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Longtime existence of the Lagrangian mean curvature flow

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Abstract. Given a compact Lagrangian submanifold in flat space evolving by its mean curvature, we prove uniform $C^{2,\alpha}$-bounds in space and C^2-estimates in time for the underlying Monge-Ampère equation under weak and natural assumptions on the initial Lagrangian submanifold. This implies longtime existence and convergence of the Lagrangian mean curvature flow. In the 2-dimensional case we can relax our assumptions and obtain two independent proofs for the same result.

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1. Introduction

In symplectic geometry there is a destinguished class of immersions, known as Lagrangian submanifolds. They are important in physics and of course in pure and applied mathematics as well. E.g. minimal Lagrangian immersions in a given Calabi-Yau manifold are relevant in physics because they are related to T-duality and Mirror symmetry [9]. However, to construct minimal Lagrangian submanifolds is a great geometric and analytic challenge. Nevertheless there is a growing community working on such problems and already some very nice results have been obtained, so e.g. a version of the Bernstein problem [5]. Due to the high codimension many techniques which are useful for hypersurfaces of prescribed curvature cannot be used unchanged in the theory of Lagrangian submanifolds. In particular the mean curvature flow for Lagrangian submanifolds is much more complicated than for hypersurfaces and it is the aim of this article to close a gap in the understanding of Lagrangian graphs moving by their mean curvature.

Let (M,ω) be a $2n$-dimensional symplectic manifold with symplectic 2-form ω. An n-dimensional submanifold $L \subset M$ is called Lagrangian if

$$\omega(V,W) = 0 \qquad \forall\ V,W \in TL.$$

The most prominent examples of symplectic manifolds are Kähler manifolds (M,J,g), where

$$\omega(V,W) = g(JV,W)$$

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is the symplectic 2-form (Kähler form) induced by the Kähler metric g and the complex structure J.

If L is a compact n-dimensional manifold, $[0,T) \subset \mathbb{R} \cup \{\infty\}$ a time interval and

$$F : L \times [0,T) \to M$$

a smooth family of immersions into a Kähler-Einstein manifold (M,J,g), such that

$$L_0 := F(L,0)$$

is Lagrangian and such that F satisfies the mean curvature flow equation

$$\frac{dF}{dt} = \vec{H} \tag{1.1}$$

($\vec{H}$ = mean curvature vector along $L_t := F(L,t)$), then it is well-known that (1.1) preserves the Lagrangian condition, so that L_t is Lagrangian $\forall\ t \in [0,T)$ (see e.g. [7].)

We write $F_t(x)$ instead of $F(x,t)$. The mean curvature form H of Lagrangian submanifolds in Kähler manifolds (M,J,g) is related to $\vec{H}$ through

$$H(V) = \omega(\vec{H}, V)\,.$$

If (M,J,g) is Kähler-Einstein, then H is closed and any locally defined function α with

$$d\alpha = H \tag{1.2}$$

is called a Lagrangian angle.

In [8] we proved the following longtime existence and convergence result for the Lagrangian mean curvature flow in Kähler-Einstein manifolds of nonpositive scalar curvature:

Proposition 1.1. *Let L be a compact manifold and let $F_0 : L \to L_0 \subset M$ be a smooth immersion of L as a Lagrangian submanifold into a Kähler-Einstein manifold (M,J,g) that is either compact or complete with bounded curvature quantities. Further assume that $[0,T),\ 0 < T \leq \infty$ is the maximal time interval on which the Lagrangian mean curvature flow (1.1) admits a smooth solution. Then the following is true:*

(a) Assume there exists a constant $C_0 < \infty$ such that

$$\max_{L_t} |A|^2 \leq C_0, \quad \forall\, t \in [0,T),$$

where $|A|^2$ is the squared norm of the second fundamental tensor A. Then for any $m \geq 0$ there exists a constant $C_m < \infty$ depending on m, L_0, M such that

$$\max_{L_t} |\nabla^m A|^2 \leq C_m, \quad \forall\, t \in [0,T).$$

(b) If $T < \infty$*, then*

$$\limsup_{t \to T} \left\{ \max_{L_t} |A|^2 \right\} = \infty.$$

(c) If in addition to (a) the initial mean curvature form of L_0 *is exact, the ambient Kähler-Einstein manifold has nonpositive Ricci curvature and the induced Riemannian metrics* $F_t^* g$ *on* L *are all uniformly equivalent, then* $T = \infty$ *and the Lagrangian submanifolds* L_t *converge smoothly and exponentially to a smooth compact minimal Lagrangian immersion* $L_\infty \subset M$*.*

In general one cannot expect longtime existence results without extra assumptions on the initial Lagrangian submanifold. In [8] we considered Lagrangian submanifolds generated by symplectic maps and proved convergence to minimal Lagrangian maps under very natural and sharp conditions for the Lagrangian angle α. Here, we will give another sufficient condition for longtime existence in flat ambient manifolds that entirely differs from those conditions. In particular we will not need the oscillation condition

$$\operatorname{osc}(\alpha) \le \frac{\pi}{2}$$

that was important there.

The crucial condition in this paper can be stated in terms of certain symmetric bilinear forms S. To explain these forms suppose (M, g, J) is a $2n$-dimensional Kähler-Einstein manifold with compatible complex structure J, i.e.

$$\omega(V, W) = g(JV, W)$$

is a symplectic 2-form (Kähler form.) Let us denote the Levi-Civita connection on M by D. We will consider tensors $S \in \Gamma(T^*M \otimes T^*M)$ that satisfy the following conditions

$$S(V,W) = S(W,V) \quad \text{(Symmetry)}, \tag{1.3}$$

$$S(JV,W) = S(V,JW) \quad \text{(Anti-compatibility)}, \tag{1.4}$$

$$DS = 0 \quad \text{(Parallelity)} \tag{1.5}$$

and denote the set of tensors satisfying Eqs. (1.3), (1.4) and (1.5) by $\Sigma(M)$. Note that if $S \in \Sigma(M)$, then $\bar{S}$ defined by

$$\bar{S}(V, W) := -S(JV, W)$$

also belongs to $\Sigma(M)$. An example in $\mathbb{R}^{2n}$ is given by $S(V, W) := \langle \bar{V}, W \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the euclidean inner product and the bar is complex conjugation. Since these conditions are very similar to the structures on hyperKähler manifolds it is clear that one must expect a special holonomy for the underlying manifold, indeed the existence of such a bilinear form S is so strong that it implies:

Proposition 1.2. *Let* $S \in \Sigma(M)$ *be non-degenerate. Then* g *is flat.*

A proof of the proposition will be given in the appendix where we will also discuss $\Sigma(\mathbb{R}^{2n})$ in more detail. Here we only mention that this fact mainly hinges on condition (1.5) and that this is the only reason why we restrict our considerations to flat ambient manifolds.

The main theorem states:

Theorem 1.3. *Assume $F_0 : L \to (M, g, J)$ is a compact Lagrangian immersion into a flat manifold such that there exists a tensor $S \in \Sigma(M)$ with*

$$F_0^* S(V, V) > 0 \qquad \forall\, V \in TL,\, V \neq 0\,. \tag{1.6}$$

Then there exist constants $c_1, c_2 > 0$ such that

$$\limsup_{t\to T} \left\{ \max_{L_t} |\vec{H}| \right\} \leq c_1,$$

$$\limsup_{t\to T} \left\{ \max_{L_t} |d^\dagger H| \right\} \leq c_2,$$

on the maximal time interval $[0, T)$, where a smooth solution of (1.1) exists ($d^\dagger H$ is shorthand for $\nabla^i H_i$). Moreover, all induced metrics $g_t := F_t^* g$ are uniformly equivalent to the initial metric g_0. If in addition $n = 2$ or

$$F_0^* \bar{S}(V, V) > 0 \qquad \forall\, V \in TL,\, V \neq 0\,, \tag{1.7}$$

then $T = \infty$ and the Lagrangian submanifolds converge smoothly to a flat Lagrangian submanifold.

Condition (1.6) implies that the Lagrangian submanifold can be written as a graph over a flat Lagrangian subspace so that the n height functions are given by the n components of the gradient of a smooth function u and such that the eigenvalues of the Hessian of u are uniformly bounded by a constant depending on S. We discuss this in Sect. 2. Condition (1.7) will imply convexity of u. This is possible even for compact Lagrangian submanifolds. An example can be given as follows:

Example 1.4. Let $u : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$u(x) := \frac{a}{2}|x|^2 + b\sum_{i=1}^{n} \cos(x^i)$$

with two constants a, b that satisfy $a > b > 0, a + b < 1$ (see also Fig. 1). Then the gradient graph

$$\begin{aligned} F &: \mathbb{R}^n \to \mathbb{R}^n \oplus i\mathbb{R}^n = \mathbb{C}^n \\ F(x) &:= \big(x, y := Du(x)\big) \end{aligned}$$

describes the universal cover of a compact Lagrangian submanifold in flat space (since $F(\mathbb{R}^n)$ is invariant under translations in $\mathbb{R}^{2n}$) and the Hessian of u is

$$u_{ij} = \mathrm{diag}\big(a - b\cos(x^1), \ldots, a - b\cos(x^n)\big)$$

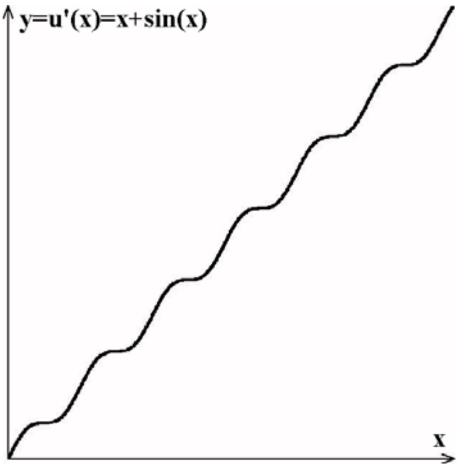

Fig. 1. The gradient graph of $u(x) = \frac{x^2}{2} - cos(x)$

so that all eigenvalues λ of u_{ij} satisfy

$$0 < \lambda < 1.$$

In particular

$$F^*S(V,V) = \langle \overline{DF(V)}, DF(V)\rangle > 0$$

and

$$F^*\bar{S}(V,V) = \langle \overline{JDF(V)}, DF(V)\rangle > 0$$

for all $V \in T\mathbb{R}^n$ (the bar denotes complex conjugation) so that conditions (1.6) and (1.7) are satisfied with $S(X,Y) := \langle \bar{X}, Y\rangle$.

In particular we get the following corollary

Corollary 1.5. *Assume $u : \mathbb{R}^n \to \mathbb{R}$ is a smooth strictly convex function such that all eigenvalues of Hess(u) are bounded by* 1 *and such that $F(x) := \big(x, Du(x)\big)$ is the universal cover of a compact Lagrangian submanifold in $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$. Then the Lagrangian mean curvature flow deforms L into a flat plane.*

(1.6) and (1.7) roughly mean that the Lagrangian submanifold lies between two different Lagrangian planes. Equation (1.1) and the Lagrangian condition then lead to a parabolic Monge-Ampère type equation (2.10) for u which describes the flow (see Sect. 2 for details) and the theorem implies that under condition (1.6) one gets uniform C^2-estimates for u both in space and time. Assumption (1.7) on the other hand guarantees that the parabolic operator is concave so that we can use the $C^{2,\alpha}$-estimates in space due to Krylov (for example Sect. 5.5 in [6]). This eventually gives longtime existence and convergence. In case $n = 2$ we can drop condition (1.7) and obtain the $C^{2,\alpha}$-estimates in two different ways, from the better regularity theory for nonlinear equations in 2 variables and also from a direct estimate of the full second fundamental form. We will discuss both proofs. It is unknown whether (1.7) is redundant in any dimension.

In this paper we will use the maximum principle for tensors due to Hamilton [3,4]. Since there is no monotonicity formula for tensors so far, we cannot drop the compactness condition here, although we believe that this can be done under appropriate growth conditions as e.g. in the well known paper by Ecker and Huisken [1] where hypersurfaces in $\mathbb{R}^{n+1}$ represented as graphs are deformed by their mean curvature. However, for the mean curvature flow in higher codimension one cannot expect the same results as in [1] and a condition like in our theorem is natural in this context. The special nature of the Lagrangian mean curvature flow allows to obtain longtime existence and convergence merely from uniform $C^{1,\alpha}$-estimates for the evolving maps $F : L \to \mathbb{R}^{2n}$ because the quasilinear parabolic system given by Eq. (1.1) can be integrated to the fully nonlinear parabolic equation of Monge-Ampère type (2.10) and $C^{1,\alpha}$-estimates for F correspond to $C^{2,\alpha}$-estimates of that equation.

This work has begun while I visited the CAS in Beijing, China in 2001 and was completed at the Max-Planck-Institute for Mathematics in the Sciences in Leipzig, Germany. I would like to thank Prof. Li Jiayu for his invitation to China and his great hospitality and Prof. Jürgen Jost at the MPI for the opportunity to finish this work.

2. Preliminaries

2.1. The Lagrangian mean curvature flow in $\mathbb{R}^{2n}$

In this section we explain the notation and recall elementary equations for the Lagrangian mean curvature flow in $\mathbb{R}^{2n}$. To begin, assume that $(\mathbb{R}^{2n}, g, J, \omega)$ is the euclidean space with compatible complex structure $J = i$ and the standard symplectic form ω. Local coordinates on $\mathbb{R}^{2n}$ will be denoted by $(y^\alpha)_{\alpha=1,\dots,2n}$ whereas local coordinates for a Lagrangian submanifold L will be denoted by $(x^i)_{i=1,\dots,n}$. Moreover, we use the Einstein convention to sum over repeated indices, the sum is taken from 1 to $2n$ for greek indices and from 1 to n for latin minuscles. If

$$F : L \to M$$

is an immersion, then we write

$$F_i^\alpha := \frac{\partial F^\alpha}{\partial x^i}\ \text{, with } F^\alpha = y^\alpha(F),$$
$$F_{ij}^\alpha := \frac{\partial^2 F^\alpha}{\partial x^i \partial x^j}.$$

The metric $g = \langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n}$ can locally be written as

$$g = g_{\alpha\beta}\, dy^\alpha \otimes dy^\beta$$

and then

$$F^* g = g_{ij}\, dx^i \otimes dx^j$$

with

$$g_{ij} := g_{\alpha\beta}\, F_i^\alpha F_j^\beta$$

is the induced Riemannian metric on L. By the Lagrangian condition we have

$$\omega_{ij} := \omega_{\alpha\beta} F_i^\alpha F_j^\beta = 0.$$

We also set

$$J = J_\beta^\alpha \frac{\partial}{\partial y^\alpha} \otimes dy^\beta,$$

$$\nu_i^\alpha := J_\beta^\alpha F_i^\beta$$

and

$$\nu_i := \nu_i^\alpha \frac{\partial}{\partial y^\alpha} = J\left(\frac{\partial F}{\partial x^i}\right),$$

so that ν_i is normal along L. The induced connection on tensor bundles over L will be denoted by ∇. Then the second fundamental tensor A is the covariant derivative of the differential

$$dF = F_i^\alpha dx^i \otimes \frac{\partial}{\partial y^\alpha},$$

i.e.

$$A = \nabla dF$$

and we set

$$A_{ij}^\alpha := \nabla_i F_j^\alpha.$$

The second fundamental form $h \in \Gamma(T^*L \otimes T^*L \otimes T^*L)$ is the tensor given by the components

$$h_{ijk} := -\omega_{\alpha\beta} F_i^\alpha A_{jk}^\beta$$

and by definition the mean curvature form $H = H_i dx^i$ is

$$H_i := g^{kl} h_{ikl}$$

so that H is related to $\overrightarrow{H}$ by

$$H(V) = \omega(\overrightarrow{H}, V).$$

We summarize the relevant equations for the second fundamental form in the following Lemma (compare with [7] and [8].)

Lemma 2.1. *The second fundamental form $h = h_{ijk} dx^i \otimes dx^j \otimes dx^k$ of a Lagrangian immersion into $\mathbb{R}^{2n}$ satisfies*

a) $h_{ijk} = h_{jik} = h_{jki}$,
b) $R_{ijkl} = h_{ik}{}^n h_{njl} - h_{il}{}^n h_{njk}$,
c) $\nabla_i h_{jkl} - \nabla_j h_{ikl} = 0$,
d) $\nabla_i H_j = \nabla_j H_i$.

Here, R_{ijkl} is the curvature symbol of F^*g w.r.t. coordinates $(x^i)_{i=1,\dots,n}$ and we have set

$$h_{il}{}^n := h_{ilm} g^{mn}.$$

Remark 2.2. In the sequel we will often rise and lower indices w.r.t. the metric tensor g_{ij}. From Lemma 2.1 d) (the traced Codazzi equation) it follows that the mean curvature form H is closed.

Let us introduce the following symmetric tensors

$$a := a_{ij}\, dx^i \otimes dx^j$$
$$b := b_{ij}\, dx^i \otimes dx^j$$

with

$$a_{ij} := g^{kl} H_k h_{lij} = H^l h_{lij}$$
$$b_{ij} := h_i{}^{mn} h_{mnj}\,.$$

It follows that

$$|A|^2 = g_{\alpha\beta} g^{ik} g^{jl} A^\alpha_{ij} A^\beta_{kl} = h^{ijk} h_{ijk} = g^{ij} b_{ij} \tag{2.1}$$

and

$$g^{ij} a_{ij} = |H|^2. \tag{2.2}$$

In addition, the Ricci curvature is given by

$$R_{ij} = a_{ij} - b_{ij}\,. \tag{2.3}$$

We recall the evolution equations under the mean curvature flow for crucial geometric objects on L (compare with Lemma 5 in [8].)

Lemma 2.3. *If $F : L \times [0, T) \to \mathbb{R}^{2n}$ is a smooth family of Lagrangian immersions that evolves according to (1.1), then*

a) $\dfrac{d}{dt} g_{ij} = -2a_{ij}$

b) $\dfrac{d}{dt} d\mu = -|H|^2 d\mu$

c) $\dfrac{d}{dt} H_i = \nabla_i d^\dagger H = \Delta H_i - R_i{}^l H_l$

d) $\dfrac{d}{dt} h_{ijk} = \Delta h_{ijk} - R_i{}^l h_{ljk} - R_j{}^l h_{lki} - R_k{}^l h_{lij} - 2h_{in}{}^m h_{jm}{}^l h_{kl}{}^n$

e) $\dfrac{d}{dt} |H|^2 = \Delta |H|^2 - 2|\nabla H|^2 + 2|a_{ij}|^2$

f) $\dfrac{d}{dt} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|b_{ij}|^2 + 2|R_{ijkl}|^2$

Here, $d\mu$ denotes the volume element on L w.r.t. $F^* g$.

For a proof of these equations we refer to [7] and [8].
In addition we will use the evolution equation of $d^\dagger H$

Lemma 2.4.
$$\frac{d}{dt}d^\dagger H = \Delta d^\dagger H + 4a^{ij}\nabla_i H_j.$$

Proof. This is a direct consequence of Lemma 2.3 a) and c). □

Now let $S \in \Sigma(M)$ be one of the tensors described above. We set

$$S_{ij} := S_{\alpha\beta}F_i^\alpha F_j^\beta$$

and

$$\dot{F}^\alpha := \frac{d}{dt}F^\alpha.$$

We need an evolution equation for $F^*S = S_{ij}\,dx^i \otimes dx^j$ and compute

$$\begin{aligned}\frac{d}{dt}S_{ij} &= \frac{d}{dt}(S_{\alpha\beta}F_i^\alpha F_j^\beta)\\ &= D_\gamma S_{\alpha\beta}\dot{F}^\gamma F_i^\alpha F_j^\beta + S_{\alpha\beta}\nabla_i\dot{F}^\alpha F_j^\beta + S_{\alpha\beta}F_i^\alpha\nabla_j\dot{F}^\beta\\ &= -S_{\alpha\beta}\big(\nabla_i(H^l\nu_l^\alpha)F_j^\beta + F_i^\alpha\nabla_j(H^l\nu_l^\beta)\big)\\ &= -\nabla_i H^l S_{\alpha\beta}\nu_l^\alpha F_j^\beta - \nabla_j H^l S_{\alpha\beta}F_i^\alpha\nu_l^\beta\\ &\quad -H^l S_{\alpha\beta}(\nabla_i\nu_l^\alpha F_j^\beta + \nabla_j\nu_l^\beta F_i^\alpha)\,.\end{aligned}$$

From $\nabla_i F_l^\alpha = A_{il}^\alpha = -h_{il}^k\nu_k^\alpha$, $DJ = 0$ and $J^2 = -\mathrm{Id}$ we obtain

$$\nabla_i\nu_l^\alpha = h_{il}^k F_k^\alpha\,, \tag{2.4}$$

so that

$$\begin{aligned}\frac{d}{dt}S_{ij} &= -\nabla_i H^l S_{\alpha\beta}\nu_l^\alpha F_j^\beta - \nabla_j H^l S_{\alpha\beta}F_i^\alpha\nu_l^\beta\\ &\quad -H^l S_{\alpha\beta}(h_{il}^k F_k^\alpha F_j^\beta + h_{jl}^k F_k^\beta F_i^\alpha)\\ &= -\nabla_i H^l S_{\alpha\beta}\nu_l^\alpha F_j^\beta - \nabla_j H^l S_{\alpha\beta}F_i^\alpha\nu_l^\beta\\ &\quad -a_i^k S_{kj} - a_j^k S_{ki}.\end{aligned} \tag{2.5}$$

Next we compute ΔS_{ij}:

$$\begin{aligned}\nabla_k S_{ij} &= \nabla_k(S_{\alpha\beta}F_i^\alpha F_j^\beta)\\ &= D_\gamma S_{\alpha\beta}F_k^\gamma F_i^\alpha F_j^\beta + S_{\alpha\beta}(\nabla_k F_i^\alpha F_j^\beta + F_i^\alpha\nabla_k F_j^\beta)\\ &= -h_{ki}^l S_{\alpha\beta}\nu_l^\alpha F_j^\beta - h_{kj}^l S_{\alpha\beta}F_i^\alpha\nu_l^\beta\end{aligned}$$

and then

$$\begin{aligned}\Delta S_{ij} &= -\nabla^k h_{ki}^l S_{\alpha\beta}\nu_l^\alpha F_j^\beta - \nabla^k h_{kj}^l S_{\alpha\beta}F_i^\alpha\nu_l^\beta\\ &\quad -h_i^{lk}S_{\alpha\beta}(h_{kl}^m F_m^\alpha F_j^\beta - h_{kj}^m\nu_l^\alpha\nu_m^\beta)\\ &\quad -h_j^{lk}S_{\alpha\beta}(-h_{ki}^m\nu_m^\alpha\nu_l^\beta + F_i^\alpha h_{kl}^m F_m^\beta).\end{aligned} \tag{2.6}$$

From the Codazzi equation (Lemma 2.1 c)) we deduce

$$\nabla^k h^l_{ki} = \nabla^l H_i = \nabla_i H^l$$

and (1.4) implies

$$S_{\alpha\beta}\nu^\alpha_m \nu^\beta_l = -S_{ml}\,,$$

so that

$$\begin{aligned}\Delta S_{ij} = &-\nabla_i H^l S_{\alpha\beta}\nu^\alpha_l F^\beta_j - \nabla_j H^l S_{\alpha\beta} F^\alpha_i \nu^\beta_l\\ &-b^k_i S_{kj} - b^k_j S_{ki}\\ &-2h^{lk}_i h^m_{kj} S_{lm}.\end{aligned}$$

Inserting this into the above expression for $\frac{d}{dt}S_{ij}$ we finally get

Lemma 2.5. *$F^*S = S_{ij}dx^i \otimes dx^j$ satisfies the evolution equation*

$$\frac{d}{dt}S_{ij} = \Delta S_{ij} - R^l_i S_{lj} - R^l_j S_{li} + 2h^{km}_i h^n_{jk} S_{mn}.$$

We define the function

$$s := g^{ij} S_{ij}.$$

Then the evolution equations for g^{ij} and S_{ij} immediately imply

Lemma 2.6. *The quantity s satisfies*

$$\frac{d}{dt}s = \Delta s + 4b^{ij} S_{ij}.$$

2.2. The underlying parabolic equation of Monge-Ampère type

We will now see that in particular any Lagrangian $L \subset \mathbb{R}^{2n}$ satisfying condition (1.6) must be a graph over an n-plane sitting in $\mathbb{R}^{2n}$.

To see this let $\sigma_S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the endomorphism associated to S, i.e. $\langle \sigma_S(V), W\rangle = S(V,W)$. σ_S must be independent of $y \in \mathbb{R}^{2n}$ because we assumed that S is parallel. If λ is an eigenvalue of σ_S, then $-\lambda$ must also be an eigenvalue because $\sigma_S J = -J\sigma_S$ and J maps the eigenspace belonging to λ to that belonging to $-\lambda$. Since $F^*S > 0$ and L is n-dimensional, we conclude that there exist exactly n positive eigenvalues $\sigma_1, \ldots, \sigma_n$ and n negative eigenvalues $-\sigma_1, \ldots, -\sigma_n$. Consequently we can split $\mathbb{R}^{2n}$ into the direct sum

$$\mathbb{R}^{2n} = X \oplus Y,$$

where X is the linear hull of all eigenvectors e belonging to positive eigenvalues and $Y := JX$ is the linear hull of eigenvectors f belonging to negative eigenvalues. In addition $F^*S > 0$ implies that L can be written as a graph over X, i.e. there exist n functions $u_1, \ldots, u_n : \mathbb{R}^n \to \mathbb{R}$ and an orthonormal frame $e_1, \ldots, e_n$ spanning X such that

$$\begin{aligned}F &: \mathbb{R}^n \to \mathbb{R}^{2n}\\ F(x) &:= x^i e_i + \delta^{ij} u_i f_j,\end{aligned}$$

gives an immersion of L (with $f_j := Je_j$.)

The tangent vectors $F_i := \frac{\partial F}{\partial x^i}$ are

$$F_i = e_i + \delta^{kl} u_{ki} f_l \tag{2.7}$$

Here, L is a compact Lagrangian in a flat manifold M and we see that the universal cover $\tilde{L}$ of L must be $\mathbb{R}^n$ immersed as a Lagrangian into $\mathbb{R}^{2n}$, the universal cover of M.

From the Lagrangian condition we further deduce that there exists a function $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$u_k = \frac{\partial u}{\partial x^k}$$

holds for any $k = 1, \ldots, n$ and on all of $\mathbb{R}^n$.

As in [7] we may then transform the mean curvature flow into a parabolic equation for u, because

$$\frac{d}{dt} F = \overrightarrow{H} = -H^m \nu_m = -H^m (f_m - \delta^{lp} u_{lm} e_p)$$

implies the equations

$$\frac{dx^i}{dt} = H^m \delta^{li} u_{lm} \tag{2.8}$$

and

$$\delta^{ip} \frac{du_p}{dt} = -H^i \tag{2.9}$$

so that in view of

$$\frac{du_p}{dt} = \frac{\partial u_p}{\partial t} + u_{pl} \frac{dx^l}{dt}$$

we obtain the equation

$$P[u] := -\alpha - \frac{\partial u}{\partial t} = 0 \tag{2.10}$$

with $d\alpha = H$, where the Lagrangian angle α of L can locally be written as either

$$\alpha = -\text{arctan}\left(\frac{a}{b}\right),$$

or as

$$\alpha = \text{arctan}\left(\frac{b}{a}\right)$$

depending on whether

$$a := \text{Im}\left(\det_{\mathbb{C}}(\delta_{kl} + iu_{kl})\right)$$

or

$$b := \text{Re}\left(\det_{\mathbb{C}}(\delta_{kl} + iu_{kl})\right)$$

is nonzero. Either a or b must be nonzero as long as L is a graph because for the induced metric

$$g_{ij} = \delta_{ij} + \delta^{kl} u_{ik} u_{jl}$$

we compute

$$\det(g_{ij}) = a^2 + b^2.$$

If all induced metrics $F_t^* g$ of the flow would be uniformly equivalent one would obtain uniform C^2-estimates w.r.t. the space variables x of (2.10) and it would also imply that the solution of (2.10) remains a graph.

A simple calculation shows that

$$\frac{\partial P[u]}{\partial u_{ij}} = g^{ij}$$

which means that $P[u]$ is always parabolic. We will also need the second derivatives of $P[u]$ and compute

$$\begin{aligned}
\frac{\partial^2 P[u]}{\partial u_{ij} \partial u_{kl}} &= \frac{\partial g^{ij}}{\partial u_{kl}} \\
&= -g^{is} g^{jt} \frac{g_{st}}{\partial u_{kl}} \\
&= -g^{is} g^{jt} \big(\delta^{pq} u_{ps} \delta_q{}^k \delta_t{}^l + \delta^{pq} u_{qt} \delta_p{}^k \delta_s{}^l \big) \\
&= -g^{is} g^{jl} \delta^{pk} u_{ps} - g^{il} g^{jt} \delta^{qk} u_{qt} \\
&= -\big(g^{is} g^{jl} + g^{il} g^{js} \big) \delta^{pk} u_{ps}.
\end{aligned}$$

If at a fixed point p we consider a basis $b_1, \dots, b_n$ of eigenvectors for u_{ij} such that u_{ij} becomes diagonal at p with $u_{ij} = \text{diag}(\lambda_1, \dots, \lambda_n)$, then for any symmetric tensor v_{kl} we obtain

$$\frac{\partial^2 P[u]}{\partial u_{ij} \partial u_{kl}} v_{ij} v_{kl} = -2 \sum_{i,j} \frac{\lambda_i}{(1 + \lambda_i^2)(1 + \lambda_j^2)} (v_{ij})^2.$$

Thus we have shown

Lemma 2.7. *The operator $P[u] = -\alpha - \frac{\partial u}{\partial t}$ is concave, if u is strictly convex.*

This lemma will become important later when we show $C^{2,\alpha}$-regularity of (2.10).

We take a closer look at conditions (1.6) and (1.7). One has

$$S_{ij} = S(F_i, F_j) = \langle \sigma(e_i), e_j \rangle + \langle \sigma(f_l), f_q \rangle \delta^{kl} u_{ki} \delta^{pq} u_{qj}$$

so that

$$S_{ij} = \sigma_i \delta_{ij} - \sum_{k=1}^{n} \sigma_k u_{ki} u_{kj}, \tag{2.11}$$

where we do not sum over i in $\sigma_i \delta_{ij}$.

In addition

$$
\begin{aligned}
\bar{S}_{ij} &= \langle J\sigma(F_i), F_j\rangle = -\langle \sigma(JF_i), F_j\rangle \\
&= -\langle \sigma(f_i - \delta^{kl}u_{ki}e_l), e_j + \delta^{pq}u_{pj}f_q\rangle) \\
&= \sigma_i u_{ij} + \sigma_j u_{ji}, \qquad (2.12)
\end{aligned}
$$

where again there is no sum over i and j on the RHS.

Lemma 2.8. *u is strictly convex if $F^*\bar{S}$ is positive definite.*

Proof. Assume there exists a nonpositive eigenvalue λ of u_{ij} and let V be an eigenvector so that

$$\sum_{j=1}^n u_{ij}V^j = \lambda \sum_{j=1}^n \delta_{ij}V^j.$$

Then from (2.12)

$$
\begin{aligned}
F^*\bar{S}(V,V) &= \sum_{i,j=1}^n (\sigma_i u_{ij} + \sigma_j u_{ji})V^iV^j \\
&= 2\lambda \sum_{i=1}^n \big(\sigma_i V^i \sum_{j=1}^n \delta_{ij}V^j\big) \\
&= 2\lambda \sum_{i=1}^n \sigma_i (V^i)^2 \le 0 \quad \text{because } \sigma_i > 0
\end{aligned}
$$

which is a contradiction. □

3. Proof of the main theorem

In this section we will prove our main theorem. For this purpose we need the next lemma.

Lemma 3.1. *Assume there exists an $\varepsilon > 0$ and $S \in \Sigma(M)$ such that*

$$M_{ij} := S_{ij} - \varepsilon g_{ij} > 0 \qquad (3.1)$$

holds on a compact L at $t = 0$. Then this is also true for $t \in [0,T)$.

Proof. First we need an evolution equation for M_{ij}. From Lemma 2.5 and Lemma 2.3 a) we obtain

$$
\begin{aligned}
\frac{d}{dt}M_{ij} &= \Delta M_{ij} - R_i^l S_{lj} - R_j^l S_{li} + 2h_i^{km}h_{jk}^n S_{mn} + 2\varepsilon a_{ij} \\
&= \Delta M_{ij} - R_i^l M_{lj} - R_j^l M_{li} + 2\varepsilon b_{ij} + 2h_i^{km}h_{jk}^n S_{mn}.
\end{aligned}
$$

We use the maximum principle for tensors proven in [3] (see also [4] for a better proof.) To prove that (3.1) is preserved we must only show that

$$N_{ij}V^iV^j \ge 0$$

for any null eigenvector V of M_{ij} that occurs for the first time, where

$$N_{ij} := -R_i^l M_{lj} - R_j^l M_{li} + 2\varepsilon b_{ij} + 2h_i^{km} h_{jk}^n S_{mn}\,.$$

If for the first time there exists a null eigenvector V of M_{ij}, then

$$M_{ij}V^i = 0$$

and

$$M_{ij}W^iW^j \geq 0 \qquad \forall\ W \in TL$$

But then

$$N_{ij}V^iV^j = 2\varepsilon b_{ij}V^iV^j + 2h_i^{km}h_{jk}^n S_{mn}V^iV^j.$$

The quadratic tensors $b_{ij} = h_i^{kl}h_{jkl}$ and $h_i^{km}V^ih_{jk}^nV^j$ are positive semidefinite and since $M_{ij} \geq 0$ implies $S_{mn} \geq \varepsilon g_{mn}$ we deduce

$$\begin{aligned} N_{ij}V^iV^j &\geq 2\varepsilon b_{ij}V^iV^j + 2\varepsilon h_i^{km}h_{jk}^n g_{mn}V^iV^j \\ &= 4\varepsilon b_{ij}V^iV^j \geq 0. \end{aligned}$$

□

Lemma 3.2. *If there exists a positive constant c, such that*

$$S_{ij} - cH_iH_j > 0 \tag{3.2}$$

at $t = 0$*, then this remains true* $\forall\ t \in [0, T)$.

Proof. Again we use the maximum principle for tensors. Here we set $M_{ij} := S_{ij} - cH_iH_j$. Then the evolution equations for S_{ij} and H_i imply

$$\begin{aligned} \frac{d}{dt}M_{ij} &= \Delta S_{ij} - R_i^l S_{lj} - R_j^l S_{li} + 2h_i^{km}h_{jk}^n S_{mn} \\ &\quad -cH_j(\Delta H_i - R_i^l H_l) - cH_i(\Delta H_j - R_j^l H_l) \\ &= \Delta M_{ij} + 2c\nabla^k H_i \nabla_k H_j - R_i^l M_{lj} - R_j^l M_{li} \\ &\quad +2h_i^{km}h_{jk}^n S_{mn} \end{aligned}$$

and here we set

$$N_{ij} = 2c\nabla^k H_i \nabla_k H_j - R_i^l M_{lj} - R_j^l M_{li} + 2h_i^{km}h_{jk}^n S_{mn}.$$

As in Lemma 3.1 we choose the first time where a null eigenvector V of M_{ij} occurs so that

$$\begin{aligned} &M_{ij}V^i = 0 \\ &M_{ij}W^iW^j \geq 0\ \forall\ W \in TL\,. \end{aligned}$$

Then

$$\begin{aligned} N_{ij}V^iV^j &= 2c|\nabla^k H_iV^i|^2 + 2h_i^{km}V^ih_{jk}^nV^jS_{mn} \\ &\geq 2ch_i^{km}V^ih_{jk}^nV^jH_mH_n \\ &= 2ca_i^kV^ia_{jk}V^j \geq 0 \end{aligned}$$

because $a_i^kV^ia_{jk}V^j = |\tilde{V}|^2$ with $\tilde{V}^k = a_i^kV^i$. □

Lemma 3.3. *With the assumptions made in Theorem 1.3 there exists a constant* c_2 *such that*

$$|d^\dagger H| \leq c_2 \tag{3.3}$$

holds for $t \in [0, T)$.

Proof. We define the function

$$f := |H|^2 + d^\dagger H - cs$$

with a positive constant c to be determined. Then Lemma 2.3 e), Lemma 2.4 and Lemma 2.6 imply

$$\frac{d}{dt} f = \Delta f - 2|\nabla H|^2 + 2|a_{ij}|^2 + 4a^{ij} \nabla_i H_j - 4cb^{ij} S_{ij}$$

and with Cauchy-Schwarz

$$\frac{d}{dt} f \leq \Delta f + 4|a_{ij}|^2 - 4cb^{ij} S_{ij}.$$

Since $|a_{ij}|^2 = b^{ij} H_i H_j$, we obtain

$$\frac{d}{dt} f \leq \Delta f + 4b^{ij}(H_i H_j - cS_{ij})$$

and from $b_{ij} \geq 0$, $S_{ij} > 0$ at $t = 0$ and Lemma 3.2 it follows

$$f \leq B$$

for some constant $B > 0$ and all $t \in [0, T)$. So we found a uniform upper bound for $d^\dagger H$, if we can prove that s is bounded. But since $S = S_{\alpha\beta} dy^\alpha \otimes dy^\beta$ is parallel, we see that $S_{\alpha\beta}$ must be constant in cartesian coordinates for the flat manifold (M, g, J, ω). Thus there exists a positive constant σ such that

$$S_{\alpha\beta} - \sigma g_{\alpha\beta} < 0$$

as a tensor. This implies

$$S_{ij} - \sigma g_{ij} = (S_{\alpha\beta} - \sigma g_{\alpha\beta}) F_i^\alpha F_j^\beta < 0$$

and then also

$$g^{ij} S_{ij} < \sigma g^{ij} g_{ij} = \sigma n. \tag{3.4}$$

In the same way we can proceed with the function $f := |H|^2 - d^\dagger H - cs$ and obtain the lower bound for $d^\dagger H$. □

We can now prove Theorem 1.3.

Proof of Theorem 1.3. Since there exists a positive constant ε with $S_{ij} - \varepsilon g_{ij} > 0$ at $t = 0$, we can find a small positive constant c so that the symmetric tensor

$$S_{ij} - cH_iH_j$$

is positive definite at $t = 0$. Then Lemma 3.2 implies that

$$|H|^2 \leq \frac{1}{c} g^{ij} S_{ij} = \frac{s}{c} \qquad \forall\, t \in [0, T).$$

Since s is bounded (compare with the proof of Lemma 3.3) this proves the uniform bound of $|H|^2$ and by Lemma 3.1 that all induced metrics stay uniformly equivalent. In addition, Lemma 3.3 is just the uniform bound of $|d^\dagger H|$. It remains to prove longtime existence and convergence in case $n = 2$ or under the extra condition (1.7). For this we recall the underlying Monge-Ampère equation

$$P[u] = -\alpha - \frac{\partial u}{\partial t} = 0$$

from Sect. 2 which is the nonparametric version of the Lagrangian mean curvature flow. Since the Lagrangian angle α depends only on second order derivatives of u and all induced metrics $g_{ij} = \delta_{ij} + \delta^{kl} u_{ik} u_{jl}$ are uniformly equivalent, we deduce uniform C^2-estimates in space directions for (2.10). In addition we have

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= -\frac{\partial \alpha}{\partial t} \\
&= -\frac{d\alpha}{dt} + \frac{d\alpha}{dx^i}\frac{dx^i}{dt} \\
&= -\frac{d\alpha}{dt} + H_i H^m \delta^{li} u_{lm} \quad \text{with (1.2) and (2.8)} \\
&= -d^\dagger H + H_i H^m \delta^{li} u_{lm} \quad \text{with (1.2) and Lemma 2.3 c).}
\end{aligned}$$

Since we already proved uniform bounds for $|H|^2$, $|d^\dagger H|$ and $|D^2 u|$, we get uniform C^2-estimates in time as well. To obtain longtime existence we need uniform $C^{1,\alpha}$-estimates in time and uniform $C^{2,\alpha}$-estimates in space for some $\alpha > 0$. Hence it remains to prove $C^{2,\alpha}$-bounds in space. So far we did not exploit condition (1.7). From Lemma 2.8, Lemma 2.7 and Lemma 3.1 applied to $\bar{S}$ we conclude that the operator $P[u]$ is concave for all t and the results in [6] imply uniform $C^{2,\alpha}$-estimates in x for some $\alpha > 0$. Standard Schauder estimates then give C^∞-estimates both in space and time. In particular the full norm of the second fundamental form is uniformly bounded and we may apply Proposition (1.1) to get convergence. The compactness of L implies that the limit manifold must be flat. In case $n = 2$ we can drop condition (1.7) because if we freeze time, we may regard F as a solution of the elliptic system

$$\Delta F = \frac{d}{dt} F = \overrightarrow{H}$$

with bounded RHS and the uniform C^1-estimates for F and the regularity theory for equations in two variables (e.g. see [2] Sect. 12) give uniform $C^{1,\alpha}$-estimates for F which amounts to uniform $C^{2,\alpha}$-bounds for u since $F(x) = \big(x, Du(x)\big)$. □

We want to give a more direct proof of Theorem 1.3 in case $n = 2$. For this we establish a uniform bound for $|A|^2$.

Let $p = p(s)$ be a function depending on s only and that has to be determined later. We set

$$f := p|A|^2$$

and compute the evolution equation for f

$$\begin{aligned}\frac{d}{dt} f &= p(\Delta|A|^2 - 2|\nabla A|^2 + 2|b_{ij}|^2 + 2|R_{ijkl}|^2)\\ &\quad +p'|A|^2(\Delta s + 4b^{ij}S_{ij})\,,\end{aligned}$$

where we used Lemma 2.3 e), Lemma 2.6 and we have set $p' = \frac{\partial p}{\partial s}$. Then

$$\Delta f = (p'\Delta s + p''|\nabla s|^2)|A|^2 + 2p'\langle\nabla s, \nabla|A|^2\rangle + p\Delta|A|^2$$

gives

$$\begin{aligned}\frac{d}{dt} f &= \Delta f - 2p|\nabla A|^2 - p''|\nabla s|^2|A|^2 - 2p'\langle\nabla s, \nabla|A|^2\rangle\\ &\quad +2p(|b_{ij}|^2 + |R_{ijkl}|^2 + \frac{2p'}{p}|A|^2 b^{ij}S_{ij}).\end{aligned}$$

To proceed we observe that for $n = \dim(L) = 2$ we have

$$\begin{aligned}R_{ijkl} &= \frac{R}{2}(g_{ik}g_{jl} - g_{il}g_{jk}),\\ R_{ij} &= \frac{R}{2}g_{ij} = a_{ij} - b_{ij},\\ R &= |H|^2 - |A|^2,\\ |R_{ijkl}|^2 &= R^2,\end{aligned}$$

so that

$$\begin{aligned}|b_{ij}|^2 &= -R_{ij}(b^{ij} + a^{ij}) + |a_{ij}|^2\\ &= -\frac{R}{2}(|A|^2 + |H|^2) + |a_{ij}|^2\\ &\le \frac{|A|^4}{2} + c_1(|A|^2|H|^2 + |H|^4)\end{aligned}$$

for some positive constant c_1.

In the same way we have

$$\begin{aligned}b^{ij}S_{ij} &= \frac{|A|^2 - |H|^2}{2}s + a^{ij}S_{ij}\\ &\ge \frac{|A|^2}{2}s - c_2(|H|^2 + |H||A|)s\,,\end{aligned}$$

if we assume that there are positive constants c_3, c_4 such that

$$c_4 s g_{ij} \ge S_{ij} \ge c_3 s g_{ij} \tag{3.5}$$

(we note here, that by (3.4) and Lemma 3.1 (3.5) is valid $\forall\ t \in [0, T)$, if we assume that $S_{ij} > \varepsilon g_{ij}$ for some $\varepsilon > 0$ at $t = 0$.)

Lemma 3.4. *Assume $S_{ij} > \varepsilon g_{ij}$ for some $\varepsilon > 0$ holds on L at $t = 0$ and that $\dim(L) = 2$. Then there exists a smooth vector field V and a positive constant c such that $f := \frac{|A|^2}{s^2}$ satisfies*

$$\frac{d}{dt}f \leq \Delta f + \langle V, \nabla f\rangle + \frac{c}{s^2} - \frac{1}{2}f|A|^2.$$

Proof. By Lemma 3.1 and (3.4) we know that (3.5) is valid $\forall\ t \in [0, T)$. Let us choose $p := \frac{1}{s^2}$ in the above expression for $f = p|A|^2$.

Since

$$|h_{ijk}\nabla_l h_{mns} - h_{mns}\nabla_l h_{ijk}| \geq 0$$

we get

$$2|A|^2|\nabla A|^2 \geq \frac{1}{2}|\nabla|A|^2|^2$$

and since in addition

$$\nabla f = p'|A|^2\nabla s + p\nabla|A|^2,$$

there exists some vector field V so that for $n = 2$

$$\begin{aligned}\frac{d}{dt}f &\leq \Delta f + \langle V, \nabla f\rangle + \frac{3}{2}\frac{(p')^2}{p}|A|^2|\nabla s|^2 - p''|A|^2|\nabla s|^2\\ &\quad +2p(|b_{ij}|^2 + |R_{ijkl}|^2 + \frac{2p'}{p}|A|^2 b^{ij}S_{ij})\\ &= \Delta f + \langle V, \nabla f\rangle + 2p(|b_{ij}|^2 + |R_{ijkl}|^2 + \frac{2p'}{p}|A|^2 b^{ij}S_{ij})\\ &\leq \Delta f + \langle V, \nabla f\rangle + 2p\left(\frac{3}{2}|A|^4 + c_5(|A|^2|H|^2 + |H|^4) + 2\frac{p'}{p}|A|^2 b^{ij}S_{ij}\right)\end{aligned}$$

so that

$$\begin{aligned}\frac{d}{dt}f \leq \Delta f + \langle V, \nabla f\rangle + 2p\Big\{&\frac{3}{2}|A|^4 + c_5(|A|^2|H|^2 + |H|^4)\\ &-2|A|^4 + 4c_2(|H|^2 + |H||A|)|A|^2\Big\},\end{aligned} \tag{3.6}$$

where we assumed that (3.5) is valid. Now by Lemma 3.1 and Lemma 3.2 we know that (3.5) is valid and $|H|$ uniformly bounded, if we assume that $S_{ij} > \varepsilon g_{ij}$ for some $\varepsilon > 0$ at $t = 0$. We apply Schwarz inequality to (3.6) and are done. □

Lemma 3.5. *Assume $n = 2$ and that there exists an $\varepsilon > 0$ such that $S_{ij} > \varepsilon g_{ij}$ at $t = 0$. Then the quantity*

$$\frac{|A|^2}{s^2}$$

is uniformly bounded on $[0, T)$.

Proof. From Lemma 3.4 we know that

$$\frac{d}{dt}f \le \Delta f + \langle V, \nabla f\rangle + \frac{c}{s^2} - \frac{f}{2}|A|^2$$

But since $s = g^{ij}S_{ij} > \varepsilon g^{ij}g_{ij} = \varepsilon n$ we obtain also

$$\frac{d}{dt}f \le \Delta f + \langle V, \nabla f\rangle + \frac{c}{n^2\varepsilon^2} - \frac{n^2\varepsilon^2}{2}f^2$$

and by the maximum principle f must be uniformly bounded from above. □

We can now give a uniform upper bound for $|A|^2$: From Lemma 3.5 we get

$$|A|^2 = fs^2 \le c_6 s^2$$

for some constant c_6 and because s is also bounded from above by (3.4) we get a uniform bound for $|A|^2$. In view of Proposition 1.1 this gives the proof of Theorem 1.3 in case $n = 2$. □

4. Appendix

Here we will give a proof of Proposition 1.2. Therefore let $S \in \Sigma(M)$ be a tensor satisfying (1.3)–(1.5). To any such bilinear form we can associate an endomorphism

$$\sigma_S : TM \to TM$$

by setting $\sigma_S\, V := g^{\alpha\beta}S_{\alpha\gamma}V^\gamma \frac{\partial}{\partial y^\beta}$. Therefore

$$S(V, W) = g(\sigma_S V, W)$$

and (1.3)–(1.5) imply

(4.1) $$g(\sigma_S V, W) = g(\sigma_S W, V) \qquad \text{(Symmetry)},$$

(4.2) $$\sigma_S \circ J = -J \circ \sigma_S \qquad \text{(Anti-compatibility)},$$

(4.3) $$D\sigma_S = 0 \qquad \text{(Parallelity)}.$$

Conversely, if σ is an endomorphism satisfying (4.1)–(4.3), then $S := g(\sigma\cdot, \cdot)$ defines an element in $\Sigma(M)$.

Remark 4.1. Moreover, if σ satisfies (4.1)–(4.3), then $J\sigma$ satisfies these relations too. Conditions (4.1)–(4.3) are very similar to the conditions for the existence of a hyper-Kähler structure on M, i.e. another complex structure K that satisfies

(4.4) $$g(KV, W) = -g(KW, V),$$

(4.5) $$K \circ J = -J \circ K,$$

(4.6) $$DK = 0,$$

(4.7) $$K^2 = -\text{Id}.$$

It is well known that hyper-Kähler metrics are Ricci-flat and that the existence of a hyper-Kähler manifold gives restrictions on the holonomy of M. Here, we do not require anything for σ^2 and the signs in (4.1) resp. (4.4) differ.

Lemma 4.2. *Let (M, g) be a Riemannian manifold and assume that σ satisfies (4.1) and (4.3). If R^M denotes the Riemannian curvature tensor on M, then one has*

$$R^M(\sigma V, W, X, Y) = -R^M(\sigma W, V, X, Y), \quad \forall\, V, W, X, Y. \tag{4.8}$$

$$R^M(\sigma X, W, Y, W) = R^M(\sigma Y, W, X, W), \quad \forall\, X, Y, W. \tag{4.9}$$

Proof. The first equation follows from (4.3) and (4.1) because

$$0 = D_V D_W \sigma - D_W D_V \sigma - D_{[V,W]}\sigma = R^M(V, W)\sigma.$$

For the second equation we use (4.8) and the first Bianchi identity

$$\begin{aligned} R^M(\sigma X, W, Y, W) &= -R^M(\sigma W, X, Y, W) \\ &= R^M(\sigma W, Y, W, X) + R^M(\sigma W, W, X, Y) \\ &= R^M(\sigma W, Y, W, X) \\ &= -R^M(\sigma Y, W, W, X) \\ &= R^M(\sigma Y, W, X, W) \end{aligned}$$

□

Lemma 4.3. *Let (M, J, g) be a Kähler manifold and assume that σ satisfies (4.1)–(4.3). Then*

$$R^M(\sigma X, W, Y, W) = \frac{1}{2} R^M(\sigma JX, Y, W, JW), \quad \forall\, X, Y, W. \tag{4.10}$$

Proof. In a first step we compute

$$\begin{aligned} R^M(\sigma X, W, Y, W) &= R^M(J\sigma JX, W, Y, W) \qquad \text{from } J^2 = -\mathrm{Id} \text{ and (4.2)} \\ &= -R^M(\sigma JX, JW, Y, W) \\ &= R^M(\sigma JX, Y, W, JW) + R^M(\sigma JX, W, JW, Y) \\ &= R^M(\sigma JX, Y, W, JW) + R^M(\sigma JX, W, JY, W). \end{aligned}$$

Now, by remark 4.1 we can apply (4.9) to $\tilde{\sigma} := \sigma J$ and obtain

$$\begin{aligned} R^M(\sigma JX, W, JY, W) &= R^M(\sigma J^2 Y, W, X, W) \\ &= -R^M(\sigma Y, W, X, W) \\ &= -R^M(\sigma X, W, Y, W), \end{aligned}$$

so that

$$\begin{aligned} R^M(\sigma X, W, Y, W) &= R^M(\sigma JX, Y, W, JW) + R^M(\sigma JX, W, JY, W) \\ &= R^M(\sigma JX, Y, W, JW) - R^M(\sigma X, W, Y, W). \end{aligned}$$

□

Corollary 4.4. *Under the assumptions made in Lemma 4.3 we also have*

$$R^M(\sigma X, JW, Y, JW) = R^M(\sigma X, W, Y, W), \quad \forall\, X, Y, W. \tag{4.11}$$

Proof. This follows from (4.10) if we replace W by JW. □

Lemma 4.5. *Let (M, J, g) be a Kähler manifold and assume that σ satisfies (4.1)–(4.3). Then*

$$R^M(X, Y, V, W) = 0, \quad \forall\, X, Y, V, W \notin ker(\sigma).$$

Proof. Let λ, μ be two different eigenvalues of σ and assume that $V \in \mathrm{Eig}(\lambda)$, $W \in \mathrm{Eig}(\mu)$. We apply (4.8) and obtain

$$\lambda R^M(V, W, X, Y) = \mu R^M(V, W, X, Y)$$

so that

$$R^M(V, W, X, Y) = 0, \quad \forall\, V \in \mathrm{Eig}(\lambda), W \in \mathrm{Eig}(\mu), \forall\, X, Y \in TM. \tag{4.12}$$

With this and the first Bianchi identity we need to show only

$$R^M(V_1, V_2, V_3, V_4) = 0$$

whenever $V_1, \ldots, V_4$ belong to the same eigenspace $\mathrm{Eig}(\lambda)$ of a nonzero eigenvalue λ. We let $X := V_1, W := V_2, Y := V_3$ and use (4.11) to obtain

$$R^M(V_1, JV_2, V_3, JV_2) = R^M(V_1, V_2, V_3, V_2).$$

The LHS vanishes because of (4.12) and $JV_2 \in \mathrm{Eig}(-\lambda)$. So

$$R^M(V_1, V_2, V_3, V_2) = 0, \quad \forall\, V_1, V_2, V_3 \in \mathrm{Eig}(\lambda).$$

But then

$$\begin{aligned} 0 &= R^M(V_1, V_2 + V_4, V_3, V_2 + V_4) \\ &= R^M(V_1, V_2, V_3, V_4) + R^M(V_1, V_4, V_3, V_2) + 0 + 0 \end{aligned}$$

gives

$$\begin{aligned} R^M(V_1, V_2, V_3, V_4) &= -R^M(V_1, V_4, V_3, V_2) \\ &= R^M(V_1, V_3, V_2, V_4) + R^M(V_1, V_2, V_4, V_3), \end{aligned}$$

where we used Bianchi's identity in the last step. Hence

$$R^M(V_1, V_2, V_3, V_4) = \frac{1}{2} R^M(V_1, V_3, V_2, V_4), \quad \forall\, V_1, V_2, V_3, V_4 \in \mathrm{Eig}(\lambda).$$

Applying the last identity once again we find

$$R^M(V_1, V_2, V_3, V_4) = \frac{1}{4} R^M(V_1, V_2, V_3, V_4), \quad \forall\, V_1, V_2, V_3, V_4 \in \mathrm{Eig}(\lambda)$$

and consequently $R^M(V_1, V_2, V_3, V_4) = 0$. □

Proof of Proposition 1.2. This is now a direct consequence of Lemma 4.5. □

References

1. Ecker, K., Huisken, G.: Mean curvature evolution of entire graphs. Ann. of Math. **130** (2), 453–471 (1989)
2. Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Grundlehren der mathematischen Wissenschaften **224**, 2nd edn. Springer, 1983
3. Hamilton, R.: Three-manifolds with positive Ricci curvature. J. Differential Geom. **17**, 255–306 (1982)
4. Hamilton, R.: Four-manifolds with positive curvature operator. J. Differential Geom. **24**, 153–179 (1986)
5. Jost, J., Xin, Y.-L.: A Bernstein theorem for special lagrangian graphs. Preprint no. 4/2001, MPI for Math. in the Sciences, Leipzig (2001)
6. Krylov, N.V.: Nonlinear elliptic and parabolic equations of the second order. Mathematics and its applications, Reidel Publishing Company, 1987
7. Smoczyk, K.: Der Lagrangesche mittlere Krümmungsfluss. (The Lagrangian mean curvature flow). (German) Leipzig: Univ. Leipzig (Habil.-Schr.), 102 pp (2000)
8. Smoczyk, K.: Angle theorems for the Lagrangian mean curvature flow. Math. Z. **240**, 849–883 (2002)
9. Strominger, A., Yau, S.-T., Zaslow, E.: Mirror Symmetry is T-duality. Nuclear Phys. **B 479**, 243–259 (1996)