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On the multi-string solutions of the self-dual static Einstein-Maxwell-Higgs system

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Abstract. In this paper we prove existence and precise decay estimates at infinity of solutions to the Bogomol'nyi system of the static Einstein equations coupled with the Maxwell-Higgs fields with translational symmetry in one direction. The equations model cosmic strings(or superconducting strings) in equilibrium state. The Higgs fields of our solutions, in particular, tend to the symmetric vacuum at infinity. The construction of our solution is by the perturbation type of argument combined with the implicit function theorem.

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1 Introduction and the Main Theorem

Let us consider the $(3+1)$ dimensional Lorentzian manifold $(\mathcal{M}, g_{\mu\nu})$, where $g_{\mu\nu}$ is a metric with signature given by $(-,+,+,+)$. We denote $g^{\mu\nu}$ for the inverse matrix of $g_{\mu\nu}$. We raise and lower the tensor indices by $g^{\mu\nu}$ and $g_{\mu\nu}$. On this manifold let us introduce the Lagragian,

$$\mathcal{L} = \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + \frac{1}{2} g^{\mu\nu} (D_\mu \phi)(D_\nu \phi)^* + \frac{1}{8}(|\phi|^2 - \sigma^2)^2, \tag{1.1}$$

where ϕ is a cross section on a $U(1)$-line bundle, called Higgs field, $A = A_\mu dx^\mu$ is a (gauge) connection 1-form, called the Maxwell field, $F = dA = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is a (gauge) curvature 2-form, and $D = d - iA$ is a (gauge) covariant derivative. We denote ϕ^* as the complex conjugation of ϕ. $\sigma > 0$ is called the symmetry breaking parameter. Let $\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}(\partial_\nu g_{\alpha\mu} + \partial_\mu g_{\alpha\nu} - \partial_\alpha g_{\mu\nu})$ be the Christoffel symbol, representing the Levi-Civita connection on $(\mathcal{M}, g_{\mu\nu})$, and let

$$R^\mu_{\nu\rho\tau} = \partial_\tau \Gamma^\mu_{\nu\rho} - \partial_\rho \Gamma^\mu_{\nu\tau} + \Gamma^\mu_{\rho\alpha}\Gamma^\alpha_{\tau\nu} - \Gamma^\mu_{\tau\alpha}\Gamma^\alpha_{\tau\nu}$$

be the Riemann curvature tensor on the manifold. Let $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ and $R = R^\alpha_\alpha$ be the Ricci tensor and the scalar curvature of the manifold respectively. Let $G > 0$ be the gravitational constant. Then, the Einstein equations coupled with the Maxwell-Higgs fields are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{1.2}$$

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where the energy-momentum tensor $T_{\mu\nu}$ given by

$$T_{\mu\nu} = g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} + \frac{1}{2}[(D_\mu\phi)(D_\nu\phi)^* + (D_\nu\phi)(D_\mu\phi)^*] - g_{\mu\nu}\mathcal{L}, \tag{1.3}$$

coupled with the matter equations,

$$\frac{1}{\sqrt{|g|}}D_\mu(g^{\mu\nu}\sqrt{|g|}D_\nu\phi) = \frac{1}{2}(|\phi|^2 - \sigma^2)\phi, \tag{1.4}$$

and

$$\frac{1}{\sqrt{|g|}}\partial_\alpha(g^{\mu\nu}g^{\alpha\beta}\sqrt{|g|}F_{\nu\beta}) = \frac{i}{2}g^{\mu\nu}[\phi(D_\nu\phi)^* - \phi^*(D_\nu\phi)], \tag{1.5}$$

where we denoted $g = \det(g_{\mu\nu})$. We assume that our metric is static and translational invatiant along a spatial direction, say along the x_3 axis. More precisely, we assume our metric is of the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dx_3^2 + \gamma_{ij}dx^i dx^j,$$

where $\partial_t\gamma_{ij} = \partial_3\gamma_{ij} = 0$, and $\mathcal{M} = R^2 \times \mathcal{M}_2$. We also assume that our matter fields A_μ, ϕ depend on x_1, x_2, the coordinates of $\mathcal{M}_2$, and $A_\mu = (0, A_1, A_2, 0)$. We denote below $A = (A_1, A_2)$. In this case it is known ([13,18]) that the system (1.2)–(1.5) posess the self-dual equations,

$$K_\gamma = 8\pi G\mathcal{E}, \tag{1.6}$$

$$(D_j \pm i\varepsilon_j^k D_k)\phi = 0, \tag{1.7}$$

$$\varepsilon^{jk}F_{jk} \pm (|\phi|^2 - \sigma^2) = 0, \tag{1.8}$$

where K_γ is the Gaussian curvature of $(\mathcal{M}_2, \gamma_{ij})$, $\mathcal{E} = T_{00}$ is the energy density, ε_{jk} is the Levi-Civita skew-symmmetric tensor with the normalization $\varepsilon_{12} = \sqrt{\gamma}$, where $\gamma = \det(\gamma_{ij})$. The Bogomol'nyi system, (1.6)–(1.8) represents a model for cosmic strings (or superconducting strings) in equilibrium ([10,19]). We further assume that our reduced manifold, $(\mathcal{M}_2, \gamma_{ij})$ is conformally flat, namely there exists a function η such that

$$\gamma_{ij} = e^\eta\delta_{ij}. \tag{1.9}$$

Following [20], we make a scale transform, $x \mapsto \frac{x}{\sigma}, \phi \mapsto \sigma\phi, A_j \mapsto \sigma A_j$. Then, the energy and the Gaussian curvature transform as $\mathcal{E} \mapsto \sigma^4\mathcal{E}, K_\gamma \mapsto \sigma^2 K_\gamma$. Then, following standard Jaffe-Taubes' procedure [9], we represent

$$\phi = \exp\left(\frac{u}{2} + i\sum_{j=1}^m n_j Arg(z - z_j)\right),$$

where the zero set of ϕ, $\mathbb{Z}(\phi) = \{z_j\}_{j=1}^m \subset \mathbb{C} = \mathbb{R}^2$ is prescribed together with their multiplicities $\{n_j\}_{j=1}^m$. We can thus reduce further the system (1.6)–(1.9) into the semilinear elliptic system for (u, η)

$$\Delta u = e^\eta(e^u - 1) + 4\pi\sum_{j=1}^m n_j\delta(z - z_j), \tag{1.10}$$

$$\Delta(\eta + ae^u) = ae^\eta(e^u - 1), \tag{1.11}$$

where we set

$$a = 4\pi G\sigma^2. \tag{1.12}$$

The system (1.10)–(1.11) is our basic equations to solve in the following sections. We want to solve (1.10)–(1.11) under the finite energy condition

$$\int_{\mathbb{R}^2} \mathcal{E}e^\eta dx < \infty, \qquad \int_{\mathbb{R}^2} K_\gamma e^\eta dx < \infty. \tag{1.13}$$

Here we note that, in terms of u, η, $\mathcal{E}$, K_γ and F_{12} have the representations,

$$K_\gamma = -\frac{1}{2}e^{-\eta}\Delta\eta = a\mathcal{E}, \qquad F_{12} = -\frac{1}{2}e^\eta(e^u - 1).$$

A solution pair (u, η) satisfying (1.10)–(1.11) generates a static finite energy solution (ϕ, A, g) of (1.6)–(1.8), (and thus solutions of (1.2)–(1.5)) called a multi-string solution. In particular, we consider the two types of solutions of (1.10)–(1.13) distinguished by the boundary conditions for u at infinity:

$$u(x) \to 0 \qquad \text{as} \qquad |x| \to \infty, \tag{1.14}$$

and

$$u(x) \to -\infty \qquad \text{as} \qquad |x| \to \infty. \tag{1.15}$$

Physically, (1.14) implies that the Higgs field, $\phi(x)$ has the asymmetric vacuum ($|\phi(x)| = 1$) at infinity, while (1.15) implies that the Higgs field satisfies the symmetric vacuum ($|\phi(x)| = 0$) at infinity. Mathematical study of the system (1.10)–(1.11) is extensively done in [6,16,21]. We also mention that recently there are many mathematical studies on the similar type of equations arising from other vortex models (see [1–5,11–15,17] and, in particular [20] for a comprehensive survey of the subject.). In [6] it is found that the necessary condition for existence of solution of (1.10)–(1.13) is $0 < aN < 2$, and under the assumption $0 < aN < 1$, general (nonradial) multi-string solutions satisfying (1.14) are constructed in [21]. In this paper, we construct a family of solution to (1.10)–(1.13) satisfying the condition (1.15) in the full range $0 < aN < 2$. Our method of construction is a variation of the perturbation type of argument, which has been developed in a series of papers [1–3]. In order to formulate our main theorem we introduce some functions. Given $\varepsilon > 0$, and $\delta \in \mathbb{C} = \mathbb{R}^2$, we define

$$\rho^I_{\varepsilon,\delta}(z) := \frac{8^{\frac{1}{a}}\varepsilon^{2N+2}\prod_{j=1}^m |z - z_j|^{2n_j}}{a^{\frac{1}{a}}(1 + |\varepsilon z + \delta|^2)^{\frac{2}{a}}}, \tag{1.16}$$

and

$$\rho^{II}_{\varepsilon,\delta}(z) := \frac{8\varepsilon^2}{a(1 + |\varepsilon z + \delta|^2)^2}. \tag{1.17}$$

where $z = x_1 + ix_2$. We also introduce the associated functions

$$\rho_1(r) := \frac{8^{\frac{1}{a}} r^{2N}}{a^{\frac{1}{a}}(1+r^2)^{\frac{2}{a}}}, \tag{1.18}$$

and

$$\rho_2(r) := \frac{8}{a(1+r^2)^2}, \tag{1.19}$$

where $r = |z|$. Below we set $f(t) = (a+1)\rho_1(t)\rho_2(t)$. Then, the function $w_1(r)$ is defined by

$$w_1(r) := \varphi_0(r) \left\{ \int_0^r \frac{\phi_f(s) - \phi_f(1)}{(1-s)^2} ds + \frac{\phi_f(1)r}{1-r} \right\} \tag{1.20}$$

with

$$\phi_f(r) := \left(\frac{1+r^2}{1-r^2} \right)^2 \frac{(1-r)^2}{r} \int_0^r \varphi_0(t) t f(t) dt,$$

and

$$\varphi_0(r) := \frac{1-r^2}{1+r^2},$$

where $\phi_f(1)$ and $w_1(1)$ are defined as limits of $\phi_f(r)$ and $w_1(r)$ as $r \to 1$. We also define

$$w_2 := aw_1 - a\rho_1. \tag{1.21}$$

The following is our main theorem.

Theorem 1.1 *Let* $\{n_j\}_{j=1}^m \subset \mathbb{N}$ *and* $\{z_j\}_{j=1}^m \in \mathbb{R}^2$ *be given. We set* $N = \sum_{j=1}^m n_j$. *Suppose* $0 < aN < 2$. *Then, there exists a constant* $\varepsilon_1 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_1)$ *there exists a family of solutions to (1.6)–(1.8),* $(\phi_\varepsilon, A^\varepsilon, \gamma_{ij}^\varepsilon)$ *satisfying the finite energy condition (1.13). Moreover, the solutions we constructed have the following properties:*

(i) The Higgs fields ϕ_ε *has zeros at* $\{z_j\}_{j=1}^m$ *with multiplicities* $\{n_j\}_{j=1}^m$ *respectively.*

(ii) The functions $\phi_\varepsilon, \gamma_{ij}^\varepsilon$ *have the representations*

$$\phi_\varepsilon(z) = \exp\left(\frac{u_\varepsilon}{2} + i \sum_{j=1}^m n_j Arg(z - z_j) \right), \tag{1.22}$$

and

$$\gamma_{ij}^\varepsilon = e^{\eta_\varepsilon} \delta_{ij}, \qquad i, j = 1, 2 \tag{1.23}$$

with

$$u_\varepsilon(z) = \ln \rho^I_{\varepsilon, \delta_\varepsilon^*}(z) + \varepsilon^2 w_1(\varepsilon|z|) + \varepsilon^2 v_\varepsilon^*(\varepsilon z), \tag{1.24}$$

and

$$\eta_\varepsilon(z) = \ln \rho^{II}_{\varepsilon,\delta^*_\varepsilon}(z) + \varepsilon^2 w_2(\varepsilon|z|) + \varepsilon^2 \xi^*_\varepsilon(\varepsilon z), \tag{1.25}$$

where $\delta^*_\varepsilon \to 0$ *as* $\varepsilon \to 0$*, and*

$$w_1(\varepsilon|z|) = -\kappa_1 \ln|z| + O(1), \tag{1.26}$$
$$w_2(\varepsilon|z|) = -\kappa_2 \ln|z| + O(1) \tag{1.27}$$

as $|z| \to \infty$ *with*

$$\kappa_1 := \frac{(a+1)8^{1+\frac{1}{a}}(1-aN)N!}{a^{2+\frac{1}{a}} \prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)}, \tag{1.28}$$

and

$$\kappa_2 := \frac{(a+1)8^{1+\frac{1}{a}}(1-aN)N!}{a^{1+\frac{1}{a}} \prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)}. \tag{1.29}$$

The functions v^*_ε *and* ξ^*_ε *in (1.24), (1.25) satisfy*

$$\sup_{z\in\mathbb{R}^2} \frac{|v^*_\varepsilon(\varepsilon z)| + |\xi^*_\varepsilon(\varepsilon z)|}{\ln(|z|+1)} \le o(1) \qquad \text{as } \varepsilon \to 0. \tag{1.30}$$

(iii) There exist constants $C_1 = C_1(G,\sigma), C_2 = C_2(G,\sigma)$ *and functions* $\beta_1(\varepsilon)$*,* $\beta_2(\varepsilon)$ *defined on a small neighborhood of* $\varepsilon = 0$ *such that*

$$\ln|\phi_\varepsilon(z)|^2 = u_\varepsilon(z) = \left[2N - \frac{4}{a} - \beta_1(\varepsilon)\right]\ln|z| + o(\ln|z|) \quad \text{as } |z| \to \infty. \tag{1.31}$$

$$|D_1\phi_\varepsilon|^2 + |D_2\phi_\varepsilon|^2 \le \frac{C_1}{|z|^{\frac{4}{a}-2N+\beta_1(\varepsilon)}} + o\left(\frac{1}{|z|^{\frac{4}{a}-2N+\beta_1(\varepsilon)}}\right) \quad \text{as } |z| \to \infty, \tag{1.32}$$

$$\eta_\varepsilon(z) = [-4 - \beta_2(\varepsilon)]\ln|z| + o(\ln|z|) \qquad \text{as } |z| \to \infty. \tag{1.33}$$

The Gaussian curvature has the decaying property,

$$\left|K^\varepsilon_\gamma(x) - \frac{a}{2}\right| = O(e^{u_\varepsilon - \eta_\varepsilon}) \qquad \text{as } |z| \to \infty, \tag{1.34}$$

and determined by comparison of decays between u_ε *and* η_ε *as described above. In the above the functions* $\beta_1(\varepsilon), \beta_2(\varepsilon)$ *satisfy*

$$\lim_{\varepsilon\to 0} \frac{\beta_1(\varepsilon)}{\varepsilon^2} = \kappa_1, \qquad \lim_{\varepsilon\to 0} \frac{\beta_2(\varepsilon)}{\varepsilon^2} = \kappa_2.$$

(iv) *The corresponding magnetic flux, total gravitational curvature, and the energy of the matter part are given by*

$$\int_{\mathbb{R}^2} F_{12}^{\varepsilon} dx = 4\pi \left(N - \frac{1}{a} \right) + \pi \kappa_1 \varepsilon^2 + o(\varepsilon^2), \tag{1.35}$$

$$\int_{\mathbb{R}^2} K_{\gamma}^{\varepsilon} e^{\eta_\varepsilon} dx = 4\pi + \pi \kappa_2 \varepsilon^2 + o(\varepsilon^2), \tag{1.36}$$

and

$$\int_{\mathbb{R}^2} \mathcal{E} e^{\eta_\varepsilon} dx = \frac{1}{G} \left[1 + \frac{\kappa_2}{4} \varepsilon^2 + o(\varepsilon^2) \right] \tag{1.37}$$

as $\varepsilon \to 0$ *respectively.*

Remarks.

(i) We note $\kappa_1, \kappa_2 > 0$ for $0 < aN < 1$, and $\kappa_1, \kappa_2 < 0$ for $1 < aN < 2$. Thus $aN = 1$ corresponds to the "critical" case similarly to the solutions constructed in [6,21].

(ii) Even in the range $0 < aN < 1$ our multi-string solutions are different from those constructed in [21], since our solution satisfy the boundary condition (1.15), not (1.14).

(iii) We compare our decay estimates with the well-known results on the topological solutions in [18]. From (1.10) and (1.11) we find that

$$\Delta(au - \eta - ae^u - 2a \sum_{j=1}^{m} n_j \ln |z - z_j|) = 0.$$

Thus, for both the topological and the nontopological solutions we can set the harmonic function $h(z) = au - \eta - ae^u - 2a \sum_{j=1}^{m} n_j \ln |z - z_j| =$ Constant. Hence,

$$\lim_{|z| \to \infty} \frac{\eta(z)}{\ln |z|} = -2aN + a \lim_{|z| \to \infty} \frac{u(z)}{\ln |z|}. \tag{1.38}$$

The formula (1.38) holds for both the topological and the nontopological solutions. For the topological solutions, we have $\lim_{|z| \to \infty} \frac{u(z)}{\ln |z|} = 0$, and

$$\lim_{|z| \to \infty} \frac{\eta(z)}{\ln |z|} = -2aN,$$

which holds for general topological solutions. Namely, for any topological solution there should be obvious dependence of the decay of η on the total string number N. For the nontopological solutions, in particular, for our family of solutions $(u_\varepsilon, \eta_\varepsilon)$ constructed in Theorem 1.1, we derive from (1.31)

$$\lim_{|z| \to \infty} \frac{u_\varepsilon(z)}{\ln |z|} = 2N - \frac{4}{a} - \kappa_1 \varepsilon^2 + o(\varepsilon^2),$$

Hence,

$$\lim_{|z|\to\infty} \frac{\eta_\varepsilon(z)}{\ln|z|} = -4 - a\kappa_1\varepsilon^2 + o(\varepsilon^2) = -4 - \kappa_2\varepsilon^2 + o(\varepsilon^2),$$

and, we obtain $\lim_{\varepsilon\to 0}\lim_{|z|\to\infty} \frac{\eta_\varepsilon(z)}{\ln|z|} = -4$, which has no dependence on N. This is not surprising, since, as will be clear in the next section, our solution η_ε is a perturbation of $\ln\rho_2$, which is smooth everywhere, and does not have any dependence on the vortices.

2 Functional formulation

Let us set $\alpha \in (0, \frac{1}{2})$ throughout this paper. Following [1], we introduce the Banach spaces X_α and Y_α as

$$X_\alpha = \{u \in L^2_{loc}(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} (1+|x|^{2+\alpha})|u(x)|^2 dx < \infty\}$$

equipped with the norm $\|u\|^2_{X_\alpha} = \int_{\mathbb{R}^2}(1+|x|^{2+\alpha})|u(x)|^2 dx$, and

$$Y_\alpha = \{u \in W^{2,2}_{loc}(\mathbb{R}^2) \mid \|\Delta u\|^2_{X_\alpha} + \left\| \frac{u(x)}{1+|x|^{1+\frac{\alpha}{2}}} \right\|^2_{L^2(\mathbb{R}^2)} < \infty\}$$

equipped with the norm $\|u\|^2_{Y_\alpha} = \|\Delta u\|^2_{X_\alpha} + \left\|\frac{u(x)}{1+|x|^{1+\frac{\alpha}{2}}}\right\|^2_{L^2(\mathbb{R}^2)}$. We first recall the following proposition proved in [1].

Proposition 2.1 *Let Y_α be the function space introduced above. Then we have the followings.*

(i) If $v \in Y_\alpha$ is a harmonic function, then $v \equiv constant$.
(ii) There exists a constant $C_1 > 0$ such that for all $v \in Y_\alpha$

$$|v(x)| \le C_1\|v\|_{Y_\alpha}(\ln^+|x| + 1), \qquad \forall x \in \mathbb{R}^2,$$

where we denote $\ln^+|x| = \max\{\ln|x|, 0\}$.

Next, given $\varepsilon > 0$, and $\delta \in \mathbb{C} = \mathbb{R}^2$, we consider the functions $\rho^I_{\varepsilon,\delta}(z), \rho^{II}_{\varepsilon,\delta}(z)$ introduced in (1.16), (1.17) respectively. We note that $\rho^I_{\varepsilon,\delta}, \rho^{II}_{\varepsilon,\delta}$ are solutions of the equations

$$\Delta \ln\rho^I_{\varepsilon,\delta} = -\rho^{II}_{\varepsilon,\delta} + 4\pi\sum_{j=1}^m n_j\delta(z-z_j) \tag{2.1}$$

$$\Delta \ln\rho^{II}_{\varepsilon,\delta} = -a\rho^{II}_{\varepsilon,\delta} \tag{2.2}$$

The key idea is that we can view a solution (u,η) of the system (1.10)–(1.13) together with (1.15) as a perturbed one from $(\ln\rho^I_{\varepsilon,\delta}, \ln\rho^{II}_{\varepsilon,\delta})$ in an appropriate sense. We set

$$u - \ln\rho^I_{\varepsilon,\delta} = \hat{u}, \qquad \eta - \ln\rho^{II}_{\varepsilon,\delta} = \hat{\eta} \tag{2.3}$$

Then, $\hat{u}, \hat{\eta}$ satisfy

$$\Delta\hat{u} = \rho^I_{\varepsilon,\delta}\rho^{II}_{\varepsilon,\delta}e^{\hat{u}+\hat{\eta}} - \rho^{II}_{\varepsilon,\delta}e^{\hat{\eta}} + \rho^{II}_{\varepsilon,\delta}, \tag{2.4}$$

$$\Delta(\hat{\eta} + a\rho^I_{\varepsilon,\delta}e^{\hat{u}}) = a\rho^I_{\varepsilon,\delta}\rho^{II}_{\varepsilon,\delta}e^{\hat{u}+\hat{\eta}} - a\rho^{II}_{\varepsilon,\delta}(e^{\hat{\eta}} - 1) \tag{2.5}$$

Next, we make a scaling transform $z \to z/\varepsilon$, and set

$$\tilde{u}(z) = \hat{u}\left(\frac{z}{\varepsilon}\right), \qquad \tilde{\eta}(z) = \hat{\eta}\left(\frac{z}{\varepsilon}\right), \tag{2.6}$$

and

$$g^I_\varepsilon(z,\delta) = \frac{1}{\varepsilon^2}\rho^I_{\varepsilon,\delta}\left(\frac{z}{\varepsilon}\right), \qquad g^{II}_\varepsilon(z,\delta) = \frac{1}{\varepsilon^2}\rho^{II}_{\varepsilon,\delta}\left(\frac{z}{\varepsilon}\right). \tag{2.7}$$

Below we denote $r = |z|$, then we find

$$\lim_{\varepsilon\to 0} g^I_\varepsilon(z,0) = \frac{8^{\frac{1}{a}}r^{2N}}{a^{\frac{1}{a}}(1+|z|^2)^{\frac{2}{a}}} = \rho_1(r), \tag{2.8}$$

and

$$\lim_{\varepsilon\to 0} g^{II}_\varepsilon(z,0) = \frac{8}{a(1+|z|^2)^2} = \rho_2,(r) \tag{2.9}$$

where $\rho_1(r), \rho_2(r)$ are introduced in (1.18), (1.19) respectively. Then, we find

$$\Delta\tilde{u} = \varepsilon^2 g^I_\varepsilon(z,\delta)g^{II}_\varepsilon(z,\delta) - g^{II}_\varepsilon e^{\tilde{\eta}} + g^{II}_\varepsilon(z,\delta), \tag{2.10}$$

$$\Delta(\tilde{\eta} + a\varepsilon^2 g^I_\varepsilon(z,\delta)e^{\tilde{u}}) = a\varepsilon^2 g^I_\varepsilon(z,\delta)g^{II}_\varepsilon(z,\delta) - ag^{II}_\varepsilon(e^{\tilde{\eta}} - 1). \tag{2.11}$$

For further transform of the equations we consider $w_1(r), w_2(r)$ defined in (1.20), (1.21) respectively, which will be shown below to satisfy the systems of linear ordinary differential equations,

$$\Delta w_1 + \rho_2 w_2 - \rho_1\rho_2 = 0 \tag{2.12}$$

$$\Delta w_2 + a\rho_2 w_2 + a\Delta\rho_1 - a\rho_1\rho_2 = 0 \tag{2.13}$$

Lemma 2.1 *Let κ_1, κ_2 be the numbers introduced in (1.28), (1.29) respectively. Then, the functions w_1, w_2 are solutions in Y_α of the system (2.12)–(2.13), which satisfy the following asymptotic formula*

$$w_1(r) = -\kappa_1 \ln r + O(1), \tag{2.14}$$

$$w_2(r) = -\kappa_2 \ln r + O(1). \tag{2.15}$$

as $r = |x| \to \infty$.

Proof. From (2.12) $\times a-$ (2.13) we obtain

$$\Delta(aw_1 - w_2 - a\rho_1) = 0.$$

We seek w_1, w_2 with $aw_1 - w_2 - a\rho_1 \in Y_\alpha$. Then, it follows that $aw_1 - w_2 - a\rho_1 =$constant by ([1], Proposition 1.1). We choose this constant$= 0$. Then, $\rho_2 w_2 = a\rho_2 w_1 - a\rho_1\rho_2$. Substituting this into (2.12) we obtain the following reduced system for w_1, w_2.

$$\Delta w_1 + a\rho_2 w_1 = (a+1)\rho_1\rho_2, \tag{2.16}$$

$$w_2 = aw_1 - a\rho_1. \tag{2.17}$$

Let us set $f(r) = (a+1)\rho_1\rho_2$. Then, it is found in [1] that the ordinary differential equation (2.18) has a solution $w_1(r) \in Y_\alpha$ given by the formula (1.20). From the formula (1.20) we find that

$$w_1(r) = \varphi_0(r)\int_2^r \left(\frac{1+s^2}{1-s^2}\right)^2 \frac{I(s)}{s}ds + \text{(bounded function of } r\text{)} \tag{2.18}$$

as $r \to \infty$, where

$$I(s) = (a+1)\int_0^s \varphi_0(t)t\rho_1(t)\rho_2(t)dt.$$

Since $\varphi_0(r) \to -1$ as $r \to \infty$, (2.14) follows if we show

$$\begin{aligned} I = I(\infty) &= (a+1)\int_0^\infty \varphi_0(r)r\rho_1(r)\rho_2(r)dr \\ &= \frac{(a+1)8^{1+\frac{1}{a}}(1-aN)N!}{a^{2+\frac{1}{a}}\prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)}(=\kappa_1). \end{aligned}$$

Indeed, substituting $r^2 = t$ in the integrand of I, we have

$$\begin{aligned} I &= \frac{4(a+1)8^{\frac{1}{a}}}{a^{1+\frac{1}{a}}}\int_0^\infty \frac{(1-t)t^N}{(1+t)^{3+\frac{2}{a}}}dt \\ &= \frac{4(a+1)8^{\frac{1}{a}}}{a^{1+\frac{1}{a}}}\left[\int_0^\infty \frac{t^N}{(1+t)^{3+\frac{2}{a}}}dt - \int_0^\infty \frac{t^{N+1}}{(1+t)^{3+\frac{2}{a}}}dt\right] \\ &= \frac{4(a+1)8^{\frac{1}{a}}}{a^{1+\frac{1}{a}}}\left[\frac{N!}{\prod_{k=2-N}^{2}\left(\frac{2}{a}+k\right)} - \frac{(N+1)!}{\prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)}\right] \\ &= \frac{(a+1)8^{1+\frac{1}{a}}(1-aN)N!}{a^{2+\frac{1}{a}}\prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)}. \end{aligned} \tag{2.19}$$

The formula (2.15), on the other hand, follows from (2,17) combined with (2.18). This completes the proof of Lemma 2.1. □

Now, we change of variables $\tilde{u} \to \varepsilon^2(v+w_1)$, $\tilde{\eta} \to \varepsilon^2(\xi+w_2)$ in (2.10)–(2.11) to get

$$\Delta v = g^I_\varepsilon g^{II}_\varepsilon e^{\varepsilon^2(v+\xi+w_1+w_2)} - \frac{1}{\varepsilon^2} g^{II}_\varepsilon (e^{\varepsilon^2(\xi+w_2)} - 1) - \Delta w_1, \tag{2.20}$$

$$\Delta(\xi + a g^I_\varepsilon e^{\varepsilon^2(v+\xi)}) = a g^I_\varepsilon g^{II}_\varepsilon e^{\varepsilon^2(v+\xi+w_1+w_2)} - \frac{a}{\varepsilon^2} g^{II}_\varepsilon (e^{\varepsilon^2(\xi+w_2)} - 1) - \Delta w_2. \tag{2.21}$$

We introduce functional

$$P = (P_1, P_2) : Y_\alpha \times Y_\alpha \times \mathbb{R}^2 \times (-\varepsilon_0, \varepsilon_0) \to X_\alpha \times X_\alpha,$$

where P_1, P_2 are defined by

$$P_1(v,\xi,\delta,\varepsilon) = \Delta v - g^I_\varepsilon g^{II}_\varepsilon e^{\varepsilon^2(v+\xi+w_1+w_2)} + \frac{1}{\varepsilon^2} g^{II}_\varepsilon (e^{\varepsilon^2(\xi+w_2)} - 1) + \Delta w_1, \tag{2.22}$$

$$P_2(v,\xi,\delta,\varepsilon) = \Delta(\xi + a g^I_\varepsilon e^{\varepsilon^2(v+\xi)}) - a g^I_\varepsilon g^{II}_\varepsilon e^{\varepsilon^2(v+\xi+w_1+w_2)} + \frac{a}{\varepsilon^2} g^{II}_\varepsilon (e^{\varepsilon^2(\xi+w_2)} - 1) + \Delta w_2. \tag{2.23}$$

The parameter ε_0 is chosen so small so that $P(\cdot)$ is well-defined from $Y_\alpha \times Y_\alpha \times \mathbb{R}^2 \times (-\varepsilon_0, \varepsilon_0)$ into $X_\alpha \times X_\alpha$. In particular we note that our condition $0 < aN < 2$ implies that $g^I_\varepsilon(z) = O(|z|^{2N-\frac{4}{a}}) = o(1)$. By standard procedure similar to the case of [1] we can check that there exists such $\varepsilon_0 > 0$. We note, particular, that due to the conditions (2.14) and (2.15) we can have the continuous extension of $P(\cdot,\cdot,\cdot,\varepsilon)$ up to $\varepsilon = 0$ by definition $P(0,0,0,0) = 0$. Then, finding a solution of (1.10)–(1.11) is reduced to that of finding an implicit function

$$\varepsilon \mapsto (v_\varepsilon, \xi_\varepsilon, \delta_\varepsilon)$$

satisfying

$$P(v_\varepsilon, \xi_\varepsilon, \delta_\varepsilon, \varepsilon) = 0.$$

We note that once a family of solutions $\{(v^*_\varepsilon, \xi^*_\varepsilon, \delta^*_\varepsilon)\}$ is found, then our solution (u, η) of the system (1.10)–(1.11) is recovered by the formula,

$$u(x) = \ln \rho^I_{\varepsilon,\delta^*_\varepsilon}(x) + \varepsilon^2 w_1(\varepsilon|x|) + \varepsilon^2 v^*_\varepsilon(\varepsilon x), \tag{2.24}$$

and

$$\eta(x) = \ln \rho^{II}_{\varepsilon,\delta^*_\varepsilon}(x) + \varepsilon^2 w_2(\varepsilon|x|) + \varepsilon^2 \xi^*_\varepsilon(\varepsilon x). \tag{2.25}$$

We note here that although the nonlinear functional $P(\cdot,\cdot,\cdot,\varepsilon)$ itself is well defined at $\varepsilon = 0$ by continuous extension remarked above, the formula (2.24), (2.25) are defined only for $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$, since our change of variables (2.6) is not defined at $\varepsilon = 0$.

3 Proof of the Main Theorem

Let us introduce functions $\varphi_\pm$ defined by

$$\varphi_+(r,\theta) = \frac{r\cos\theta}{1+r^2}, \quad \varphi_-(r,\theta) = \frac{r\sin\theta}{1+r^2}. \tag{3.1}$$

We can easily obtain by direct computation

$$\lim_{\varepsilon\to 0} \frac{\partial g^I_\varepsilon(z,\delta)}{\partial\delta_1}\bigg|_{\delta=0} = -\frac{4}{a}\rho_1\varphi_+, \quad \lim_{\varepsilon\to 0} \frac{\partial g^I_\varepsilon(z,\delta)}{\partial\delta_2}\bigg|_{\delta=0} = -\frac{4}{a}\rho_1\varphi_-,$$

$$\lim_{\varepsilon\to 0} \frac{\partial g^{II}_\varepsilon(z,\delta)}{\partial\delta_1}\bigg|_{\delta=0} = -4\rho_2\varphi_+, \quad \lim_{\varepsilon\to 0} \frac{\partial g^{II}_\varepsilon(z,\delta)}{\partial\delta_2}\bigg|_{\delta=0} = -4\rho_2\varphi_-.$$

Using these results, we compute the linearized operator, $\mathcal{A}[\cdot]$ defined by

$$\mathcal{A}[u,\eta,\beta]{:=}P'_{(v,\xi,\delta)}(0,0,0,0)[u,\eta,\beta]{=}(L_1[u,\eta]+M_1[\beta], L_2[u,\eta]{+}M_2[\beta]), \tag{3.2}$$

where

$$L_1[u,\eta] = \Delta u + \rho_2\eta, \qquad L_2[u,\eta] = \Delta\eta + a\rho_2\eta, \tag{3.3}$$

$$M_1[\beta] = \frac{4}{a}[(a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_+\beta_1 + \frac{4}{a}[(a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_-\beta_2, \tag{3.4}$$

and

$$\begin{aligned} M_2[\beta] = 4\{[(a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_+ - \Delta(\rho_1\varphi_+)\}\,\beta_1 \\ +4\{[(a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_- - \Delta(\rho_1\varphi_-)\}\,\beta_2. \end{aligned} \tag{3.5}$$

(3.6)

Here we set $\beta = (\beta_1,\beta_2) \in \mathbb{R}^2$. For the linearized operator $\mathcal{A}[\cdot]$ we will establish the following key lemma.

Lemma 3.1 *The operator $\mathcal{A} : Y^2_\alpha \times \mathbb{R}^2 \to X^2_\alpha$ given by (3.2)–(3.5) is onto. Moreover, kernel of $\mathcal{A}$ is given by*

$$Ker\mathcal{A} = Span\{(1,0); (\frac{\varphi_\pm}{a},\varphi_\pm); (\frac{\varphi_0}{a},\varphi_0)\} \times \{(0,0)\}. \tag{3.7}$$

Thus if we decompose $Y^2_\alpha \times \mathbb{R}^2 = U_\alpha \oplus Ker\mathcal{A}$, where we set $U_\alpha = (Ker\mathcal{A})^\perp$, then $\mathcal{A}$ is an isomorphism from U_α onto X^2_α.

In order to prove the above lemma we first recall the following lemma, which is established in [1].

Lemma 3.2 *Let L_2 be the operator defined in (3.3), then*

$$KerL_2 = \textit{Span}\{\varphi_+, \varphi_-, \varphi_0\}. \tag{3.8}$$

Moreover, we have

$$ImL_2 = \{f \in X_\alpha | \int_{\mathbb{R}^2} f\varphi_\pm dx = 0\}. \tag{3.9}$$

Next, we need the following:

Proposition 3.1 *Let w_2 solve (2.16)–(2.17), then*

$$I_\pm = \int_{\mathbb{R}^2} \left\{[(a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_\pm^2 - \Delta(\rho_1\varphi_\pm)\varphi_\pm\right\} dx > 0. \tag{3.10}$$

Proof. From (3.8) and (3.3) we have

$$\Delta\varphi_+ = -a\rho_2\varphi_+.$$

By integration by part we obtain

$$\begin{aligned} I_\pm &= \int_{\mathbb{R}^2} \left\{[(a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_\pm^2 - \rho_1\varphi_\pm\Delta\varphi_\pm\right\} dx \\ &= \int_{\mathbb{R}^2} [(2a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_\pm^2 dx. \end{aligned} \tag{3.11}$$

Now, we prove (3.10) for I_+. The case of I_- is similar. Below we list useful formulas, which can be checked by elementary computations.

$$\varphi_+^2\rho_2 = \frac{1}{16}\cos^2\theta L_2\rho_2, \quad \varphi_-^2\rho_2 = \frac{1}{16}\sin^2\theta L_2\rho_2, \tag{3.12}$$

$$\varphi_+^2 = \frac{a}{8}r^2\rho_2\cos^2\theta, \quad \varphi_-^2 = \frac{a}{8}r^2\rho_2\sin^2\theta \tag{3.13}$$

$$\Delta\rho_2 = a(2r^2-1)\rho_2^2. \tag{3.14}$$

Using (3.12)–(3.14), and integrating by parts, we transform the integral as follows.

$$\begin{aligned} I_+ &= \int_0^\infty \int_0^{2\pi} \left\{\frac{a(2a+1)}{8}r^2\rho_1\rho_2^2 - \frac{a}{16}(L_2w_2)\rho_2\right\} r\cos^2\theta d\theta dr \\ &= \pi\int_0^\infty \left\{\frac{a(2a+1)}{8}r^2\rho_1\rho_2^2 - \frac{a}{16}(a\rho_1\rho_2 - a\Delta\rho_1)\rho_2\right\} rdr \\ &= \pi\int_0^\infty \left\{\frac{a(2a+1)}{8}r^2\rho_1\rho_2^2 - \frac{a^2}{16}\rho_1\rho_2^2 + \frac{a^2}{16}\rho_1\Delta\rho_2\right\} rdr \\ &= \frac{a(a+1)\pi}{16}\int_0^\infty [2(a+1)r^2 - a]\rho_1\rho_2^2 rdr \\ &= \frac{4(a+1)8^{\frac{1}{a}}\pi}{a^{1+\frac{1}{a}}}\int_0^\infty \frac{[2(a+1)r^2-a]r^{2N+1}}{(1+r^2)^{\frac{2}{a}+4}}dr \quad \text{(Setting } r^2 = t) \end{aligned}$$

$$= \frac{2(a+1)8^{\frac{1}{a}}\pi}{a^{1+\frac{1}{a}}} \int_0^\infty \left[\frac{2(a+1)t^{N+1}}{(1+t)^{\frac{2}{a}+4}} - \frac{at^N}{(1+t)^{\frac{2}{a}+4}} \right] dt$$
$$= \frac{2(a+1)8^{\frac{1}{a}}\pi}{a^{1+\frac{1}{a}}} \left[\frac{2(a+1)(N+1)!}{\prod_{k=2-N}^{3}\left(\frac{2}{a}+k\right)} - \frac{aN!}{\prod_{k=3-N}^{3}\left(\frac{2}{a}+k\right)} \right]$$
$$= \frac{2(a+1)(3a+2)8^{\frac{1}{a}}N \cdot N!\pi}{a^{1+\frac{1}{a}}\prod_{k=2-N}^{3}\left(\frac{2}{a}+k\right)} > 0. \tag{3.15}$$

This completes the proof of Proposition (3.1). □

We are now ready to prove Lemma 3.1.

Proof of Lemma 3.1. Given $(f_1, f_2) \in X_\alpha^2$, we want first to show that there exists $(v, \eta) \in Y_\alpha^2$, $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$\mathcal{A}(v, \eta, \beta_1, \beta_2) = (f_1, f_2), \tag{3.16}$$

which can be rewritten as

$$\begin{aligned} &\Delta v + \rho_2\eta + \frac{4}{a}[(a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_+\beta_1 \\ &\quad + \frac{4}{a}[(a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_-\beta_2 = f_1, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} &\Delta\eta + a\rho_2\eta + 4\{[(a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_+ - \Delta(\rho_1\varphi_+)\}\beta_1 \\ &\quad + 4\{[(a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_- - \Delta(\rho_1\varphi_-)\}\beta_2 = f_2. \end{aligned} \tag{3.18}$$

Let us set

$$\beta_1 = \frac{1}{4I_+}\int_{\mathbb{R}^2} f_2\varphi_+ dx, \qquad \beta_2 = \frac{1}{4I_-}\int_{\mathbb{R}^2} f_2\varphi_- dx, \tag{3.19}$$

where $I_\pm > 0$ is defined in (3.9). We introduce $\tilde{f}$ by

$$\tilde{f}_2 = f_2 - \beta_1\varphi_+ - \beta_2\varphi_-. \tag{3.20}$$

Using the fact

$$\int_0^{2\pi} \varphi_+\varphi_- d\theta = 0, \tag{3.21}$$

we find easily

$$\int_{\mathbb{R}^2} \tilde{f}_2\varphi_\pm dx = 0. \tag{3.22}$$

Hence, by (3.9) there exists $\eta \in Y_\alpha$ such that $\Delta\eta + a\rho_2\eta = \tilde{f}_2$. Thus we have found $(\eta, \beta_1, \beta_2) \in Y_\alpha \times \mathbb{R}^2$ satisfying (3.18). Given such (η, β_1, β_2), in order to construct $v \in Y_\alpha$ satisfying (3.17), we consider the following equation, obtained by (3.17)$\times a-$(3.18),

$$\Delta(av - \eta + 4\rho_1\varphi_+\beta_1 + 4\rho_1\varphi_-\beta_2) = af_1 - f_2. \tag{3.23}$$

Obviously, the function

$$v(x) = \frac{1}{2\pi a}\int_{\mathbb{R}^2} \ln(|x-y|)(af_1(y) - f_2(y))dy + \frac{1}{a}(\eta - 4\rho_1\varphi_+\beta_1 - 4\rho_1\varphi_-\beta_2) \tag{3.24}$$

satisfies (3.23), and belongs to Y_α. We have just finished the proof that $\mathcal{A} : Y_\alpha^2 \times \mathbb{R}^2 \to X_\alpha^2$ is onto.
We now show that Ker$\mathcal{A} = Span\{(1,0); (\frac{\varphi_\pm}{a}, \varphi_\pm); (\frac{\varphi_0}{a}, \varphi_0)\} \times \{(0,0)\}$. Let us consider the equations

$$\Delta v + \rho_2\eta + \frac{4}{a}[(a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_+\beta_1 + \frac{4}{a}[(a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_-\beta_2 = 0, \tag{3.25}$$

and

$$\Delta\eta + a\rho_2\eta + 4\left\{[(a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_+ - \Delta(\rho_1\varphi_+)\right\}\beta_1 + 4\left\{[(a+1)\rho_1\rho_2 - aw_2\rho_2]\varphi_- - \Delta(\rho_1\varphi_-)\right\}\beta_2 = 0. \tag{3.26}$$

Taking $L^2(\mathbb{R}^2)$ inner product of (3.26) with $\varphi_\pm$, and using (3.8), (3.21) and (3.9), we find $\beta_1 = \beta_2 = 0$. Thus, (3.26) implies $\eta \in KerL_2 = Span\{\varphi_\pm, \varphi_0\}$, where we used the fact (3.8). When η takes each one of $0, \varphi_\pm, \varphi_0$ we find that the solution $v \in Y_\alpha$ of $\Delta v + \rho_2\eta = 0$ is given by $1, \varphi_\pm/a, \varphi_0/a$ respectively. This completes the proof of the lemma. □

We are now ready to prove our main theorem.

Proof of Theorem 1.1. Let us set $U_\alpha = (Ker\mathcal{A})^\perp$. Then, Lemma 3.1 shows that $P'_{(v,\xi,\beta)}(0,0,0,0) : U_\alpha \to X_\alpha \times X_\alpha$ is an isomorphism for $\alpha \in (0, \frac{1}{2})$. Then, the standard implicit function theorem (see e.g. [22]), applied to the functional $P : U_\alpha \times (-\varepsilon_0, \varepsilon_0) \to X_\alpha \times X_\alpha$, implies that there exists a constant $\varepsilon_1 \in (0, \varepsilon_0)$ and a continuous function $\varepsilon \mapsto \psi_\varepsilon^* := (v_\varepsilon^*, \xi_\varepsilon^*, \delta_\varepsilon^*)$ from $(0, \varepsilon_1)$ into a neighborhood of 0 in U_α such that

$$P(v_\varepsilon^*, \xi_\varepsilon^*, \delta_\varepsilon^*, \varepsilon) = (0,0), \quad \text{for all } \varepsilon \in (0, \varepsilon_1).$$

Let (u, η) be the functions recovered by the formula (2.24) and (2.25) . Then $\gamma_{jk} = e^\eta\delta_{jk}$, and (A, ϕ) defined by the formulas,

$$\phi(x) = \sigma \exp\left(\tfrac{1}{2}u(x) + i\textstyle\sum_{j=1}^m n_j Arg(z - z_j)\right),$$
$$A_1 = \mathrm{Re}(2i\partial_z^* \ln\phi), \qquad A_2 = \mathrm{Im}(2i\partial_z^* \ln\phi),$$

where we denoted $\partial_z^* = (\partial_1 + i\partial_2)/2$, form a solution (ϕ, A, g) of the original system (1.2)–(1.5). By standard elliptic regularity estimates (see e.g. [8]) one can easily check that (ϕ, A, g) is smooth. We now prove the decay estimates in Theorem 1.1. We first obtain easily

$$|\phi(x)|^2 = \sigma^2 e^{u(x)} = O(|x|^{2N-\frac{4}{a}-\kappa_1\varepsilon^2+o(\varepsilon^2)}), \tag{3.27}$$

and

$$e^{\eta(x)} = O(|x|^{-4-\kappa_2\varepsilon^2+o(\varepsilon^2)}). \tag{3.28}$$

From (1.10) we have the integral representation,

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y| e^{\eta(y)}(e^{u(y)}-1)dy + 2\sum_{j=1}^m n_j \ln|x-z_j| + C$$

for some constant C. Since $u(x) \le 0$ for $x \in \mathbb{R}^2$ (by the maximum principle applied to (1.10)), taking derivative of u we obtain

$$|\nabla u(x)|^2 \le C\left(\int_{\mathbb{R}^2} \frac{e^{\eta(y)}}{|x-y|}dy\right)^2 + C\sum_{j=1}^m \frac{1}{|x-z_j|^2} = O(1) \tag{3.29}$$

as $|x| \to \infty$. This estimate, combined with $|D_1\phi|^2 + |D_2\phi|^2 = \frac{1}{2}e^u|\nabla u|^2$, gives

$$|D_1\phi|^2 + |D_2\phi|^2 = O(e^u) = O(|x|^{2N-\frac{4}{a}-\kappa_1\varepsilon^2+o(\varepsilon^2)}). \tag{3.30}$$

From the rescaled form of (1.8) we note

$$F_{12} = -\frac{1}{2}e^{\eta}(e^u - 1). \tag{3.31}$$

Thus,

$$|F_{12}(x)| = O(e^{\eta}) = O(|x|^{-4-\kappa_2\varepsilon^2+o(\varepsilon^2)}). \tag{3.32}$$

In order to estimate the decay of the Gaussian curvature we first note that $K_\gamma = -\frac{1}{2}e^{-\eta}\Delta\eta$ for $\gamma_{ij} = e^{\eta}\delta_{ij}$. Now, from (1.11), (1.10)

$$\begin{aligned}\Delta\eta &= -a|\nabla u|^2 e^u - a\Delta u e^u + a e^{\eta}(e^u - 1)\\ &= -a|\nabla u|^2 e^u - a e^{u+\eta}(e^u - 1) + a e^{\eta}(e^u - 1),\end{aligned}$$

and

$$K_\gamma = \frac{a}{2}|\nabla u|^2 e^{u-\eta} + \frac{a}{2}e^u(e^u - 1) - \frac{a}{2}(e^u - 1).$$

Thus, using (3.29), we obtain

$$\left|K_\gamma(x) - \frac{a}{2}\right| = O(e^{u-\eta}) \qquad \text{as } |x| \to \infty. \tag{3.33}$$

This provides (1.34). We now prove (1.35)–(1.37). From (3.31) and (1.10), using the Gauss theorem, we deduce

$$\int_{\mathbb{R}^2} F_{12}dx = -\frac{1}{2}\lim_{R\to\infty}\oint_{S_R}\frac{\partial u}{\partial r}ds + 2\pi N, \tag{3.34}$$

where we set $S_R = \{x \in \mathbb{R}^2 | \quad |x| = R\}$. For our solution, $u(x) = u_\varepsilon(x)$ given by (2.24), we compute

$$\begin{aligned}\oint_{S_R}\frac{\partial u_\varepsilon}{\partial r}ds &= \oint_{S_R}\frac{\partial}{\partial r}\ln\rho^I_{\varepsilon,\delta^*_\varepsilon}ds + \varepsilon^2\oint_{S_R}\frac{\partial w_1(\varepsilon|z|)}{\partial r}ds + \varepsilon^2\oint_{S_R}\frac{\partial u^*_\varepsilon(\varepsilon x)}{\partial r}ds\\ &= I_1 + \varepsilon^2 I_2 + \varepsilon^2 I_3.\end{aligned} \tag{3.35}$$

Following [1] (pp. 135–138) we easily compute

$$I_1 = -4\pi(N - \frac{2}{a}) + O\left(\frac{1}{R}\right), \tag{3.36}$$

and

$$\begin{aligned}I_2 &= -2\pi(a+1)\int_0^\infty \varphi_0 t\rho_1\rho_2 dt + O\left(\frac{1}{R}\right)\\ &= \frac{-2\pi(a+1)8^{1+\frac{1}{a}}(1-aN)N!}{a^{2+\frac{1}{a}}\prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)} + O\left(\frac{1}{R}\right)\end{aligned} \tag{3.37}$$

as $R \to \infty$, where we used the result of the computation in the proof of Lemma 2.1, and finally

$$\sup_{R>0}|I_3| \le C\|v^*_\varepsilon\|_{Y_\alpha} \le C\|\psi^*_\varepsilon\|_{U_\alpha} \to 0 \tag{3.38}$$

as $\varepsilon \to 0$ due to the continuity of $\varepsilon \mapsto \psi^*_\varepsilon$ in U_α on $(-\varepsilon_1, \varepsilon_1)$. Combining (3.35)–(3.38) with (3.34), (3.33) we obtain (1.35). Similarly to the above by the Gauss theorem we compute

$$\begin{aligned}\int_{\mathbb{R}^2} K_\gamma e^\eta dx &= -\frac{1}{2}\int_{\mathbb{R}^2}\Delta\eta dx = -\frac{1}{2}\lim_{R\to\infty}\oint_{S_R}\frac{\partial\eta}{\partial r}ds\\ &= -\frac{1}{2}\lim_{R\to\infty}\oint_{S_R}\frac{\partial}{\partial r}\ln\rho^{II}_{\varepsilon,\alpha^*_\varepsilon}ds\\ &\quad -\frac{\varepsilon^2}{2}\lim_{R\to\infty}\oint_{S_R}\frac{\partial w_2(\varepsilon|z|)}{\partial r}ds - \frac{\varepsilon^2}{2}\int_{\mathbb{R}^2}\Delta\xi^*_\varepsilon(\varepsilon x)dx\\ &= J_1 + \varepsilon^2 J_2 + \varepsilon^2 J_3.\end{aligned} \tag{3.39}$$

Similarly to the case of magnetic flux we easily compute

$$J_1 = 4\pi. \tag{3.40}$$

From the relation (2.16), (2.17) between w_1 and w_2, and using (3.37), we obtain

$$J_2 = \frac{\pi a(a+1)8^{1+\frac{1}{a}}(1-aN)N!}{a^{1+\frac{1}{a}}\prod_{k=1-N}^{2}\left(\frac{2}{a}+k\right)}. \tag{3.41}$$

Similarly to I_3 above we have

$$|J_3| \leq C\|\xi_\varepsilon^*\|_{Y_\alpha} \leq C\|\psi_\varepsilon^*\|_{U_\alpha} \to 0 \tag{3.42}$$

as $\varepsilon \to 0$. Combination of (3.38) and (3.4) together with Proposition 2.1 imply (1.30). This completes the proof of Theorem 1.1. □

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