1) $F \in C^2(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$ and is 1-periodic in x, y,
(F_2) $F(x, y, 0) = 0 = F(x, y, 1)$ and $F(x, y, z) > 0$ for $x, y \in \mathbb{R}$ and $z \in (0, 1)$,
(F_3) $F(x, y, z) \geq 0$ for $x, y, z \in \mathbb{R}$.

See also Alessio, Jeanjean, and Montecchiari [2]–[3] for some related work. The solutions $u \equiv 0, 1$ of (PDE) are called pure states. Mixed states are solutions with $0 < u < 1$ that are asymptotic in some sense to the pure states. Minimization arguments were used in [1] to establish the existence of a large number of mixed states. In particular, [1] obtained two basic types of such minima: (a) solutions periodic in one spatial variable and heteroclinic in the other, e.g. v heteroclinic in x from 0 to 1 and $\widehat{v}$ heteroclinic in x from 1 to 0; and (b) solutions heteroclinic in one spatial variable from 0 to 1 or 1 to 0 and heteroclinic in the other variable to a pair

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of solutions of type (a). As was indicated in [1], the existence of such heteroclinics is strongly related to results of Bangert [4] which in turn extend work of Moser [5].

For $i, j \in \mathbb{Z}$, set $\tau_i u(x, y) = u(x-i, y)$ and $\sigma_j u(x, y) = u(x, y-j)$. Note that τ_i and σ_j map solutions to solutions. Hence, in the spirit of the theory of chaotic dynamical systems, more complex solutions which we called multibump solutions were constructed 'near' formal chains of solutions of type (a) or type (b). Here 'near' should be interpreted as close in some norm on a large region. For example, infinitely many 2-bump solutions of (PDE) homoclinic to 0 in x, and periodic in y were found near the formal two-chain obtained by gluing v and $\widehat{v}$.

To construct the solutions of type (b) and some families of corresponding multi-bump solutions, further conditions were imposed on F and on the class of minima of type (a). The goal of this paper is to remove the extra condition on F and weaken the restriction on the class of minima.

To be more precise, taking y to be the 1-periodic variable, the class of minima heteroclinic in x from 0 to 1 will be denoted by $\mathcal{M}(0,1)$ and those heteroclinic in x from 1 to 0 will be denoted by $\mathcal{M}(1,0)$. Observe that $v \in \mathcal{M}(0,1)$ implies $\tau_j v \in \mathcal{M}(0,1)$ for all $j \in \mathbb{Z}$ and likewise for $\mathcal{M}(1,0)$. It was shown in [1] that the elements of $\mathcal{M}(0,1)$ are (i) 'monotone' in x in the sense that $v \in \mathcal{M}(0,1)$ implies that $v < \tau_{-1} v$, and are (ii) ordered, i.e. $v, w \in \mathcal{M}(0,1)$ implies $v \equiv w$, $v < w$, or $v > w$.

To get basic solutions of type (b), it was assumed that:

$(*)$ $\mathcal{M}(0,1)$ contains an adjacent pair v, w with $v < w$.

Condition $(*)$ is equivalent to saying there does not exist a continuum of solutions of (PDE) in (the weak closure of) $\mathcal{M}(0,1)$ joining 0 and 1, i.e. there are gaps in the ordered set $\mathcal{M}(0,1)$. Then as was shown by a minimization argument in [1], there are solutions of (PDE) heteroclinic in x from 0 to 1 and in y from v to w. Let $\mathcal{M}(v,w)$ denote this set of solutions. As above it consists of functions 'monotone' in y ($u < \sigma_{-1} u$) and is ordered. Likewise reversing the roles of v and w in $\mathcal{M}(v,w)$ produces a corresponding set $\mathcal{M}(w,v)$.

The basic solutions of type (a) were used to construct more complex solutions, so-called multibump solutions in x, near formal heteroclinic chains of the basic ones. No further assumptions beyond $(*)$ and its analogue for $\mathcal{M}(1,0)$ are needed to do this. Analogous results were found for the basic solutions of type (b). However more assumptions were needed in [1] to do this. Suppose first that

$$\text{(Ma)} \qquad \begin{aligned} \mathcal{M}(0,1) &= \{\tau_k v \mid k \in \mathbb{Z}\} \\ \mathcal{M}(1,0) &= \{\tau_i \widehat{v} \mid i \in \mathbb{Z}\}. \end{aligned}$$

This is the nicest case one can hope to deal with in y. Further assume

$$\text{(Mb)} \qquad \mathcal{M}(v, \tau_{-1} v) \text{ and } \mathcal{M}(\tau_{-1} v, v) \text{ contain gaps .}$$

In [1], it was shown that (Ma) and (Mb) imply there exist monotone in y, $|j|$ bump solutions heterolinic in y from v to $\tau_j v$ (and near $\tau_1 v, \ldots, \tau_{j-1} v$ for $j > 0$ or near $\tau_{-1} v, \ldots, \tau_{-j+1} v$ for $j < 0$ on large intermediate regions) for any $j \in \mathbb{Z}$.

Nonmonotone multibump solutions in y were also constructed but under the further hypothesis:

(F_4) F is even in y.

The main goal of this paper is to weaken (Ma)–(Mb) and to drop (F_4). This requires introducing new ideas and methods. In Sect. 2, some results from [1] will be recalled, and compactness properties of minimizing sequences and the regularity of their weak limits and the asymptotic behavior of the limits will be studied. Section 3 contains some results that will be employed in comparison arguments and the analysis of asymptotic behavior in the later sections. These useful tools will be applied first in Sect. 4 where the condition

$$(**) \qquad \mathcal{M}(v, w) \text{ and } \mathcal{M}(w, v) \text{ have gaps}$$

is assumed but (F_4) is not required. It is shown how to find the simplest kind of nonmonotone (in y) multibump solutions homoclinic from v to v and near w over a large intermediate y strip. In fact there are infinitely many such solutions. Next Sect. 5 deals with replacing (Ma)–(Mb) by $(*)$ and $(**)$. Note that without (Ma) or a similar condition, it is possible that there are $\varphi, \psi \in \mathcal{M}(0, 1)$ with $\varphi < \psi$ and there are no gaps between φ and ψ. Even if φ, ψ are adjacent members of $\mathcal{M}(0, 1)$, $\mathcal{M}(\varphi, \psi)$ may not have gaps. Nevertheless in Sect. 5 it will be shown that if $(*)$ and $(**)$ are satisfied, there exist monotone multibump solutions heteroclinic in y from v to $\tau_{-j}w$ for any $j \in \mathbb{N}$. Lastly in Sect. 6, with the aid of the basic cases treated in §4–5, the existence of more general nonmonotone multibump heteroclinics in y will be discussed.

2 Some preliminaries

In this section, some results of a technical nature will be presented. They will be required repeatedly in the remainder of this paper. The results address compactness, regularity, and asymptotic questions that arise in going from minimizing sequences to solutions for various variational problems.

By way of motivation, we begin by discussing at the qualitative level some of the difficulties involved in trying to find heteroclinics of type (b) and corresponding multibump solutions of (PDE) via minimization arguments. To begin one needs a functional and class of admissible functions which contains the class of solutions we seek. The first difficulty is in choosing an appropriate functional. The Lagrangian for (PDE) is

$$(2.1) \qquad L(u) = \frac{1}{2}|\nabla u|^2 + F(x, y, u) \geq 0$$

and the corresponding functional is

$$\mathcal{I}(u) = \int_{\mathbb{R}^2} L(u)dxdy.$$

If $\mathcal{C}$ denotes the class of $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ functions possessing the asymptotic properties of type (b) solutions, $\mathcal{I}$ is infinite for any $u \in \mathcal{C}$. There is no general recipe to handle such situations, but this difficulty was overcome in [1] by replacing $\mathcal{I}$ by a 'renormalized' functional, J, which was obtained – oversimplifying somewhat – by subtracting a term that is infinite from $\mathcal{I}$. Thus – see Proposition 2.6 below – J is finite valued and bounded from below on a subclass, $\mathcal{C}_1$, of $\mathcal{C}$.

Next to obtain solutions of type (b), J is minimized over $\mathcal{C}_1$. The form and properties of J – see Proposition 2.6 – imply a minimizing sequence, (u_k), for this problem is bounded in the Hilbert space $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$. Consequently a subsequence of u_k converges weakly in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ to some $U \in W^{1,2}_{\text{loc}}(\mathbb{R}^2)$. Proposition 2.10 shows more is true: minimizing sequences for J with various choices of $\mathcal{C}_1$ can be chosen so that they converge strongly in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ to U. One might call this fact the analogue of verifying the classical Palais-Smale condition in the current setting. The usual procedure now is to show that U has whatever asymptotic behavior is built into $\mathcal{C}_1$ and in fact that (i) $U \in \mathcal{C}_1$, and (ii) U is a solution of (PDE). Proposition 2.10 shows that even without knowing (i) and (ii), U satisfies (PDE) in subdomains of $\mathbb{R}^2$ which avoid any constraints that $\mathcal{C}_1$ may require. Verifying (i) and (ii) involves the use of comparison arguments which will be carried out in Sect. 3.

In this section the notation and some of the existence results of [1] will be recalled. In particular the functional, J, will be introduced and its properties will be studied. To begin, set

$$\begin{aligned} \Gamma(0,1) = \Bigg\{ & u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2,\mathbb{R}) \mid u \text{ is 1-periodic in } y, \\ & \int_0^1 \left(\int_{k-1}^k u^2 dx \right) dy \to 0 \text{ as } k \to -\infty, \\ & \int_0^1 \left(\int_{k-1}^k (u-1)^2 dx \right) dy \to 0 \text{ as } k \to \infty \Bigg\}, \end{aligned}$$

i.e. $\Gamma(0,1)$ consists of candidates for solutions of (PDE) that are 1-periodic in y and heteroclinic in the above L^2 sense in x from 0 to 1. Similarly let $\Gamma_n(0,1)$ denote the related class of functions n-periodic in y. With L as in (2.1), let

$$\begin{aligned} I(u) &= \int_0^1 \left(\int_{\mathbb{R}} L(u) dx \right) dy, \\ I_n(u) &= \int_0^n \left(\int_{\mathbb{R}} L(u) dx \right) dy, \\ c(0,1) &= \inf_{u \in \Gamma(0,1)} I(u), \end{aligned} \tag{2.2}$$

and

$$c_n(0,1) = \inf_{u \in \Gamma_n(0,1)} I_n(u). \tag{2.3}$$

It was proved in [1] that

Theorem 2.4. *Let F satisfy (F_1)–(F_3). Then:*

1° *There is a* $v \in \Gamma(0,1)$ *such that* $I(v) = c(0,1)$.
2° *Any such* v *is a classical solution of (PDE) satisfying* $0 < v < \tau_{-1}v < 1$, $\|v\|_{C^2([k-1,k]\times[0,1]} \to 0$ *as* $k \to -\infty$ *and* $\|v-1\|_{C^2([k-1,k]\times[0,1])} \to 0$ *as* $k \to \infty$.
3° $\mathcal{M}(0,1) \equiv \{u \in \Gamma(0,1) \mid I(u) = c(0,1)\}$ *is an ordered set, i.e.* $v, w \in \mathcal{M}(0,1)$ *implies* $v \equiv w$, $v < w$, *or* $v > w$.
4° $\mathcal{M}_n(0,1) \equiv \{u \in \Gamma_n(0,1) \mid I_n(u) = c_n(0,1)\} = \mathcal{M}(0,1)$.
5° *The above statements are also valid with the roles of* 0 *and* 1 *interchanged.*

To continue, assume:

$(*)$ There exist adjacent members $v < w$ of $\mathcal{M}(0,1)$.

In the simplest case, $w = \tau_{-1}v$. To find solutions of (PDE) that are heteroclinic in y from v to w requires an appropriate functional. To define the renormalized functional, J, used for this purpose in [1], for $p, q \in \mathbb{Z}, p \le q$, set $S_p = \mathbb{R}\times[p, p+1]$ and for $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$, let

$$J_{p,q}(u) \equiv \sum_{i=p}^{q}\left(\int_{S_i} L(u)dxdy - c(0,1)\right).$$

Then define

$$J(u) = \varliminf_{p\to-\infty} J_{p,0}(u) + \varliminf_{q\to\infty} J_{1,q}(u). \tag{2.5}$$

It will always be assumed for what follows that F satisfies (F_1)–(F_3). The next result contains the basic properties of J.

Proposition 2.6. *Assume* $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ *and* $\varphi \le u \le \psi$ *for* $\varphi, \psi \in \mathcal{M}(0,1)$.

1° *Then there is a constant* $K \ge 0$ *independent of* u *and* $p, q \in \mathbb{Z}$ *such that*

$$-K \le J_{p,q}(u) \le J(u) + 2K.$$

2° *If* $J(u) < \infty$, *and there is a* $\phi \in \mathcal{M}(0,1)$ *such that* $\sigma_{-p}u \to \phi$ *in* $L^2_{\mathrm{loc}}(S_0)$ *as* $p \to \infty$ *(resp.* $p \to -\infty$*), then*

$$\|u-\phi\|_{W^{1,2}(S_p)} \to 0 \ \text{ as } p \to \infty \text{ (resp. } p \to -\infty)$$

and

$$\lim_{p\to\infty} J_{1,p}(u) \equiv J_{1,\infty}(u)$$

exists (resp. $\lim_{p\to-\infty} J_{-p,0}(u) \equiv J_{-\infty,0}(u)$ *exists), i.e. the* $\liminf$ *in (2.5) is a limit.*
3° *If* $u_0 \in W^{1,2}_{loc}(\mathbb{R}^2)$, $\varphi \le u_0 \le \psi$, *and* $J(u_0) \le M$, *then for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *independent of* $u, u_0, p \in \mathbb{Z}$ *such that*

$$\|u-u_0\|_{W^{1,2}(S_p)} < \delta \Rightarrow |J_{p,p}(u) - J_{p,p}(u_0)| < \varepsilon.$$

Proof. The proof of 1^o is essentially given in Proposition 3.3 and Lemma 3.4 of [1], using the fact that by 3^o of Theorem 2.4, there is a $q \in \mathbb{N}$ such that $\psi \le \tau_{-q}\varphi$. Therefore, since $0 \le \varphi \le \psi \le 1$, for $\alpha \ge 1$

$$\int_{\mathbb{R}} |\psi - \varphi|^{\alpha} dx \le \int_{\mathbb{R}} (\tau_{-q}\varphi - \varphi)dx = \lim_{s\to\infty} \int_{-s}^{s} (\tau_{-q}\varphi - \varphi)dx = q. \tag{2.7}$$

The proof of 2^o is given in Proposition 3.5 of [1] but with the assumption that $\sigma_{-p}u$ converges in $L^2_{\rm loc}(S_0)$ to v and w respectively as $p \to -\infty, \infty$. Once this is known, the remainder of the proof of Proposition 3.5 of [1] can be used here unchanged. To establish 3^o note that for any $j \in \mathbb{N}$,

$$\left| \int_{S_p} (F(x,y,u) - F(x,y,u_0))dxdy \right| \le M_1 \int_p^{p+1} \left(\int_{-j}^{j} |u-u_0|dx \right) dy + M_1 \int_{(\mathbb{R}\setminus[-j,j])\times[p,p+1]} |\psi-\varphi|dxdy. \tag{2.8}$$

Due to (2.7), the second term on the right can be made arbitrarily small for j large enough. The first term is then bounded by a multiple of $\|u-u_0\|_{W^{1,2}(S_p)}$. Likewise

$$\int_{S_p} \frac{1}{2}(|\nabla u|^2 - |\nabla u_0|^2)dxdy \le \frac{1}{2}\|\nabla(u-u_0)\|_{L^2(S_p)}\|\nabla(u-u_0) + 2\nabla u_0\|_{L^2(S_p)} \tag{2.9}$$

can be made arbitrarily small for small enough $\|u - u_0\|_{W^{1,2}(S_p)}$, since $\|\nabla u_0\|_{L^2(S_p)}$ is bounded due to $J(u_0) < \infty$ and 1^o.

In the remaining sections of this paper the question of minimizing J on some subset of $W^{1,2}_{\rm loc}(\mathbb{R}^2)$ will be encountered repeatedly. The next result which establishes some compactness properties of minimizing sequences will be useful in treating such minimization problems.

Proposition 2.10. *1^o Suppose $\mathcal{Y} \subset W^{1,2}_{\rm loc}(\mathbb{R}^2)$ satisfies*

(Y_1) There are $\varphi, \psi \in \mathcal{M}(0,1)$ such that $\varphi \le u \le \psi$ for all $u \in \mathcal{Y}$.

Let (u_k) be a sequence in $\mathcal{Y}$ with $J(u_k) \le M$ for some $M > 0$ and all $k \in \mathbb{N}$. Then there is a subsequence of (u_k) and a $U \in W^{1,2}_{\rm loc}(\mathbb{R}^2)$ such that u_k converges to U weakly in $W^{1,2}_{\rm loc}(\mathbb{R}^2)$, strongly in $L^2(S_i)$ for all $i \in \mathbb{Z}$, and pointwise a.e. along the subsequence.

2^o Suppose $\mathcal{Y}$ also satisfies:

(Y_2) If $u \in \mathcal{Y}$, U is as given by 1^o, and $\chi = \chi(u)$ is defined by

$$\begin{aligned} \chi &= u && y \le q \\ &= (y-q)U + (q+1-y)u && q \le y \le q+1 \\ &= U && q+1 \le y \le p \\ &= (y-p)u + (p+1-y)U && p \le y \le p+1 \\ &= u && p+1 \le y \end{aligned}$$

where $q, p \in \mathbb{Z}$, then there is a $p_0 \in \mathbb{N}$ such that whenever $-q = p \geq p_0$, $\chi \in \mathcal{Y}$. Define

$$c(\mathcal{Y}) = \inf_{u \in \mathcal{Y}} J(u). \tag{2.11}$$

If $c(\mathcal{Y}) < \infty$ and (u_k) is a minimizing sequence for (2.11), then there is a subsequence of (u_k) such that

$$\lim_{k\to\infty} \|u_k - U\|_{W^{1,2}(S_i)} = 0, \quad i \in \mathbb{Z}.$$

Proof. Since $c(\mathcal{Y}) < \infty$, by 1^o of Proposition 2.6, there is an $M > 0$ such that

$$J_{p,q}(u_k) \leq M + 2K. \tag{2.12}$$

for all $p \leq q \in \mathbb{Z}$ and $k \in \mathbb{N}$. The form of $J_{p,q}$ and (2.12) imply (u_k) is bounded in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$. Therefore there is a $U \in W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ such that along a subsequence, u_k converges to U weakly in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$, strongly in $L^2_{\text{loc}}(\mathbb{R}^2)$, and pointwise a.e. To see that $u_k \to U$ in $L^2(S_i)$ for all $i \in \mathbb{Z}$ (along a subsequence), note that for any $j \in \mathbb{N}$, by (Y_1),

$$\begin{aligned}\int_{\mathbb{R}} \int_i^{i+1} |u_k - U|^2 dxdy \leq \int_{-j}^{j} \int_i^{i+1} |u_k - U|^2 dxdy \\ + \int_{\mathbb{R}\setminus[-j,j]} \int_i^{i+1} |\psi - \varphi|^2 dxdy.\end{aligned} \tag{2.13}$$

The $L^2_{\text{loc}}(\mathbb{R}^2)$ convergence of u_k to U implies the first term on the right in (2.13) goes to 0 as $k \to \infty$ for any $j \in \mathbb{N}$. The second term on the right in (2.13) goes to 0 as $j \to \infty$ due to (2.7). Combining these facts yields

$$\|u_k - U\|_{L^2(S_i)} \to 0 \qquad i \in \mathbb{Z} \tag{2.14}$$

as $k \to \infty$ along our subsequence. Thus 1^o is satisfied.

To verify 2^o, observe that $J_{p,q}$ is weakly lower-semicontinuous on $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ since if $f_k \to f$ weakly in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$, for any $i \in \mathbb{N}$,

$$\begin{aligned}\int_{-i}^{i} \int_p^{q+1} L(f) dxdy &\leq \varliminf_{k\to\infty} \int_{-i}^{i} \int_p^{q+1} L(f_k) dxdy \\ &\leq \varliminf_{k\to\infty} \int_{\mathbb{R}} \int_p^{q+1} L(f_k) dxdy = \varliminf_{k\to\infty} J_{p,q}(f_k) + (q+1-p)c(0,1).\end{aligned} \tag{2.15}$$

Thus letting $i \to \infty$,

$$J_{p,q}(f) \leq \varliminf_{k\to\infty} J_{p,q}(f_k). \tag{2.16}$$

Now define

$$\delta_p^2 = \varliminf_{s\to\infty} J_{p,p}(u_s) - J_{p,p}(U). \tag{2.17}$$

The right-hand side of (2.17) is nonnegative by the weak lowersemicontinuity just established so the definition makes sense. In (3.74)–(3.82) of [1], it was shown that

$$\sum_{p\in\mathbb{Z}} \delta_p^2 < \infty \tag{2.18}$$

and

$$\varliminf_{k\to\infty} \|u_k - U\|_{W^{1,2}(S_p)} \le \sqrt{2}\delta_p. \tag{2.19}$$

We claim

$$\delta_p = 0, \quad p \in \mathbb{Z} \tag{2.20}$$

provided that (Y_2) holds. To verify (2.20), set $\chi_k = \chi(u_k)$, χ as in 2^o with $q = -p$. Then for $p \ge p_0$,

$$\begin{aligned} c(\mathcal{Y}) \le J(\chi_k) &= J_{-\infty,-p}(u_k) + J_{-p+1,p-1}(U) \\ &+ J_{p,\infty}(u_k) + J_{-p,-p}(\chi_k) - J_{-p,-p}(u_k) \\ &+ J_{p,p}(\chi_k) - J_{p,p}(u_k). \end{aligned} \tag{2.21}$$

Passing to a subsequence of (u_k) for which (2.19) holds with $\varliminf$ replaced by $\lim$, there is a $\gamma_p > 0$ with $\gamma_p \to 0$ as $p \to \infty$, such that

$$|J_{-p,-p}(\chi_k) - J_{-p,-p}(u_k)| + |J_{p,p}(\chi_k) - J_{p,p}(u_k)| \le \gamma_p \tag{2.22}$$

due to 3^o of Proposition 2.6. By (2.17) and (2.21),

$$\begin{aligned} c(\mathcal{Y}) \le J(u_k) &+ \varliminf_{s\to\infty} J_{-p+1,p-1}(u_s) \\ &- J_{-p+1,p-1}(u_k) - \sum_{-p+1}^{p-1} \delta_i^2 + \gamma_p. \end{aligned} \tag{2.23}$$

Letting $k \to \infty$ shows

$$\sum_{-p+1}^{p-1} \delta_i^2 \le \gamma_p \tag{2.24}$$

and letting $p \to \infty$ yields (2.20) and completes the proof of Proposition 2.20.

A common method to find a solution of a partial differential equation is to formulate a minimization problem on some class of admissible functions, $\mathcal{Y}$, and prove there exists a minimizer in $\mathcal{Y}$ and then show it is a solution of the PDE. The next result provides a criterion for showing the limit, U, of a minimizing sequence in $\mathcal{Y}$ is a solution of (PDE) without knowing $U \in \mathcal{Y}$ let alone whether U minimizes the functional. It will be used repeatedly in later sections.

Proposition 2.25. *Suppose $\mathcal{Y} \subset W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ satisfies (Y_1)–(Y_2), $c(\mathcal{Y}) < \infty$, and*

(Y_3) There is a minimizing sequence (u_k) for (2.11) such that for some $r > 0$ and $z \in \mathbb{R}^2$,

$$c(\mathcal{Y}) \le J(u_k + t\varphi) + \delta_k \tag{2.26}$$

with $\delta_k \to 0$ as $k \to \infty$, for all smooth φ with support in $B_r(z)$ and $|t| \le t_0(\varphi)$. Then the weak limit U of (u_k) given by Proposition 2.10 satisfies (PDE) in $B_r(z)$.

Proof. If (u_k) is a minimizing sequence for (2.11), define ε_k by

$$J(u_k) = c(\mathcal{Y}) + \varepsilon_k. \tag{2.27}$$

Then $\varepsilon_k \to 0$ as $k \to \infty$. Therefore by (2.26),

$$c(\mathcal{Y}) \le J(u_k) = c(\mathcal{Y}) + \varepsilon_k \le J(u_k + t\varphi) + \varepsilon_k + \delta_k$$

or

$$J(u_k) \le J(u_k + t\varphi) + \varepsilon_k + \delta_k. \tag{2.28}$$

Suppose $B_r(z) \subset \mathbb{R} \times [p, q+1]$ for some $p, q \in \mathbb{Z}$ with $p \le q$. Then (2.28) implies

$$J_{p,q}(u_k) \le J_{p,q}(u_k + t\varphi) + \varepsilon_k + \delta_k. \tag{2.29}$$

Letting $k \to \infty$ and using Propositions 2.6, 2.10 yields

$$J_{p,q}(U) \le J_{p,q}(U + t\varphi) \tag{2.30}$$

or

$$\int_{B_r(z)} L(U)dxdy \le \int_{B_r(z)} L(U + t\varphi)dxdy. \tag{2.31}$$

Standard elliptic arguments then show U is a solution of (PDE) in $B_r(z)$.

To conclude this section, we recall one of the main results of [1] establishing the existence of solutions of (PDE) heteroclinic in x from 0 to 1 and in y from v to w where v and w are a pair of adjacent members of $\mathcal{M}(0,1)$ as well as a technical result from [1]. Thus assuming that $(*)$ is satisfied for $\mathcal{M}(0,1)$, let

$$\widehat{\Gamma}(v, w) = \{u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2) \mid v \le u \le \sigma_{-1}u \le w, v \not\equiv u \not\equiv w\}$$

and

$$\widehat{c}(v, w) = \inf_{u \in \widehat{\Gamma}(v,w)} J(u). \tag{2.32}$$

Then we have:

Proposition 2.33. *If (F_1)–(F_3) and $(*)$ are satisfied and $u \in \widehat{\Gamma}(v, w)$ with $J(u) < \infty$, then $\|u - v\|_{W^{1,2}(S_i)} \to 0$ as $i \to -\infty$, $\|u - w\|_{W^{1,2}(S_i)} \to 0$ as $i \to \infty$.*

Theorem 2.34. *Suppose (F_1)–(F_3) and $(*)$ are satisfied. Then there exists a $\widehat{U} \in \widehat{\Gamma}(v, w)$ such that $J(\widehat{U}) = \widehat{c}(v, w)$. Moreover, any such $\widehat{U}$ is a classical solution of (PDE) with $v < \widehat{U} < \sigma_{-1}\widehat{U} < w$ and $\|\widehat{U} - v\|_{C^2(S_i)} \to 0$ as $i \to -\infty$, $\|\widehat{U} - w\|_{C^2(S_i)} \to 0$ as $i \to \infty$.*

Remark 2.35. Similarly (where the notation is self explanatory) there exists a $\widehat{V} \in \widehat{\Gamma}(w, v)$ such that $J(\widehat{V}) = \widehat{c}(w, v)$, etc.

3 Comparison results

In this section, several results will be presented that will be used in the existence arguments of Sect. 4. In particular they will be useful in comparison arguments and in asymptotic analysis. Let $v \in \mathcal{M}(0,1)$ and define

$$\Gamma(v,v) = \big\{u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2) \mid \tau_1 v \le u \le \tau_{-1} v \ \text{ and } \\ \|u - v\|_{L^2(S_i)} \to 0 \ \text{ as } |i| \to \infty\big\}$$

and

$$c(v,v) = \inf_{u \in \Gamma(v,v)} J(u). \tag{3.1}$$

Set

$$\mathcal{M}(v,v) = \{u \in \Gamma(v,v) \mid J(u) = c(v,v)\}.$$

Theorem 3.2. *Assume F satisfies (F_1)–(F_3). Then*

1° $c(v,v) = 0$.
2° If $u \in \mathcal{M}(v,v)$, then u is a solution of (PDE).
3° $\mathcal{M}(v,v) = \{v\}$.

Proof. Since $v \in \Gamma(v,v)$ and $J(v) = 0$,

$$c(v,v) \le 0. \tag{3.3}$$

To get the reverse inequality and therefore 1^o, it suffices to show

$$J(u) \ge 0 \tag{3.4}$$

for any $u \in \Gamma(v,v)$ with $J(u) < \infty$. For $k \in \mathbb{N}$, define

$$\begin{aligned} u_k &= v, \quad y \le -k \\ &= (y+k)u + (-k+1-y)v, \quad -k \le y \le -k+1 \\ &= u, \quad -k+1 \le y \le k-1 \\ &= (y-k+1)v + (k-y)u, \quad k-1 \le y \le k \\ &= v, \quad k \le y. \end{aligned} \tag{3.5}$$

Thus $u_k \in \Gamma(v,v)$. Set $w_k = u_k\big|_{\mathbb{R}\times[-k-1,k+1]}$ and extended as a $2k+2$ periodic function of y. Then by 4^o of Theorem 2.4,

$$0 \le J_{-k-1,k}(w_k) = J_{-k,k-1}(w_k) = J_{-k,k-1}(u_k) = J(u_k). \tag{3.6}$$

But

$$\begin{aligned} J(u_k) = J(u) &+ J_{-k,-k}(u_k) - J_{-k,-k}(u) \\ &+ J_{k-1,k-1}(u_k) - J_{k-1,k-1}(u) - J_{-\infty,-k-1}(u) \\ &- J_{k,\infty}(u) \equiv J(u) - R_k(u) \end{aligned}$$

or

$$R_k(u) \le J(u). \tag{3.7}$$

Thus to verify (3.4), it suffices to prove that $R_k(u) \to 0$ as $k \to \infty$. By 2^o of Proposition 2.6,

$$J_{-\infty,-k-1}(u),\ J_{k,\infty}(u) \to 0 \ \text{ as } k \to \infty.$$

To estimate the remaining terms, consider e.g.

$$J_{k-1,k-1}(u_k) = J_{k-1,k-1}(u_k) - J_{k-1,k-1}(v).$$

Since $u \in \Gamma(v,v)$, $u_k - v \to 0$ in $W^{1,2}(S_{k-1})$ as $k \to \infty$ via 2^o of Proposition 2.6. Thus 3^o of Proposition 2.6 with $u = u_k, u_0 = v, p = k-1$ implies $J_{k-1,k-1}(u_k) \to 0$, and consequently $R_k(u) \to 0$, as $k \to \infty$, and 1^o is proved.

Remark 3.8. If $\tau_1 v$, $\tau_{-1} v$ in the definition of $\Gamma(v,v)$ are replaced by any $\varphi, \psi \in \mathcal{M}(0,1)$, with $\varphi < v < \psi$, the result and proof of 1^o remain unchanged.

Next to verify 2^o, set $\mathcal{Y} = \Gamma(v,v)$. Then $\mathcal{Y}$ satisfies (Y_1)–(Y_2) and by 1^o, $c(\mathcal{Y}) = 0$. Moreover let $r > 0$ and $z \in \mathbb{R}^2$. Suppose φ is smooth and has support in $B_r(z)$. If $u \in \mathcal{M}(v,v)$, $\tau_1 v \le u \le \tau_{-1} v$. Therefore for t_0 small enough, $\tau_2 v < u + t\varphi < \tau_{-2} v$ for $|t| \le t_0$. Thus by Remark 3.8 and 1^o, (2.26) holds and $\mathcal{Y}$ satisfies (Y_3). Consequently by Proposition 2.25, any $u \in \mathcal{M}(v,v)$ is a solution of (PDE).

Lastly to prove 3^o, an argument essentially due to Moser [2] will be employed. Observe for $u \in \mathcal{M}(v,v)$, that $\sigma_{-1} u \in \mathcal{M}(v,v)$. If $u = \sigma_{-1} u$, then $u = v$ by definition of $\Gamma(v,v)$. Otherwise we claim (i) $u < \sigma_{-1} u$, or (ii) $u > \sigma_{-1} u$. If not, set $\varphi = \max(u, \sigma_{-1} u)$ and $\psi = \min(u, \sigma_{-1} u)$. Then $\varphi \ge \psi$, there are points ξ, η such that $\varphi(\xi) = \psi(\xi)$ and $\varphi(\eta) > \psi(\eta)$ and $\varphi, \psi \in \Gamma(v,v)$. For any $i \in \mathbb{Z}$,

$$\int_{S_i} (L(\varphi) + L(\psi)) dx dy = \int_{S_i} (L(u) + L(\sigma_{-1} u)) dx dy. \tag{3.9}$$

Therefore, by 2^o of Proposition 2.6,

$$J(\varphi) + J(\psi) = J(u) + J(\sigma_{-1} u) = 0. \tag{3.10}$$

Since by 1^o, $J(\varphi)$, $J(\psi) \ge 0$, (3.10) shows $\varphi, \psi \in \mathcal{M}(v,v)$ and therefore are solutions of (PDE). But $f \equiv \varphi - \psi$ satisfies the linear elliptic PDE:

$$-\Delta f + af = -bf \tag{3.11}$$

where $a = \max(A, 0)$, $b = \min(A, 0)$ and

$$\begin{aligned} A &= \frac{F_u(x,y,\varphi(x,y) - F_u(x,y,\psi(x,y))}{\varphi(x,y) - \psi(x,y)} \qquad \text{if } \varphi(x,y) > \psi(x,y) \\ &= F_{uu}(x,y,\varphi(x,y)) \qquad \text{if } \varphi(x,y) = \psi(x,y). \end{aligned}$$

Thus A, a, b are continuous on $\mathbb{R}^2$. Since $f \ge 0$ in $\mathbb{R}^2$, by the Maximum Principle if f is somewhere 0, then $f \equiv 0$. But $f(\xi) = 0$ and $f(\eta) > 0$. Thus we have a contradiction and (i) or (ii) must hold. The argument is the same in either case, so suppose (i) holds. Then for all $j \in \mathbb{N}$,

$$\sigma_j u < u < \sigma_{-j} u \tag{3.12}$$

and letting $j \to \infty$ shows

$$v \leq u \leq v. \tag{3.13}$$

Thus $\mathcal{M}(v, v) = \{v\}$ and 3^o is proved.

The next result is an application of Theorems 2.34 and 3.2.

Corollary 3.14. *Let v and w be adjacent numbers of $\mathcal{M}(0,1)$. Then* $\widehat{c}(v, w) + \widehat{c}(w, v) > 0$.

Proof. Let $\widehat{U} \in \widehat{\Gamma}(v, w)$ and $\widehat{V} \in \widehat{\Gamma}(w, v)$ as provided by Theorem 2.34 and Remark 2.35. Set $\varphi = \max(\widehat{U}, \widehat{V})$ and $\psi = \min(\widehat{U}, \widehat{V})$. Then $\varphi \in \Gamma(w, w)$ and $\psi \in \Gamma(v, v)$. Since $\psi = \widehat{U}$ for $x \in [0, 1]$ and y near $-\infty$, $\psi \neq v$. Thus by 1^o of Theorem 3.2, $J(\psi) > 0$. Similarly $J(\varphi) > 0$. But as in (3.9)–(3.10),

$$J(\varphi) + J(\psi) = J(\widehat{U}) + J(\widehat{V}) = \widehat{c}(v, w) + \widehat{c}(w, v) \tag{3.15}$$

and the corollary follows.

A disadvantage of Theorem 2.34 is that the characterization of $\widehat{c}(v, w)$ that it provides requires working with functions u that are monotone in the sense that $u \leq \sigma_{-1} u$. A less restrictive class of functions is needed for some of the arguments that will be employed in what follows. The next result addresses this point and extends Theorem 2.34. Define

$$\Gamma(v, w) = \big\{u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2) \mid v \leq u \leq w, \|u - v\|_{L^2(S_i)} \to 0 \text{ as } i \to -\infty, \\ \text{and } \|u - w\|_{L^2(S_i)} \to 0 \text{ as } i \to \infty\big\}$$

and

$$c(v, w) = \inf_{u \in \Gamma(v,w)} J(u). \tag{3.16}$$

Theorem 3.17. *Let F satisfy (F_1)–(F_3) and ($*$) hold. Then there exists $U \in \Gamma(v, w)$ such that $J(U) = c(v, w)$. Any such U is a classical solution of (PDE) with $v < U < \sigma_{-1} U < w$, and $\|U - v\|_{C^2(S_i)} \to 0$ as $i \to -\infty$, $\|U - w\|_{C^2(S_i)} \to 0$ as $i \to \infty$. Moreover $c(v, w) = \widehat{c}(v, w)$.*

Proof. Let (u_k) be a minimizing sequence for (3.16). Observing that $J(u_k) = J(\sigma_{-p} u_k)$ for all $p \in \mathbb{Z}$, (u_k) can be normalized so that for $i \in \mathbb{Z}$, $i < 0$,

$$\int_i^{i+1} \left(\int_0^1 (u_k - v)^2 dx \right) dy \leq \frac{1}{2} \int_0^1 \int_0^1 (w - v)^2 dx dy, \\ < \int_0^1 \int_0^1 (u_k - v)^2 dx dy. \tag{3.18}$$

Since $\Gamma(v, w)$ satisfies (Y_1)–(Y_2) of Sect. 2, by Proposition 2.10, it can be assumed there is a $U \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ such that $u_k \to U$ weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$, strongly in $L^2(S_i)$ for all $i \in \mathbb{Z}$, and pointwise a.e., as $k \to \infty$. Therefore $v \le U \le w$. Also since (u_k) is a minimizing sequence for (3.16), there is an $M > 0$ such that

$$J(u_k) \le M. \tag{3.19}$$

Hence by Proposition 2.6, for all $p \le q$ in $\mathbb{Z}$,

$$J_{p,q}(u_k) \le M + 2K \tag{3.20}$$

so letting $k \to \infty$ shows (3.20) holds for U. Thus letting $p \to -\infty$, $q \to \infty$,

$$J(U) \le M + 2K. \tag{3.21}$$

We claim (Y_3) of Sect. 2 is satisfied for any $z \in \mathbb{R}^2$ and $r > 0$ and therefore by Proposition 2.25, U is a solution of (PDE). To verify this, suppose φ is smooth with support in $B_r(z)$. Set $f_k = \max(u_k + t\varphi, w)$ and $g_k = \min(u_k + t\varphi, w)$. Since $v \le u_k \le w$, for $t_0 = t_0(\varphi)$ sufficiently small, $\tau_1 v \le u_k + t\varphi \le \tau_{-1} w$ for $|t| \le t_0$, so by Remark 3.8, it can be assumed that $f_k \in \Gamma(w, w)$. Therefore by 1^o of Theorem 3.2,

$$J(f_k) \ge 0. \tag{3.22}$$

Note also that $g_k \le w$, and

$$\|g_k - v\|_{L^2(S_i)} \to 0,\ t \to -\infty;\ \|g_k - w\|_{L^2(S_i)} \to 0,\ i \to \infty. \tag{3.23}$$

Now as in (3.9)–(3.10), by (3.22)

$$J(g_k) \le J(f_k) + J(g_k) = J(u_k + t\varphi). \tag{3.24}$$

Set $\chi_k = \max(g_k, v)$ and $\psi_k = \min(g_k, v)$. Then with the aid of (3.23), $\chi_k \in \Gamma(v, w)$ and $\psi_k \in \Gamma(v, v)$. Hence by 1^o of Theorem 3.2 and (3.9)–(3.10) again

$$J(\chi_k) \le J(\chi_k) + J(\psi_k) = J(g_k). \tag{3.25}$$

Combining (3.24)–(3.25) and using (3.16) shows

$$\begin{aligned} c(v, w) \le J(u_k) \equiv c(v, w) + \varepsilon_k \le \\ \le J(\chi_k) + \varepsilon_k \le J(u_k + t\varphi) + \varepsilon_k \end{aligned} \tag{3.26}$$

where $\varepsilon_k \to 0$ as $k \to \infty$. Thus (Y_3) holds and U is a solution of (PDE) as is $\sigma_{-1} U$.

To complete the proof, it suffices to show that

(A) $U \le \sigma_{-1} U$ and $U \in \widehat{\Gamma}(v, w)$,
(B) $J(U) \le c(v, w)$.

Indeed (A) implies

$$J(U) \geq \widehat{c}(v,w). \tag{3.27}$$

Since $c(v,w) \leq \widehat{c}(v,w)$ (see Theorem 2.34), (3.27) and (B) together with Theorem 2.34 provide the remaining assertions of Theorem 3.17.

To establish (A), set $\Phi_k = \max(u_k, \sigma_{-1}u_k)$ and $\Psi_k = \min(u_k, \sigma_{-1}u_k)$. Then $\Phi_k, \Psi_k \in \Gamma(v,w)$ and as above,

$$J(\Phi_k) + J(\Psi_k) = J(u_k) + J(\sigma_{-1}u_k) = 2J(u_k) \to 2c(v,w),\ k \to \infty. \tag{3.28}$$

Therefore (Φ_k), (Ψ_k) are also minimizing sequences for (3.16). By Proposition 2.10 and the continuity of $\max(\cdot,\cdot)$, $\min(\cdot,\cdot)$ on $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$, $\Phi_k \to \Phi = \max(U, \sigma_{-1}U)$ and $\Psi_k \to \Psi = \min(U, \sigma_{-1}U)$ as $k \to \infty$. Moreover by (3.22)–(3.26), Φ and Ψ are solutions of (PDE). By their definition, $\Phi \geq \Psi$. Hence by the argument centered about (3.11), either (a) $\Phi(z) \equiv \Psi(z)$ for all $z \in \mathbb{R}^2$, or (b) $\Phi(z) > \Psi(z)$ for all $z \in \mathbb{R}^2$. If (a) holds, $U \equiv \sigma_{-1}U$, i.e. U is 1-periodic in y. Passing to a limit in (3.18) shows for $i < 0$

$$\begin{aligned} \int_i^{i+1} \left(\int_0^1 (U-v)^2 dx \right) dy &\leq \frac{1}{2} \int_0^1 \int_0^1 (w-v)^2 dx dy \\ &\leq \int_0^1 \int_0^1 (U-v)^2 dx dy. \end{aligned} \tag{3.29}$$

Therefore $v \not\equiv U \not\equiv w$. Recalling that $v \leq U \leq w$, it follows that $U \in \Gamma(0,1)$ so

$$c(0,1) < I(U). \tag{3.30}$$

But then $J(U) = \infty$, contrary to (3.21). Thus (a) fails and (b) must hold.

By (b), either (c) $U > \sigma_{-1}U$ or (d) $U < \sigma_{-1}U$. Since (c) is incompatiable with (3.29) (with $i = -1$), (d) must hold and $U \in \widehat{\Gamma}(v,w)$. Thus (A) is verified.

To prove (B), since $U \in \widehat{\Gamma}(v,w)$,

$$\|U - v\|_{W^{1,2}(S_i)} \to 0, \quad i \to -\infty \tag{3.31}$$

and

$$\|U - w\|_{W^{1,2}(S_i)} \to 0, \quad i \to \infty \tag{3.32}$$

by Proposition 2.33. Let $\varepsilon > 0$ and set $T_i = S_{i-1} \cup S_i \cup S_{i+1}$. Since $u_k \to U$ in $W^{1,2}(S_i)$ for all $i \in \mathbb{Z}$, by (3.31)–(3.32),

$$\|u_k - v\|_{W^{1,2}(T_{-p})},\ \ \|u_k - w\|_{W^{1,2}(T_p)} \leq \varepsilon \tag{3.33}$$

for all $p \geq p_0(\varepsilon)$ and $k \geq k_0(p)$. Since $u_k \in \Gamma(v,w)$,

$$\|u_k - v\|_{W^{1,2}(T_{-q})},\ \ \|u_k - w\|_{W^{1,2}(T_q)} \leq \varepsilon \tag{3.34}$$

for all large q. Define

$$f_k = u_k, \qquad y \le p-1 \\ = w, \qquad p \le y \le p+1 \\ = u_k, \qquad p+2 \le y \le q-1 \\ = w, \qquad q \le y \le q+1 \\ = u_k, \qquad q+2 \le y \tag{3.35}$$

and interpolate in the intermediate y intervals as in (3.5). Then by (3.33)–(3.34),

$$|J_{p,q}(u_k) - J_{p,q}(f_k)| \le \kappa(\varepsilon) \tag{3.36}$$

where $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since $f_k\big|_{\mathbb{R}\times[p,q+1]}$ extends naturally to a $W^{1,2}_{\rm loc}(\mathbb{R}^2)$ $(q+1-p)$-periodic function of y, by 4^o of Theorem 2.4,

$$J_{p,q}(f_k) \ge 0. \tag{3.37}$$

Writing

$$J_{1,\infty}(u_k) = J_{1,p-1}(u_k) + J_{p,q}(u_k) + J_{q+1,\infty}(u_k), \tag{3.38}$$

by (3.36)–(3.37),

$$J_{1,\infty}(u_k) \ge J_{1,p-1}(u_k) - \kappa(\varepsilon) + J_{q+1,\infty}(u_k). \tag{3.39}$$

Thus letting $q \to \infty$ gives

$$J_{1,\infty}(u_k) \ge J_{1,p-1}(u_k) - \kappa(\varepsilon). \tag{3.40}$$

Combining (3.40) with the analogous result for $J_{-\infty,0}(u_k)$ produces

$$J(u_k) \ge J_{-p+1,p-1}(u_k) - 2\kappa(\varepsilon). \tag{3.41}$$

Let $k \to \infty$ and use the $W^{1,2}(S_i)$ convergence of u_k to U to get

$$c(v,w) = \lim_{k\to\infty} J(u_k) \ge J_{-p+1,p-1}(U) - 2\kappa(\varepsilon). \tag{3.42}$$

Since (3.42) is true for all large p,

$$c(v,w) \ge J(U) - 2\kappa(\varepsilon). \tag{3.43}$$

Finally letting $\varepsilon \to 0$ yields

$$c(v,w) \ge J(U). \tag{3.44}$$

Thus (B) has been verified and the proof of Theorem 3.17 is complete.

Note that by Theorem 3.17, Corollary 3.14 becomes

$$c(v,w) + c(w,v) > 0. \tag{3.45}$$

As a consequence of this fact and Theorems 3.2 and 3.17, the next result gives, roughly, a more quantitative version of Theorem 3.2.

Corollary 3.46. *Let* $(*)$ *be satisfied and* $u \in \Gamma(v,v)$ *(resp.* $u \in \Gamma(w,w)$*) with* $v \le u \le w$. *If* $\gamma > 0$ *and*

$$\|u-v\|_{W^{1,2}(X_0)} \ge \gamma \qquad \text{(resp. } \|u-w\|_{W^{1,2}(X_0)} \ge \gamma\text{)}, \tag{3.47}$$

for $X_0 = \cup_{j=-2}^{2} S_j$, *then there exists a* $\beta = \beta(\gamma) > 0$ *(and independent of* u*) such that* $J(u) \ge \beta$.

Proof. The $\Gamma(v,v)$ case will be proved; the proof of the $\Gamma(w,w)$ case is essentially the same. Set

$$\mathcal{Y} = \{u \in \Gamma(v,v) \mid v \le u \le w \text{ and } u \text{ satisfies (3.47)}\}$$

and

$$c(\mathcal{Y}) = \inf_{u\in\mathcal{Y}} J(u). \tag{3.48}$$

Certainly

$$0 \le c(v,v) \le c(\mathcal{Y}) < \infty.$$

If $c(\mathcal{Y}) > 0$, the corollary is proved with $\beta(\gamma) = c(\mathcal{Y})$. Thus suppose that $c(\mathcal{Y}) = 0$. Let (u_k) be a minimizing sequence for (3.48). Then

$$J(u_k) \to 0, \quad k \to \infty. \tag{3.49}$$

But then (u_k) is also a minimizing sequence for (3.1) and as was shown in the proof of Theorem 3.2, $\Gamma(v,v)$ satisfies (Y_1)–(Y_3). Therefore by Propositions 2.10 and 2.25, it can be assumed that $u_k \to P \in W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ weakly in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$, and $u_k - P \to 0$ in $W^{1,2}(S_i)$ for all $i \in \mathbb{Z}$. Moreover P satisfies (3.47) and is a solution of (PDE). Set $\Phi_k = \max(u_k, \sigma_{-1}u_k)$ and $\Psi_k = \min(u_k, \sigma_{-1}u_k)$. Then $\Phi_k, \Psi_k \in \Gamma(v,v)$ so as in (3.28) (with $c(v,w)$ replaced by 0) and the argument following it: (i) Φ_k and Ψ_k converge to $\Phi = \max(P, \sigma_{-1}P)$, $\Psi = \min(P, \sigma_{-1}P)$ respectively; (ii) Φ, Ψ are solutions of (PDE) with $\Phi \ge \Psi$; and (iii) either (a) $\Phi(z) \equiv \Psi(z)$ for $z \in \mathbb{R}^2$ or (b) $\Phi(z) > \Psi(z)$ for all $z \in \mathbb{R}^2$.

If (a) holds, $P = \sigma_{-1}P$ so $P \in \Gamma(0,1)$. Moreover $v \le P \le w$ and as in (3.19)–(3.21),

$$J(P) < \infty. \tag{3.50}$$

Hence by $(*)$ and $P \in \Gamma(0,1)$, $J(P) = 0$ and $P \equiv v$ or $P \equiv w$. Since P satisfies (3.47), $P \equiv v$ is excluded. Thus if (a) holds, $P \equiv w$. If (b) is satisfied, either $P > \sigma_{-1}P$ or $\sigma_{-1}P > P$. The argument is similar in either event so assume

$$\sigma_{-1}P > P. \tag{3.51}$$

Then $P \in \widehat{\Gamma}(v,w)$ and

$$\|P-w\|_{W^{1,2}(T_i)} \to 0, \quad i \to \infty. \tag{3.52}$$

Observe that (3.52) also holds in case (a). Thus to show that $c(\mathcal{Y}) > 0$, it suffices to prove that (3.52) is not possible.

Let $\varepsilon > 0$. Since $u_k - P \to 0$ in $W^{1,2}(T_i)$ for all $i \in \mathbb{Z}$, by (3.52), $q = q(\varepsilon) \in \mathbb{N}$ can be chosen so that for all large k,

$$\|u_k - w\|_{W^{1,2}(T_q)} \le \varepsilon. \tag{3.53}$$

Define

$$\begin{aligned} f_k &= u_k, \quad y \le q-1 \\ &= w, \qquad q \le y \le q+1 \\ &= u_k, \quad q+2 \le y \end{aligned} \tag{3.54}$$

with the usual interpolation inbetween. Therefore as in (3.36), there is a $\mu(s) \to 0$ as $s \to 0$ such that

$$|J(u_k) - J(f_k)| \le \mu(\varepsilon). \tag{3.55}$$

Choose ε so small that

$$\mu(\varepsilon) < \frac{1}{2}(c(v,w) + c(w,v)). \tag{3.56}$$

Thus by (3.55)–(3.56), for large k

$$J(f_k) \le J(u_k) + \frac{1}{2}(c(v,w) + c(w,v)). \tag{3.57}$$

Defining

$$\begin{aligned} g_k &= f_k, \quad y \le q \\ &= w, \qquad y \ge q \end{aligned} \tag{3.58}$$

and

$$\begin{aligned} h_k &= w, \qquad y \le q \\ &= f_k, \quad y \ge q, \end{aligned} \tag{3.59}$$

then

$$J(f_k) = J(g_k) + J(h_k). \tag{3.60}$$

But $g_k \in \Gamma(v,w)$ and $h_k \in \Gamma(w,v)$. Therefore (3.60) implies for large k,

$$c(v,w) + c(w,v) \le J(f_k) \tag{3.61}$$

so by (3.57), (3.61), and (3.45),

$$0 < \frac{1}{2}c(v,w) + c(w,v) \le J(u_k) \tag{3.62}$$

which is contrary to (3.49). Hence $c(\mathcal{Y}) > 0$ and Corollary 3.46 is proved.

The next result plays a crucial role in the construction of solutions of (PDE) in the later sections. Set $X_i = \bigcup_{j=-2}^{2} S_{i+j}$.

Proposition 3.63. *Let v, w be as given by (∗) and $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ with $v \le u \le w$ and $J(u) \le M < \infty$. Then for any $\sigma > 0$, there is an $\ell_0 = \ell_0(\sigma) \in \mathbb{N}$ with ℓ_0 independent of u such that whenever $\ell \in \mathbb{N}$ and $\ell \ge \ell_0$, there exists $i_\ell \in (-\ell+2, \ell-2)$ and $\varphi_\ell \in \{v, w\}$ satisfying*

$$\|u - \varphi_\ell\|_{L^2(X_{i_\ell})} \le \sigma. \tag{3.64}$$

Proof. Arguing indirectly, if the Proposition is false, for some $\sigma > 0$, there is a sequence (u_k) satisfying

$$v \le u_k \le w, \tag{3.65}$$

$$J(u_k) \le M, \tag{3.66}$$

and

$$\|u_k - \varphi\|_{L^2(X_i)} \ge \sigma \tag{3.67}$$

for $\varphi = v, w$ and all $i \in (-k, k) \cap \mathbb{Z}$. By Proposition 2.6, (u_k) is bounded in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$. Hence by 1^o of Proposition 2.10 (and $\mathcal{Y} = \{(u_k)\}$, there is a $U^* \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ such that along a subsequence $u_k \to U^*$ weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$, and $u_k - U^* \to 0$ in $L^2(S_i)$ for all $i \in \mathbb{Z}$, and pointwise a.e. Moreover

$$v \le U^* \le w, \tag{3.68}$$

$$-K \le J(U^*) \le M + 2K \tag{3.69}$$

as in (3.21), and

$$\|U^* - \varphi\|_{L^2(X_i)} \ge \sigma \tag{3.70}$$

for $\varphi = v, w$ and all $i \in \mathbb{Z}$.

It will be shown that the existence of such a U^* is not possible. Set

$$\Gamma(v, U^*) = \big\{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2) \mid v \le u \le w, \|v - u\|_{L^2(S_i)} \to 0 \text{ as } i \to -\infty, \\ \|U^* - u\|_{L^2(S_i)} \to 0 \text{ as } i \to \infty\big\}.$$

Then $\Gamma(v, U^*) \ne \phi$, e.g. setting

$$\begin{aligned} \overline{u} &= v, \quad y \le 0 \\ &= yU^* + (1-y)v, \quad 0 < y < 1 \\ &= U^*, \qquad y \ge 1 \end{aligned} \tag{3.71}$$

shows $\overline{u} \in \Gamma(v, U^*)$. Taking $\mathcal{Y} = \Gamma(v, U^*)$, $\mathcal{Y}$ satisfies (Y_1)–(Y_2). Define

$$c(v, U^*) = \inf_{u \in \Gamma(v, U^*)} J(u). \tag{3.72}$$

Then by (3.69), (3.71) and Proposition 2.6,

$$-K \le c(v, U^*) \le J(\overline{u}) < \infty. \tag{3.73}$$

Let (φ_k) be a minimizing sequence for (3.72). Then there is an $i_k \in \mathbb{N}$ such that for all $i \geq i_k$,

$$\|\varphi_k - U^*\|_{L^2(X_i)} \leq \frac{\sigma}{3}. \tag{3.74}$$

Since $J(\varphi_k) = J(\sigma_{-i_k}\varphi_k)$, $\sigma_{-i_k}\varphi_k$ is also a minimizing sequence for (3.72). Therefore by Proposition 2.10, it can be assumed that there is a $\Phi \in W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ such that $\sigma_{-i_k}\varphi_k \to \Phi$ weakly in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ and in $W^{1,2}(S_i)$ for all $i \in \mathbb{Z}$. Hence by (3.70) and (3.74),

$$\begin{aligned}\|\sigma_{-i_k}\varphi_k - \varphi\|_{L^2(X_i)} &= \|\varphi_k - \varphi\|_{L^2(X_{i+i_k})}\\ &\geq \|U^* - \varphi\|_{L^2(X_{i+i_k})} - \|\varphi_k - U^*\|_{L^2(X_{i+i_k})}\\ &\geq \frac{2}{3}\sigma, \qquad i \geq 0.\end{aligned} \tag{3.75}$$

Consequently

$$\|\Phi - \varphi\|_{L^2(X_i)} \geq \frac{2}{3}\sigma, \quad i \geq 0. \tag{3.76}$$

Repeating the argument of (3.22)–(3.26) for the current setting shows $\mathcal{Y}$ satisfies (Y_3) for all $z \in \mathbb{R}^2$. Hence Φ is a solution of (PDE) on $\mathbb{R}^2$.

A second solution of (PDE) will be produced next. Let $U \in \Gamma(v, w)$ be a solution of (PDE) as given by Theorem 3.17. Thus $\sigma_{-j}U \in \Gamma(v, w)$ is also a solution for any $j \in \mathbb{Z}$. Therefore U can be normalized so that

$$\|U - v\|_{L^2(X_0)} < \frac{\sigma}{3}. \tag{3.77}$$

Set $\psi_k = \max(\sigma_{-i_k}\varphi_k, U)$ and $\chi_k = \min(\sigma_{-i_k}\varphi_k, U)$. Therefore

$$J(\chi_k) + J(\psi_k) = J(\varphi_k) + J(U). \tag{3.78}$$

But $\psi_k \in \Gamma(v, w)$ and $\chi_k \in \Gamma(v, U^*)$. Consequently

$$J(U) = c(v, w) \leq J(\psi_k) \tag{3.79}$$

and by (3.78),

$$J(\chi_k) \leq J(\varphi_k). \tag{3.80}$$

Thus (χ_k) is also a minimizing sequence for (3.72) and as for (φ_k), it can be assumed that χ_k converges in $W^{1,2}(X_i)$, for all $i \in \mathbb{Z}$, to a solution $\chi = \min(\Phi, U)$ of (PDE). Set $\Psi = \Phi - \chi$. Thus $\Psi \geq 0$ and by the Maximum Principle argument centered around (3.11) again, either (a) $\Psi > 0$ on $\mathbb{R}^2$ or (b) $\Psi \equiv 0$. By (3.76), Φ is not close to w in $L^2(X_i)$ for large i while $\|U - w\|_{C^2(X_i)} \to 0$ as $i \to \infty$. Hence for large i, there are points where $\chi = \Phi$ and $\Psi = 0$. On the other hand, by (3.76)–(3.77) there are points in X_0 where $\chi = U$ and $\Psi > 0$. Thus neither (a) nor (b) are possible. This contradiction establishes Proposition 3.63.

Corollary 3.81. *Under the hypotheses of Proposition 3.63, if there is a y_1 such that u is a solution of (PDE) for $y \geq y_1$ (resp. $y \leq y_1$), then*

$$\|u-\varphi\|_{W^{1,2}(X_i)} \to 0 \ \text{ as } i \to \infty$$

for some $\varphi \in \{v, w\}$ (resp.

$$\|u-\psi\|_{W^{1,2}(X_i)} \to 0 \ \text{ as } i \to -\infty$$

for some $\psi \in \{v, w\}$).

Proof. By Proposition 3.63, there is a $\varphi \in \{v, w\}$ independent of σ having the property that for any small σ, there is a sequence $(s_i(\sigma))$ with $s_i(\sigma) \to \infty$ as $i \to \infty$ such that $\|u-\varphi\|_{L^2(X_{s_i(\sigma)})} \leq \sigma$. With φ so determined, we claim

$$\|u-\varphi\|_{L^2(X_i)} \to 0 \ \text{ as } i \to \infty. \tag{3.82}$$

If not, there is a $\gamma > 0$ and sequence (p_i) with $p_i \to \infty$ as $i \to \infty$ such that

$$\|u-\varphi\|_{L^2(X_{p_i})} \geq \gamma. \tag{3.83}$$

Passing to a subsequence if necessary, it can be assumed that $p_{i+1}-p_i > 2\ell_0(\sigma)+4$ with ℓ_0 as given by Proposition 3.63. Applying Proposition 3.63 with $(-\ell+2, \ell-2)$ replaced by $(p_i+4, p_{i+1}-4)$ gives q_i in this interval such that

$$\|u-\varphi\|_{L^2(X_{q_i})} \leq \sigma. \tag{3.84}$$

We claim (3.84) implies there is a constant $M_2 > 0$ such that

$$\|u-\varphi\|_{W^{1,2}(T_{q_i})} \leq M_2\|u-\varphi\|_{L^2(X_{q_i})} \leq M_2\sigma. \tag{3.85}$$

Assuming (3.85) for the moment, for $i \in \mathbb{N}$, define

$$\begin{aligned} f_i &= u, && y \leq q_i - 1 \\ &= \varphi, && q_i \leq y \leq q_i + 1 \\ &= u, && q_i + 2 \leq y \leq q_{i+1} - 1 \\ &= \varphi, && q_{i+1} \leq y \leq q_{i+1} + 1 \\ &= u, && q_{i+1} + 2 \leq y \end{aligned} \tag{3.86}$$

with the usual interpolation in the remaining intervals. Then as in (3.36), there is a $\kappa(s) \to 0$ as $s \to 0$ such that

$$|J_{q_i,q_{i+1}}(u) - J_{q_i,q_{i+1}}(f_i)| \leq \kappa(\sigma). \tag{3.87}$$

Set

$$\begin{aligned} h_i &= \varphi, && y \leq q_i \\ &= f_i, && q_i \leq y \leq q_{i+1} \\ &= \varphi, && q_{i+1} \leq y. \end{aligned} \tag{3.88}$$

Then $h_i \in \Gamma(\varphi, \varphi)$ and (3.83) holds. Therefore by Corollary 3.46,

$$J(h_i) \geq \beta(\gamma). \tag{3.89}$$

But

$$J(h_i) = J_{q_i, q_{i+1}}(f_i) \tag{3.90}$$

so by (3.87)–(3.90),

$$J_{q_i, q_{i+1}}(u) \geq \beta(\gamma) - \kappa(\sigma). \tag{3.91}$$

By (3.85), it can also be assumed that

$$|J_{q_{i+1}, q_{i+1}}(u)| \leq \kappa(\sigma). \tag{3.92}$$

Choose σ so small that

$$4\kappa(\sigma) < \beta(\gamma). \tag{3.93}$$

Therefore

$$J_{q_i, q_{i+1}-1}(u) \geq \frac{1}{2}\beta(\gamma). \tag{3.94}$$

Writing

$$J(u) = J_{-\infty, q_1-1}(u) + \sum_{1}^{n} J_{q_i, q_{i+1}-1}(u) + J_{q_{n+1}, \infty}(u) \tag{3.95}$$

with n free for the moment, and recalling that $J(u) \leq M$, by (3.94),

$$M \geq J_{-\infty, q_1-1}(u) + \frac{n}{2}\beta(\gamma) + J_{q_{n+1}, \infty}(u). \tag{3.96}$$

Further applying 1^o of Proposition 2.6 yields

$$M + 2K \geq \frac{n}{2}\beta(\gamma). \tag{3.97}$$

But (3.97) cannot hold for large n. Thus we have a contradiction and (3.82) is valid. Hence by (3.85),

$$\|u - \varphi\|_{W^{1,2}(T_i)} \to 0, \qquad i \to \infty.$$

Now to complete the proof of the Proposition, it remains to verify (3.85). Set $f = u - \varphi$. Then by (3.11), f satisfies

$$-\Delta f + Af = 0 \tag{3.98}$$

where $\|A\|_{L^\infty(\mathbb{R}^2)} \leq M_1 < \infty$. Let $\eta \in C^1(\mathbb{R}^2)$ having support in X_i, with $0 \leq \eta \leq 1$, $|\nabla \eta| \leq 2$, $\eta(x, y) = 1$ if $(x, y) \in T_i$ and $|x| < R$, and $\eta = 0$ if $|x| > R + 1$. Multiplying (3.98) by $\eta^2 f$ and integrating by parts yields

$$0 = \int_{X_i} (\eta^2 |\nabla f|^2 + 2\eta f \nabla \eta \cdot \nabla f + A\eta^2 f^2) dxdy. \tag{3.99}$$

Therefore

$$\begin{aligned}(3.100)\qquad &\frac{1}{2}\int_{T_i\cap\{|x|\le R\}}|\nabla f|^2dxdy\\ &\le\frac{1}{2}\int_{X_i}\eta^2|\nabla f|^2dxdy\le(M_1+8)\int_{X_i}f^2dxdy.\end{aligned}$$

Letting $R\to\infty$ yields

$$(3.101)\qquad \int_{T_i}|\nabla f|^2dxdy\le 2(M_1+8)\int_{X_i}f^2dxdy.$$

Hence

$$(3.102)\qquad \int_{T_i}(|\nabla f|^2+f^2)dxdy\le(2M_1+17)\int_{X_i}f^2dxdy$$

from which (3.85) follows with $M_2=(2M_1+17)^{1/2}$.

The next result refines the convergence given by Corollary 3.81.

Corollary 3.103. *Under the hypotheses of Corollary 3.81,* $\|u-\varphi\|_{C^2(S_i)}\to 0$ *as* $i\to\infty$ *(resp.* $\|u-\psi\|_{C^2(S_i)}\to 0$ *as* $i\to-\infty$*).*

Proof. Since $|u-\varphi|\le 1$ and $u-\varphi$ satisfies (3.98) with $\|A\|_{L^\infty(\mathbb{R}^2)}\le M_1$, for any $p>2$, the L^p_{loc} elliptic estimates imply for any $z\in S_i$,

$$\begin{aligned}(3.104)\ \|u-\varphi\|_{W^{2,p}(B_1(z))}&\le M_3\|u-\varphi\|_{L^p(B_2(z))}\\ &\le M_3\|u-\varphi\|^{2/p}_{L^2(B_2(z))}\le M_3\|u-\varphi\|^{2/p}_{L^2(X_i)}\end{aligned}$$

where the constant M_3 is independent of i and z in S_i. Hence for large p, the Sobolev Embedding Theorem and Corollary 3.81 imply

$$(3.105)\qquad \|u-\varphi\|_{C^1(S_i)}\to 0,\quad i\to\infty.$$

The interior Schauder estimates further imply for any fixed $\alpha\in(0,1)$, there is a constant M_4 such that

$$(3.106)\qquad \|u-\varphi\|_{C^{2,\alpha}(B_1(z))}\le M_4$$

for all $z\in\mathbb{R}^2$. Finally (3.105)–(3.106) and local interpolation inequalities (independent of $z\in S_i$) show

$$(3.107)\qquad \|u-\varphi\|_{C^2(S_i)}\to 0,\qquad i\to\infty.$$

4 Two bump solutions

In this section, the tools of Sects. 2–3 will be used to establish the existence of an infinitude of solutions of (PDE) that are homoclinic in y to $v \in \mathcal{M}(0,1)$. A further nondegeneracy condition in the spirit of $(*)$ is required. To formulate it, let v, w be as given by $(*)$ and

$$\mathcal{M}(v,w) = \{u \in \Gamma(v,w) \mid J(u) = c(v,w)\}.$$

Then by Theorem 3.17, $\mathcal{M}(v,w) \neq \phi$ and similarly for

$$\mathcal{M}(w,v) = \{u \in \Gamma(w,v) \mid J(u) = c(w,v)\}.$$

Proposition 4.1. *$\mathcal{M}(v,w)$ and $\mathcal{M}(w,v)$ are ordered sets.*

Proof. The proof is essentially the same as that of 3^o of Theorem 2.4 and involves arguments used above so we will be brief. If $U, V \in \mathcal{M}(v,w)$, $\Phi = \max(U,V)$ and $\Psi = \min(U,V) \in \Gamma(v,w)$ and as in (3.9)–(3.10),

$$J(\Phi) + J(\Psi) = J(U) + J(V) = 2c(v,w). \tag{4.2}$$

Hence $\Phi, \Psi \in \mathcal{M}(v,w)$ and $\Phi \geq \Psi$. By the argument centered around (3.11), either $\Phi \equiv \Psi$ in which case $U \equiv V$ or $\Phi > \Psi$ in which case $U > V$ or $V > U$. Thus $\mathcal{M}(v,w)$ and similarly $\mathcal{M}(w,v)$ is an ordered set.

The nondegeneracy condition required of these sets is:

$$\mathcal{M}(v,w) \text{ and } \mathcal{M}(w,v) \text{ are not continua.} \tag{$**$}$$

Thus as for $\mathcal{M}(0,1)$ under $(*)$, there are gaps in $\mathcal{M}(v,w)$, i.e. there exist adjacent members of these sets. To obtain solutions of (PDE) homoclinic to v, a constrained minimization argument will be employed. Of course constraints were involved in the definitions of the classes of functions $\Gamma(\cdot,\cdot)$ introduced in Sect. 3, but in the current setting, there are also integral constraints. Although the technicalities are rather different, the spirit is that of the variational approach to chaotic dynamics as in work of Mather [6] and others, e.g. [7].

To set up the current minimization problem, for $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ define

$$\begin{cases} \rho_-(u) = \|u - v\|_{L^2(S_0)} \\ \rho_+(u) = \|w - u\|_{L^2(S_0)} \end{cases} \tag{4.3}$$

and observe that by Proposition 4.1, ρ_- is strictly increasing on $\mathcal{M}(v,w)$ and on $\mathcal{M}(w,v)$, and ρ_+ is strictly decreasing on these sets. Set $\overline{\rho} = \|w - v\|_{L^2(S_0)}$ and choose constants $\rho_i \in (0, \overline{\rho})$, $i = 1, \dots, 4$ such that

$$\begin{cases} \rho_1 \notin \rho_-(\mathcal{M}(v,w)), & \rho_2 \notin \rho_+(\mathcal{M}(v,w)), \\ \rho_3 \notin \rho_+(\mathcal{M}(w,v)), & \rho_4 \notin \rho_-(\mathcal{M}(w,v)). \end{cases} \tag{4.4}$$

Let $\ell \in \mathbb{N}$ and $m \in \mathbb{Z}^4$ with

$$m_1 < m_2 < m_2 + 2\ell < m_3 < m_4. \tag{4.5}$$

The class of admissible functions here is

$$\Gamma_m \equiv \Gamma_m(v, v) \equiv \{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2) \mid u \text{ satisfies (4.6)–(4.8)}\}$$

where

$$v \le u \le w, \tag{4.6}$$

$$\begin{cases} \text{(i) } \rho_-(\sigma_{-i}u) \le \rho_1, & i = m_1 - \ell, \dots, m_1 - 1, \\ \text{(ii) } \rho_+(\sigma_{-i}u) \le \rho_2, & i = m_2, \dots, m_2 + \ell - 1, \\ \text{(iii) } \rho_+(\sigma_{-i}u) \le \rho_3, & i = m_3 - \ell, \dots, m_3 - 1, \\ \text{(iv) } \rho_-(\sigma_{-i}u) \le \rho_4, & i = m_4, \dots, m_4 + \ell - 1 \end{cases} \tag{4.7}$$

$$\|u - v\|_{L^2(S_i)} \to 0, \quad |i| \to \infty. \tag{4.8}$$

Set

$$c_m \equiv c_m(v, v) \equiv \inf_{u \in \Gamma_m} J(u). \tag{4.9}$$

Then we have

Theorem 4.10. *Suppose F satisfies (F_1)–(F_3) and ($*$) and ($**$) hold. Then for each ℓ sufficiently large, there is a $U_m \in \Gamma_m$ such that $J(U_m) = c_m$. If in addition, $m_2 - m_1$ and $m_4 - m_3$ are also sufficiently large, U_m is a solution of (PDE) and*

$$\|U_m - v\|_{C^2(S_i)} \to 0 \ \text{ as } |i| \to \infty. \tag{4.11}$$

Remark 4.12. The conditions in (4.7) force U_m to be close to v, w in $L^2(S_i)$ for certain ranges of i. The same methods as used in the proof of Theorem 4.10 allow this to be strengthened to closeness in $W^{1,2}(S_i)$, and to increase the range of admissible i. For instance U_m can be forced to shadow w in $\| \cdot \|_{W^{1,2}}$ for $m_2 \le i \le m_3$, and v for $i \le m_1$ and $i \ge m_4$.

Proof of Theorem 4.10. The proof consists of several steps. Let (u_k) be a minimizing sequence for (4.9). Then there is an $M > 0$ such that

$$J(u_k) \le M, \quad k \in \mathbb{N}. \tag{4.13}$$

Since Γ_m satisfies (Y_1)–(Y_2), by Proposition 2.10 it can be assumed that $u_k \to U_m \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$, strongly in $W^{1,2}(S_i)$ for all $i \in \mathbb{Z}$ and pointwise a.e. Hence U_m satisfies (4.6)–(4.7) and as in (3.20)–(3.21),

$$J(U_m) \le M + 2K. \tag{4.14}$$

It will now be shown that: (A) U_m is a solution of (PDE) outside of the integral constraint regions; (B) For ℓ sufficiently large, there is an X_i in each integral constraint region such that U_m satisfies (PDE) in X_i; (C) U_m satisfies (4.8) and consequently $U_m \in \Gamma_m$; (D) $J(U_m) = c_m$; (E) for $m_2 - m_1$ and $(m_4 - m_3)$ sufficiently large, U_m satisfies (PDE) in the constraint regions; (F) (4.11) holds.

Proof of (A). For z not in a constraint region and $r = r(z)$ small, (Y_3) is satisfied as in the proof of Theorem 3.17 – see (3.21)–(3.26) so (A) holds via Proposition 2.25.

Proof of (B). Let $\mathcal{R}$ be one of the integral constraint regions. Choose σ so that

$$0 < \sigma < \min_{1 \le j \le 4} \{(\rho_j, \overline{\rho} - \rho_j)\}. \tag{4.15}$$

Assume $\ell \ge \ell_0(\sigma)$ as given by Proposition 3.63. Then there is an $X_i \subset \mathcal{R}$ such that

$$\|U_m - \varphi_i\|_{L^2(X_i)} \le \sigma. \tag{4.16}$$

where φ_i is v or w. The choice of σ in (4.15) shows $\varphi_i = v$ if X_i corresponds to ρ_1 or ρ_4 and $\varphi_i = w$ if X_i corresponds to ρ_2 or ρ_3. E.g. if X_i corresponds to ρ_2 and $\varphi_i = v$, by (4.16) and (4.7)(ii),

$$\begin{aligned} \sigma &\ge \|U_m - v\|_{L^2(X_i)} \ge \|U_m - v\|_{L^2(S_i)} \\ &\ge \overline{\rho} - \|U_m - w\|_{L^2(S_i)} \ge \overline{\rho} - \rho_2, \end{aligned} \tag{4.17}$$

contrary to (4.15). Thus (4.16) shows the corresponding constraint in (4.7) is satisfied with strict inequality. This implies for any $z \in X_i$ and $r > 0$ such that $B_r(z) \subset X_i$, (Y_3) holds. Indeed if φ is smooth with support in $B_r(z)$ and $u_k \in \Gamma_m$, then one can truncate $u_k + t\varphi$ as in the proof of (A) for $|t|$ small, so (2.26) is valid. Thus by Proposition 2.25 again, U_m is a solution of (PDE) in X_i for some X_i in each integral constraint region.

Proof of(C). By Corollary 3.81 with $y_1 = m_4 + \ell$,

$$\|U_m - \varphi\|_{W^{1,2}(X_j)} \to 0, \quad j \to \infty. \tag{4.18}$$

where φ is v or w. If $\varphi = v$, this case is proved. Thus the possibility that $\varphi = w$ must be excluded. Suppose $\varphi = w$ in (4.18). Then there is a $p > m_4 + \ell$ such that

$$\|U_m - v\|_{L^2(S_p)} \ge \frac{3}{4}\overline{\rho}. \tag{4.19}$$

Since $u_k \to U_m$ in $W^{1,2}(S_p)$,

$$\|u_k - v\|_{L^2(S_p)} \ge \frac{1}{2}\overline{\rho} \tag{4.20}$$

for large k.

Let X_i be as given by Proposition 3.63, with $i \in (m_4 + 2, m_4 + \ell - 2)$. Then as in the proof of (B),

$$\|U_m - v\|_{L^2(X_i)} \le \sigma. \tag{4.21}$$

Since U_m and v are solutions of (PDE), by (3.85),

$$\|U_m - v\|_{W^{1,2}(T_i)} \le M_2\sigma. \tag{4.22}$$

Again using the convergence of u_k to U_m, for k large,

(4.23)
$$\|u_k - v\|_{W^{1,2}(T_i)} \le 2M_2\sigma.$$

Now we will cut and paste as in the proof of Corollary 3.81. As in (3.86) define

(4.24)
$$\begin{aligned} f_k &= u_k, \quad y \le i-1 \\ &= v, \qquad i \le y \le i+1 \\ &= u_k, \quad i+2 \le y \le q_k - 1 \\ &= v, \qquad q_k \le y \le q_k + 1 \\ &= u_k, \quad q_k + 2 \le y \end{aligned}$$

with the usual interpolation for the remaining y intervals. In (4.24), $q_k > p$ is chosen so large that

(4.25)
$$\|u_k - v\|_{W^{1,2}(T_{q_k})} \le M_2\sigma.$$

Then as in (3.87)

(4.26)
$$|J_{i,q_k}(u_k) - J_{i,q_k}(f_k)| \le \kappa(\sigma).$$

Set

(4.27)
$$\begin{aligned} h_k &= v, \qquad y \le i \\ &= f_k, \quad i \le y \le q_k \\ &= v, \qquad q_k \le y \end{aligned}$$

so $h_k \in \Gamma(v, v)$ and by (4.27) and (4.20),

(4.28)
$$J(h_k) = J_{i,q_k}(f_k) \ge \beta\left(\frac{1}{2}\overline{\rho}\right),$$

with β as in Corollary 3.46.

Hence by (4.26)–(4.28),

(4.29)
$$J(u_k) \ge J_{-\infty,i-1}(u_k) + \beta\left(\frac{1}{2}\overline{\rho}\right) - \kappa(\sigma) + J_{q_{k+1},\infty}(u_k).$$

On the other hand, setting

(4.30)
$$\begin{aligned} g_k &= u_k, \quad y \le i-1 \\ &= v, \qquad i \le y \le q_k + 1 \\ &= u_k, \quad q_k + 2 \le y, \end{aligned}$$

with the usual interpolation inbetween, it can be assumed that

(4.31)
$$|J_{i-1,i-1}(u_k) - J_{i-1,i-1}(g_k)| + |J_{q_k+1,q_k+1}(u_k) - J_{q_k+1,q_k+1}(g_k)| \le \kappa(\sigma).$$

Thus by (4.30)–(4.31),

$$J_{-\infty,i-1}(u_k) + J_{q_k+1,\infty}(u_k) \geq J(g_k) - \kappa(\sigma) \tag{4.32}$$

so combining (4.29) and (4.32) yields

$$J(u_k) \geq J(g_k) + \beta\left(\frac{1}{2}\overline{\rho}\right) - 2\kappa(\sigma). \tag{4.33}$$

Observing that $g_k \in \Gamma_m$ and choosing σ so small that

$$4\kappa(\sigma) < \beta\big(\frac{1}{2}\overline{\rho}\big), \tag{4.34}$$

(4.33) shows that (u_k) is not a minimizing sequence for (4.9). This contradiction implies $\varphi = v$ in (4.18). Similarly $U_m \to v$ in $W^{1,2}(X_i)$ as $i \to -\infty$ and (C) is proved.

Proof of (D). To begin, since by (C), $U_m \in \Gamma_m$,

$$J(U_m) \geq c_m. \tag{4.35}$$

Let $\varepsilon > 0$. Then p can be chosen so that

$$\|U_m - v\|_{W^{1,2}(X_i)} \leq \frac{\varepsilon}{2}, \quad |i| \geq p. \tag{4.36}$$

Since (u_k) converges to U_m in $W^{1,2}(X_i)$ for all $i \in \mathbb{Z}$, for $k \geq k_0(p)$,

$$\|u_k - U_m\|_{W^{1,2}(X_{\pm p})} \leq \frac{\varepsilon}{2} \tag{4.37}$$

so by (4.36),

$$\|u_k - v\|_{W^{1,2}(X_{\pm p})} \leq \varepsilon. \tag{4.38}$$

Fixing k, since $u_k \in \Gamma_m$, for $|q| \geq q_0(k)$,

$$\|u_k - v\|_{W^{1,2}(X_q)} \leq \varepsilon. \tag{4.39}$$

With $q > p$, let f_k be as in (4.24) with i replaced by p and q_k by q. Thus as in (4.26) it can be assumed that

$$|J_{p,q}(u_k) - J_{p,q}(f_k)| \leq \kappa(\varepsilon). \tag{4.40}$$

Since $f_k\big|_{\mathbb{R}\times[p,q+1]}$ extends naturally as a $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ $(q+1-p)$-periodic function in y, by 4^o of Theorem 2.4,

$$J_{p,q}(f_k) \geq 0. \tag{4.41}$$

Writing

$$J_{1,\infty}(u_k) = J_{1,p-1}(u_k) + J_{p,q}(u_k) + J_{q,\infty}(u_k), \tag{4.42}$$

and using (4.40), (4.41) gives

$$J_{1,\infty}(u_k) \geq J_{1,p-1}(u_k) - \kappa(\varepsilon) + J_{q,\infty}(u_k). \tag{4.43}$$

Since

$$J_{q,\infty}(u_k) \to 0. \tag{4.44}$$

as $q \to \infty$ by Proposition 2.6, letting $q \to \infty$ in (4.43) shows

$$J_{1,\infty}(u_k) \geq J_{1,p-1}(u_k) - \kappa(\varepsilon). \tag{4.45}$$

Adding (4.45) to the related inequality for $J_{-\infty,0}(u_k)$,

$$J(u_k) \geq J_{-p+1,p-1}(u_k) - 2\kappa(\varepsilon). \tag{4.46}$$

Letting $k \to \infty$, the $W^{1,2}(S_i)$ convergence of u_k to U_m yields

$$c_m = \lim_{k\to\infty} J(u_k) \geq J_{-p+1,p-1}(U_m) - 2\kappa(\varepsilon) \tag{4.47}$$

by 3^o of Proposition 2.6. First letting $p \to \infty$ and then using that $\varepsilon > 0$ is arbitrary produces

$$c_m \geq J(U_m). \tag{4.48}$$

Thus (4.35) and (4.48) imply (D).

Proof of (E). If U_m satisfies all of the integral constraints (4.7) with strict inequality, then as in (B) or (A), it is a solution of (PDE) everywhere in the constraint region. Strict inequality will be shown for (4.7)(i)–(ii); the other cases are handled similarly.

Suppose there is equality in (4.7)(i) or (ii). Thus

$$\|U_m - \varphi\|_{L^2(S_i)} = \rho \tag{4.49}$$

where $(\varphi, \rho) = (v, \rho_1)$ or (w, ρ_2). By Proposition 3.63, there are strips X_s, X_t with $s \in [m_1 - \ell + 2, m_1 - 3] \cap \mathbb{Z}$ and $t \in [m_2 + 2, m_2 + \ell - 3] \cap \mathbb{Z}$ such that

$$\|U_m - v\|_{L^2(X_s)}, \ \ \|U_m - w\|_{L^2(X_t)} \leq \sigma. \tag{4.50}$$

Hence by (3.85),

$$\|U_m - v\|_{W^{1,2}(T_s)}, \ \ \|U_m - w\|_{W^{1,2}(T_t)} \leq M_2\sigma. \tag{4.51}$$

Two possibilities arise:

(i) S_i lies between X_s and X_t.
(ii) S_i does not lie between X_s and X_t.

Each possibility will be excluded by a comparison argument. Taking $\sigma < \rho_i$ elliminates $s-1 \le i \le s+1, t-1 \le i \le t+1$ due to (4.50).

Case (ii) will be treated first. The two remaining subcases are: $i < s-1$ or $i > t+1$. Suppose $i > t+1$. By Proposition 3.63 and (3.85) again, there is a $q \in [m_3 - \ell + 2, m_3 - \ell - 3] \cap \mathbb{Z}$ such that

(4.52) $$\|U_m - w\|_{W^{1,2}(X_q)} \le M_2\sigma.$$

Define

(4.53) $$\begin{aligned} f &= U_m, \quad y \notin T_t \cup T_q \\ &= w, \qquad y \in S_t \cup S_q \end{aligned}$$

with the usual interpolation in the remaining four strips. As in e.g. (3.86) there is a $\kappa(\varepsilon)$ with $\kappa \to 0$ as $\varepsilon \to 0$ such that

(4.54) $$|J(U_m) - J(f)| \le \kappa(\sigma).$$

Set

(4.55) $$\begin{aligned} g &= w, \qquad y \le t \\ &= f, \qquad t \le y \le q+1 \\ &= w, \qquad q+1 \le y \end{aligned}$$

and

(4.56) $$\begin{aligned} h &= f, \qquad y \le t \\ &= w, \qquad t \le y \le q+1 \\ &= f, \qquad q+1 \le y. \end{aligned}$$

Then

(4.57) $$J(f) = J(g) + J(h).$$

Note that $g \in \Gamma(w,w)$ and

(4.58) $$\|g - w\|_{L^2(S_i)} = \|U_m - w\|_{L^2(S_i)} = \rho_2.$$

Consequently by Corollary 3.46,

(4.59) $$J(g) \ge \beta(\rho_2).$$

The function $h \in \Gamma_m$ and by (4.54), (4.57)–(4.59), and (D),

(4.60) $$J(U_m) = c_m \ge \beta(\rho_2) + c_m - \kappa(\sigma).$$

Further choosing σ so small that

(4.61) $$2\kappa(\sigma) < \min_{1 \le j \le 4} \beta(\rho_j),$$

shows $i > t+1$ is not possible.

If $i < s-1$, choose $q < i-5$ so that (4.52) holds with w replaced by v. Likewise replacing w by v, t by q, q by s and ρ_2 by ρ_1 in the rest of the argument from (4.52) to (4.61) shows this case is also impossible.

Next suppose that case (i) occurs. An additional comparison result is needed. It plays the role for $\Gamma(v,w)$ and $\Gamma(w,v)$ that Corollary 3.46 does for Theorem 3.2. Define

$$\Lambda(v,w) = \left\{u \in \Gamma(v,w) \mid \|u-v\|_{L^2(S_0)} = \rho_1 \quad \text{or} \quad \|u-w\|_{L^2(S_0)} = \rho_2\right\}.$$

Set

$$d(v,w) = \inf_{u\in\Lambda(v,w)} J(u). \tag{4.62}$$

Define $\Lambda(w,v)$ and $d(w,v)$ similarly. Then we have

Proposition 4.63. *$d(v,w) > c(v,w)$ and $d(w,v) > c(w,v)$.*

The proof of Proposition 4.63 will be postponed until the completion of (F). To complete the discussion of case (i), define

$$\begin{aligned} f &= U_m, \quad y \notin T_s \cup T_t \\ &= v, \qquad y \in S_s \\ &= w, \qquad y \in S_t \end{aligned} \tag{4.64}$$

with the usual interpolation for the remaining y intervals. Then as in earlier such cases,

$$|J_{s,t}(U_m) - J_{s,t}(f)| \le \kappa(\sigma). \tag{4.65}$$

Choose σ so small that

$$3\kappa(\sigma) < \min(d(v,w) - c(v,w), d(w,v) - c(w,v)) \equiv \mu. \tag{4.66}$$

Define

$$\begin{aligned} g &= v, \qquad y \le s \\ &= f, \qquad s \le y \le t+1 \\ &= w, \qquad t+1 \le y. \end{aligned} \tag{4.67}$$

Then $g \in \Lambda(v,w)$ so

$$J(g) = J_{s,t}(g) = J_{s,t}(f) \ge d(v,w). \tag{4.68}$$

Thus by (4.65) and (4.68),

$$J_{s,t}(U_m) \ge d(v,w) - \kappa(\sigma). \tag{4.69}$$

Let $U \in \mathcal{M}(v,w)$ as given by Theorem 2.34 or 3.17. Define

$$\begin{aligned} h &= U_m, \quad y \le s-1 \ \text{ and } \ y \ge t+2 \\ &= \sigma_{-q} U, \quad s \le y \le t+1 \end{aligned} \tag{4.70}$$

where q is free for the moment, with the usual interpolation in the intermediate regions. Choose $\varepsilon \in (0, \mu/3)$. Then for $m_2 - m_1$ sufficiently large, $q \in \mathbb{Z}$ can be chosen so that $h \in \Gamma_m$,

$$J_{s,t}(h) \leq c(v, w) + \varepsilon, \tag{4.71}$$

and

$$\|\sigma_{-q} U - v\|_{W^{1,2}(X_s)}, \quad \|\sigma_{-q} U - w\|_{W^{1,2}(X_t)} \leq \sigma. \tag{4.72}$$

Therefore

$$J(U_m) = c_m \leq J(h) = J_{-\infty,s-2}(U_m) + J_{s-1,t+1}(h) + J_{t+2,\infty}(U_m). \tag{4.73}$$

By (4.51), (4.71)–(4.73) and 3^o of Proposition 2.6,

$$J_{s,t}(U_m) \leq c(v, w) + \varepsilon + \kappa(\sigma) \tag{4.74}$$

which together with (4.69) yields

$$\mu \leq d(v, w) - c(v, w) \leq \varepsilon + 2\kappa(\sigma) < \frac{\mu}{3} + 2\kappa(\sigma). \tag{4.75}$$

But (4.75) is contrary to (4.66). Thus case (i) cannot occur and (E) is proved.

Proof of (F). It has already been established that $\|U_m - v\|_{W^{1,2}(X_i)} \to 0$ as $i \to \infty$. The C^2 convergence then follows immediately from Corollary 3.103.

This completes the proof of Theorem 4.10 except for the proof of Proposition 4.63 which will be carried out next.

Proof of Proposition 4.63. The first inequality will be proved, the second proof is the same. Since $\Lambda(v, w) \subset \Gamma(v, w)$,

$$d(v, w) \geq c(v, w). \tag{4.76}$$

To see that inequality is not possible, let (u_k) be a minimizing sequence for (4.62). Then by Propositions 2.6, 2.10, and 2.25, it can be assumed that u_k converges to $P \in W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ with P a solution of (PDE) for $y \notin [0, 1]$ and

$$\|P - v\|_{L^2(S_0)} = \rho_1 \quad \text{or} \quad \|P - w\|_{L^2(S_0)} = \rho_2. \tag{4.77}$$

Moreover by Corollary 3.81

$$\|P - \varphi\|_{W^{1,2}(X_i)}, \quad \|P - \psi\|_{W^{1,2}(X_j)} \to 0 \tag{4.78}$$

as $i \to \infty$, $j \to -\infty$ for some $\varphi, \psi \in \{v, w\}$. Suppose e.g. $\psi = w$. Let $\varepsilon > 0$. Then there is a $t \in -\mathbb{N}$ such that for all $k \geq k_0(t)$,

$$\|u_k - w\|_{W^{1,2}(X_t)} \leq \varepsilon \tag{4.79}$$

and for any such k, there is a $q = q(k) \in \mathbb{N}$ such that

$$\|u_k - w\|_{W^{1,2}(X_q)} \leq \varepsilon. \tag{4.80}$$

Define f_k as in (4.53) with u_k replacing U_m. Then (4.54)–(4.57) hold for the current setting with g now g_k, h now h_k, $\sigma = \varepsilon$, and (4.58) becomes

$$\|g_k - w\|_{L^2(S_0)} = \|u_k - w\|_{L^2(S_0)}. \tag{4.81}$$

Thus if $\|P - w\|_{L^2(S_0)} = \rho_2$, it can be assumed that

$$\|u_k - w\|_{L^2(S_0)} = \rho_2$$

while if $\|P - v\|_{L^2(S_0)} = \rho_1$,

$$\|u_k - w\|_{L^2(S_0)} \geq \overline{\rho} - \rho_1.$$

Thus in any event,

$$\|g_k - w\|_{L^2(S_0)} \geq \min(\rho_2, \overline{\rho} - \rho_1) \equiv \gamma. \tag{4.82}$$

Thus

$$J(g_k) \geq \beta(\gamma). \tag{4.83}$$

Now $h_k \in \Gamma(v, w)$ and as in (4.60),

$$J(u_k) \geq \beta(\gamma) + c(v, w) - \kappa(\varepsilon). \tag{4.84}$$

Choosing ε so small that

$$2\kappa(\varepsilon) \leq \beta(\gamma), \tag{4.85}$$

and letting $k \to \infty$ in (4.84) yields

$$d(v, w) \geq c(v, w) + \frac{1}{2}\beta(\gamma) \tag{4.86}$$

so the Proposition is proved for this case. If $\varphi = v$, a similar argument yields (4.86) with γ replaced by $\min(\rho_1, \overline{\rho} - \rho_2)$.

Lastly suppose that $\psi = v$ and $\varphi = w$. Then $P \in \Lambda(v, w)$ so $J(P) \geq d(v, w)$. An argument as e.g. in (D) of the proof of Theorem 4.10 shows $J(P) = d(v, w)$. If $d(v, w) = c(v, w)$, then since $P \in \Gamma(v, w)$, by Theorem 3.17, P is a solution of (PDE). But P satisfies (4.77) and by the choice of ρ_1 and ρ_2, this is not possible for a solution of (PDE) in $\Gamma(v, w)$. Hence $d(v, w) > c(v, w)$ for this case also and Proposition 4.63 is proved.

Remark 4.87. The argument used to prove step (E) in Theorem 4.10 also provides an upper bound for $\|U_m - w\|_{L^2(S_i)}$ for S_i in the unconstrained portion of $\mathbb{R} \times [m_2, m_3]$. Namely set

$$\rho^* = \max_{m_2 + \ell \leq i \leq m_3 - \ell + 1} \|U_m - w\|_{L^2(S_i)}, \tag{4.88}$$

Then with $\kappa(\sigma)$ as in (4.54),

$$\beta(\rho^*) \leq \kappa(\sigma). \tag{4.89}$$

Indeed if the maximum in (4.88) is achieved when $i = p$, then $t < p < q$ with t defined before (4.50) and q before (4.52). Following the argument from (4.52)–(4.59) with ρ_2 replaced by ρ^*, (4.60) becomes

$$c_m \geq \beta(\rho^*) + c_m - \kappa(\sigma) \tag{4.90}$$

and (4.89) is satisfied.

5 Monotone multibump solutions

In [1], multibump solutions of (PDE) which were monotone in y, i.e. $u < \sigma_{-1}u$ or $u > \sigma_{-1}u$ were constructed under the assumption that

$$(\mathcal{M}) \qquad \mathcal{M}(0,1) = \{\tau_n v \mid n \in \mathbb{Z}\}.$$

In this section, such 2-bump solutions will be constructed under a milder condition. The approach also generalizes to obtain k-bump solutions of (PDE). The condition ($\mathcal{M}$) will be replaced by the condition ($*$) that $\mathcal{M}(0,1)$ contains gaps. Thus whenever $v < w$ corresponds to such a gap, by Theorem 2.34 or Theorem 3.17, there is a solution of (PDE), monotone in y and heteroclinic in y from v to w. By Proposition 4.1, $\mathcal{M}(v,w)$, the set of such minimizers of J on $\Gamma(v,w)$, is an ordered set. It will be further assumed that ($**$) holds, i.e. $\mathcal{M}(v,w)$ also has gaps. Since $\tau_{\pm 1} : \mathcal{M}(0,1) \to \mathcal{M}(0,1)$ and $\sigma_{\pm 1} : \mathcal{M}(v,w) \to \mathcal{M}(v,w)$, under ($*$) and ($**$), $\mathcal{M}(0,1)$ and $\mathcal{M}(v,w)$ possess infinitely many gaps.

To formulate the main result in this section, suppose v_1, w_1 and v_2, w_2 are a pair of adjacent numbers of $\mathcal{M}(0,1)$ with

$$(5.1) \qquad v_1 < w_1 \le v_2 < w_2.$$

The simplest such situation occurs when ($\mathcal{M}$) holds and $w_1 = \tau_{-1}v_1 = v_2$, $w_2 = \tau_{-2}v_1$. Choose

$$s_i, t_i \in \left(\int_0^1 \int_0^1 v_i dxdy, \int_0^1 \int_0^1 w_i dxdy \right),$$

$i = 1, 2$ so that

$$s_i \ne \int_0^1 \int_0^1 h_i dxdy \ne t_i$$

for all $h_i \in \mathcal{M}(v_i, w_i)$ and

$$(5.2) \qquad C_i = \left\{ h \in \mathcal{M}(v_i, w_i) \mid s_i < \int_0^1 \int_0^1 hdxdy < t_i \right\} \ne \emptyset.$$

Then two bump solutions will be constructed which are close to some $h_1 \in \mathcal{C}_1$ on a prescribed y interval and to $h_2 \in \mathcal{C}_2$ on another prescribed y interval. The class of admissible functions that are needed to do this will be defined next.

Given $n \in \mathbb{Z}^2$, $n = (n_1, n_2)$, $n_1 + 4 \le n_2$, let

$$(5.3) \qquad \widehat{Y}_n = \Big\{ u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2) \mid v_1 \le u \le \sigma_{-1}u \le w_2,$$
$$s_i \le \int_{n_i}^{n_i+1} \left(\int_0^1 u_i dx \right) dy \le t_i,\ i = 1, 2, \text{ where}$$
$$u_1 = \min(u, w_1),\ u_2 = \max(u, v_2) \Big\}$$

and

$$\widehat{b}_m = \inf_{u \in \widehat{Y}_m} J(u). \tag{5.4}$$

The main result of this section is

Theorem 5.5. *There exists $\widehat{U}_m \in \widehat{Y}_m$ such that $J(\widehat{U}_m) = \widehat{b}_m$. If $n_2 >> n_1$, then for any such $\widehat{U}_m$,*

$$\|\widehat{U}_m - v_1\|_{W^{1,2}(S_{-n})} \to 0, \quad \|\widehat{U}_m - w_2\|_{W^{1,2}(S_n)} \to 0 \quad \text{as } n \to \infty, \tag{5.6}$$

and

$$\widehat{U}_m \quad \text{satisfies (PDE).} \tag{5.7}$$

Moreover given $\rho, R > 0$, for $n_2 - n_1$ possibly still larger, there exist $U_i \in C_i$, $i = 1, 2$ such that

$$\|\widehat{U}_m - \sigma_{n_1} U_1\|_{W^{1,2}(S_n)} \leq \rho \qquad \text{for } n \leq n_1 + R, \tag{5.8}$$

$$\|\widehat{U}_m - \sigma_{n_2} U_2\|_{W^{1,2}(S_n)} \leq \rho \qquad \text{for } n \geq n_2 - R. \tag{5.9}$$

Remark 5.10. In Sect. 4, $U_m \in \Gamma_m$ implies that U_m is close to v, w over certain y intervals due to the integral constraints, so U_m is clearly a multi-bump solution. In this section (5.8), (5.9) play the same role, as is illustrated in the following claims which are verified at the end of the proof of Theorem 5.5. Note that C_1, C_2 have smallest and largest elements. Thus given $\rho > 0, T > 0$, if $n_2 - n_1$ is large enough, there is an $N = N(\rho)$ (independent of $\widehat{U}_m$ and T) such that

$$\begin{cases} \|\widehat{U}_m - v_1\|_{W^{1,2}(S_n)} \leq \rho & \text{for } n \leq n_1 - N(\rho), \\ \|\widehat{U}_m - w_1\|_{W^{1,2}(S_n)} \leq \rho & \text{for } n_1 + N(\rho) \leq n \leq n_1 + N(\rho) + T, \\ \|\widehat{U}_m - v_2\|_{W^{12}(S_n)} \leq \rho & \text{for } n_2 - N(\rho) - T \leq n \leq n_2 - N(\rho) \\ \|\widehat{U}_m - w_2\|_{W^{1,2}(S_n)} \leq \rho & \text{for } n_2 + N(\rho) \leq n. \end{cases} \tag{5.11}$$

In addition

$$\begin{cases} \|(\widehat{U}_m - w_1)_-\|_{W^{1,2}(S_n)} \leq \rho & \text{for } n_1 + N(\rho) \leq n, \\ \|(\widehat{U}_m - v_2)_+\|_{W^{1,2}(S_n)} \leq \rho & \text{for } n \leq n_2 - N(\rho). \end{cases} \tag{5.12}$$

Proof of Theorem 5.5. The fact that $\widehat{b}_m \in \mathbb{R}$, the existence of a minimizer $\widehat{U}_m$, and (5.6) follow by a mild variation on the proof of the analogous results in Theorem 5.1 of [1]. We refer to [1] for the details. Next it will be shown that $\widehat{U}_m$ satisfies (PDE). Towards this end, it will be proved that the integral constraints in (5.3) with $u = \widehat{U}_m$ hold with strict inequality. This requires some preparation.

If $w_1 < v_2$, set

$$\widetilde{\Gamma}(w_1, v_2) = \{u \in \Gamma(w_1, v_2) \mid w_1 \not\equiv u \leq \sigma_{-1} u \not\equiv v_2\} \tag{5.13}$$

and

$$c = \inf_{u \in \widetilde{\Gamma}(w_1, v_2)} J(u). \tag{5.14}$$

As with $\widehat{b}_m$, $c \in \mathbb{R}$. If $w_1 = v_2$, $\widetilde{\Gamma}(w_1, v_2)$ and c are not needed. Given $\delta > 0$, take $\sigma_{-n_1}\alpha \in \mathcal{C}_1$, $\beta \in \widetilde{\Gamma}(w_1, v_2)$, $\sigma_{-n_2}\gamma \in \mathcal{C}_2$, with $J(\beta) \leq c + \delta$. If $n_2 >> n_1$, then there exist $a, b \in \mathbb{Z}$, $n_2 >> b >> a >> n_1$ such that

$$A = \begin{cases} \alpha & y \leq a \\ w_1 & a+1 \leq y \leq a+2 \\ \beta & a+3 \leq y \leq b-3 \\ v_2 & b-2 \leq y \leq b-1 \\ \gamma & b \leq y \end{cases}$$

and extended to the remaining y intervals as in the previous sections, satisfies

$$J(A) \leq J(\alpha) + J(\beta) + J(\gamma) + \delta \leq c(v_1, w_1) + c + c(v_2, w_2) + 2\delta, \tag{5.15}$$

due to Proposition 2.6. Note that $A \in \widehat{Y}_m$ by Theorem 3.17 and (5.2), so

$$\widehat{b}_m \leq c(v_1, w_1) + c + c(v_2, w_2) + 2\delta. \tag{5.16}$$

Given $u \in \widehat{Y}_m$ such that $J(u) = \widehat{b}_m$, note that

$$\begin{cases} f_1 = \min(u, w_1) \in \Gamma(v_1, w_1), \\ f_2 = \min(v_2, \max(u, w_1)) \in \widetilde{\Gamma}(w_1, v_2), \\ f_3 = \max(u, v_2) \in \Gamma(v_2, w_2) \end{cases} \tag{5.17}$$

by (5.6). Now assume one of the integral constraints in (5.3) holds with equality, say

$$s_1 = \int_{n_1}^{n_1+1} \left(\int_0^1 f_1 dx \right) dy. \tag{5.18}$$

We will show this leads to a contradiction. Let

$$\Lambda_1 = \left\{ u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}) \mid v_1 \leq u \leq \sigma_{-1}u \leq w_1, \right.$$
$$\left. s_1 = \int_0^1 \int_0^1 u dx dy \right\}$$

and

$$d_1 = \inf_{u \in \Lambda_1} J(u).$$

The analogue of Proposition 4.63 here is

Lemma 5.19. $d_1 > c(v_1, w_1)$.

Proof. Due to

$$\int_0^1 \int_0^1 v_1 dxdy < s_1 < \int_0^1 \int_0^1 w_1 dxdy,$$

$\Lambda_1 \subset \widehat{\Gamma}(v_1, w_1)$, so by Theorem 3.17,

$$d_1 \geq \widehat{c}(v_1, w_1) = c(v_1, w_1). \tag{5.20}$$

Assume $d_1 = \widehat{c}(v_1, w_1)$ so there exists $(u_n) \subset \Lambda_1$ with $J(u_n) \to \widehat{c}(v_1, w_1)$. The results of Sect. 3 of [1] or §4 here imply that on a subsequence $u_n \to u \in \widehat{\Gamma}(v_1, w_1)$ weakly in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$, and in $L^2_{\text{loc}}(\mathbb{R}^2)$ with $J(u) = \widehat{c}(v_1, w_1)$. However

$$s_1 = \int_0^1 \int_0^1 u_k dxdy. \tag{5.21}$$

Thus (5.21) holds with u_k replaced by u which contradicts the definition of s_1 since $u \in \mathcal{M}(v_1, w_1)$ by Theorems 2.34, 3.17. Thus Lemma 5.19 is proved.

Returning to the proof of Theorem 5.5, note that $\sigma_{-n_1} f_1 \in \Lambda_1$, so

$$J(f_1) \geq d_1. \tag{5.22}$$

However (5.17) and (5.22) imply

$$\begin{aligned} \widehat{b}_m = J(u) &= J(f_1) + J(\max(u, w_1)) = J(f_1) + J(f_2) + J(f_3) \\ &\geq d_1 + c + c(v_2, w_2). \end{aligned} \tag{5.23}$$

Choosing δ small enough, (5.23) contradicts Lemma 5.19 due to (5.16). Thus (5.18) is impossible.

The other 3 integral constraints are similarly shown to hold for u with strict inequality. With these inequalities in hand, one can argue as in [1, Proposition 3.8] to establish that u is a solution of (PDE), i.e. (5.7). Indeed we need only take r small enough so that necessary perturbations do not violate the strict inequality of the integral constraints, which is possible since $0 \leq v_1 \leq w_2 \leq 1$.

It remains to prove (5.8)–(5.9) for $u - \widehat{U}_m$. Proceeding as above, but without assuming (5.18), and replacing d_1 by $J(f_1)$ in (5.23), it follows from (5.16) and (5.17) that

$$J(f_1) \leq c(v_1, w_1) + 2\delta, \tag{5.24}$$

with $f_1 \in \Gamma(v_1, w_1)$. Thus given $\sigma > 0$, the following lemma implies, for large enough $n_2 - n_1$ (i.e. small enough δ),

$$\|f_1 - U\|_{W^{1,2}(S_n)} \leq \sigma \text{ for all } n \in \mathbb{Z}, \tag{5.25}$$

for some $U \in \mathcal{M}(v_1, w_1)$.

Lemma 5.26. *Let $\varepsilon > 0$. Then there is a $\delta > 0$ such that whenever $u \in \Gamma(v_1, w_1)$ with $J(u) \leq c(v_1, w_1) + \delta$, there exists $U \in \mathcal{M}(v_1, w_1)$ such that*

$$\|u - U\|_{W^{1,2}(X_n)} \leq \varepsilon \quad \textit{for all } n \in \mathbb{Z}.$$

Proof. Assume the result is false. Then for some $\varepsilon > 0$ and sequence $(u_k) \subset \Gamma(v_1, w_1)$ with $J(u_k) \to c(v_1, w_1)$ as $k \to \infty$, and any $U \in \mathcal{M}(v_1, w_1)$ there exists $j_k = j_k(U) \in \mathbb{Z}$ such that

$$\|u_k - U\|_{W^{1,2}(X_{j_k})} > \varepsilon. \tag{5.27}$$

By integer translation in y it can be assumed that

$$\int_i^{i+1} \Big(\int_0^1 (u_k - v_1)dx \Big) dy \le c_0 \equiv \frac{\int_0^1 \int_0^1 (w_1 - v_1)dxdy}{2} \quad \text{for all } i < 0, \tag{5.28}$$

and

$$\int_0^1 \int_0^1 (u_k - v_1)dxdy \ge c_0. \tag{5.29}$$

As in the proof of Theorem 3.17, on a subsequence $u_k \to u \in \mathcal{M}(v_1, w_1)$ weakly in $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$, in $L^2_{\text{loc}}(\mathbb{R}^2)$, and pointwise a.e., with $v_1 < u < \sigma_{-1}u < w_1$. In addition (5.28) and (5.29) hold with u_k replaced by u, and by Proposition 2.10,

$$\|u_k - u\|_{W^{1,2}(X_i)} \to 0 \quad \text{as } k \to \infty \text{ for all } i \in \mathbb{Z}. \tag{5.30}$$

Together $u \in \mathcal{M}(v_1, w_1)$ and (5.27) imply

$$\|u_k - u\|_{W^{1,2}(X_{j_k})} \ge \varepsilon \tag{5.31}$$

for $j_k = j_k(u)$, so extracting a subsequence it can be assumed that $j_k \to \infty$ or $j_k \to -\infty$ due to (5.30). Assume $j_k \to \infty$, the other case being similar. Given $\delta > 0$, choose $q \in \mathbb{Z}$ such that

$$\|u - w_1\|_{W^{1,2}(X_i)} \le \delta \quad \text{for } i \ge q. \tag{5.32}$$

This is possible due to $u \in \mathcal{M}(v_1, w_1)$. Thus for large enough k,

$$\|u_k - w_1\|_{W^{1,2}(X_q)} \le 2\delta. \tag{5.33}$$

Define $f_k \in \Gamma(v_1, w_1), g_k \in \Gamma(v_1, w_1), h_k \in \Gamma(w_1, w_1)$ as in (3.54)-(3.59) so

$$|J(u_k) - J(f_k)| \le \mu(\delta), \qquad J(f_k) = J(g_k) + J(h_k). \tag{5.34}$$

Thus

$$J(f_k) \le J(u_k) + \mu(\delta) \le c(v_1, w_1) + 2\mu(\delta) \le J(g_k) + 2\mu(\delta) \tag{5.35}$$

for large k, since $J(u_k) \to c(v_1, w_1)$, which combined with (5.34) implies

$$J(h_k) \le 2\mu(\delta) \le \frac{\beta(\varepsilon/2)}{2} \tag{5.36}$$

for small enough δ, and β as in Corollary 3.46. However $h_k \in \Gamma(w_1, w_1)$, so by Corollary 3.46 and (5.31), for $\delta < \varepsilon/2$,

$$J(h_k) \geq \beta\left(\frac{\varepsilon}{2}\right) \tag{5.37}$$

which contradicts (5.36) and completes the proof of the Lemma.

Returning to the proof of Theorem 5.5, note from the choice of s_i and t_i, the order property of $\mathcal{M}(v_1, w_1)$, and methods of Theorem 3.17, that there is a least element $h_1 \in \mathcal{M}(v_1, w_1)$ for which

$$t_1 < c_1 \equiv \int_0^1 \int_0^1 h_1 dxdy,$$

and a greatest element $h_0 \in \mathcal{M}(v_1, w_1)$ for which $s_1 > c_0 \equiv \int_0^1 \int_0^1 h_0 dxdy$. Therefore, since $u \in \widehat{Y}_m$ with $u_1 = f_1$, u_1 as in (5.3), if σ, U are as in (5.25) with σ chosen so $\sigma < \min(s_1 - c_0, c_1 - t_1)$, then for $U_1 = \sigma_{-n_1} U$ it follows from (5.25) that

$$\begin{aligned}\int_0^1 \int_0^1 U_1 dxdy &\leq \int_{n_1}^{n_1+1} \left(\int_0^1 f_1 dx\right) dy + \int_{n_1}^{n_1+1} \left(\int_0^1 |U - f_1| dx\right) dy \\ &\leq t_1 + \sigma < t_1 + (c_1 - t_1) = c_1\end{aligned}$$

and similarly

$$\int_0^1 \int_0^1 U_1 dxdy > c_0$$

so

$$U_1 \in \mathcal{C}_1. \tag{5.38}$$

Let U_{max} be the largest element of $\mathcal{C}_1$, which exists as above so

$$U_1 \leq U_{max} < \sigma_{-1} U_{max} \leq w_1 \tag{5.39}$$

due to the strict ordering of $\mathcal{M}(v_1, w_1)$.

Given $\alpha > 0$, choose $N > 0$ such that

$$\int_0^1 \left(\int_{\mathbb{R} \setminus [-N+1, N-1]} (w_1 - v_1)^2 dx\right) dy \leq \alpha^2. \tag{5.40}$$

Given $R > 0$, let $s = \min(w_1 - U_{max})/4$ on $[-N-1, N+1] \times [R+1, R+2]$, so

$$0 < 4s \leq w_1 - U_{max} \leq w_1 - U_1 \quad \text{on } [-N-1, N+1] \times (-\infty, R+2] \tag{5.41}$$

due to (5.39) and $U_1 \leq \sigma_{-1} U_1$. We claim for small enough σ that

$$u < w_1 \quad \text{on } [-N, N] \times (-\infty, n_1 + R + 1] \tag{5.42}$$

so (5.25) implies

$$\|u-U\|_{W^{1,2}([-N,N]\times[k,k+1])} \le \sigma \ \text{ for } k \le n_1 + R. \tag{5.43}$$

To confirm the claim, set $\varphi = (f_1 - U)^+$ so

$$\begin{cases} \varphi = 0 \quad \text{on } \{u \le U\}, \\ \varphi \ge 4s \text{ on } \{u \ge w_1\} \cap ([-N-1, N+1] \times (-\infty, n_1 + R + 2] \end{cases} \tag{5.44}$$

by (5.41). Assume the claim is false so for some $k \le n_1 + R$, there exists $p \in [-N,N] \times [k,k+1]$ with $\varphi(p) \ge 4s$. Moreover (5.25) implies

$$\text{meas}(\{\varphi \ge s\} \cap (T_k)) \le \frac{1}{s^2} \int_{T_k} \varphi^2 dxdy \le \frac{3\sigma^2}{s^2}. \tag{5.45}$$

Recall u, U satisfy (PDE) with $0 \le u,\ U \le 1$ so $|\nabla u|$, $|\nabla U| \le M$ for some $M \in \mathbb{R}$. Set $r = \min(1, s/(2M))$, so $B_r(p) \subset T_k$. It can be assumed that σ is so small that

$$\sigma < rs. \tag{5.46}$$

Then $\pi r^2 > 3\sigma^2/s^2$. If $B_r(p) \subset \{\varphi \ge s\} \cap T_k$, by (5.45),

$$\pi r^2 \le \text{meas}(\{\varphi \ge s\} \cap T_k) \le \frac{3\sigma^2}{s^2} \tag{5.47}$$

contrary to (5.46). Thus there is $q_1 \in B_r(p)$ such that $\varphi(q_1) < s$. Take q_2, q_3 on the line segment joining p, q_1 such that $\varphi(q_2) = s$, $\varphi(q_3) = 3s$, and $s \le \varphi \le 3s$ on the line segment ℓ joining q_2, q_3. Thus $\varphi = u - U$ on ℓ due to (5.44), and $2s/r \le 2s/|q_3 - q_2| \le |\nabla\varphi(q)| \le |\nabla u(q)| + |\nabla U(q)| \le 2M$ for some $q \in \ell$, contradicting the definition of r, so the claim is verified.

To establish the analogue of (5.43) on $(\mathbb{R} \setminus [-N,N]) \times [k,k+1]$, estimate as in (3.98)-(3.102) with $f = u - U$, $\eta = 1$ in $[N-1, \overline{R}] \times [k,k+1]$, and $\eta = 0$ outside of $[N-1, \overline{R}+1] \times [k-1, k+2]$, letting $\overline{R} \to \infty$. Combining this with the analogous estimate for negative x leads to

$$\|u-U\|_{W^{1,2}((\mathbb{R}\setminus(-N,N))\times[k,k+1]} \le c_1\alpha \tag{5.48}$$

with $c_1 = (6M_1 + 51)^{1/2}$, due to (5.40), since $|u - U| \le w_1 - v_1$.

Take $\sigma \le \alpha$ so (5.43), (5.48) imply

$$\|u-U\|_{W^{1,2}(S_k)} \le c_2\alpha \ \text{ for } k \le n_1 + R, \tag{5.49}$$

and $c_2 = (6M_1 + 52)^{1/2}$. Thus (5.49) yields (5.8), (5.9) follows similarly, and Theorem 5.5 is established.

To confirm the claims in Remark 5.10, given $\rho > 0$, choose $N(\rho)$ such that

$$\begin{cases} \|U_1 - v_1\|_{W^{1,2}(\mathbb{R}\times[k,k+1])} \le \frac{\rho}{2} & \text{for } k \le -N(\rho), \\ \|U_1 - w_1\|_{W^{1,2}(\mathbb{R}\times[k,k+1])} \le \frac{\rho}{2} & \text{for } k \ge N(\rho). \end{cases} \tag{5.50}$$

In fact it is not hard to see that $N(\rho)$ can be made uniform over all $U_1 \in \mathcal{C}_1$. Note that $(u - w_1)_- = f_1 - w_1$ so half of (5.12) follows from (5.50) and (5.25) with $\sigma \le \rho/2$, since $U_1 = \sigma_{-n_1}U$. Take $R = N(\rho) + T$ and $\alpha = \rho/(2c_2)$ in (5.49) to get (5.8). Then (5.8) with ρ replaced by $\rho/2$ combined with (5.50) gives the first two inequalities in (5.11). The remaining inequalities follow from an analogous argument involving v_2, w_2.

6 K-bump solutions

The goal of this section is to discuss the differences between two-bump solutions of (PDE) and general k-bump solutions, and to sketch the details needed to construct k-bump solutions. Two-bump solutions of the types obtained in Sects. 4 and 5 are somewhat special in comparison to general k-bump solutions, those of Sect. 4 due to the fact that the solutions are restricted to lie between adjacent elements v, w of $\mathcal{M}(0,1)$, and those of Sect. 5 due to the monotonicity condition, and the fact that the asymptotic limits v_1, w_2 as $y \to \pm\infty$ are upper and lower bounds for solutions, with $v_i, w_i, i = 1, 2$ being pairs of adjacent elements in $\mathcal{M}(0,1)$ satisfying $v_1 < w_1 \leq v_2 < w_2$. The monotonicity condition was introduced to significantly simplify technicalities in Sect. 4, but is not natural in the general k-bump setting. The other restrictions allow one to show that the two-bump solutions are in fact minimizers of the variational problem used to generate the solutions, something that seems not to be true in general for k-bump solutions.

The symbolic dynamics of k-bump solutions is controlled by the availability of gaps in sets $\mathcal{M}(v,w)$, v, w adjacent elements of $\mathcal{M}(0,1)$. If there exists one such gap, then there are an infinite number of such gaps and one can generate a rich class of monotone k-bump solutions using the methods of Sect. 5. If there exists an adjacent pair v, w for which both $\mathcal{M}(v,w)$ and $\mathcal{M}(w,v)$ have gaps, then one can use the methods of Sect. 4, to generate a class of k-bump solutions lying between v and w with very general symbolic dynamics.

The difference between the solutions constructed in Sects. 4 and 5, and general k-bump solutions is probably best seen by considering the asymptotic behavior of solutions as $y \to \pm\infty$. One cannot precisely control the asymptotic behaviour of limits of minimizing sequences in general, although one can do so in an approximate sense. The essential reason for the difference appears to be the fact that one can carefully control asymptotic behavior in regions between adjacent elements of $\mathcal{M}(0,1)$ but not elsewhere. These vague statements will be clarified in Theorem 6.14 and the remarks following it. Some preliminaries are needed.

Let $v_i, w_i, i = 1\ldots, k$ be pairs of adjacent elements of $\mathcal{M}(0,1)$ for $k \in \mathbb{N}, k > 1$ (with either $v_i < w_i$ or $w_i < v_i$ being possible), and $\varphi = \min\{v_i, w_i, i = 1, \ldots, k\}$, $\psi = \max\{v_i, w_i, i = 1, \ldots, k\}$ so $\varphi, \psi \in \mathcal{M}(0,1)$. It will be assumed that

$$\mathcal{M}(v_i, w_i) \text{ has a gap}, i = 1, \ldots, k, \tag{6.1}$$

and that one of the following cases occurs for each $i = 1, \ldots, k-1$,

$$v_{i+1} = w_i, \quad w_{i+1} = v_i, \tag{6.2}$$

$$v_i < w_i \leq v_{i+1} < w_{i+1}, \tag{6.3}$$

$$v_i > w_i \geq v_{i+1} > w_{i+1}. \tag{6.4}$$

Let

$$\rho(u,v) = \|u - v\|_{L^2(S_0)}.$$

Note that $\rho(u, v_i)$ (resp. $\rho(u, w_i)$) is strictly increasing (resp. decreasing) on $\mathcal{M}(v_i, w_i)$ if $v_i < w_i$, and strictly decreasing (resp. increasing) on $\mathcal{M}(v_i, w_i)$ if $v_i > w_i$. Choose constants $\rho_i, i = 1, \ldots, 2k$ such that

$$\text{(6.5)} \qquad \begin{cases} \rho_i \in (0, \overline{\rho}), & \overline{\rho} = \min_{1 \le j \le k} \|w_i - v_i\|_{L^2(S_0)} \\ \rho_{2i-1} \notin \rho(\mathcal{M}(v_i, w_i), v_i), & i = 1, \ldots, k, \\ \rho_{2i} \notin \rho(\mathcal{M}(v_i, w_i), w_i), & i = 1, \ldots, k, \end{cases}$$

Also as in Sect. 5, choose $s_i, t_i, i = 1, \ldots, k$ in the open interval with endpoints $\int_0^1 \int_0^1 v_i dxdy, \int_0^1 \int_0^1 w_i dxdy$ so that

$$s_i \neq \int_0^1 \int_0^1 h_i dxdy \neq t_i$$

for all $h_i \in \mathcal{M}(v_i, w_i)$, and

$$\text{(6.6)} \qquad C_i = \left\{ h \in \mathcal{M}(v_i, w_i) \mid s_i < \int_0^1 \int_0^1 hdxdy < t_i \right\} \neq \emptyset.$$

Let $\ell \in \mathbb{N}$ and $m \in (2\mathbb{Z})^{2k}$ with $m_{2i-1} + 2 < m_{2i}, i = 1, \ldots, k$, and

$$m_{2i} + 2\ell < m_{2i+1}, \quad i = 1, \ldots, k-1,$$

and define

$$n_i = \frac{m_{2i-1} + m_{2i}}{2}, \quad i = 1, \ldots, k.$$

Also let

$$\mathcal{Y}_m = \{u \in W^{1,2}(\mathbb{R}^2) \mid u \text{ satisfies (6.7)-(6.11)}\}$$

where

$$\text{(6.7)} \qquad \varphi \le u \le \psi,$$

and for $i = 1, \ldots, k$, defining

(6.8)
$$U(u, v, w) = \max(\min(u, w), v), \quad U(u, w, v) = U(u, v, w), \text{ for any } v \le w,$$

$$\text{(6.9)} \qquad \begin{cases} \rho(U(\sigma_{-j} u, v_i, w_i), v_i) \le \rho_{2i-1}, \; j = m_{2i-1} - \ell, m_{2i-1} - 1 \\ \rho(U(\sigma_{-j} u, v_i, w_i), w_i) \le \rho_{2i}, \quad j = m_{2i}, m_{2i} + \ell - 1, \end{cases}$$

$$\text{(6.10)} \qquad s_i \le \int_{n_i}^{n_i+1} \left(\int_0^1 U(u, v_i, w_i)\, dx \right) dy \le t_i,$$

and

$$\text{(6.11)} \qquad \|u - v_1\|_{L^2(S_{-i})} \to 0, \quad \|u - w_k\|_{L^2(S_i)} \to 0 \text{ as } i \to \infty.$$

Finally define

$$y_m = \inf_{u \in \mathcal{Y}_m} J(u). \tag{6.12}$$

The following notation is used to describe the asymptotic behaviour displayed by limits of minimizing sequences. Given $v \in \mathcal{M}(0,1)$ let $C(v)$ be the maximal collection of elements of $\mathcal{M}(0,1)$ containing v and having no gaps, so $C(v)$ is a maximal continuum in $\mathcal{M}(0,1)$. By assumption $\mathcal{M}(0,1)$ contains adjacent elements, that is it has gaps. More precisely due to the fact that $\tau_q \mathcal{M}(0,1) = \mathcal{M}(0,1)$ for $q \in \mathbb{Z}$ there is a gap between $\tau_{-1}v, v$ even though they may not be adjacent in $\mathcal{M}(0,1)$ and likewise between $v, \tau_1 v$. Thus $s(v), l(v)$, the smallest and largest elements of $C(v)$, exist due to the order property of $\mathcal{M}(0,1)$ and the methods of the proof of Theorem 3.17. Also

$$\tau_{-1}v < s(v) \le v \le l(v) < \tau_1 v. \tag{6.13}$$

The main result of this section can now be stated.

Theorem 6.14. *Suppose F satisfies (F_1)-(F_3) and (6.1)-(6.4) hold. Then $y_m \in \mathbb{R}$. If $(u_n) \subset \mathcal{Y}_m$ with $J(u_n) \to y_m$, then on a subsequence $u_n \to u$ weakly in $W^{1,2}(\mathbb{R}^2)$, and strongly in $W^{1,2}(S_i)$ for $i \in \mathbb{Z}$, with u satisfying (6.7)-(6.10), and for some $v, w \in \mathcal{M}(0,1)$*

$$\|(u - l(v))_+\|_{W^{1,2}(S_i)} \to 0, \quad \|(u - s(v))_-\|_{W^{1,2}(S_i)} \to 0 \tag{6.15}$$

as $i \to -\infty$, and

$$\|(u - l(w))_+\|_{W^{1,2}(S_i)} \to 0, \quad \|(u - s(w))_-\|_{W^{1,2}(S_i)} \to 0 \tag{6.16}$$

as $i \to \infty$.

Moreover given $\epsilon > 0, N > 0$, if $\ell, m_{i+1} - m_i$ are sufficiently large, then u is a solution of (PDE), and if $v_1 < w_1$ (resp. $v_1 > w_1$), then $v \le v_1$ (resp. $v \ge v_1$) and

(6.17)
$$\|(s(v) - s(v_1))_-\|_{W^{1,2}(S_0)} \le \epsilon, \quad \text{(resp. } \|(l(v) - l(v_1))_+\|_{W^{1,2}(S_0)} \le \epsilon\text{)},$$

and $w \ge w_k$ (resp. $w \le w_k$), and

(6.18)
$$\|(l(w) - l(w_k))_+\|_{W^{1,2}(S_0)} \le \epsilon, \quad \text{(resp. } \|(s(w) - s(w_k))_-\|_{W^{1,2}(S_0)} \le \epsilon\text{)}.$$

In addition there exist $U_i \in C_i, i = 1, \dots, k$ such that

$$\|u - \sigma_{n_i} U_i\|_{W^{1,2}(S_j)} \le \epsilon, \quad n_i - N \le j \le n_i + N. \tag{6.19}$$

Remark 6.20. Suppose that $k = 3$, $v_1 = w_2 < w_1 = v_2$ and $v_1 = \ell(v_1) > s(v_1) \ge v_3 > w_3$. Then we cannot prove that $\|u - v_1\|_{L^2(S_i)} \to 0$ as $i \to -\infty$ but only the milder statement (6.15). If there is a gap on both sides of v_1, or as in Sects. 4,5, $v_1 \in \{\varphi, \psi\}$, then $s(v) = l(v) = v_1$. If the analogous condition holds for w_k as well, then $u \in \mathcal{Y}_m$ and $J(u) = y_m$. More generally if $C(v_1) = \{v_1\}$ and $\ell, m_{i+1} - m_i \ge N(\epsilon)$, then $\|u - v_1\|_{W^{1,2}(S_i)} \le \epsilon$ for $i << 0$.

Remark 6.21. Contrary to what occurs in Sect. 4, the integral constraints (6.9) do not indicate that u is a true multi-bump solution because the constraints only imply that $U(u, v_i, w_i)$ is close to v_i, w_i on appropriate y intervals. For this reason the shadowing result (6.19) is emphasized since, as in Sect. 5, it implies the following closeness condition. Given $\epsilon > 0, T > 0$, if $\ell, m_{i+1} - m_i$ are large enough,

$$(6.22)\qquad \begin{cases} \|u - v_i\|_{W^{1,2}(S_i)} \le \epsilon & \text{for } n_i - N(\epsilon) - T \le i \le n_i - N(\epsilon), \\ \|u - w_i\|_{W^{1,2}(S_i)} \le \epsilon & \text{for } n_1 + N(\epsilon) \le i \le n_1 + N(\epsilon) + T \end{cases}$$

with $N(\epsilon)$ independent of T and u.

Remark 6.23. The basic methods of Sect. 4 allow one to show that U_m shadows w for $m_2 \le y \le m_3$. Unless $w_1 = v_2$, or more generally w_1 is 'close' to v_2, this is not possible in §5. In the present context the first type of behaviour is possible if (6.2) holds, but only if $C(w_i)$ is 'small'. This can be established using the 'one sided' interpolation methods introduced in the proof of Theorem 6.14. As in Sect. 5 such control is not possible where (6.3), (6.4) hold unless $w_i = v_{i+1}$ or w_i is close to v_{i+1}.

The following generalization of Corollary 3.46 is introduced as a first step towards establishing Theorem 6.14. For $q \in \mathbb{N}$, let

$$(6.24)\qquad \Gamma_q(v, v) = \{u \in W^{1,2}_{loc}(\mathbb{R}^2) \mid \tau_q v \le u \le \tau_{-q} v \text{ and } \|u - v\|_{L^2(S_i)} \to 0 \text{ as } |i| \to \infty\}.$$

Proposition 6.25. *Assume $\mathcal{M}(0,1)$ has a gap and $u \in \Gamma_q(v, v)$ for some $q \in \mathbb{N}, v \in \mathcal{M}(0,1)$. If $\gamma > 0$ and*

$$(6.26)\qquad \|(u - l(v))_+\|_{W^{1,2}(X_0)} \ge \gamma \quad (\textit{ resp. } \|u - s(v))_-\|_{W^{1,2}(X_0)} \ge \gamma)$$

then there exists $\beta = \beta(\gamma)$ (independent of u) such that $J(u) \ge \beta$.

Proof. Assume for $q \in \mathbb{N}, v \in \mathcal{M}(0,1)$ that the result is false, so there exist $\gamma > 0$ and $u_n \in \Gamma_q(v, v)$ with $J(u_n) \to 0$, and

$$(6.27)\qquad \|(u_n - l(v))_+\|_{W^{1,2}(\mathbb{R}\times[-2,3])} \ge \gamma.$$

As in the proof of Corollary 3.46, $u_n \to u$, a solution of (PDE) such that

$$\|u_n - u\|_{W^{1,2}(\mathbb{R}\times[k,k+1])} \to 0 \text{ for all } k \in \mathbb{Z},$$

and thus

$$(6.28)\qquad \|(u - l(v))_+\|_{W^{1,2}(\mathbb{R}\times[-2,3])} \ge \gamma.$$

Say $l(v) \le v_1 < v_2 \le \tau_{-1} v$, with v_1, v_2 adjacent in $\mathcal{M}(0,1)$. However for $\psi_n = \max(u_n, v_2)$, $\varphi_n = U(u_n, v_1, v_2)$ (recall 6.8), $\chi_n = \min(u_n, v_1)$,

$$(6.29)\qquad J(u_n) = J(\psi_n) + J(\varphi_n) + J(\chi_n) \ge J(\varphi_n),$$

since $\psi_n \in \Gamma_q(v_2, v_2)$, $\chi_n \in \Gamma_q(v, v)$ imply $J(\psi_n) \geq 0$, $J(\chi_n) \geq 0$ by Remark 3.8. Also $\varphi_n \in \Gamma(v_1, v_1)$ implies $J(\varphi_n) \geq 0$, so (6.29) and $J(u_n) \to 0$ imply $J(\varphi_n) \to 0$. However, $v_1 \leq \varphi_n \leq v_2$ so Corollary 3.46 implies $\|\varphi_n - v_1\|_{W^{1,2}(\mathbb{R}\times[k,k+1])} \to 0$ uniformly in k. Note $\varphi_n \to U(u, v_1, v_2)$ so $U(u, v_1, v_2) = v_1$, i.e. $u \leq v_1$. Thus by the definition of $l(v)$, $u \leq l(v)$, contradicting (6.28).

The following two results generalize Corollary 3.81, and will be used to establish the asymptotic behaviour in Theorem 6.14.

Proposition 6.30. *Assume $\varphi, \psi \in \mathcal{M}(0,1)$ and v, w are adjacent elements of $\mathcal{M}(0,1)$ with $\varphi \leq v < w \leq \psi$. If $u \in W^{1,2}_{loc}(\mathbb{R}^2)$ with $\varphi \leq u \leq \psi$, $J(u) < \infty$, and there exists y_1 such that u is a solution of (PDE) for $y \geq y_1$ (resp. $y \leq y_1$), then*

(6.31)
$$\|(u - w)_-\|_{W^{1,2}(S_i)} \to 0$$

or

(6.32)
$$\|(u - v)_+\|_{W^{1,2}(S_i)} \to 0$$

as $i \to \infty$ (resp. $i \to -\infty$), but not both.

Corollary 6.33. *Assume $\varphi, \psi \in \mathcal{M}(0,1)$, $\varphi < \psi$, and $u \in W^{1,2}_{loc}(\mathbb{R}^2)$ with $\varphi \leq u \leq \psi$, $J(u) < \infty$ and there exists y_1 such that u is a solution of (PDE) for $y \geq y_1$ (resp. $y \leq y_1$). Then for some $\phi \in \mathcal{M}(0,1)$*

(6.34)
$$\|(u - l(\phi))_+\|_{W^{1,2}(S_i)} \to 0$$

and

(6.35)
$$\|(u - s(\phi))_-\|_{W^{1,2}(S_i)} \to 0$$

as $i \to \infty$ (resp. $i \to -\infty$).

Proof of Proposition 6.30. Recall $U = U(u, v, w) = \max(\min(u, w), v) \in W^{1,2}_{loc}(\mathbb{R}^2)$and $v \leq U \leq w$. Also

(6.36)
$$M \geq J(u) = J(\max(u, w)) + J(U) + J(\min(u, v))$$

so

(6.37)
$$J(U) \leq M + 2K$$

by Proposition 2.6. By Proposition 3.63, there exists a sequence $q_i \to \infty$ as $i \to \infty$ and $\phi \in \{v, w\}$ such that

(6.38)
$$\sigma_i = \|U - \phi\|_{L^2(X_{q_i})} \to 0 \text{ as } i \to \infty.$$

Consider the case $\phi = v$, the other case being similar. If (6.32) does not hold, there exists $\gamma > 0$ such that

(6.39)
$$\|(u - v)_+\|_{W^{1,2}(S_i)} \geq \gamma$$

for a sequence of i going to infinity. Given $\alpha > 0$, choose $N > 0$ such that

$$\int_0^1 \left(\int_{\mathbb{R}\setminus[-N,N]} (\psi - v)^2 dx \right) dy \le \alpha^2. \tag{6.40}$$

We claim $u < w$ on $[-N, N] \times [q_i - 1, q_i + 2]$ for large i. To see this let

$$s = \min_{[-N-1,N+1]\times[0,1]} \frac{w - v}{4} \tag{6.41}$$

and $f = U - v$. Thus $f = 0$ on $\{u \le v\}$ and $f = w - v \ge 4s$ on $\{u \ge w\} \cap \{|x| \le N + 1\}$. Arguing as after (5.44) with σ, T_k, ϕ, U, and (5.25) replaced by σ_i, X_{q_i}, f, v, and (6.38) verifies the claim. Thus (6.38) and (6.40) imply

$$\int_{T_{q_i}} (u - v)_+^2 dxdy \le \sigma_i^2 + 3\alpha^2 \le 4\alpha^2 \tag{6.42}$$

for large i.

Estimating as in (3.98) to (3.102) with $f = u-v$, but replacing $\eta^2 f$ by $\eta^2 f_+$ and using $\nabla f_+ = \nabla f \chi_{\{f>0\}}$, $f f_+ = f_+^2$ (where $\chi_E(x) = 1$ for $x \in E$, $\chi_E(x) = 0$ for $x \notin E$), one gets

$$\|(u - v)_+\|_{W^{1,2}(S_{q_i})} \le c\|(u - v)_+\|_{L^2(T_{q_i})} \tag{6.43}$$

with $c = (2M_1 + 17)^{1/2}$. Combined with (6.42) this gives

$$\|(u - v)_+\|_{W^{1,2}(S_{q_i})} \le 2c\alpha \tag{6.44}$$

for large i.

Noting that α can be made as small as desired and that $l(v) = v$ since v, w are adjacent in $\mathcal{M}(0, 1)$ with $v < w$, argue as in the proof of Corollary 3.81 but use Proposition 6.25 instead of Corollary 3.46. Using (6.39), this leads to a contradiction as in the proof of Corollary 3.81. Thus (6.32) is verified. The case $\phi = w$ leads to (6.31).

Assume both (6.31), (6.32) hold. Note $0 \le (u - w)_+ \le (u - v)_+$ so $\|(u - w)\|_{L^2(S_i)} \to 0$. Similarly $\|(u - v)\|_{L^2(S_i)} \to 0$, contradicting $v < w$.

Proof of Corollary 6.33. If $\psi \in C(\varphi)$, then take $\phi = \varphi$, so $s(\phi) \le \varphi \le u \le \psi \le l(\phi)$, and $(u - s(\phi))_- = 0, (u - l(\phi))_+ = 0$, and the result holds. If $\psi \notin C(\varphi)$, there exists a pair v, w of adjacent elements in $\mathcal{M}(0, 1)$ with $\varphi \le v < w \le \psi$. For any such pair, exactly one of (6.31), (6.32) holds. Let $\mathcal{G}$ denote the collection of such pairs, $\mathcal{V}$ the set of v's, and $\mathcal{W}$ the set of w's. Set

$$W = \sup_{w\in\mathcal{W}} w; \quad V = \inf_{v\in\mathcal{V}} v.$$

Then $W, V \in \mathcal{M}(0, 1)$, $W = s(\psi) \le \psi$ and $V = \ell(\varphi) \ge \varphi$. If

$$\|(u - W)_-\|_{W^{1,2}(S_n)} = \|(u - s(\psi))_-\|_{W^{1,2}(S_n)} \to 0, \quad n \to \infty, \tag{6.45}$$

then Corollary 6.33 follows since

$$\|(u - \ell(\psi))_+\|_{W^{1,2}(S_n)} = 0, \quad n \in \mathbb{Z}.$$

Likewise if

$$\|(u-V)_+\|_{W^{1,2}(S_n)} = \|(u-\ell(\varphi))_+\|_{W^{1,2}(S_n)} \to 0, \quad n\to\infty, \tag{6.46}$$

the Corollary follows since

$$\|(u-s(\varphi))_-\|_{W^{1,2}(S_n)} = 0, \quad n\in\mathbb{Z}.$$

Thus suppose (6.45)–(6.46) fail to hold. By Proposition 6.30, this is impossible if $\mathcal{G}=\{(V,W)\}$. Consequently $\widetilde{\mathcal{G}}=\mathcal{G}\setminus\{(V,W)\}\neq\phi$.

If $\widetilde{\mathcal{G}}$ is a finite set, $V\in\mathcal{V}, W\in\mathcal{W}$. Also $(V,w^*),(v_*,W)\in\mathcal{G}$ for some $w^*,v_*\in\mathcal{M}(0,1)$. Thus by Proposition 6.30 again

$$\|(u-v_*)_+\|_{W^{1,2}(S_n)},\ \ \|(u-w^*)_-\|_{W^{1,2}(S_n)}\to 0, \quad n\to\infty.$$

Set

$$\underline{v}=\inf\{v\in\mathcal{V}\mid \text{(6.32) holds for } v\} \tag{6.47}$$

and

$$\overline{w}=\sup\{w\in\mathcal{W}\mid \text{(6.31) holds for } w\}. \tag{6.48}$$

Therefore since (6.45) and (6.46) fail to hold, $W>\overline{w}\geq w^*$ and $v_*\geq\underline{v}>V$. Since $\mathcal{G}$ is finite, there is a $\underline{w}\in\mathcal{W}$ such that $(\underline{v},\underline{w})\in\mathcal{G}$. By Proposition 6.30, $\underline{w}$ does not satisfy (6.31). Therefore $\underline{w}>\underline{v}\geq\overline{w}$. By the definitions (6.47)–(6.48), there cannot be a pair $(v,w)\in\mathcal{G}$ with $\overline{w}\leq v<w\leq\underline{v}$. Therefore $\overline{w}=s(\underline{v})$ and Corollary 6.33 is satisfied with $\overline{w}=s(\underline{v})$ and $\underline{v}=\ell(\underline{v})$.

Finally suppose $\widetilde{\mathcal{G}}$ is an infinite set. Again define $\underline{v}$ and $\overline{w}$ by (6.47) and (6.48). Then there is a sequence $(v_n)\subset\mathcal{V}$ with $v_i\downarrow\underline{v}\in\mathcal{M}(0,1)$ uniformly on compact sets.

Note, for $R>0$

$$\begin{aligned}\|(u-\underline{v})_+\|_{L^2(T_n)} &\leq \|(u-v_i)_+\|_{L^2(T_n)}\\ &+(3\|(v_i-\underline{v})_+\|^2_{L^2([-R,R]\times[0,1])}+3\|\psi-\varphi\|^2_{L^2((\mathbb{R}\setminus[-R,R])\times[0,1])})^{1/2}.\end{aligned} \tag{6.49}$$

On the right-hand side of (6.49), the last term can be made arbitrarily small by taking R large as in (2.7). Fixing such an R, the second term is arbitrarily small for large i due to $v_i\downarrow\underline{v}\in\mathcal{M}(0,1)$ uniformly on compact sets. Again fixing such an i, the first term goes to zero as $n\to\infty$ since (6.32) holds for $v=v_i$. Thus the left hand side of (6.49) goes to zero as $n\to\infty$, so by the analogue of (6.43),

$$\|(u-\underline{v})_+\|_{W^{1,2}(S_n)}\to 0 \text{ as } n\to\infty. \tag{6.50}$$

As in the case where $\mathcal{G}$ is finite, there cannot be a pair $(v,w)\in\mathcal{G}$ with $\overline{w}\leq v<w\leq\underline{v}$. Thus $\overline{w}=s(\underline{v})$ and as in (6.50) one has

$$\|(u-\overline{w})_-\|_{W^{1,2}(S_n)}\to 0 \text{ as } n\to\infty. \tag{6.51}$$

Hence Corollary 6.33 is verified with $\phi=\underline{v}=\ell(\underline{v})$.

Proof of Theorem 6.14. As in previous results it is easy to construct an admissible function to establish that $y_m < \infty$. In addition $-\infty < y_m$ due to Proposition 2.6. The techniques from previous sections imply that a minimizing sequence u_n converges on a subsequence to u weakly in $W^{1,2}_{loc}(\mathbb{R}^2)$, strongly in $W^{1,2}(S_i)$ for all $i \in \mathbb{Z}$, and pointwise almost everywhere, and that u is a solution of (PDE) for $y < m_1 - \ell$ and $y > m_k + \ell$. Thus (6.15), (6.16) follow from Corollary 6.33.

Statement (6.17) will now be verified assuming $v_1 < w_1$. The other case, and (6.18) are established in a similar manner. Note that for $\rho > 0$, if ℓ is large enough, then for some $i_0, m_2 + 2 \le i_0 \le m_2 + l - 3$,

$$\|U(u, v_1, w_1) - w_1\|_{L^2(X_{i_0})} \le \rho, \tag{6.52}$$

due to (6.9) and Proposition 3.63, since as in the proof of Theorem 2.4, $J(u) \le M + 2K$ so

$$\begin{aligned} M + 2K \ge J(u) &= J(\min(u, v_1)) + J(U(u, v_1, w_1)) + J(\max(u, w_1)) \\ &\ge J(U(u, v_1, w_1)) - 2K, \end{aligned} \tag{6.53}$$

by Proposition 2.6, i.e. $J(U(u, v_1, w_1)) \le M + 4K$.

But as before, u is a solution of (PDE) for $i_0 - 2 < y < i_0 + 3$ if ρ is small enough, since then the relevant integral constraints are strict due to (6.52). Thus an estimate as in (6.40)-(6.44) implies

$$\|(u - w_1)_-\|_{W^{1,2}(T_{i_0})} \le c_1\rho,$$

which gives

$$\|(u_n - w_1)_-\|_{W^{1,2}(T_{i_0})} \le 2c_1\rho \tag{6.54}$$

for large n due to the convergence properties of u_n.

The following ‘one sided interpolation’ illustrates the difference between interpolation arguments in previous sections and those needed here. The need for such a one sided argument is due to the fact that u_n is not bounded by v_1, w_1 as is the case in Sect. 4 for example. Note that $u_n = (u_n - w_1)_- + \max(u_n, w_1)$. The one sided interpolation is carried out by replacing $(u_n - w_1)_-$ in this decomposition by

$$f_n = \begin{cases} (u_n - w_1)_- & \text{if } \ i_0 + 2 \le y \\ \qquad 0 & \text{if } \ i_0 \le y \le i_0 + 1 \\ (u_n - w_1)_- & \text{if } \ y \le i_0 - 1 \end{cases}, \tag{6.55}$$

extended as usual.

Define $\widetilde{u}_n = f_n + \max(u_n, w_1)$ so $\widetilde{u}_n \in \mathcal{Y}_m$, and $u_n - \widetilde{u}_n = (u_n - w_1)_- - f_n$. From (6.54), estimates from previous sections imply

$$|J(u_n) - J(\widetilde{u}_n)| \le c_2\rho. \tag{6.56}$$

Let

$$\psi_n = \begin{cases} \qquad v_1 & \text{if } \ i_0 \le y \\ \min(\widetilde{u}_n, v_1) & \text{if } \ y \le i_0 \end{cases}, \tag{6.57}$$

noting for $i_0 \le y \le i_0 + 1$ that $\widetilde{u}_n \ge w_1 > v_1$ so $\min(\widetilde{u}_n, v_1) = v_1$. Also let

$$\varphi_n = \begin{cases} \widetilde{u}_n & \text{if } i_0 \le y \\ \max(\widetilde{u}_n, v_1) & \text{if } y \le i_0 \end{cases} \tag{6.58}$$

so

$$\begin{aligned} J(\psi_n) + J(\varphi_n) &= J(\widetilde{u}_n) \le J(u_n) + c_2\rho \\ &\le y_m + 2c_2\rho \le J(\varphi_n) + 2c_2\rho, \end{aligned} \tag{6.59}$$

for large n, since $\varphi_n \in \mathcal{Y}_m$. Thus

$$J(\psi_n) \le 2c_2\rho. \tag{6.60}$$

Consequently, given $\epsilon > 0$, choose $\epsilon_0 > 0$, such that if $h \in \mathcal{M}(0,1)$ with $l(h) < s(v_1)$, then

$$\|s(v_1) - l(h)\|_{W^{1,2}(S_0)} \le \epsilon_0 \Rightarrow \|s(v_1) - s(h)\|_{W^{1,2}(S_0)} \le \epsilon. \tag{6.61}$$

This is possible since $s(v_1)$ is either the upper element in a pair forming a gap in $\mathcal{M}(0,1)$, in which case no such h can exist for small enough ϵ_0, or $s(v_1)$ is the increasing limit of such gaps.

If $\rho = \rho(\epsilon_0)$ and $n \ge N(\rho)$, then

$$\|(u_n - s(v_1))_-\|_{W^{1,2}(S_i)} \le \frac{\epsilon_0}{2} \quad \text{for } i \le i_0 - 2 \tag{6.62}$$

by (6.60) and Proposition 6.25, since $\psi_n \in \Gamma_q(v_1, v_1)$ for some $q \in \mathbb{N}$, and $(\psi_n - s(v_1))_- = (\min(u_n, v_1) - s(v_1))_- = (u_n - s(v_1))_-$ for $y \le i_0 - 1$. Let $n \to \infty$ in (6.62) to get

$$\|(u - s(v_1))_-\|_{W^{1,2}(S_i)} \le \frac{\epsilon_0}{2} \quad \text{for } i \le i_0 - 2. \tag{6.63}$$

It can be assumed that $l(v) < s(v_1)$ otherwise $v \le s(v) \le s(v_1)$ and (6.17) is trivially true. Thus $0 \le s(v_1) - l(v) \le (u - l(v))_+ - (u - s(v_1))_-$ so (6.63), the first limit in (6.15), and (6.61) imply $\|s(v_1) - s(v)\|_{W^{1,2}(S_0)} \le \epsilon$ for large i, which confirms (6.17).

We now verify the statement in Theorem 6.14 that $v \le v_1$ in the present case where $v_1 < w_1$. Assume to the contrary $v > v_1$ so $s(v) \ge w_1$. From the second limit in (6.15) we see that $\|(u_n - w_1)_-\|_{W^{1,2}(T_{i_1})}$ is small for large i_1, and $n \ge N_1(i_1)$. In addition we know that $\|u_n - v_1\|_{W^{1,2}(T_i)}$ is small for $i \ge N_2(n)$ due to the asymptotic conditions in the definition of $\mathcal{Y}_m$ and Proposition 2.6. Also $\|(u_n - v_1)_+\|_{W^{1,2}(T_{i_2})}$ is small for some $i_2, m_1 - \ell \le i_2 \le m_1$ due to arguments leading to (6.54). One can then do a standard interpolation on the interval $i \le y \le i+1$ for $i << i_1$, and one sided interpolations on $i_1 - 1 \le y \le i_1 + 2$ and $i_2 - 1 \le y \le i_2 + 2$ to define a function

$$\widetilde{u}_n = \begin{cases} u_n & \text{if } i_2 + 2 \le y \\ \min(u_n, v_1) & \text{if } i_2 \le y \le i_2 + 1 \\ u_n & \text{if } i_1 + 2 \le y \le i_2 - 1 \\ \max(u_n, w_1) & \text{if } i_1 \le y \le i_1 + 1 \\ u_n & \text{if } i + 1 \le y \le i_1 - 1 \\ v_1 & \text{if } y \le i \end{cases},$$

extended in the usual way. Consequently

$$J_{-\infty,i_2}(U(\widetilde{u}_n, v_1, w_1)) \geq c(v_1, w_1) + c(w_1, v_1) > 0. \tag{6.64}$$

Let

$$\hat{u}_n = \begin{cases} \widetilde{u}_n & \text{if } i_2 \leq y \\ \min(\widetilde{u}_n, v_1) & \text{if } y \leq i_2 \end{cases}.$$

Then $\hat{u}_n \in \mathcal{Y}_m$ and $J(\widetilde{u}_n) \leq J(u_n) + c\rho \leq y_m + 2c\rho \leq J(\hat{u}_n) + 2c\rho$ for large n. Thus $J_{-\infty,i_2}(\max(\widetilde{u}_n, v_1)) \leq 2c\rho$. However $\max(\widetilde{u}_n, v_1) = 0$ for $i_2 \leq y \leq i_2+1$ and $y \leq i$ so $J_{-\infty,i_2}(\max(\widetilde{u}_n, w_1)) \geq 0$. Consequently $J_{-\infty,i_2}(U(\widetilde{u}_n, v_1, w_1)) \leq J_{-\infty,i_2}(\max(\widetilde{u}_n, v_1)) \leq 2c\rho$, which contradicts (6.64) for small ρ.

All that remains is to verify that u is a solution of (PDE) in all of $\mathbb{R}^2$ and the shadowing estimate (6.19). These follow from the same basic proof. Given $p \in \mathbb{N}, 1 \leq p \leq k$, the approach is to consider a maximal set of $i \in \mathbb{N}$ on which condition (6.3) (resp. (6.4)) holds. To simplify notation assume that

$$v_1 > w_1 = v_2 < w_2 \leq v_3 < w_3 \leq \cdots \leq v_{k-1} < w_{k-1} = v_k > w_k.$$

All other cases are treated the same except the one where instead $v_1 < w_1$ and/or $v_k < w_k$. However in the latter case the estimates are simpler for some p since the one sided 'outer most' interpolations can be replaced by two sided interpolations as in earlier sections due to the asymptotic properties of elements of $\mathcal{Y}_m$.

As before if $\ell, m_{i+1} - m_i$ are sufficiently large there exist integers $\alpha_i, i = 1, 2, 2k-1, 2k$ such that $(u - v_1)_-$, $(u - w_1)_+$, $(u - v_k)_-$, and $(u - w_k)_+$ are small in $W^{1,2}(X_{\alpha_i})$ for $i = 1, 2, 2k-1, 2k$ respectively. So for large n the same is true with u_n replacing u. One can then use one sided interpolations to generate $\widetilde{u}_n \in \mathcal{Y}_m$ such that $|J(u_n) - J(\widetilde{u}_n)| \leq c\rho$, for arbitrarily small ρ, and $\widetilde{u}_n \geq v_1 > w_1$ on S_{α_1}, $\widetilde{u}_n \leq w_1$ on S_{α_2}, $\widetilde{u}_n \geq v_k$ on $S_{\alpha_{2k-1}}$, and $\widetilde{u}_n \leq w_k$ on $S_{\alpha_{2k}}$. Let

$$r_n = \begin{cases} w_1 & \text{if } \alpha_{2k-1} \leq y \\ \min(\widetilde{u}_n, w_n) & \text{if } \alpha_1 \leq y \leq \alpha_{2k-1} \\ w_1 & \text{if } y \leq \alpha_1 \end{cases}$$

noting that $\min(\widetilde{u}_n, w_n) = w_1$ on $S_{\alpha_1} \cup S_{\alpha_{2k-1}}$, so $r_n \in \Gamma(v_1, v_1)$ and $J(r_n) \geq 0$. Define

$$\psi_n = \begin{cases} \widetilde{u}_n & \text{if } \alpha_{2k-1} \leq y \\ \max(\widetilde{u}_n, w_n) & \text{if } \alpha_1 \leq y \leq \alpha_{2k-1} \\ \widetilde{u}_n & \text{if } y \leq \alpha_1 \end{cases}$$

noting that $\max(\widetilde{u}_n, w_n) = \widetilde{u}_n$ on $S_{\alpha_1} \cup S_{\alpha_{2k-1}}$, so $\psi_n \in \mathcal{Y}_m$ and $J(\psi_n) \leq J(\psi_n) + J(r_n) = J(\widetilde{u}_n)$.

Similarly let

$$s_n = \begin{cases} v_k & \text{if } \alpha_{2k} \leq y \\ \max(\psi_n, v_k) & \text{if } \alpha_2 \leq y \leq \alpha_{2k} \\ v_k & \text{if } y \leq \alpha_2 \end{cases}$$

noting that $\max(\psi_n, v_k) = v_k$ on $S_{\alpha_2} \cup S_{\alpha_{2k}}$, so $s_n \in \Gamma(v_k, v_k)$ and $J(s_n) \geq 0$. Also define

$$\varphi_n = \begin{cases} \psi_n & \text{if } \alpha_{2k} \leq y \\ \min(\psi_n, v_k) & \text{if } \alpha_2 \leq y \leq \alpha_{2k} \\ \psi_n & \text{if } y \leq \alpha_2 \end{cases}$$

noting that $\min(\psi_n, v_k) = \psi_n$ on $S_{\alpha_2} \cup S_{\alpha_{2k}}$, so $\varphi_n \in \mathcal{Y}_m$ and $J(\varphi_n) \leq J(\varphi_n) + J(s_n) = J(\psi_n) \leq J(\widetilde{u}_n)$.

Note that $\varphi = w_1$ on S_{α_2} and $\varphi = v_k$ on $S_{\alpha_{2k-1}}$, so if $m_{i+1} - m_i$ are large enough, $i = 1, \ldots, 2k-1$, one can redefine φ_n for $\alpha_2 \leq y \leq \alpha_{2k-1} + 1$ by pasting together various minimizers as in Sect. 5 to generate a function $h_n \in \mathcal{Y}_m$ such that

$$J(\varphi) \leq J(\widetilde{u}_n) \leq J(u_n) + c\rho \leq y_m + 2c\rho \leq J(h_m) + 2c\rho \tag{6.65}$$

for large n, and

(6.66)

$$J_{\alpha_2,\alpha_{2k-1}}(\varphi) \leq J_{\alpha_2,\alpha_{2k-1}}(h_n) + 2c\rho \leq \sum_{i=2}^{k-1} c(v_i, w_i) + \sum_{i=2}^{k-2} c(w_i, v_{i+1}) + 3c\rho.$$

Let

$$\phi_n = \begin{cases} v_k & \text{if } \alpha_{2k-1} \leq y \\ \varphi_n & \text{if } \alpha_2 \leq y \leq \alpha_{2k-1} \\ v_k & \text{if } y \leq \alpha_2 \end{cases}$$

so

$$J(\phi_n) = J_{\alpha_2,\alpha_{2k-1}}(\varphi) \tag{6.67}$$

and note

$$J(\phi) = \sum_{i=2}^{k-1} J(U(\phi_n, v_i, w_i)) + \sum_{i=2}^{k-2} J(U(\phi_n, w_i, v_{i+1}) + 3c\rho \tag{6.68}$$

with

$$J(U(\phi_n, v_i, w_i)) \geq c(v_i, w_i), \quad J(U(\phi_n, w_i, v_{i+1}) \geq c(w_i, v_{i+1}). \tag{6.69}$$

Thus applying all but $J(U(\phi_n, v_p, w_p)) \geq c(v_p, w_p)$, for some p, $2 \leq p \leq k-1$, to (6.68), and combining the result with (6.66), (6.67) leads to

$$J(U(\phi_n, v_p, w_p)) \leq 3c\rho. \tag{6.70}$$

Therefore by Lemma 5.26 there exists $U_n \in \mathcal{M}(v_p, w_p)$ such that

$$\|U_n - U(\phi_n, v_p, w_p)\|_{W^{1,2}(S_i)} \leq \epsilon(3c\rho) \quad \text{for all } i \in \mathbb{Z}. \tag{6.71}$$

The constraint (6.10) implies $U_n \in C_p$, so if $m_{i+1} - m_i$ are large enough (depending only on C_p, since C_p has a smallest and largest element), then $\rho(\sigma_{-j} U_n, v_p) \leq \rho_{2p-1}/3$, $m_{2p-1} - \ell \leq j \leq m_{2p-1} - 1$. Thus for ρ small enough $\rho(U(\sigma_{-j}\phi_n, v_p, w_p)) \leq \rho_{2p-1}/2$. However $U(u_n, v_p, w_p) = U(\phi_n, v_p, w_p)$ on S_j so the first set of constraints in (6.9) holds for u with $i = p$. The second set is verified similarly, as is (6.10), since (6.10) holds with strict inequality for the largest and smallest elements of C_p. Thus u is a solution of (PDE). Previous methods imply that $U_n \to U \in C_p$ in $W^{1,2}(S_i)$ so variations on (6.38)–(6.42) with $f = U(u, v_p, w_p) - U$, and (6.38) replaced by $\|U - U(u, v_p, w_p)\|_{W^{1,2}(S_i)} \le \epsilon(3c\rho)$, imply $v_p < u < w_p$ for $n_p - N \le y \le n_p + N, \rho \le \rho(N)$, and (6.19) is verified.

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