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Homoclinic solutions for a semilinear elliptic equation with an asymptotically linear nonlinearity

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Abstract. We study the following semilinear elliptic equation

$$-\Delta u + b(x)u = f(u), \quad x \in \mathbf{R}^N,$$

where b is periodic and f is assumed to be asymptotically linear. The purpose of this paper is to establish the existence of infinitely many homoclinic type solutions for this class of nonlinearities.

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1. Introduction

In this paper we study the semilinear elliptic equation

$$-\Delta u + b(x)u = f(u), \quad x \in \mathbf{R}^N \tag{1.1}$$

under the following basic assumptions on the potential:

(b_0) $b \in C(\mathbf{R}^N, \mathbf{R})$ and there exists a $b_0 > 0$ so that $b(x) \geq b_0$ for a.e $x \in \mathbf{R}^N$.
(b_1) $b(x_1, \dots, x_N)$ is T_i periodic in x_i, $i = 1, \dots, N$.

During the past decade a variational method was developed to establish the existence of homoclinic type multibump solutions. After the initial ground breaking work by Séré [15], Coti Zelati, Ekeland and Séré [2], and Coti Zelati and Rabinowitz [3], there are many papers that utilize this method for both Hamiltonian systems and semilinear elliptic equations (see [12,13] and references therein). However, in all the results mentioned above, it is assumed that the nonlinearity f satisfies a superlinear growth condition, i.e. $f(s)/s \to \infty$ as $|s| \to \infty$. The purpose of this paper is to adapt this technique to a different class of nonlinearities. More precisely, we make the following assumptions on f:

(f_0) $f \in C^1(\mathbf{R}, \mathbf{R})$, $f(0) = 0$.
(f_1) $f(s) = o(s)$ as $|s| \to 0$.
(f_2) There exists an $a \in (0, \infty)$ so that $\frac{f(s)}{s} \to a$ as $|s| \to \infty$ and $a > \inf \sigma(-\Delta + b)$ (here σ denotes the spectral set).

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(f_3) $H(s) := \frac{1}{2}f(s)s - F(s) \geq 0$ for all $s \in \mathbf{R}$, where $F(s) := \int_0^s f(t)\,dt$.
(f_4) There exists a $\delta_0 >$ such that $f(s) \geq b_0 - \delta_0 \Rightarrow H(s) \geq \delta_0$.

Thus f is asymptotically linear. There are many recent existence and multiplicity results for asymptotically linear problems on $\mathbf{R}^N$, see for instance [1,5,6,8,16–20,23]. Our goal is to establish the existence of infinitely many homoclinic type solutions to (1.1).

In order to properly state our main result, we define the energy functional associated with (1.1):

$$I : W^{1,2}(\mathbf{R}^N) \to \mathbf{R}, \quad I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} F(u)\,dx,$$

where F is the primitive of f and

$$\|u\|^2 := \int_{\mathbf{R}^N} |\nabla u|^2 + b(x)u^2\,dx.$$

Remark 1.2. By (b_0) and (b_1), there exists a $\overline{b} < \infty$ so that $b(x) \leq \overline{b}$ for all $x \in \mathbf{R}^N$. Thus it is clear that $\|\cdot\|$ is equivalent to,

$$\|u\|^2_{W^{1,2}(\mathbf{R}^N)} := \int_{\mathbf{R}^N} |\nabla u|^2 + u^2\,dx,$$

the standard norm on $W^{1,2}(\mathbf{R}^N)$.

It is well known that under (f_0)-(f_2), $I \in C^1(W^{1,2}(\mathbf{R}^N), \mathbf{R})$, and critical points of I correspond to classical solution of (1.1) satisfying $u(x) \to 0$ as $|x| \to \infty$. Set $\mathcal{K} := \{u \in W^{1,2}(\mathbf{R}^N) : I'(u) = 0\}$. We also use the following notation to denote the level sets of I: $I^b := \{u \in W^{1,2}(\mathbf{R}^N) : I(u) \leq b\}$, $I_a := \{u \in W^{1,2}(\mathbf{R}^N) : I(u) \geq a\}$ and $I_a^b := I^b \cap I_a$. Then $\mathcal{K}^b := \mathcal{K} \cap I^b$, $\mathcal{K}_a := \mathcal{K} \cap I_a$ and $\mathcal{K}_a^b = \mathcal{K} \cap I_a^b$.

From results in for instance [5,6,20], it is clear that under (b_0) and (f_1)-(f_2), I possess a Mountain Pass geometry, i.e.

$$\Gamma := \left\{g \in C([0,1], W^{1,2}(\mathbf{R}^N)) : g(0) = 0,\;\; g(1) \in I^0\backslash\{0\}\right\} \neq \emptyset.$$

We may therefore define the Mountain Pass level

$$c := \inf_{g \in \Gamma} \sup_{t \in [0,1]} I(g(t)) > 0.$$

Recall that $(u_n) \subset W^{1,2}(\mathbf{R}^N)$ is a Palais-Smale sequence ($(PS)_d$ for short) of I if $I(u_n) \leq d$ and $I'(u_n) \to 0$. I satisfies the Palais-Smale condition if any such sequence contains a convergent subsequence.

We also note that, due to (b_1), I and I' are invariant under the discrete translations

$$\tau_k u(x) := u(x_1 + k_1T_1, \ldots, x_N + k_NT_N),$$

for any $k = (k_1, \ldots, k_N) \in \mathbf{Z}^N$. This leads to the conclusion that the Palais-Smale condition fails at every level. It is therefore not immediately clear wether c is a critical value. We exploit this lack of compactness, together with the assumption,

$(*)$ there exists an $\alpha > 0$ so that $\mathcal{K}^{c+\alpha}/\mathbf{Z}^N$ is finite,

to obtain:

Theorem 1.3. *Assume* (b_0)-(b_1), (f_0)-(f_4), *and* $(*)$. *Then* $\mathcal{K}^{kc+\alpha}_{kc-\alpha}/\mathbf{Z}^N$ *is infinite for all* $k \in \mathbf{N}/\{1\}$.

This mimics the result obtained by Coti Zelati and Rabinowitz [3,4] for a semilinear elliptic equation with a superlinear, subcritical nonlinearity. In order to prove Theorem 1.3, we adopt the techniques used in the above mentioned papers.

Finally we highlight some of the major differences between our case and the superlinear case considered in [4]. First is the lack of an *a priori* bound for Palais-Smale sequences. However, to overcome this difficulty, an argument based on the Concentration Compactness Lemma of P.L. Lions [9] was developed by L. Jeanjean [5].

Also, in the superlinear case, it is easily verified that there exists a $\underline{c} > 0$ so that

$$I(v) \geq \underline{c} \quad \forall v \in \mathcal{K}\backslash\{0\}.$$

This plays a crucial rule in determining the exact behavior of Palais-Smale sequences and the nature of the non-compactness due to the translation invariance. Roughly, it is shown that there exists a finite dichotomy of any $(PS)_d$ sequence into a collection of translated non-trivial critical points with energy less than d. The above lower bound ensures that the energy is exhausted in a finite number of steps. See [4] for more details. Since, in our case, the existence of such a uniform bound is not immediately apparent, we will use a different approach, based on the Concentration-Compactness Principle, to obtain a similar result.

After establishing the required properties for Palais-Smale sequences, the rest of the construction is very similar. This is due to the fact that the arguments used in the later part of [4] depends largely on the behavior of the nonlinearity near 0, which, due to (f_1), is the same for both cases. However, there are some minor adjustments and simplifications, in which case full details will be provided.

This paper is organized as follows: In Section 2, we collect all the important properties of Palais-Smale sequences. Section 3 sets up a suitable existence criterion for our main result, Theorem 1.3. Section 4 contains the bulk of the construction used in the contradiction argument outlined in Section 3.

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Further notations and conventions

For any $r > 0$, set $B_r(y) := \{x \in \mathbf{R}^N : |x-y| \leq r\}$ and to simplify notation we let $B_r := B_r(0)$. If $K \subset W^{1,2}(\mathbf{R}^N)$, then $N_r(K) := \{u \in W^{1,2}(\mathbf{R}^N) : \|u - K\| \leq r\}$ and, for $w \in W^{1,2}(\mathbf{R}^N)$, $\mathcal{B}_r(w) := \{u \in W^{1,2}(\mathbf{R}^N) : \|u - w\| \leq r\}$. Let $\overline{\tau}_j := (j_1 T_1, \dots, j_N T_N) \in \mathbf{R}^N$, where $j \in \mathbf{Z}^N$. For any $\Omega \subset \mathbf{R}^N$, we set $\tau_j \Omega := \{x + \overline{\tau}_j : x \in \Omega\}$. We will also make use of

$$\|u\|^2_\Omega := \int_\Omega |\nabla u|^2 + b(x)u^2\, dx,$$

to denote the restriction of $\|\cdot\|$ to $\Omega \subset \mathbf{R}^N$. All other norms will be distinguished by a proper subscript. Subsequences of (u_n) will still be denoted by (u_n).

2. Behavior of Palais-Smale sequences

This section concerns itself with the properties of sequences $(u_n) \subset W^{1,2}(\mathbf{R}^N)$ satisfying

$$I(u_n) \leq d, \;\; I'(u_n) \to 0, \tag{2.1}$$

where $0 < d < \infty$.

By (f_1) and (f_2), there exists a $C_f > 0$ such that

$$\left| \frac{f(s)}{s} \right| \leq C_f \quad \forall s \in \mathbf{R}. \tag{2.2}$$

Thus for any $\epsilon > 0$ and $2 < p \leq 2^*$ (here $2^* := 2N/N-2$ if $N \geq 3$ and $2^* = \infty$ if $N = 2$), by (f_1)-(f_2), there exists a $C_\epsilon > 0$ such that

$$f(s) \leq \epsilon s + C_\epsilon s^{p-1} \quad \forall s \in \mathbf{R}. \tag{2.3}$$

Lemma 2.4. *There exists a $\delta_1 > 0$, such that if $(u_n) \subset W^{1,2}(\mathbf{R}^N)$, satisfy*

$$I'(u_n) \to 0,$$

then, up to a subsequence, either $\|u_n\| \geq \delta_1$ or $\|u_n\| \to 0$ as $n \to \infty$.

Proof. By (2.3) there exists a $C_{\frac{b_0}{2}} > 0$ such that

$$f(s)s \leq \frac{b_0}{2}|s|^2 + C_{\frac{b_0}{2}}|s|^{2^*} \quad \forall s \in \mathbf{R}.$$

Thus for any $u \in W^{1,2}(\mathbf{R}^N)$,

$$\begin{aligned} I'(u)u &= \|u\|^2 - \int_{\mathbf{R}^N} f(u)u\,dx \\ &\geq \frac{1}{2}\|u\|^2 - C_2\|u\|^{2^*}, \end{aligned}$$

where we used the continuity of the embedding $W^{1,2}(\mathbf{R}^N) \hookrightarrow L^{2^*}(\mathbf{R}^N)$ and Remark 1.2. Therefore, there exists a $\delta_1 > 0$ such that

$$\frac{1}{4}\|u\|^2 \leq I'(u)u \quad \text{for } \|u\| < \delta_1. \tag{2.5}$$

Let $(u_n) \subset W^{1,2}(\mathbf{R}^N)$ satisfy $I'(u_n) \to 0$ as $n \to \infty$. Then, up to a subsequence, either

$$\limsup_{n\to\infty} \|u_n\| < \delta_1 \;\; \text{or} \;\; \liminf_{n\to\infty} \|u_n\| \geq \delta_1.$$

For the first option, we have by (2.5) that

$$\frac{1}{4}\|u_n\|^2 \leq o(1) \cdot \|u_n\|$$

for n large and therefore $\liminf_{n\to\infty} \|u_n\| = 0$. □

Corollary 2.6. (i) *If* $u \in \mathcal{K}\backslash\{0\}$*, then* $\|u\| \geq \delta_1$.
(ii) *If* $(u_n) \subset W^{1,2}(\mathbf{R}^N)$*,* $I'(u_n) \to 0$ *and* $\liminf_{n\to\infty} I(u_n) \neq 0$*, then* $\|u_n\| \geq \delta_1$.

Corollary 2.7. *There exists a* $\delta_2 > 0$ *such that* $\|v\|_{L^2(\mathbf{R}^N)} \geq \delta_2$ *for all* $v \in \mathcal{K}\backslash\{0\}$.

Proof. Suppose this is false, then there exists a $(v_n) \subset \mathcal{K}\backslash\{0\}$ such that

$$\|v_n\|_{L^2(\mathbf{R}^N)} \to 0.$$

By (2.2) this implies that,

$$\|v_n\|^2 = I'(v_n)v_n + \int_{\mathbf{R}^N} f(v_n)v_n\,dx \leq C_f\|v_n\|^2_{L^2(\mathbf{R}^N)} \to 0,$$

which contradicts Corollary 2.6 (i). □

Next, we show that for any $(u_n) \subset W^{1,2}(\mathbf{R}^N)$ satisfying (2.1), there exists a $M > 0$, depending only on d, so that $\|u_n\| \leq M$, i.e. (PS) sequences are bounded in $W^{1,2}(\mathbf{R}^N)$.

Proposition 2.8. *If* $(u_n) \subset W^{1,2}(\mathbf{R}^N)$ *satisfy*

$$I(u_n) \leq d, \quad I'(u_n) \to 0,$$

then there exists a $M = M(d) > 0$ *such that* $\|u_n\| \leq M$.

Using the translation invariance of I, our arguments are similar to the techniques used in [5]. In order to prove Proposition 2.8, we need the following results. For any $(u_n) \subset W^{1,2}(\mathbf{R}^N)$, $q \in \mathbf{N}$ and $r > 0$ define the family of concentration functions:

$$Q_{n,q}(r) := \sup_{\{y_1,\dots,y_q\}\in\mathbf{R}^N} \int_{\cup_{i=1}^q B_r(y_i)} u_n^2\,dx.$$

We state the following Concentration Compactness result (see [22, Lemma 4.1]).

Lemma 2.9. *Suppose* $(u_n) \in W^{1,2}(\mathbf{R}^N)$ *satisfies* $\|u_n\|^2_{L^2(\mathbf{R}^N)} \leq M < \infty$*, then, up to a subsequence,* $\lim_{n\to\infty} Q_{n,q}(r)$ *exists for all* $q \in \mathbf{N}$*,* $r > 0$*, and*

(i) (λ_i)*, defined by*

$$\lambda_1 := \lim_{m\to\infty}\lim_{n\to\infty} Q_{n,1}(m) \ \text{ and for } q > 1,$$

$$\lambda_q := \lim_{m\to\infty}\lim_{n\to\infty} Q_{n,q}(m) - \sum_{i=1}^{q-1}\lambda_i,$$

is a nonnegative, nonincreasing sequence satisfying

$$\lim_{q\to\infty}\sum_{i=1}^{q}\lambda_i \leq M.$$

(ii) For each $\lambda_i > 0$, there exists a $(y_{i,n}) \subset \mathbf{R}^N$ satisfying

$$|y_{i,n} - y_{j,n}| \to \infty, \quad \forall i \neq j,$$

and given any $q \geq 1$ with $\lambda_q > 0$, for any $\epsilon > 0$, there exists a $R > 0$ such that

$$\limsup_{n\to\infty} \sum_{i=1}^{q} \left| \lambda_i - \int_{B_r(y_{i,n})} u_n^2 \, dx \right| < \epsilon,$$

for a all $r \geq R$.

Remark 2.10. From the proof of Lemma 2.9, we note that, for $\lambda_q > 0$ and $R > 0$, the sequences $(y_{i,n}) \subset \mathbf{R}^N$ are chosen so that

$$\limsup_{n\to\infty} \int_{\cup_{i=1}^{q} B_r(y_{i,n})} u_n^2 \, dx = \lim_{n\to\infty} Q_{n,q}(r),$$

for all $r \geq R$. We also note that Lemma 2.9 is a generalization of P.L. Lions Concentration Compactness Lemma [9] and is a reformulation of a result in [10]. If $\lambda_1 = 0$, the sequence (u_n), *vanishes*. If, on the other hand, $\lambda_1 > 0$, then (u_n) is *non-vanishing*.

Proposition 2.11. *If $(u_n) \subset W^{1,2}(\mathbf{R}^N)$ satisfies*

$$\|u_n\| \geq \delta_1, \;\; I(u_n) \leq d, \;\; I'(u_n) \to 0,$$

then $w_n := u_n / \|u_n\|$ is non-vanishing.

Proof. Seeking a contradiction, suppose (w_n) vanishes, i.e.,

$$\lim_{n\to\infty} \sup_{y\in\mathbf{R}^N} \int_{B_R(y)} w_n^2 \, dx = 0, \quad \forall R > 0. \tag{2.12}$$

Observe that

$$\int_{\mathbf{R}^N} f(u_n)u_n \, dx = \|u_n\|^2 + o(1) \cdot \|u_n\|.$$

Thus,

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \frac{f(u_n)}{u_n} w_n^2 \, dx = 1. \tag{2.13}$$

Set

$$K_n := \{x \in \mathbf{R}^N : \frac{f(u_n(x))}{u_n(x)} \leq b_0 - \frac{1}{2}\delta_0\}.$$

Since $1 = \|w_n\|^2 \geq b_0 \|w_n\|^2_{L^2(\mathbf{R}^N)}$,

$$\int_{K_n} \frac{f(u_n)}{u_n} w_n^2 \, dx \leq \left(b_0 - \frac{1}{2}\delta_0\right) \int_{K_n} w_n^2 \, dx \leq \frac{1}{b_0}\left(b_0 - \frac{1}{2}\delta_0\right) < 1,$$

thus, by (2.13),

$$\lim_{n\to\infty}\int_{\mathbf{R}^N\setminus K_n}\frac{f(u_n)}{u_n}w_n^2\,dx>0. \tag{2.14}$$

Next we show that $m\left(\mathbf{R}^N\setminus K_n\right)\to\infty$ (here m denotes the Lebesque measure on $\mathbf{R}^N$). Since (w_n) is bounded in $W^{1,2}(\mathbf{R}^N)$, it follows from (2.12) and a lemma of P.L. Lions (see for instance [21, Lemma 1.21]), that $w_n\to 0$ in $L^{2q}(\mathbf{R}^N)$, where $1<q<2^*/2$. Thus if $m\left(\mathbf{R}^N\setminus K_n\right)<\infty$,

$$\begin{aligned}\int_{\mathbf{R}^N\setminus K_n}\frac{f(u_n)}{u_n}w_n^2\,dx &\le C_f\int_{\mathbf{R}^N\setminus K_n}w_n^2\,dx\\ &\le C_f\left(\int_{\mathbf{R}^N\setminus K_n}|w_n|^{2q}\,dx\right)^{\frac{1}{q}}\left(m\left(\mathbf{R}^N\setminus K_n\right)\right)^{1/q'}\to 0,\end{aligned}$$

which contradicts (2.14). By (f_3),

$$d\ge I(u_n)-I'(u_n)u_n=\int_{\mathbf{R}^N}H(u_n)\,dx\ge\int_{\mathbf{R}^N\setminus K_n}H(u_n)\,dx.$$

Since $H(u_n)\ge\delta_0$ on $\mathbf{R}^N\setminus K_n$, this implies that

$$d\ge\int_{\mathbf{R}^N\setminus K_n}H(u_n)\,dx\ge\delta_0 m\left(\mathbf{R}^N\setminus K_n\right)\to\infty,$$

which yields the desired contradiction. □

Corollary 2.15. *If* $(u_n)\subset W^{1,2}(\mathbf{R}^N)$ *satisfy*

$$I(u_n)\le d,\ \ I'(u_n)\to 0,$$

then either (i) $\|u_n\|\to 0$ *or, (ii)* (u_n) *is non-vanishing.*

Proof. Suppose (i) does not hold, then by Lemma 2.4, $\|u_n\|\ge\delta_1$. For contradiction suppose (ii) fails, i.e. (u_n) vanishes. Since

$$|w_n|=\frac{|u_n|}{\|u_n\|}\le\frac{1}{\delta_1}|u_n|,$$

this implies that (w_n) vanishes, which contradicts Proposition 2.11. □

Proof of Proposition 2.8. Seeking a contradiction suppose $\|u_n\|\to\infty$. Set $w_n:=u_n/\|u_n\|$. By Proposition 2.11, (w_n) is non-vanishing, i.e. there exists $\rho_0>0$, $R_0>0$ and $(y_n)\subset\mathbf{R}^N$ such that

$$\lim_{n\to\infty}\int_{B_{R_0}(y_n)}w_n^2\,dx\ge\rho_0. \tag{2.16}$$

For each component (y_n^i), $i = 1, \dots, N$, set $k_n^i := [\![y_n^i / T_i]\!]$, where $[\![x]\!]$ denotes the integer part of x. Then

$$\left| \left(k_n^1 T_1, \dots, k_n^N T_N\right) - y_n \right| \leq \left(\sum_{i=1}^N T_i^2 \right)^{1/2} =: T$$

and therefore

$$\lim_{n\to\infty} \int_{B_{R_0+T}} \left(\tau_{k_n} w_n\right)^2 \, dx \geq \lim_{n\to\infty} \int_{B_{R_0}(y_n)} w_n^2 \, dx \geq \rho_0. \tag{2.17}$$

Set $w_n^1 := \tau_{k_n} w_n$ and $u_n^1 := \tau_{k_n} u_n$. Using the invariance of $\|\cdot\|$ and I under the translations τ:

$$\left.\begin{aligned} \|w_n^1\| &= \|w_n\| = 1 \\ I(u_n^1) &= I(u_n) \leq d \\ I'(u_n^1) &= I'(u_n) \to 0. \end{aligned}\right\} \tag{2.18}$$

Thus, up to a subsequence, $w_n^1 \rightharpoonup w$ in $W^{1,2}(\mathbf{R}^N)$, $w_n^1 \to w$ in $L^2_{loc}(\mathbf{R}^N)$. This, together with (2.17), implies that $w \neq 0$. Next we show that,

$$-\Delta w + b(x) w = a w, \quad x \in \mathbf{R}^N, \tag{2.19}$$

i.e., a is an eigenvalue of the operator $-\Delta + b$. This would contradict the fact that under (b_0)-(b_1), the Schrödinger operator $-\Delta + b$ has only purely continuous spectrum [14, Theorem XIII.100]. In order to show (2.19) it suffices to show

$$\int_{\mathbf{R}^N} \nabla w \nabla \varphi + b(x) w \varphi \, dx = a \int_{\mathbf{R}^N} w \varphi \, dx \quad \forall \varphi \in C_0^\infty(\mathbf{R}^N). \tag{2.20}$$

Fixing an arbitrary $\varphi \in C_0^\infty(\mathbf{R}^N)$, (2.18) implies that

$$\int_{\mathbf{R}^N} \nabla w_n^1 \nabla \varphi + b(x) w_n^1 \varphi \, dx = o(1) + \int_{\mathbf{R}^N} \frac{f(u_n^1)}{u_n^1} w_n^1 \varphi \, dx.$$

By virtue of the weak convergence,

$$\int_{\mathbf{R}^N} \nabla w_n^1 \nabla \varphi + b(x) w_n^1 \varphi \, dx \to \int_{\mathbf{R}^N} \nabla w \nabla \varphi + b(x) w \varphi \, dx,$$

and to complete (2.20), we only need to establish

$$\int_{\mathbf{R}^N} \frac{f(u_n^1)}{u_n^1} w_n^1 \varphi \, dx \to a \int_{\mathbf{R}^N} w \varphi \, dx. \tag{2.21}$$

Clearly $w_n^1 \varphi \to w\varphi$ in $L^1(\mathbf{R}^N)$. Set $\mathcal{N} := \{x \in \mathbf{R}^N : w(x)\varphi(x) \neq 0\}$. Since we may assume that $\varphi \neq 0$, $m(\mathcal{N}) > 0$. For any $x \in \mathcal{N}$, $u_n^1(x) \to \infty$ which implies that

$$\frac{f(u_n^1(x))}{u_n^1(x)} \to a \quad \forall x \in \mathcal{N},$$

and (2.21) follows from [20, Lemma A.1]. This completes the proof of Proposition 2.8.

We end this section with an important result that describes the lack of compactness of I. First we need the following lemma:

Lemma 2.22. *If $(u_n) \subset W^{1,2}(\mathbf{R}^N)$ satisfy $I'(u_n) \to 0$, and $u_n \rightharpoonup v$ in $W^{1,2}(\mathbf{R}^N)$, then $v \in \mathcal{K}$ and $I'(u_n - v) \to 0$.*

Proof. The fact that $v \in \mathcal{K}$ is easily verified, see for instance [20]. We also note that, in order to show $I'(u_n - v) \to 0$, it suffices to show

$$\int_{\mathbf{R}^N} (f(u_n - v) - f(u_n) + f(v))\, w\, dx = o(1) \cdot \|w\|,$$

for all $w \in W^{1,2}(\mathbf{R}^N)$. Under (f_1) and (f_2), this is the content of [20, Lemma 3.7]. □

Proposition 2.23. *Suppose $(u_n) \subset W^{1,2}(\mathbf{R}^N)$ satisfy*

$$\liminf_{n\to\infty} I(u_n) > 0, \quad I(u_n) \le d, \quad I'(u_n) \to 0,$$

then there exist $l = l(d) \in \mathbf{N}$, $\{v_i\}_{i=1}^l \in \mathcal{K}^d/\{0\}$ and corresponding $\{k_{i,n}\}_{i=1}^l \in \mathbf{Z}^N$ such that

$$\|u_n - \sum_{i=1}^{l} \tau_{k_{i,n}} v_i\| \to 0,$$

and

$$|k_{i,n} - k_{j,n}| \to \infty \quad \forall i \ne j.$$

Proof. By Corollary 2.6, $\|u_n\| \ge \delta_1$ which implies that (ii) of Corollary 2.15 holds, i.e. (u_n) is non-vanishing. Thus, by Lemma 2.9 and Remark 2.10, there exist constants $\rho_1, m_1 > 0$ and a sequence $(y_{1,n}) \subset \mathbf{R}^N$ such that

$$\lim_{n\to\infty} \int_{B_m(y_{1,n})} u_n^2\, dx = \lim_{n\to\infty} Q_{n,1}(m) \ge \rho_1, \tag{2.24}$$

for all $m \ge m_1$. As in the proof of Proposition 2.8, setting $\overline{k}_{1,n}^i = [\![y_{1,n}^i / T_i]\!]$ and $u_{1,n} = \tau_{\overline{k}_{1,n}} u_n$ yields

$$\int_{B_m} (u_{1,n})^2\, dx \le \int_{B_m(y_{1,n})} u_n^2\, dx \le \int_{B_{m+T}} (u_{1,n})^2\, dx. \tag{2.25}$$

Since $I(u_{1,n}) = I(u_n) \le d$ and $I'(u_{1,n}) = I'(u_n) \to 0$, by Proposition 2.8 $\|u_{1,n}\| \le M(d)$. Thus, up to a subsequence, $u_{1,n} \rightharpoonup v_1$ in $W^{1,2}(\mathbf{R}^N)$, $u_{1,n} \to v_1$ in $L^2_{loc}(\mathbf{R}^N)$. By (2.24), (2.25) and Lemma 2.22, $v_1 \in \mathcal{K}\backslash\{0\}$. Also, by (f_3) and Fatou's Lemma

$$d = \liminf_{n\to\infty} \int_{\mathbf{R}^N} H(u_{1,n})\, dx \ge \int_{\mathbf{R}^N} H(v_1)\, dx = I(v_1),$$

thus $v_1 \in \mathcal{K}^d \backslash \{0\}$. Next, observe that

$$\lim_{m\to\infty} \left(\lim_{n\to\infty} \int_{B_m} (u_{1,n})^2\, dx \right) \le \lambda_1 \le \lim_{m\to\infty} \left(\lim_{n\to\infty} \int_{B_{m+T}} (u_{1,n})^2\, dx \right),$$

i.e., $\lambda_1 = \|v_1\|^2_{L^2(\mathbf{R}^N)}$. Set $u_{2,n} := u_{1,n} - v_1$. By Lemma 2.22, $I'(u_{2,n}) \to 0$ and therefore, by Corollary 2.15, either (i) $\|u_{2,n}\| \to 0$ or (ii) $(u_{2,n})$ is non-vanishing. If (i) occurs, then we have completed the proof with $l = 1$ and $k_{1,n} = -\overline{k}_{1,n}$.

Thus suppose (ii) occurs, i.e. there exist constants $\rho_2, m_2 > 0$ and a sequence $(z_n) \in \mathbf{R}^N$ such that

$$\lim_{n\to\infty} \int_{B_m(z_n)} (u_{2,n})^2\, dx \geq \rho_2,$$

for all $m \geq m_2$. Clearly $|z_n| \to \infty$. Let $\epsilon > 0$ be arbitrary. Choose a $N = N(\epsilon)$ such that

$$\int_{B_m(z_N)} (v_1)^2\, dx - \epsilon \leq \int_{B_m(z_N)} u_{1,n}v_1\, dx \leq \int_{B_m(z_N)} (v_1)^2\, dx + \frac{\epsilon}{2},$$

for all $n \geq N$ where m is fixed. Increasing N, we may also assume that $\int_{B_m(z_N)} (v_1)^2\, dx \leq \frac{\epsilon}{2}$. Thus

$$\lim_{n\to\infty} \int_{B_m(z_n)} u_{1,n}v_1\, dx = \lim_{n\to\infty} \int_{B_m(z_n)} (v_1)^2\, dx = 0,$$

and therefore

$$\lim_{n\to\infty} \int_{B_m(z_n)} (u_{2,n})^2\, dx = \lim_{n\to\infty} \int_{B_m(z_n)} (u_{1,n})^2\, dx.$$

Then

$$\int_{B_m(z_n+y_{1,n})} u_n^2\, dx \leq \int_{B_m(z_n)} (u_{1,n})^2\, dx \leq \int_{B_{m+T}(z_n+y_{1,n})} u_n^2\, dx,$$

and since we may assume $y_{2,n} = z_n + y_{1,n}$, we conclude that

$$\lambda_2 = \lim_{m\to\infty} \lim_{n\to\infty} \int_{B_m(z_n)} (u_{2,n})^2\, dx.$$

As before, there exists a $(\overline{k}_{2,n}) \in \mathbf{Z}^N$ and $v_2 \in \mathcal{K}^d/\{0\}$ such that $\tau_{\overline{k}_{1,n}} u_{2,n} \rightharpoonup v_2$ and $\lambda_2 = \|v_2\|^2_{L^2(\mathbf{R}^N)}$.

Once again, setting $u_{3,n} := \tau_{\overline{k}_{2,n}} u_{2,n} - v_2$, either (i) $\|u_{3,n}\| \to 0$, or (ii) $(u_{3,n})$ is non-vanishing. Continuing this process, we obtain a sequence $(v_i) \in \mathcal{K}^d \backslash \{0\}$ with $\lambda_i = \|v_i\|^2_{L^2(\mathbf{R}^N)}$. It remains to show that this process terminates after a finite number of steps (i.e. there exists a $l \in \mathbf{N}$ such that (i) occurs after l steps). Since,

$$\|u_n\|^2_{L^2(\mathbf{R}^N)} \leq \frac{1}{b_0}\|u_n\|^2 \leq \frac{M^2(d)}{b_0},$$

Lemma 2.9 yields

$$\sum_{i=1}^{\infty} \|v_i\|^2_{L^2(\mathbf{R}^N)} \leq \frac{M^2(d)}{b_0} < \infty,$$

and we conclude by Lemma 2.7 that there exists a $l \in \mathbf{N}$ (which depends on d) such that $\|v_i\|^2_{L^2(\mathbf{R}^N)} = 0$ for all $i \geq l$. □

Remark 2.26. We note that all the results in this section are completely independent of $(*)$. Thus Palais-Smale sequences behave in a similar way when $(*)$ is replaced with certain weaker conditions.

3. A criterion for existence

Let $\mathcal{F} := \mathcal{K}^{c+\alpha}/\mathbf{Z}^N$, then under $(*)$ $\mathcal{F}$ is finite. For any $l \in \mathbf{N}$, define

$$\mathcal{T}(l) := \left\{ \sum_{i=1}^{k} \tau_{j_i} v_i : 1 \leq k \leq l, j_i \in \mathbf{Z}^N, v_i \in \mathcal{F} \right\}.$$

From a combinatorial result in [3], we deduce that $\mathcal{T}(l)$ is a discreet set, i.e.

$$\mu(l) := \inf \{\|x - y\| : x \neq y \in \mathcal{T}(l)\} > 0.$$

By Proposition 2.23, we fix a $\bar{l} \in \mathbf{N}$ such that, whenever $(u_n) \subset I_0^{c+\alpha}$ satisfy $I'(u_n) \to 0$, then $u_n \to \mathcal{T}(\bar{l})$. Also, by the discreteness property mentioned above, we set

$$\alpha_1 := \sup \left\{ \gamma < \alpha : \mathcal{K}_{c-\gamma}^{c+\gamma} = \mathcal{K}(c) \right\} > 0.$$

Proposition 3.1. *There exists a finite, non-empty $\mathcal{C} \subset \mathcal{K}(c)$ such that for all $\bar{\epsilon} \leq \frac{\alpha_1}{2}$, $r_1 \leq \frac{1}{12}\mu(\bar{l})$ and $p \in \mathbf{N}$, there exists an $\epsilon_1 \in (0, \bar{\epsilon})$ and $g_1 \in \Gamma$ satisfying*

(i) $\max_{t\in[0,1]} I(g_1(t)) \leq c + \frac{\epsilon_1}{p}$ *and*
(ii) $I(g_1(t)) \geq c - \epsilon_1 \Rightarrow g_1(t) \in N_{r_1}(\mathcal{C})$.

Since the proof of Proposition 3.1 is similar to the equivalent result obtained in [3], we will not provide any details. We do however note that Proposition 3.1 follows from a suitable deformation result. Since the Palais-Smale condition fails, we lack a positive lower bound for I' outside some neighborhood of $\mathcal{K}$ and the standard Deformation Theorem does not apply. However, due to Proposition 2.23, we have the following:

Proposition 3.2. *For any $r < \frac{1}{3}\mu(\bar{l})$ there exists a $\delta_3 > 0$ such that $\|I'(u)\| \geq \delta_3$ for all $u \in I_{c-\alpha}^{c+\alpha} \backslash N_{\frac{r}{8}}(\mathcal{T}(\bar{l}))$.*

Proof. If not, we can find a $(u_n) \subset I_{c-\alpha}^{c+\alpha} \backslash N_{\frac{r}{8}}(\mathcal{T}(\bar{l}))$ such that $I'(u_n) \to 0$. By Proposition 2.23, $u_n \to \mathcal{T}(\bar{l})$, a contradiction. □

A variant of the standard Deformation Theorem then follows:

Proposition 3.3. *If $d \in (0, c + \alpha)$, then for any $\bar{\epsilon} \in (0, \alpha]$ and $r < \frac{1}{3}\mu(\bar{l})$, there exists an $\epsilon \in (0, \bar{\epsilon})$, $\eta \in C\left([0,1] \times W^{1,2}(\mathbf{R}^N), W^{1,2}(\mathbf{R}^N)\right)$ and $\sigma \in C\left(I^{d+\epsilon}, [0,1]\right)$ such that*

1^o $\eta(0, u) = u$ *for all* $u \in W^{1,2}(\mathbf{R}^N)$,
2^o $\eta(s, u) = u$ *for all* $u \in I_{d-\bar{\epsilon}}^{d+\bar{\epsilon}}$,
3^o $I(\eta(s, \cdot))$ *is non-increasing,*
4^o $\eta(1, I^{d+\epsilon} \backslash N_r(\mathcal{K}_{d-\bar{\epsilon}}^{d+\bar{\epsilon}})) \subset I^{d-\epsilon}$,

5^o $\sigma(u) = 0$ *for all* $u \in I^{d-\epsilon} \backslash N_r(\mathcal{K}_{d-\bar{\epsilon}}^{d+\bar{\epsilon}})$ *and* $I(\eta(\sigma(u), u)) = d - \epsilon$ *for all* $u \in I_{d-\epsilon}^{d+\epsilon} \backslash N_r(\mathcal{K}_{d-\bar{\epsilon}}^{d+\bar{\epsilon}})$.
6^o $\|\eta(\sigma(u), u) - u\| \leq r$ *for all* $u \in W^{1,2}(\mathbf{R}^N)$ *and*
7^o $\eta(s, \tau_j u) = \tau_j \eta(s, u)$ *for all* $u \in W^{1,2}(\mathbf{R}^N)$, $j \in \mathbf{Z}^N$.

See [3] for a proof.

Next we prove a suitable existence criteria for our main result. We also provide the initial setup for the contradiction argument that occupy Section 4. First, we fix a $n_0 \in \mathbf{N}$ and $j_1, \ldots, j_k \in \mathbf{Z}^N$ so that

$$|j_i - j_m| \geq n_0 \quad \forall i \neq m,$$

implies,

$$\| \sum_{i=1}^{k} \tau_{j_i} v_i \| \geq \frac{1}{2} \sum_{i=1}^{k} \|v_i\| \geq \frac{k}{2} \delta_1 \quad \forall v_i \in \mathcal{C}, \tag{3.4}$$

and

$$\left| I\left(\sum_{i=1}^{k} \tau_{j_i} v_i\right) - \sum_{i=1}^{k} I(v_i) \right| = \left| I\left(\sum_{i=1}^{k} \tau_{j_i} v_i\right) - kc \right| < \frac{\alpha}{2} \quad \forall v_i \in \mathcal{C}. \tag{3.5}$$

For $l \in \mathbf{N}$, we introduce the following notation:

$$\mathcal{M}(l) := \left\{ \sum_{i=1}^{k} \tau_{l j_i} v_i : v_i \in \mathcal{C} \right\}, \quad \mathcal{M}^* := \bigcup_{l \in \mathbf{N}} \mathcal{M}(l).$$

The next couple of results are similar to their counter parts in [3,4], with similar proofs. We simply state the results here:

Proposition 3.6. *There exists a* $r_k > 0$ *such that if* $r \leq r_k$ *and* $w \in \overline{N}_r(\mathcal{M}^*)$, *then* $w \in I_{kc-\alpha}^{kc+\alpha}$.

Proposition 3.7. *Set*

$$\overline{r}_1 = \min\left(\frac{1}{12}\mu(\bar{l}), \frac{\delta_1}{2}, r_k\right), \tag{3.8}$$

then for any $r \leq \overline{r}_1$ *and* $l \in \mathbf{N}$, *either*

(i) there exists a $\hat{\delta}_l > 0$ *such that* $\|I'(w)\| \geq \hat{\delta}_l$ *for all* $w \in N_r(\mathcal{M}(l))$, *or*
(ii) there exists a $w \in \overline{N}_r(\mathcal{M}(l))$ *such that* $I'(w) = 0$, *i.e.* $\overline{N}_r(\mathcal{M}(l)) \cap \mathcal{K} \neq \emptyset$.

Set

$$\mathcal{L} := \{l \in \mathbf{N} : \text{(i) of Proposition 3.7 holds for } \mathcal{M}(l)\},$$

and

$$\mathcal{W} := \bigcup_{l \in \mathcal{L}} \mathcal{M}(l).$$

Combining Proposition 3.6 and Proposition 3.7, we can formulate the following existence criterion:

Proposition 3.9. *If* $|\mathcal{L}| < \infty$*, then* $\mathcal{K}_{kc-\alpha}^{kc+\alpha}/\mathbf{Z}^N$ *is infinite.*

Proof. Assuming $\mathcal{L}$ is finite, Proposition 3.6 and Proposition 3.7 yields

$$\mathcal{K}_{kc-\alpha}^{kc+\alpha}/\mathbf{Z}^N \cap \overline{N}_r(\mathcal{M}(l)) \neq \emptyset,$$

for all but finitely many l. Since $N_r(\mathcal{M}(l_1)) \cap N_r(\mathcal{M}(l_2)) = \emptyset$, if $|l_1 - l_2|$ is sufficiently large, the proof is complete. □

Thus, in order to prove Theorem 1.3, it suffices to show $|\mathcal{L}| < \infty$. Towards this end, we essentially follow [4]. Define the following class of functions:

$$\Gamma_k := \{G \in C([0,1]^k, W^{1,2}(\mathbf{R}^N)) : G = g_1 + \ldots + g_k, \text{ and } g_i \text{ satisfies } (g_1)\text{-}(g_3)\}$$

(g_1) $g_i \in C([0,1]^k, W^{1,2}(\mathbf{R}^N))$ for all $1 \leq i \leq k$;
(g_2) Setting $\mathbf{0}_i := (t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_k)$ and $\mathbf{1}_i := (t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_k)$, we require $g_i(\mathbf{0}_i) = 0$ and $I(g_i(\mathbf{1}_i)) < 0$ for all $1 \leq i \leq k$;
(g_3) There exist compact sets $\mathcal{S}_i \subset \mathbf{R}^N$ such that $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$ for all $i \neq j$ and $\operatorname{supp} g_i \subset \mathcal{S}_i$ for all $1 \leq i \leq k$.

Note that, if $g_i \in \Gamma$ satisfies (g_3), then $G(\mathbf{t}) = \sum_{i=1}^k g_i(t_i) \in \Gamma_k$. Set

$$c_k := \inf_{G \in \Gamma_k} \sup_{\mathbf{t} \in [0,1]^k} I(G(\mathbf{t})).$$

To show that $\mathcal{L}$ is finite, we argue in the following manner: Seeking a contradiction suppose $|\mathcal{L}| = \infty$, then we construct a $\overline{G} \in \Gamma_k$ such that

$$I(\overline{G}(\mathbf{t})) \leq kc - \epsilon,$$

where $\epsilon > 0$. This would contradict the following:

Proposition 3.10. $c_k = kc$.

The construction of such a $\overline{G}$ occupies Section 4. We close this section with the proof of Proposition 3.10.

Lemma 3.11. *Let* g_i *satisfy* (g_1)-(g_3), $1 \leq i \leq k$*. Then there exists a* $\overline{\mathbf{t}} \in [0,1]^k$ *such that*

$$I(g_i(\overline{\mathbf{t}})) \geq c,$$

for all $1 \leq i \leq k$.

Proof. See [3, Proposition 3.4].

Lemma 3.12. *For any* $u \in W^{1,2}(\mathbf{R}^N)$ *and* $\epsilon, \epsilon^* > 0$*, there exists a* $R = R(\epsilon, \epsilon^*) > 0$ *and* $u^* \in W^{1,2}(\mathbf{R}^N)$ *such that (i)* $\|u - u^*\| \leq \epsilon^*$*, (ii)* $|I(u) - I(u^*)| \leq \epsilon$*, and (iii)* $\operatorname{supp} u^* \subset B_{R+1}$.

Proof. For any $R > 0$, let $\chi_R \in C^\infty(\mathbf{R}^+, \mathbf{R})$ satisfy $|\chi_R'(s)| \leq 2$ and

$$\chi_R(s) = \begin{cases} 1 & \text{if } s \leq R \\ 0 & \text{if } s \geq R + 1. \end{cases}$$

For any $u \in W^{1,2}(\mathbf{R}^N)$ set

$$u^*(x) = \chi_R(|x|)u(x).$$

We claim that for R sufficiently large, u^* defined above satisfies properties (i)-(iii). Property (iii) is obvious. Set

$$\gamma(R) := \|u\|^2_{B_R^c}.$$

Note that

$$\begin{aligned} \|u - u^*\|^2 \leq & \left| \int_{|x|>R} |\nabla u|^2 + b(x)u^2\, dx \right| \\ & + \left| \int_{R<|x|<R+1} |\nabla \chi_R u|^2 + b(x)(\chi_R u)^2\, dx \right|. \end{aligned}$$

A calculation yields $|\nabla \chi_R u|^2 \leq 2(\chi_R' u)^2 + 2\chi_R^2 |\nabla u|$. Thus,

$$\begin{aligned} \|u - u^*\|^2 &\leq \gamma(R) + 8 \left| \int_{R<|x|<R+1} |\nabla u|^2 + (b(x) + 1)u^2\, dx \right| \\ &\leq \gamma(R) + 8\gamma(R) + \frac{8}{b_0}\gamma(R) \leq \left(\frac{8 + 9b_0}{b_0} \right) \gamma(R). \end{aligned}$$

Choosing R large enough such that $\gamma(R) < \frac{\epsilon^* b_0}{8+9b_0}$, property (i) holds. Finally, by (2.2),

$$\begin{aligned} |I(u) - I(u^*)| &\leq \left(\frac{8 + 9b_0}{b_0} \right) \gamma(R) + \left| \int_{|x|>R} F(u)\, dx \right| \\ &\quad + \left| \int_{R<|x|<R+1} F(\chi_R u)\, dx \right| \\ &\leq (\epsilon^* + 2\frac{C_f}{b_0})\gamma(R). \end{aligned}$$

Once again, choosing R large enough completes the proof. □

Proof of Proposition 3.10. Lemma 3.11 yields

$$\sup_{\mathbf{t}\in[0,1]^k} I(G(\mathbf{t})) = \sup_{\mathbf{t}\in[0,1]^k} \sum_{i=1}^{k} I(g_i(\mathbf{t})) \geq kc,$$

for all $G \in \Gamma_k$. Hence $c_k \geq kc$. Let $\epsilon > 0$ and choose a $g \in \Gamma$ so that

$$\sup_{t\in[0,1]} I(g(t)) \leq c + \frac{\epsilon}{2k}.$$

Since $[0,1]$ is compact, we may apply Lemma 3.12 to obtain a $\hat{R} > 0$ and $\hat{g} \in \Gamma$ so that supp $\hat{g}(t) \subset B_{\hat{R}+1}$ and

$$\sup_{t\in[0,1]} I(\hat{g}(t)) \leq c + \frac{\epsilon}{k}.$$

Choose $m_1, \dots, m_k \in \mathbf{Z}^N$ so that $\tau_{m_i} B_{\hat{R}+1} \bigcap \tau_{m_j} B_{\hat{R}+1} = \emptyset$ for all $i \neq j$. Then

$$G(\mathbf{t}) = \sum_{i=1}^{k} \tau_{m_i} \hat{g}(t_i) \in \Gamma_k,$$

and

$$\sup_{t\in[0,1]} I(G(\mathbf{t})) \leq \sum_{i=1}^{k} I(\hat{g}(t_i)) \leq kc + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, it follows that $c_k \leq kc$. □

4. Construction of $\overline{G}$

Assuming $|\mathcal{L}| = \infty$, we construct a $\overline{G} \in \Gamma_k$ with $\sup_{\mathbf{t}\in[0,1]^k} I(\overline{G}(\mathbf{t})) < kc$. Since this contradicts Proposition 3.10, we conclude that $|\mathcal{L}| < \infty$ and by Proposition 3.9 the proof of Theorem 1.3 would be complete.

Let

$$\overline{\epsilon} < \min\left(\frac{\alpha_1}{2}, \frac{\delta_4 r}{40}\right). \tag{4.1}$$

Step 1: The construction of G_1

For $r < \overline{r}_1$ and $\epsilon < \overline{\epsilon}$, by Proposition 3.1 there exists a $g_1 \in \Gamma$ such that

$$I(g_1(t)) \leq c + \frac{2\epsilon}{6k},$$

and

$$I(g_1(t)) \geq c - 2\epsilon \Rightarrow g_1(t) \in N_{\frac{r}{16k}}(\mathcal{C}).$$

Since $[0,1]$ is compact, we may apply Lemma 3.12 to obtain a $g_0 \in \Gamma$ and $R_0 > 0$ such that $\|g_0(t) - g_1(t)\| \leq \frac{r}{16k}$, $|I(g_0(t)) - I(g_1(t))| \leq \frac{2\epsilon}{3k}$ and supp $g_1(t) \subset B_{R_0+1}$ for all $t \in [0,1]$. Thus,

$$I(g_0(t)) \leq |I(g_0(t)) - I(g_1(t))| + |I(g_1(t))| \leq c + \frac{\epsilon}{k}, \tag{4.2}$$

and

$$I(g_0(t)) \geq c - \frac{(k+1)\epsilon}{3k} \Rightarrow \frac{2\epsilon}{3k} + I(g_1(t)) \geq c - \frac{(k+1)\epsilon}{3k}$$
$$\Rightarrow I(g_1(t)) > c - 2\epsilon \Rightarrow g_1(t) \in N_{\frac{r}{16k}}(\mathcal{C})$$
$$\Rightarrow g_0(t) \in N_{\frac{r}{8k}}(\mathcal{C}).$$

For $l \in \mathcal{L}$, set

$$G_1(\mathbf{t}) := \sum_{i-1}^{k} \tau_{lj_i} g_0(t_i).$$

For notational convenience we set $\frac{R}{2} = R_0 + 1$, and

$$\beta := \inf_{i \neq m} |\tau_{lj_i} B_{R+1} - \tau_{lj_m} B_{R+1}| . \tag{4.3}$$

We require that $\beta > \hat{\beta}$, where $\hat{\beta}$ is free for the moment. We note that, since it is assumed that $\mathcal{L}$ is infinite, $\hat{\beta}$ can be made arbitrarily large.

If $\hat{\beta} > 0$, then $G_1 \in \Gamma_k$ and

$$\operatorname{supp} G_1(\mathbf{t}) \subset \bigcup_{i=1}^{k} \tau_{lj_i} B_{\frac{R}{2}}.$$

Thus $I(G_1(\mathbf{t})) \leq kc + \epsilon$ and for any $1 \leq m \leq k$,

$$I(G_1(\mathbf{t})) \geq kc - \epsilon \Rightarrow I(g_0(t_m)) + \sum_{i \neq m}^{k} I(g_0(t_i)) \geq kc - \epsilon$$
$$\Rightarrow I(g_0(t_m)) + (k-1)c + \frac{(k-1)\epsilon}{k} \geq kc - \epsilon$$
$$\Rightarrow I(g_0(t_m)) > c - 2\epsilon \Rightarrow g_0(t_m) \in N_{\frac{r}{8k}}(\mathcal{C}),$$

which implies $G_1(\mathbf{t}) \in N_{\frac{r}{8}}(\mathcal{W})$. We complete Step 1 by summarizing the properties of $G_1 \in \Gamma_k$:

$(G_1)_1$ $I(G_1(\mathbf{t})) \leq kc + \epsilon$,
$(G_1)_2$ $I(G_1(\mathbf{t})) \geq kc - \epsilon \Rightarrow G_1(\mathbf{t}) \in N_{\frac{r}{8}}(\mathcal{W})$ and
$(G_1)_3$ $\operatorname{supp} G_1(\mathbf{t}) \subset \bigcup_{i=1}^{k} \tau_{lj_i} B_{\frac{R}{2}}$.

Step 2: Construction of G_2 via a deformation of G_1

The idea is to construct a deformation of G_1 from $I^{kc+\epsilon}$ to $I^{kc-\epsilon}$ using the gradient flow. This amounts to showing that there exists a $\eta \in C([0,1], W^{1,2}(\mathbf{R}^N))$ and $\sigma \in C(W^{1,2}(\mathbf{R}^N), [0,1])$ such that

$$\eta(0, G_1(\mathbf{t})) = G_1(\mathbf{t}) \text{ and } G_2(\mathbf{t}) := \eta(\sigma(G_1(\mathbf{t})), G_1(\mathbf{t})) \subset I^{kc-\epsilon}.$$

This follows exactly as in [4, Section 4, Step 2]. We summarize the basic properties of G_2:

$(G_2)_1$ $I(G_2(\mathbf{t})) \leq kc - \epsilon$,
$(G_2)_2$ $\|G_2(\mathbf{t}) - G_1(\mathbf{t})\| \leq \frac{5}{8}r$ and
$(G_2)_3$ $G_2(\mathbf{t}) = G_1(\mathbf{t})$ for $\mathbf{t} = \mathbf{0}_i, \mathbf{1}_i$.

At this point we note that, if also $G_2 \in \Gamma_k$, then by $(G_2)_1$ our proof would be complete. However, due to the way in which G_2 was constructed, property (g_3) may no longer be satisfied. The rest of this section concerns itself with modifying or "cutting" (G_2) into disjoint pieces without compromising the crucial energy estimate $(G_2)_1$.

Step 3: Construction of G_3 via a smooth approximation of G_2

Let $\rho \in C_0^\infty(\mathbf{R}^N)$ be a properly scaled mollifier, i.e. $\rho \geq 0$, $\int_{\mathbf{R}^N} \rho\, dx = 1$ and $\operatorname{supp} \rho \subset B_1$. Then we set

$$G^*(\mathbf{t})(x) = J_{\epsilon_*} G_2(\mathbf{t})(x) := \frac{1}{\epsilon_*^N} \int_{\mathbf{R}^N} \rho\left(\frac{x-y}{\epsilon_*}\right) G_2(\mathbf{t})(y)\, dy.$$

It is well known (see for instance [7]) that $G^* \in C([0,1]^k, W^{1,2}(\mathbf{R}^N))$, $G^*(\mathbf{t}) \in C^\infty(\mathbf{R}^N)$ and for any $\tilde{\epsilon} > 0$ there exists a $\epsilon_* > 0$ such that

$$\|G^*(\mathbf{t}) - G_2(\mathbf{t})\| \leq \tilde{\epsilon}.$$

By $(G_2)_3$ and $(G_1)_3$, we note that for $\mathbf{t} = \mathbf{0}_i, \mathbf{1}_i$,

$$\operatorname{supp} G^*(\mathbf{t}) = \operatorname{supp} J_{\epsilon_*} G_1(\mathbf{t}) \subset \bigcup_{i=1}^k \tau_{l j_i} B_{\frac{R}{2}+\epsilon_*}.$$

Thus choosing ϵ_* small enough, we may assume

$$\operatorname{supp} G^*(\mathbf{t}) \subset \bigcup_{i=1}^k \tau_{l j_i} B_R$$

for $\mathbf{t} = \mathbf{0}_i, \mathbf{1}_i$. Also note that, since $g_0(0) = 0$,

$$\operatorname{supp} G^*(\mathbf{0}_i) \subset \bigcup_{n \neq i}^k \tau_{l j_n} B_R.$$

Setting $\tilde{\epsilon} = \min\left(\frac{\epsilon}{2}, -\frac{1}{2} I(g_0(1))\right)$, we also choose $\epsilon_* << 1$ such that

$$|I(G^*(\mathbf{t})) - I(G_2(\mathbf{t}))| \leq \tilde{\epsilon} \quad \text{and} \quad \|G^*(\mathbf{t}) - G_2(\mathbf{t})\| \leq \tfrac{r}{4}.$$

Then, by $(G_2)_1$, $(G_2)_2$ and $(G_2)_3$,

$$I(G^*(\mathbf{t})) \leq kc - \frac{\epsilon}{4},\ \|G^*(\mathbf{t}) - G_1(\mathbf{t})\| \leq \frac{7}{8} r,\ I\left(G^*(\mathbf{1}_i)|_{\tau_{l j_i} B_R}\right) < 0.$$

Using the techniques of Lemma 3.12, there exits a $\hat{R} > 0$ and $G_3(\mathbf{t}) \in C_0^\infty(\mathbf{R}^N)$ such that

$$\|G_3(\mathbf{t}) - G^*(\mathbf{t})\| \leq \frac{r}{8},\ |I(G_3(\mathbf{t})) - I(G^*(\mathbf{t}))| \leq \frac{\epsilon}{4},\ \operatorname{supp} G_3(\mathbf{t}) \subset B_{\hat{R}+1},$$

for all $\mathbf{t} \in [0,1]^k$. We also choose $\hat{R}$ large enough such that $\cup_{1=1}^{k} \tau_{lj_i} B_R \subset B_{\hat{R}+1}$ and

$$\left|\partial B_{\hat{R}+1} - \tau_{lj_i} B_R\right| > \beta + 2, \text{ for all } i = 1, \dots, k. \tag{4.4}$$

We complete Step 3 by summarizing the properties of G_3:

$(G_3)_1$ $G_3(\mathbf{t}) \in C_0^\infty(\mathbf{R}^N)$ and $\operatorname{supp} G_3(\mathbf{t}) \subset B_{\hat{R}+1}$,
$(G_3)_2$ $I(G_3(\mathbf{t})) \le kc - \frac{\epsilon}{4}$,
$(G_3)_3$ $\|G_3(\mathbf{t}) - G_1(\mathbf{t})\| \le r$,
$(G_3)_4$ $\operatorname{supp} G_3(\mathbf{t}) \in \bigcup_{i=1}^{k} \tau_{lj_i} B_R$ for $\mathbf{t} = \mathbf{0}_i, \mathbf{1}_i$, $\operatorname{supp} G_3(\mathbf{0}_i) \subset \bigcup_{n \ne i}^{k} \tau_{lj_n} B_R$ and $I\left(G_3(\mathbf{1}_i)|_{\tau_{lj_i} B_R}\right) < 0$.

Step 4: Modifying G_3

Set

$$S := \{x \in B_{\hat{R}+1} : x \notin \bigcup_{i=1}^{k} \tau_{lj_i} B_R\}$$

and

$$\hat{H}(\mathbf{t}) := \{v \in W^{1,2}(S) : \|v\|_S < 4r \text{ and } u = G_3(\mathbf{t}) \text{ on } \partial S\}.$$

By $(G_3)_3$ and $(G_1)_3$,

$$\|G_3(\mathbf{t})\|_S = \|G_3(\mathbf{t}) - G_1(\mathbf{t})\|_S \le r,$$

which shows that $G_3(\mathbf{t}) \in \hat{H}(\mathbf{t})$ and $\hat{H}(\mathbf{t}) \ne \emptyset$. Define

$$I_S(v) := \frac{1}{2}\|v\|_S^2 - \int_S F(v)\, dx,$$

and

$$\hat{m}(\mathbf{t}) := \inf_{v \in \hat{H}(\mathbf{t})} I_S(v).$$

Step 4.1. There exists a unique $\hat{v} = \hat{v}(\mathbf{t}) \in \hat{H}(\mathbf{t})$, which depends continuously on $\mathbf{t}$, such that $I_S(\hat{v}) = \hat{m}(\mathbf{t})$.

We first show that if $I_S(\hat{v}) = \hat{m}(\mathbf{t})$, then $\hat{v}$ lies in the interior of $\hat{H}(\mathbf{t})$. By (2.3), there exist a $C_8 > 0$ such that

$$F(v) \le \frac{b_0}{8}|v|^2 + C_8|v|^{2^*}.$$

Thus, by the Sobolev imbedding, there exists a $K_1 > 0$ such that

$$\int_S F(v)\, dx \le \frac{1}{8}\|v\|_S^2 + C_8 K_1^{2^*} \|v\|_S^{2^*}.$$

Let $\overline{r}_2 > 0$ satisfy:

$$C_8 K_1^{2^*} (4\overline{r}_2)^{2^*-2} = \frac{1}{8},$$

then if $r < \overline{r}_2$ and $v \in \hat{H}(\mathbf{t})$,

$$\int_S F(v)\,dx \le \frac{1}{4}\|v\|_S^2,$$

which implies that

$$I_S(v) \ge \frac{1}{4}\|v\|_S^2.$$

Next, note that

$$\hat{m}(\mathbf{t}) \le I_S(G_3(\mathbf{t})) \le \frac{3}{4}\|G_3(\mathbf{t})\|_S^2 \le \frac{3}{4}r^2.$$

Thus, if $\|v\|_S \ge 2r$, then

$$I_S(v) \ge r^2 > I_S(G_3(\mathbf{t})) \ge \hat{m}(\mathbf{t}),$$

and we conclude that $\|\hat{v}\|_S < 2r$.

Let $(v_n) \subset \hat{H}(\mathbf{t})$ be a minimizing sequence, i.e. $I_S(v_n) \to \hat{m}(\mathbf{t})$. Since $\|v_n\|_S < 4r$, up to a subsequence, $v_n \rightharpoonup \hat{v}$ in $\hat{H}(\mathbf{t})$, $v_n \to \hat{v}$ in $L^2(S)$. Since $F(v) \le \frac{1}{2}C_f|v|^2$, $F(v_n) \to F(\hat{v})$ in $L^1(S)$ (see [21, Lemma A.2]) and

$$\hat{m}(\mathbf{t}) = \lim_{n\to\infty} I_S(v_n) \ge I_S(\hat{v}) \ge \hat{m}(\mathbf{t}),$$

which shows that $\hat{m}(\mathbf{t})$ is achieved.

Since $f \in C^1$ and $\partial S \in C^\infty$, standard regularity arguments show that $\hat{v} \in C^{2,\gamma}(S)$, $0 < \gamma < 1$, and is a classical solution of

$$\left.\begin{aligned} -\Delta v + b(x)v &= f(v) \quad \text{in } S \\ v &= G_3(\mathbf{t}) \text{ on } \partial S. \end{aligned}\right\} \tag{4.5}$$

Finally we show that for r sufficiently small, $\hat{v}$ is unique. This would immediately imply that $\hat{v}(\mathbf{t})$ depends continuously on $\mathbf{t}$. Seeking a contradiction, suppose $\hat{w} \ne \hat{v}$ solves (4.5). Then

$$\|\hat{v}-\hat{w}\|_S^2 = \int_S (f(\hat{v}) - f(\hat{w}))\,(\hat{v}-\hat{w})\,dx = \int_S (\hat{v}-\hat{w})^2 \int_0^1 f'(\hat{v}-t(\hat{v}-\hat{w}))\,dt\,dx$$

By (f_1)-(f_2), $f'(s) \to 0$ as $|s| \to 0$ and $f'(s) \to a$ as $|s| \to \infty$. Thus, there exists a $C_8' > 0$ such that

$$f'(s) \le \frac{b_0}{8} + C_8'|s|^{\frac{4}{N-2}}.$$

Using Hölders Inequality with $p = \frac{N}{N-2}$ and $p' = \frac{N}{2}$ yields

$$\begin{aligned} \|\hat{v} - \hat{w}\|_S^2 &\le \frac{b_0}{8}\int_S (\hat{v}-\hat{w})^2\,dx + C_8'\|\hat{v}-\hat{w}\|^2_{L^{2^*}(S)} \left(\|\hat{v}\|_{L^{2^*}(S)} + \|\hat{w}\|_{L^{2^*}(S)}\right)^{\frac{4}{N-2}} \\ &\le \frac{1}{8}\|\hat{v} - \hat{w}\|_S^2 + C_8'K_1^2\|\hat{v} - \hat{w}\|_S^2\,(K_1\|\hat{v}\|_S + K_1\|\hat{w}\|_S)^{\frac{4}{N-2}} \\ &\le \frac{1}{8}\|\hat{v} - \hat{w}\|_S^2 + C_8'K_1^2\|\hat{v} - \hat{w}\|_S^2(K_1(2r))^{\frac{4}{N-2}}. \end{aligned}$$

Let $\overline{r}_3 > 0$ satisfy

$$C_8' K_1^{2^*} (2\overline{r}_3)^{\frac{4}{N-2}} = \frac{3}{4}.$$

Then, with $r \leq \overline{r}_3$,

$$\|\hat{v} - \hat{w}\|_S^2 \leq \frac{7}{8} \|\hat{v} - \hat{w}\|_S^2,$$

which yields the desired contradiction, and completes Step 4.1.

Step 4.2. Set $D_\rho := \{x \in S : |x - \partial S| \geq \rho\}$. *Then there exists a* $\overline{K} > 0$, *depending only on* ρ *and* N, *such that*

$$\|\hat{v}\|_{L^\infty(D_\rho)} \leq \overline{K}\, \|\hat{v}\|_S.$$

Fix any arbitrary $x \in D_\rho$. Define

$$B_i := B_{\frac{i\rho}{2(j+1)}}(x), \;\; i = 1, \dots, j+1$$

where $1 \leq j < \infty$ is free for the moment. (The integer j would later be determined in terms of N.) Note that $B_i \subset S$. Since $\hat{v} \in C^{2,\gamma}(S)$ solves (4.5), the elliptic L^p_{loc} estimates (see [7, Theorem 9.11]) yields

$$\|\hat{v}\|_{W^{2,p}(B_i)} \leq K_1 \left(\|\hat{v}\|_{L^p(B_n)} + \|f(\hat{v})\|_{L^p(B_n)} \right), \tag{4.6}$$

for any $i < n \leq j+1$ and $1 < p < \infty$. Since i runs over a finite range, we can assume that the constant $K_1 > 0$ depends only on N and p.

By the Sobolev-Rellich-Kondrachov Imbedding (see [7, Theorem 7.11]) there exists a $K_2 > 0$ such that if $p > \frac{N}{2}$ then

$$\|\hat{v}\|_{L^\infty(B_i)} \leq K_2 \|\hat{v}\|_{W^{2,p}(B_i)}. \tag{4.7}$$

We will also make use of the Gagliardo-Nirenberg inequality in the following form:

$$\|\hat{v}\|_{L^t(B_i)} \leq K_3 \|\hat{v}\|^{\gamma}_{W^{2,d}(B_i)} \|\hat{v}\|^{1-\gamma}_{L^q(B_i)}, \tag{4.8}$$

where $1 \leq d, q < \infty$, $K_3 > 0$ and

$$\frac{1}{t} = \gamma \left(\frac{1}{d} - \frac{2}{N} \right) + (1-\gamma)\frac{1}{q}.$$

As before, we may assume K_2, K_3 to be independent of B_i. Finally, note that, for any $1 < p < \infty$,

$$\|f(\hat{v})\|_{L^p(B_i)} \leq C_f \|\hat{v}\|_{L^p(B_i)}. \tag{4.9}$$

Set $p_1 = 2^*$. For the remainder of this step the positive constants k_m, $\overline{k}_m$ and $\overline{K}_m$ are chosen independent of B_i.

Case 1. $N < 6$

Then $p_1 > \frac{N}{2}$. Set $j = 1$. By (4.9), (4.6) and the Sobolev Imbedding

$$\|\hat{v}\|_{W^{2,p_1}(B_1)} \leq K_1(1 + C_f)\|\hat{v}\|_{L^{p_1}(B_2)} \leq k_1 \|\hat{v}\|_{W^{1,2}(B_2)} \leq k_2 \|\hat{v}\|_S. \tag{4.10}$$

By (4.7), this implies

$$\|\hat{v}\|_{L^\infty(B_1)} \leq \overline{K}_1 \|\hat{v}\|_S.$$

Case 2. $N = 6$

Then $p_1 = \frac{N}{2}$. Set $j = 1$. Setting $t = \infty$, $d = p_1$ and $\gamma = 1$, (4.8) and (4.10) yields

$$\|\hat{v}\|_{L^\infty(B_1)} \leq \overline{K}_2 \|\hat{v}\|_S.$$

Case 3. $N > 6$

Then $p_1 < \frac{N}{2}$. Set $j = [\![\frac{N-6}{4}]\!] + 1$. Let $\frac{1}{t_1} = \frac{1}{p_1} - \frac{2}{N}$. By (4.8) with $\gamma = 1$ and (4.10)

$$\|\hat{v}\|_{L^{t_1}(B_{j+1})} \leq K_3 \|\hat{v}\|_{W^{2,p_1}(B_{j+1})} \leq k_1 \|\hat{v}\|_S.$$

Then, by (4.9) and (4.6)

$$\|\hat{v}\|_{W^{2,t_1}(B_j)} \leq \overline{k}_1 \|\hat{v}\|_S.$$

Continuing this process with,

$$\frac{1}{t_i} = \frac{1}{t_{i-1}} - \frac{2}{N}$$

yields

$$\|\hat{v}\|_{W^{2,t_i}(B_{j-i+1})} \leq \overline{k}_i \|\hat{v}\|_S.$$

Note that

$$\frac{1}{t_i} = \frac{1}{2^*} - \frac{2i}{N} = \frac{N - 2(1+2i)}{2N}$$

and therefore

$$t_j > \frac{2N}{N - 2(1 + 2(\frac{N-6}{4}))} = \frac{N}{2}.$$

We conclude by (4.7) that

$$\|\hat{v}\|_{L^\infty(B_1)} \leq K_2 \|\hat{v}\|_{W^{2,t_j}(B_1)} \leq \overline{K_3} \|v\|_S.$$

Since the above holds for all $x \in D_\rho$ and the constants $\overline{K}_i$ are independent of B_1, we have completed Step 4.2.

Step 4.3. $\hat{v}$ is exponentially small in certain annular regions contained in S.

Define the following sets:

$$\hat{M} := \{x \in \mathbf{R}^N : R + 1 \leq |x| \leq R + \beta + 1\} \text{ and } \hat{S}_i = \tau_{l j_i} \hat{M}.$$

By (4.3) and (4.4), $S_m \cap \tau_{l j_i} B_R^o = \emptyset$ for all $m \neq i$, and $\hat{S}_i \subset \overline{D_1}$ for all $i = 1, \ldots, k$. Set $\hat{S} := \cup_{i=1}^k \hat{S}_i$. Using elliptic estimates similar to [4, Section 5, Step 4] it follows that, for

$$\hat{A}_i := \tau_{l j_i} \{x \in \mathbf{R}^N : R + \frac{\beta}{2} + \frac{1}{2} \leq |x| \leq R + \frac{\beta}{2} + 1\} \subset \hat{S}_i,$$

and $\omega := \min(1, b_0)$, it holds,

$$\hat{v}^2(x) \le 2\overline{s}^2 e^{-\omega\frac{\beta}{2}} \cosh\frac{\omega}{2} \quad \text{for } x \in \hat{A}_i,\ i = 1, \dots, k. \tag{4.11}$$

This completes Step 4.3.

To complete Step 4, define

$$G_4(\mathbf{t})(x) := \begin{cases} G_3(\mathbf{t})(x) & \text{if } x \notin S \\ \hat{v}(\mathbf{t})(x) & \text{if } x \in S. \end{cases}$$

We summarize the properties of G_4:

$(G_4)_1$ By $(G_3)_2$ and the definition of $\hat{v}(\mathbf{t})$,

$$I(G_4(\mathbf{t})) \le I(G_3(\mathbf{t})) \le kc - \frac{\epsilon}{4}.$$

$(G_4)_2$ $G_4(\mathbf{t}) \in C^2(S)$, and there exists a constant $A_1 > 0$ such that

$$G_4(\mathbf{t})(x) = \hat{v}(\mathbf{t})(x) \le A_1 \overline{s} e^{-\omega\frac{\beta}{4}} \quad \text{for all } x \in \hat{A}_i,\ i = 1, \dots, k.$$

Step 5: The construction of $\overline{G}$

Set

$$\mathcal{A}_i := \tau_{l j_i}\{x \in \mathbf{R}^N : R + \frac{\beta}{2} + \frac{5}{8} \le |x| \le R + \frac{\beta}{2} + \frac{7}{8}\} \subset \hat{A}_i.$$

Let $\zeta_\beta \in C^\infty(\mathbf{R}, \mathbf{R})$ such that $|\zeta_\beta'| \le c_0$, for some positive constant c_0, and

$$\zeta_\beta(s) = \begin{cases} 1 & \text{if } s \le R + \frac{\beta}{2} + \frac{5}{8} \\ 0 & \text{if } s \ge R + \frac{\beta}{2} + \frac{7}{8}. \end{cases}$$

For $i = 1, \dots, k$, define

$$\overline{g}_i(\mathbf{t})(x) := \begin{cases} G_4(\mathbf{t})(x) & \text{if } x \in \tau_{l j_i} B_{R + \frac{\beta}{2} + \frac{5}{8}} \\ \zeta_\beta(|x - \overline{\tau}_{l j_i}|)\hat{v}(\mathbf{t})(x) & \text{if } x \in \mathcal{A}_i \\ 0 & \text{otherwise.} \end{cases}$$

Step 5.1. $\overline{g}_i(\mathbf{t})$ *satisfies* (g_1)*-*(g_3)*.*

By $(G_4)_1$, we see that $\overline{g}_i$ satisfies (g_1). Since $\hat{\beta} > 0$, $\overline{g}_i$ satisfies (g_3). If $\mathbf{t} = \mathbf{0}_i$ or $\mathbf{1}_i$, by $(G_3)_4$,

$$G_3(\mathbf{t})(x) = 0 \quad \text{for } x \in S.$$

This implies that $\hat{v}(\mathbf{t})(x) = 0$ for $x \in \partial S$, and by uniqueness $\hat{v}(\mathbf{t})(x) = 0$ for $x \in S$. Thus, for $\mathbf{t} = \mathbf{0}_i$ or $\mathbf{1}_i$,

$$\overline{g}_i(\mathbf{t})(x) = \begin{cases} G_3(\mathbf{t})(x) & \text{if } x \in \tau_{l j_i} B_R \\ 0 & \text{otherwise.} \end{cases}$$

Finally, by $(G_3)_4$, this implies that $\overline{g}_i(\mathbf{0}_i) = 0$ and $I(\overline{g}_i(\mathbf{1}_i)) < 0$, which shows that $\overline{g}_i(\mathbf{t})$ satisfies (g_2) and completes Step 5.1.

Setting

$$\overline{G}(\mathbf{t}) := \sum_{i=1}^{k} \overline{g}_i(\mathbf{t}),$$

we have just shown that $\overline{G} \in \Gamma_k$.

Step 5.2. $I(\overline{G}(\mathbf{t})) \leq kc - \frac{\epsilon}{8}$.

We first provide some additional uniform estimates for $\hat{v}$ on $\mathcal{A}_i$.

Step 5.2.1. There exists a constant $A_2 > 0$ *such that*

$$\|\hat{v}(\mathbf{t})\|_{C^1(\mathcal{A}_i)} \leq A_2 e^{\frac{-\omega\beta}{4}},$$

for all $i = 1, \ldots, k$.

Fix any arbitrary $x \in \mathcal{A}_i$. Set $\mathcal{O} := B_{\frac{1}{32}}(x)$ and $\hat{\mathcal{O}} := B_{\frac{1}{16}}(x)$. Note that $\mathcal{O} \subset\subset \hat{\mathcal{O}} \subset\subset \hat{A}_i$. By (4.6), for any $1 < p < \infty$,

$$\|\hat{v}\|_{W^{2,p}(\mathcal{O})} \leq K_1 \left(\|f(\hat{v})\|_{L^p(\hat{\mathcal{O}})} + \|\hat{v}\|_{L^p(\hat{\mathcal{O}})} \right).$$

Choose $\hat{\beta}$ large enough such that $A_1 e^{-\omega\frac{\beta}{4}} \leq 1$. Then, by $(G_4)_2$, $|\hat{v}(x)| \leq \overline{s}$ for all $x \in \hat{A}_i$ and we conclude that

$$\|f(\hat{v})\|_{L^p(\hat{\mathcal{O}})} \leq \frac{b_0}{2} \|\hat{v}\|_{L^p(\hat{\mathcal{O}})}.$$

Thus,

$$\begin{aligned}\|\hat{v}\|_{W^{2,p}(\mathcal{O})} &\leq K_1 \left(1 + \frac{b_0}{2}\right) \|\hat{v}\|_{L^p(\hat{\mathcal{O}})} \leq K_1 \left(1 + \frac{b_0}{2}\right) A_1 \overline{s} e^{-\omega\frac{\beta}{4}} (m(\hat{\mathcal{O}}))^{\frac{1}{p}} \\ &= K_2 e^{-\omega\frac{\beta}{4}},\end{aligned}$$

where K_2 depends only on N and p. Fix a $p > N$, then there exist a $K_3 > 0$ such that

$$\|\hat{v}\|_{C^1(\mathcal{O})} \leq K_3 \|\hat{v}\|_{W^{2,p}(\mathcal{O})} \leq K_4 e^{-\omega\frac{\beta}{4}}.$$

Since the above holds for all $x \in \mathcal{A}_i$, we have completed Step 5.2.1.

Set $\mathcal{A} := \cup_{i=1}^{k} \mathcal{A}_i$, and

$$\begin{aligned}I_1 &:= \left| \int_{\mathcal{A}} \frac{1}{2}(|\nabla \overline{G}(\mathbf{t})|^2 + b(x)\overline{G}(\mathbf{t})^2) - F(\overline{G}(\mathbf{t}))\, dx \right| \\ I_2 &:= \left| \int_{\mathcal{A}} \frac{1}{2}(|\nabla G_4(\mathbf{t})|^2 + b(x)G_4(\mathbf{t})^2) - F(G_4(\mathbf{t}))\, dx \right|.\end{aligned}$$

Since $\overline{G}(\mathbf{t})$ and $G_4(\mathbf{t})$ agrees everywhere except on $\mathcal{A}$, by $(G_4)_1$,

$$I(\overline{G}(\mathbf{t})) \leq kc - \frac{\epsilon}{4} + \left| I(\overline{G}(\mathbf{t})) - I(G_4(\mathbf{t})) \right| \leq kc - \frac{\epsilon}{4} + I_1 + I_2.$$

To complete Step 5.2, it suffices to show $I_1,\ I_2 \leq \frac{\epsilon}{16}$.

Step 5.2.2. $I_1 \leq \frac{\epsilon}{16}$.

Note that

$$I_1 = \left| \sum_{i=1}^{k} \left(\frac{1}{2} \|\overline{g}_i(\mathbf{t})\|_{\mathcal{A}_i}^2 - \int_{\mathcal{A}_i} F(\overline{g}_i(\mathbf{t}))\, dx \right) \right| \le \frac{1}{2} \left(1 + \frac{C_f}{b_0} \right) \sum_{i=1}^{k} \|\overline{g}_i(\mathbf{t})\|_{\mathcal{A}_i}^2 .$$

By the uniform estimates of Step 5.2.1:

$$\begin{aligned} \|\overline{g}_i(\mathbf{t})\|_{\mathcal{A}_i}^2 &= \int_{\mathcal{A}_i} |\nabla \zeta_\beta|x - \overline{\tau}_{lj_i}|\hat{v}(x)|^2 + b(x)(\zeta_\beta|x - \overline{\tau}_{lj_i}|\hat{v}(x))^2\, dx \\ &\le 2c_0 \int_{\mathcal{A}_i} |\nabla \hat{v}(x)|^2 + (\overline{b}+1)\hat{v}^2(x)\, dx \\ &\le 2c_0 \left(A_2 e^{-\omega \frac{\beta}{2}} + A_2(\overline{b}+1) e^{-\omega \frac{\beta}{2}} \right) m(\mathcal{A}_i). \end{aligned}$$

Choosing $\hat{\beta} >> 1$ such that

$$2c_0 A_2 \left(e^{-\omega \frac{\beta}{2}} + (\overline{b}+1) e^{-\omega \frac{\beta}{2}} \right) m(\mathcal{A}_i) \le \frac{\epsilon}{8k(1+\frac{C_f}{b_0})}, \tag{4.12}$$

for all $\beta \ge \hat{\beta}$, we conclude that $I_1 \le \frac{\epsilon}{16}$.

Step 5.2.3. $I_2 \le \frac{\epsilon}{16}$.

As in Step 5.2.2, we have that

$$I_2 \le \frac{1}{2} \left(1 + \frac{C_f}{b_0} \right) \sum_{i=1}^{k} \|\hat{v}(\mathbf{t})\|_{\mathcal{A}_i}^2 ,$$

and

$$\|\hat{v}(\mathbf{t})\|_{\mathcal{A}_i}^2 \le \left(A_2 e^{-\omega \frac{\beta}{2}} + A_2 \overline{b} e^{-\omega \frac{\beta}{2}} \right) m(\mathcal{A}_i).$$

If $\hat{\beta}$ satisfies (4.12), then $I_2 \le \frac{\epsilon}{16}$, and Step 5 is complete.

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